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Symmetrical Hybrid Coupled Fuzzy Fixed-Point Results on Closed Ball in Fuzzy Metric Space with Applications

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Abstract: In this research, we establish some new fixed-point results for a symmetric coupled dominated fuzzy mapping satisfying a new advanced contraction on a closed ball in the setting of complete fuzzy metric spaces. In addition, the new notion of hybrid fuzzy-graph-dominated mappings introduced in fuzzy metric spaces achieves some new advanced fuzzy fixed-point problems. Some new definitions and illustrative examples are given to validate our new findings. Lastly, to demonstrate the originality of our new results, we present an application to the Fredholm-type integral equation.

Keywords: fuzzy fixed point; advanced fuzzy contraction; closed ball; α_*-fuzzy-dominated mappings; graph contractions; Fredholm-type integral equations

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1. Introduction

Fixed-point theory is a famous field of functional analysis with a lot of applications in different fields of both pure and applied mathematics. In the metric fixed point (FP), Banach [1] first proposed the Banach contraction theorem, which has become more important with vast applications. In nonlinear analysis, FP theory is a large and active area of research. It is used to solve differential equations, integral equations, nonlinear and functional analysis, as well as other computer sciences and engineering problems. The notion of the fuzzy set, and some basic operations on the fuzzy set, was introduced by Zaddeh [2]. The fuzzy set has proven quite hopeful and fruitful in modeling human participation in human–ground intellect to achieve innovation in many fields such as data analysis, data mining, image coding explaining, and also for intelligence systems that are new notional systems to assist human-centric frames.

In 1982, Deng [3] introduced fuzzy pseudo-metric spaces and discussed two fuzzy points. Grabiec [4] and George et al. [5] proved Baire's theorem for fuzzy metric spaces (FMSs) along with other well-known metric spaces facts, including a Hausdorff topology on the FMS that Kramosil [6] and Kaleva et al. [7] proposed. Moreover, Beg et al. [8], Manro et al. [9], Qiu et al. [10], and Rehman et al. [11] discussed different FP problems and related applications in FMSs. We can create a triangle inequality that is similar to the ordinary triangle inequality by defining an ordering and an addition in the set of fuzzy numbers. Similarly, Weiss [12] and Butnariu [13] established the concept of fuzzy maps and showed many significant results in the field theory of FPs.

Heilpern [14] proved an important FP theorem for fuzzy maps that is more general than Nadler's set-valued result [15]. Inspired from Heilpern's results, FP theory for fuzzy contraction utilizing the Hausdorff metric spaces has become more important in various directions by many researchers [16–19]. Furthermore, Shamas et al. [20] presented unique

FP problems for various self-contractive mappings in FMSs by utilizing the "triangular property of the FMS". They also provided some examples to back up their conclusions. They also demonstrated an application by solving a specific situation of a second-order Fredholm integral equation (FIE). Recently, Rasham et al. [21] established the existence of fuzzy FP theorems for advanced local contraction in complete multiplicative metric spaces with applications to integral and functional equations in dynamical programming. In this paper, we prove some new symmetrical fuzzy FP theorems satisfying a generalized local contraction for a hybrid pair of fuzzy-dominated mappings in FMSs. Some new FP theorems for a couple of fuzzy-graph-dominated contraction on a closed ball in such spaces. Illustrative examples are provided in detail to validate our obtained findings. Lastly, to show the originality of our main FP theorems, we apply it to prove the existence of a common solution of FIEs. We present the definitions and outcomes that we use in the initiation.

Definition 1 [20]. *Let $\acute{X}$ be a nonempty set. A 3-tuple $(\acute{X}, M_F, *)$ is said to be an FMS, $*$ is known as a continuous t-norm, and M_F is a fuzzy set on $\acute{X} \times \acute{X} \times [0,1]$ satisfying the given conditions:*

(i) $M_F\ (x,y,t) > 0$ and $M_F\ (x,y,t) = 1$ iff $x = y$

(ii) $M_F\ (x,y,t) = M_F\ (x,y,t)$

(iii) $M_F\ (x,y,t) * M_F\ (y,z,s) \leq M_F\ (x,z,t+s)$

(iv) $M_F\ (x,y,t) : (0,\infty) \to (0,1)$ is a continuous t-norm for all $x,\ y,\ z \in \acute{X}$ and $t,s > 0$.

For $x_0 \in \acute{X}$ and $r > 0$, $\overleftarrow{B}_F(x_0,r,t) = \left\{x_1 \in X' : \frac{1}{M_F(x_0,x_1,t)} - 1 \leq r\right\}$ is the closed ball (C-bàl) in the FMS.

Definition 2 [20]. *Let $(\acute{X}, M_F, *)$ be an FMS.*

(i) A sequence $\{x_n\}$ is known as a Cauchy sequence if, for each 0 < ε < 1 and t > 0, there is $x_0 \in \mathbb{N}$ so that $M_F\ (x_m, x_n, t) > 1 - \varepsilon\ \forall\ x_m\ ,\ x_n > x_0$.

(ii) Let $(\acute{X}, M_F, *)$ be an FMS, for $x \in \acute{X}$, and the sequence x_n in $\acute{X}$. Then, $\{x_n\}$ is said to be convergent to a point $x \in \acute{X}$ if $\lim\limits_{n\to\infty} M_F\ (x_n, x, t) = 1$ for $t > 0$

(iii) If every Cauchy sequence is convergent in $\acute{X}$, then $(X, M_F\ ,*)$ is complete.

Definition 3 [17]. *Let $(\acute{X}, M_F\ ,*)$ be an FMS and $G \subseteq \acute{X}$. An element g_0 of $\acute{X}$ is very nearest to G if it gives the finest estimation in G* for *$h \in \acute{X}$, i.e.,*

$$\frac{1}{M_F(h,G,t)} - 1 = inf_{g_0 \in G}\left(\frac{1}{M_F(h,G,t)} - 1\right) = \frac{1}{M_F(h,g_0,t)} - 1$$

Definition 4 [16]. *Let $(\acute{X}, M_F\ ,*)$ be an FMS. The function $H_{M_F} : W(\acute{X}) \times W(\acute{X}) \to [0,\infty)$, given as $\frac{1}{H_{M_F}(X,Y,t)} - 1 = max\left\{Sup_{x\in X}\left(\frac{1}{M_F(x,Y,t)} - 1\right), Sup_{y\in Y}\left(\frac{1}{M_F(X,y,t)} - 1\right)\right\}$, is the M_F-Hausdorff metric. The pair $\left(W(\acute{X}), H_{M_F}\right)$ is called the M_F-Hausdorff metric space.*

Definition 5 [16]. *Let $\acute{X}$ be a nonempty set, $S : \acute{X} \to W(\acute{X})$, $G \subseteq \acute{X}$, and a function is given as $\alpha : \acute{X} \times \acute{X} \to [0,\infty)$. Then, S is said to be α_*-admissible on G if $\alpha_*(Sp,Sq) = inf\{\alpha(u,v) : u \in Sp, v \in Sq\} \geq 1$, where $\alpha(p,q) \geq 1$ for each $p,q \in G$.*

Definition 6 [17]. *Let $\acute{X}$ be a nonempty set, $S : \acute{X} \to W(\acute{X})$, $G \subseteq \acute{X}$, and a function is given as $\alpha : \acute{X} \times \acute{X} \to [0,\infty)$. Then, S is said to be α_*-dominated on G if $\alpha_*\left(x_{2i}, [S_{x2i}]_{\alpha(x_{2i})}\right) = inf\left\{\alpha(x_{2i},b) : b \in [S_{x2i}]_{\alpha(x_{2i})}\right\} \geq 1$.*

Definition 7 [17]. *Let A be a fuzzy set, it functions from $\acute{X}$ to $[0,1]$, and $W(\acute{X})$ denotes the class of entirely fuzzy sets in $\acute{X}$. If $c \in \acute{X}$, then $A(c)$ is said to be the grade membership of element c in A. Then, $[A]_\beta$ represents the β-level set of A and given by*

$$[A]_\beta = \{c : A(c) \geq \beta\} \text{ where } 0 \leq \beta \leq 1, [A]_0 = \overline{\{c : A(c) \geq 0\}}$$

Definition 8 [17]. *A fuzzy subset G of $\acute{X}$ is an approximate quantity iff its β-level set is a compact convex subset of $\acute{X}$ for each $\beta \in [0,1]$ and $sup_{e \in G} G(e) = 1$.*

Definition 9 [17]. *Let R be an arbitrary set and $\acute{X}$ be any metric space. Then, a fuzzy mapping $S : R \to W(X')$ as a fuzzy subset of $R \times X'$, $S : R \times X' \to [0,1]$ in the sense that $S(c,y) = S(c)(y)$.*

Definition 10 [17]. *Let $S : \acute{X} \to W(\acute{X})$ be a fuzzy mapping. A point $e \in \acute{X}$ is said to be a fuzzy FP of S if there exists $0 < \beta \leq 1$ so that $e \in [S(e)]_\beta$.*

Lemma 1. *Let $(\acute{X}, M_F, *)$ be an FMS. Let $(W(\acute{X}), M_F)$ be a Hausdorff-FMS on $(W(\acute{X}))$. Then, for all $Q, H \in W(\acute{X})$ and for each $a \in Q$ and $g \in T$ satisfying $\frac{1}{M_F(a,H,t)} - 1 \leq \frac{1}{M_F(a,g_a,t)} - 1$, then $\frac{1}{M_F(a,g_a,t)} - 1 \leq \frac{1}{H_{M_F}(Q,H,t)} - 1$.*

Proof. If $\frac{1}{H_{M_F}(Q,H,t)} - 1 = sup_{a \in S}\left(\frac{1}{M_F(a,H,t)} - 1\right)$, then $\frac{1}{H_{M_F}(Q,H,t)} - 1 \geq \left(\frac{1}{M_F(a,H,t)} - 1\right)$ for all $a \in Q$. As H is a closed compact set, for each $a \in \acute{X}$, there exists at most one estimate $g_a \in H$ satisfying $\left(\frac{1}{M_F(a,H,t)} - 1\right) = \left(\frac{1}{M_F(a,g_a,t)} - 1\right)$.

Now, we obtain

$$\left(\frac{1}{H_{M_F}(Q,H,t)} - 1\right) \geq \left(\frac{1}{M_F(a,g_a,t)} - 1\right).$$

Now, if

$$\left(\frac{1}{H_{M_F}(Q,H,t)} - 1\right) = sup_{g_a \in H}\left(\frac{1}{M_F(Q,g_a,t)} - 1\right) \geq sup_{a \in Q}\left(\frac{1}{M_F(a,H,t)} - 1\right)$$

it implies that

$$\left(\frac{1}{H_{M_F}(Q,H,t)} - 1\right) \geq \left(\frac{1}{M_F(a,g_a,t)} - 1\right)$$

□

2. Main Results

Let $(\acute{X}, M_F, *)$ be a complete FMS and $x_0 \in \acute{X}$ and $S, T : \acute{X} \to W(\acute{X})$ be two α_*-fuzzy-dominated mappings on $\acute{X}$. Let $x_1 \in [S(x_0)]_{\alpha(x_0)}$ be an element so that $\frac{1}{M_F\left(x_0,[S(x_0)]_{\alpha(x_0)},t\right)} - 1 = \frac{1}{M_F(x_0,x_1,t)} - 1$. Let $x_2 \in [T(x_1)]_{\beta(x_1)}$ be such that $\frac{1}{M_F\left(x_1,[T(x_1)]_{\beta(x_1)},t\right)} - 1 = \frac{1}{M_F(x_1,x_2,t)} - 1$. Let $x_3 \in [S(x_2)]_{\alpha(x_2)}$ be such that $\frac{1}{M_F\left(x_2,[S(x_2)]_{\alpha(x_2)},t\right)} - 1 = \frac{1}{M_F(x_2,x_3,t)} - 1$.

Proceeding this way, we achieve a sequence x_n of points in $\acute{X}$ so that $x_{2n+1} \in [S(x_{2n})]_{\alpha(x_{2n})}$ and $x_{2n+2} \in [T(x_{2n+1})]_{\beta(x_{2n+1})}$, where $n \in N$. In addition, $\frac{1}{M_F\left(x_{2n},[S(x_{2n})]_{\alpha(x_{2n})},t\right)} - 1 = \frac{1}{M_F(x_{2n},x_{2n+1},t)} - 1$ and $\frac{1}{M_F\left(x_{2n+1},[T(x_{2n+1})]_{\beta(x_{2n+1})},t\right)} - 1 = \frac{1}{M_F(x_{2n+1},x_{2n+2},t)} - 1$. We name this type of sequence as $\{TS(x_n)\}$, where $\{TS(x_n)\}$ is the sequence in $\acute{X}$ generated by x_0 .

Theorem 1. *Let* $(\acute{X}, M_F, *)$ *be a complete FMS. Let* $x_0 \in B_F(\overleftarrow{x_0}, r, t) \subseteq \acute{X}, \underleftarrow{\alpha} : \acute{X} \times \acute{X} \to [0, \infty)$ *and* $S, T : \acute{X} \to W(\acute{X})$ *be two fuzzy-dominated maps on* $\{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t)$ *satisfying:*

$$\frac{1}{H_{M_F}\left([S(x)]_{\alpha(x)}, [T(y)]_{\beta(y)}, t\right)} - 1 \le a\left(\frac{1}{M_F(x,y,t)} - 1\right)$$
$$+b\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(y, [T(y)]_{\beta(y)}, t\right)} - 1 + \frac{1}{M_F\left(y, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(x, [T(y)]_{\beta(y)}, t\right)} - 1\right) \tag{1}$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x, [S(x)]_{\alpha(x)}, t\right), M_F\left(y, [T(y)]_{\beta(y)}, t\right), M_F\left(y, [S(x)]_{\alpha(x)}, t\right), M_F\left(x, [T(y)]_{\beta(y)}, t\right)\right\}} - 1\right),$$

where $\alpha, \beta \in (0,1)$ *and*$x, y \in \{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t), \alpha(x, y) \ge 1$. *Moreover,*

$$\frac{1}{M_F(x_0, x_1, t)} - 1 \le (1-\beta)r \le r \tag{2}$$

whenever $a \in (0, \frac{1}{4})$, $b \in \left(0, \frac{1}{9}\right)$, $c \in \left(0, \frac{1}{13}\right)$, *and* $\beta = max\left(\frac{a+2b+c}{1-2b}, \frac{a+2b}{1-2b-c}, \frac{a+2b+c}{1-2b-c}\right) < 1$. *Then,* $\{TS(x_n)\}$ *is a sequence in* $B_F(\overleftarrow{x_0}, r, t)$ *and* $\{TS(\underleftarrow{x_n})\} \to x \in B_F(\overleftarrow{x_0}, r, t)$. *Again, if (1) holds for* x, *then* S *and* T *have a common fuzzy FP in* $B_F(x_0, r, t)$.

Proof. Consider a sequence $\{TS(x_n)\}$. Then, from (2),

$$\frac{1}{M_F(x_0, x_1, t)} - 1 \le (1-\beta)r \le r.$$

It can be proven that

$$x_1 \in B_F(\overleftarrow{x_0}, r, t).$$

Let $x_2, \ldots .. \ x_j \in B_F(\overleftarrow{x_0}, r, t)$ for $j \in N$. If j is odd, then $j = 2i+1$ for some $i \in N$. As $S, T : \acute{X} \to W(\acute{X})$ are α_*-dominated maps on $B_F(\overleftarrow{x_0}, r, t)$, $\alpha_*\left(x_{2i}, [S(x_{2i})]_{\alpha(x_{2i})}\right) \ge 1$ and $\alpha_*\left(x_{2i+1}, [T(x_{2i+1})]_{\beta(x_{2i+1})}\right) \ge 1$. This signifies $inf\left\{\alpha(x_{2i}, b) : b \in [S(x_{2i})]_{\alpha(x_{2i})}\right\} \ge 1$. In addition, $x_{2i+1} \in [S(x_{2i})]_{\alpha(x_{2i})}$, so $\alpha(x_{2i}, x_{2i+1}) \ge 1$.

Now, by applying Lemma 1,

$$\frac{1}{M_F(x_{2i+1}, x_{2i+2}, t)} - 1 \le \frac{1}{H_{M_F}\left([S(x_{2i})]_{\beta(x_{2i})}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right)} - 1$$
$$\le a\left(\frac{1}{M_F(x_{2i}, x_{2i+1}, t)} - 1\right) + b\Big(\frac{1}{M_F\left(x_{2i}, [S(x_{2i})]_{\beta(x_{2i})}, t\right)} - 1 + \frac{1}{M_F\left(x_{2i+1}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right)} - 1$$
$$+ \frac{1}{M_F\left(x_{2i+1}, [S(x_{2i})]_{\beta(x_{2i})}, t\right)} - 1 + \frac{1}{M_F\left(x_{2i}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right)} - 1$$
$$\le a\left(\frac{1}{M_F(x_{2i}, x_{2i+1}, t)} - 1\right)$$
$$+b\left(\frac{1}{M_F\left(x_{2i}, [S(x_{2i})]_{\beta(x_{2i})}, t\right)} - 1 + \frac{1}{M_F\left(x_{2i+1}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right)} - 1 + \frac{1}{M_F\left(x_{2i+1}, [S(x_{2i})]_{\beta(x_{2i})}, t\right)} - 1 + \frac{1}{M_F\left(x_{2i}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right)} - 1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x_{2i}, [S(x_{2i})]_{\beta(x_{2i})}, t\right), M_F\left(x_{2i+1}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right), M_F\left(x_{2i+1}, [S(x_{2i})]_{\beta(x_{2i})}, t\right), M_F\left(x_{2i}, [T(x_{2i+1})]_{\beta(x_{2i+1})}, t\right)\right\}} - 1\right)$$
$$= a\left(\frac{1}{M_F(x_{2i}, x_{2i+1}, t)} - 1\right)$$
$$+b\left(\frac{1}{M_F(x_{2i}, x_{2i+1}, t)} - 1 + \frac{1}{M_F(x_{2i+1}, x_{2i+2}, t)} - 1 + \frac{1}{M_F(x_{2i+1}, x_{2i+1}, t)} - 1 + \frac{1}{M_F(x_{2i}, x_{2i+2}, t)} - 1\right)$$
$$+c\left(\frac{1}{\min\{M_F(x_{2i}, x_{2i+1}, t), M_F(x_{2i+1}, x_{2i+2}, t), M_F(x_{2i+1}, x_{2i+1}, t), M_F(x_{2i}, x_{2i+2}, t)\}} - 1\right)$$

$$\leq a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)$$
$$+b\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1+\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1+\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1+\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big) \quad (3)$$
$$+c\Big(\tfrac{1}{\min\{M_F(x_{2i},x_{2i+1},t),M_F(x_{2i+1},x_{2i+2},t),M_F(x_{2i},x_{2i+2},t)\}}-1\Big).$$

Taking $M_F(x_{2i}, x_{2i+1}, t)$ as a minimum from $\{M_F(x_{2i}, x_{2i+1}, t), M_F(x_{2i+1}, x_{2i+2}, t), M_F(x_{2i}, x_{2i+2}, t)\}$, then (3) becomes

$$\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\ a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1)+2b\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)+c\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)$$
$$\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)-2b\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)\leq\ a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1)+c\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big).$$

That gives

$$\frac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\left(\frac{a+2b+c)}{(1-2b)}\right)\left(\frac{1}{M_F(x_{2i},x_{2i+1},t)}-1\right). \quad (4)$$

Taking $M_F(x_{2i+1}, x_{2i+2}, t)$ as a minimum from $\{M_F(x_{2i}, x_{2i+1}, t), M_F(x_{2i+1}, x_{2i+2}, t), M_F(x_{2i}, x_{2i+2}, t)\}$, then inequality (3) will be

$$\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\ a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)$$
$$+2b\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)+c\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)$$
$$\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)-2b\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)-c\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)$$
$$\leq a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)$$

implies that

$$\frac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\left(\frac{a+2b}{1-2b-c}\right)\left(\frac{1}{M_F(x_{2i},x_{2i+1},t)}-1\right). \quad (5)$$

Taking $M_F(x_{2i}, x_{2i+2}, t)$ as a minimum from $\{M_F(x_{2i}, x_{2i+1}, t), M_F(x_{2i+1}, x_{2i+2}, t), M_F(x_{2i}, x_{2i+2}, t)\}$, then inequality (3) will be

$$\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\ a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)+c\Big(\tfrac{1}{M_F(x_{2i},x_{2i+2},t)}-1\Big)$$

$$\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)-2b\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)-c\Big(\tfrac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\Big)\leq a\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)+2b$$
$$\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1)+c\Big(\tfrac{1}{M_F(x_{2i},x_{2i+1},t)}-1\Big)$$
$$\frac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\left(\frac{a+2b+c}{1-2b-c}\right)\left(\frac{1}{M_F(x_{2i},x_{2i+1},t)}-1\right) \quad (6)$$

Let β be the maximum term of $(\frac{a+2b+c}{1-2b}, \frac{a+2b}{1-2b-c}, \frac{a+2b+c}{1-2b-c}) < 1$. Then, from all three cases of inequalities (4)–(6), we obtain

$$\frac{1}{M_F(x_{2i+1},x_{2i+2},t)}-1\leq\beta\left(\frac{1}{M_F(x_{2i},x_{2i+1},t)}-1\right) \quad (7)$$

This signifies that $x_{2i+2}\ \epsilon\ B_F(\overleftharpoondown{x_0}, r, t)$
Similarly,

$$\frac{1}{M_F(x_{2i},x_{2i+1},t)}-1\leq\beta\left(\frac{1}{M_F(x_{2i-1},x_{2i},t)}-1\right).$$

From inequality (7), we have

$$\frac{1}{M_F(x_{2i+1}, x_{2i+2}, t)} - 1 \leq \beta \times \beta\left(\frac{1}{M_F(x_{2i-1}, x_{2i}, t)} - 1\right)$$

$$\frac{1}{M_F(x_{2i+1}, x_{2i+2}, t)} - 1 \leq \beta^2\left(\frac{1}{M_F(x_{2i-1}, x_{2i}, t)} - 1\right) \tag{8}$$

Repeating these steps, we can obtain

$$\frac{1}{M_F(x_{2i+1}, x_{2i+2}, t)} - 1 \leq \beta^{2i+1}\left(\frac{1}{M_F(x_0, x_1, t)} - 1\right). \tag{9}$$

Similarly, for $j = 2i + 2$,

$$\frac{1}{M_F(x_{2i+2}, x_{2i+3}, t)} - 1 \leq \beta^{2i+2}\left(\frac{1}{M_F(x_0, x_1, t)} - 1\right) \tag{10}$$

Thus, inequalities (9) and (10) can be written as for all $n \in N$:

$$\frac{1}{M_F(x_n, x_{n+1}, t)} - 1 \leq \beta^n\left(\frac{1}{M_F(x_0, x_1, t)} - 1\right)$$

Now,

$$\frac{1}{M_F(x_0, x_n, t)} - 1 \leq \frac{1}{M_F(x_0, x_1, t)} - 1 + \frac{1}{M_F(x_1, x_2, t)} - 1 + \ldots\ldots + \frac{1}{M_F(x_{n-1}, x_n, t)} - 1,$$

$$\begin{gathered}\tfrac{1}{M_F(x_0,x_n,t)} - 1 \leq \left(\tfrac{1}{M_F(x_0,x_1,t)} - 1\right) + \beta\left(\tfrac{1}{M_F(x_0,x_1,t)} - 1\right) \\ +\beta^2\left(\tfrac{1}{M_F(x_0,x_1,t)} - 1\right) + \ldots\ldots + \beta^n\left(\tfrac{1}{M_F(x_0,x_1,t)} - 1\right),\end{gathered}$$

$$\frac{1}{M_F(x_0, x_n, t)} - 1 \leq (1 + \beta + \beta^2 + \ldots\ldots + \beta^n)\left(\frac{1}{M_F(x_0, x_1, t)} - 1\right).$$

$$\leq \left(\frac{1-\beta^n}{1-\beta}\right)(1-\beta)r \leq r.$$

Hence, $x_n \in B_F(\overleftarrow{x_0}, r, t)$.

$$\begin{gathered}\tfrac{1}{M_F(x_n,x_{n+1},t)} - 1 \leq \beta\left(\tfrac{1}{M_F(x_{n-1},x_n,t)} - 1\right) \leq \beta^2\left(\tfrac{1}{M_F(x_{n-2},x_{n-1},t)} - 1\right) \leq \\ \ldots \leq \beta^n\left(\tfrac{1}{M_F(x_0,x_1,t)} - 1\right)\end{gathered} \tag{11}$$

Taking $n \to +\infty$, this yields from (11):

$$lim_{n\to\infty} M_F(x_n, x_{\ n+1}, t) = 1 \text{ for } t > 0 \tag{12}$$

.

As M_F is triangular, we have

$$\begin{gathered}\tfrac{1}{M_F(x_n,x_m,t)} - 1 \leq \tfrac{1}{M_F(x_n,x_{n+1},t)} - 1 + \tfrac{1}{M_F(x_{n+1},x_{n+2},t)} - 1 + \\ \ldots\ldots + \tfrac{1}{M_F(x_{m-1},x_m,t)} - 1 \\ \tfrac{1}{M_F(x_n,x_m,t)} - 1 \leq (\beta^n + \beta^{n+1} + \ldots\ldots + \beta^{m-1})\left(\tfrac{1}{M_F(x_0,x_1,t)} - 1\right).\end{gathered}$$

$$\frac{1}{M_F(x_n, x_m, t)} - 1 \leq \left(\frac{\beta^n}{1-\beta}\right)\left(\frac{1}{M_F(x_0, x_1, t)} - 1\right) \to 0 \text{ as } n \to +\infty. \tag{13}$$

Thus, the sequence $\left\{TS(x_{n)}\right\}$ is a Cauchy sequence in $B_F(\overleftarrow{x_0}, r, t)$. As $B_F(\overleftarrow{x_0}, r, t)$ is complete, $\{TS(x_n)\} \to x \in B_F(\overleftarrow{x_0}, r, t)$, by the assumption $\alpha(x_n, x) \geq 1$. Suppose that

$$\frac{1}{M_F\left(x, S(x)_{\alpha(x)}, t\right)} - 1 \geq 1.$$

$$Lim_{n\to\infty}(x, x_n, t) = 1, t > 0. \tag{14}$$

We want to show that $x\epsilon[S(x)]_{\alpha(x)}$. By using the triangular property of M_F, we have

$$\frac{1}{M_F\left(x,[S(x)]_{\alpha(x)},t\right)} - 1 \leq \left(\frac{1}{M_F(x,x_{2n+1},t)} - 1\right) + \left(\frac{1}{M_F\left(x_{2n+1},[S(x)]_{\alpha(x)},t\right)} - 1\right) \text{ for } t > 0. \tag{15}$$

Now, by the assumption $\alpha\left(x_{2n+1}, [S(x)]_{\alpha(x)}\right) > 1$, $\alpha\left(x, [T(x_{2n})]_{\beta(x_{2n})}\right) > 1$, and $\alpha\left(x_{2n}, [T(x_{2n})]_{\beta(x_{2n})}\right) > 1$, using inequality (1) and Lemma 1, we obtain $\frac{1}{M_F\left(x_{2n+1},[S(x)]_{\alpha(x)},t\right)}$
$-1 \leq \frac{1}{H_{M_F}\left([T(x_{2n})]_{\beta(x_{2n})},[S(x)]_{\alpha(x)},t\right)} - 1$

$$\leq a\left(\frac{1}{M_F(x_{2n},x,t)} - 1\right)$$

$$+b\left(\frac{1}{M_F\left(x_{2n},[T(x_{2n})]_{\beta(x_{2n})},t\right)} - 1 + \frac{1}{M_F\left(x,[S(x)]_{\alpha(x)},t\right)} - 1 + \frac{1}{M_F\left(x,[T(x_{2n})]_{\beta(x_{2n})},t\right)} - 1 + \frac{1}{M_F\left(x_{2n},[S(x)]_{\alpha(x)},t\right)} - 1\right)$$

$$+c\left(\frac{1}{min\left\{M_F\left(x_{2n},[T(x_{2n})]_{\beta(x_{2n})},t\right),M_F\left(x,[S(x)]_{\alpha(x)},t\right),M_F\left(x,[T(x_{2n})]_{\beta(x_{2n})},t\right),M_F\left(x_{2n},[S(x)]_{\alpha(x)},t\right)\right\}} - 1\right)$$

Now, taking $lim_{n\to+\infty}$, and using (12) and (14), we obtain

$$\left(\frac{1}{M_F\left(x_{2n+1}, [S(x)]_{\alpha(x)}, t\right)} - 1\right) \leq 2b\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1\right) + c\left(\frac{1}{\min\left\{1, M_F\left(x, [S(x)]_{\alpha(x)}, t\right)\right\}} - 1\right) \tag{16}$$

Hence,

$$Lim_{n\to+\infty}Sup\left(\frac{1}{M_F\left(x_{2n+1}, [S(x)]_{\alpha(x)}, t\right)} - 1\right) \leq (2b + c)\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1\right) \text{ for } t > 0 \tag{17}$$

Inequality (17) together with (15) and (14) implies that

$$\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1 \leq (2b + c)\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1\right), \text{ for } t > 0. \tag{18}$$

As $(2b + c) < 1$, one has $M_F\left(x, [S(x)]_{\alpha(x)}, t\right) = 1$. This shows that $x\epsilon[S(x)]_{\alpha(x)}$

Let $x^*\epsilon\ W$ be so that $x^*\epsilon[S(x)]_{\alpha(x)}$. Suppose that $\alpha\left(x, [S(x)]_{\alpha(x)}\right) > 1$ and $\alpha\left(x^*, [S(x^*)]_{\alpha(x^*)}\right) > 1$. Then, in view of inequality (1), we have for $t > 0$:

$$\frac{\frac{1}{M_F(x,x^*,t)}}{} \leq a\left(\frac{1}{M_F(x,x^*,t)}-1\right)$$
$$+b\left(\frac{1}{M_F\left(x,[S(x)]_{\alpha(x)},t\right)}-1+\frac{1}{M_F\left(x^*,[S(x^*)]_{\alpha(x^*)},t\right)}-1+\frac{1}{M_F\left(x,[S(x^*)]_{\alpha(x^*)},t\right)}-1+\frac{1}{M_F\left(x^*,[S(x)]_{\alpha(x)},t\right)}-1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x,[S(x)]_{\alpha(x)},t\right),M_F\left(x^*,[S(x^*)]_{\alpha(x^*)},t\right),M_F\left(x,[S(x^*)]_{\alpha(x^*)},t\right),M_F\left(x^*,[S(x)]_{\alpha(x)},t\right)\right\}}-1\right)$$
$$= a\left(\frac{1}{M_F(x,x^*,t)}-1\right)+2b\left(\frac{1}{M_F(x,x^*,t)}-1\right)+c\left(\frac{1}{\min\{1,M_F(x,x^*,t)\}}-1\right)$$
$$= (a+2b+c)\left(\frac{1}{M_F(x,x^*,t)}-1\right)$$
$$\leq (a+2b+c)^2\left(\frac{1}{M_F(x,x^*,t)}-1\right) \leq \ldots\ldots \leq (a+2b+c)^n\left(\frac{1}{M_F(x,x^*,t)}-1\right) \to 0 \text{ as } n \to +\infty.$$

Hence, we obtain $M_F(x,x^*,t)$= 1, so $x = x^*$. Thus, x^* is the unique FP of S in $B_F(x_0,r,t)$ Similarly, we can prove that T has a unique FP in $B_F(x_0,r,t)$. Hence, S and T both have a common fuzzy FP x^* in $B_F(x_0,r,t)$ and $M_F(x^*,x^*,t)=1$. □

Example 1. *Take* $\acute{X}=[0,\infty)$, *and* M_F *is an FMS from* $\acute{X}\times\acute{X}\times(0,\infty)\to(0,\infty)$ *defined as* $M_F(x,y,t)=\frac{t}{t+d(x,y)}$ *for* $t>0$ *and* $d(x,y)=|x-y|$, $\forall\ x,y\in X$. *Now, for* $p\in\acute{X}$ *and* $\alpha,\beta\in[0,1]$, $S,T:\acute{X}\to W(\acute{X})$ *is defined as*

$$(S(p))(u)=\begin{cases}\alpha\ if\ 0\leq u<\frac{p}{2}\\ \frac{\alpha}{2}\ if\ \frac{p}{2}\leq u\leq\frac{3p}{2}\\ \frac{\alpha}{4}\ if\ \frac{3p}{2}<u\leq p\\ 0\ if\ p<u<\infty,\end{cases}$$

and

$$(T(p))(u)=\begin{cases}\beta\ if\ 0\leq u\leq\frac{p}{2}\\ \frac{\beta}{3}\ if\ \frac{p}{2}<u\leq\frac{2p}{3}\\ \frac{\beta}{7}\ if\ \frac{2p}{3}<u\leq p\\ 0\ if\ p<u<\infty.\end{cases}$$

$[S(p)]_{\frac{\alpha}{2}}=\left[\frac{p}{2},\frac{3p}{2}\right]$ *and* $[T(p)]_{\frac{\beta}{3}}=\left[\frac{p}{2},\frac{2p}{3}\right]$, *where* $p\in\acute{X}$.

Let $x_0=1$, *t* = *7, and* $r=7$. *Then,* $B_F(x_0,r,t)=[0,7]$ *Now, we have*

$$\left(\frac{1}{M_F\left(x_0,[S(x_0)]_{\alpha(x_0)},t\right)}-1\right)=\frac{d\left(x_0,[S(x_0)]_{\alpha(x_0)}\right)}{t}=\frac{d\left[1,\left[\frac{1}{2},\frac{3}{2}\right]\right]}{7}$$

$$\left(\frac{1}{M_F\left(x_0,[S(x_0)]_{\alpha(x_0)},t\right)}-1\right)=\frac{d\left[1,\frac{1}{2}\right]}{7}=\frac{\left|1-\frac{1}{2}\right|}{7}=\frac{1}{14}$$

Now, $\left(\frac{1}{M_F\left(x_1,[T(x_1)]_{\beta(x_1)},t\right)}-1\right)=\frac{d\left(x_1,[T(x_1)]_{\beta(x_1)}\right)}{t}=\frac{d\left[\frac{1}{14},\left[\frac{1}{28},\frac{1}{21}\right]\right]}{7}=\frac{1}{196}$.

Thus, $\{ST(x_n)\}=\left\{1,\frac{1}{14},\frac{1}{196},\ldots\right\}$ *in* $\acute{X}$ $B_F(x_0,r,t)$ *with generator* x_0. *Define*

$$\alpha(p,q)=\begin{cases}1\ p>q\\ 0\ otherwise.\end{cases}$$

Now, for $p,q\in B_F(x_0,r,t)\cap\{ST(x_n)\}$ *and*$\alpha(p,q,t)\geq1$, *by using*$[S(p)]_{\frac{\alpha}{2}}$ *and*$[T(p)]_{\frac{\beta}{3}}$ *in (1), we have*

$$\frac{1}{H_{M_F}\left([S(p)]_{\frac{\alpha}{2}},[T(p)]_{\frac{\beta}{3}},t\right)}-1\le a\left(\frac{1}{M_F(x,y,t)}-1\right)$$

$$+b\left(\frac{1}{M_F\left(x,[S(p)]_{\frac{\alpha}{2}},t\right)}-1+\frac{1}{M_F\left(y,[T(p)]_{\frac{\beta}{3}},t\right)}-1+\frac{1}{M_F\left(y,[S(p)]_{\frac{\alpha}{2}},t\right)}-1+\frac{1}{M_F\left(x,[T(p)]_{\frac{\beta}{3}},t\right)}-1\right)$$

$$+c\left(\frac{1}{\min\left\{M_F\left(x,[S(p)]_{\frac{\alpha}{2}},t\right),M_F\left(y,[T(p)]_{\frac{\beta}{3}},t\right),M_F\left(y,[S(p)]_{\frac{\alpha}{2}},t\right),M_F\left(x,[T(p)]_{\frac{\beta}{3}},t\right)\right\}}-1\right).$$

$$\frac{1}{H_{M_F}\left([S(p)]_{\frac{\alpha}{2}},[T(p)]_{\frac{\beta}{3}},t\right)}-1=max\left\{\frac{1}{sup_{a\epsilon[S(p)]_{\frac{\alpha}{2}}}M_F\left(a,[T(p)]_{\frac{\beta}{3}},t\right)}-1,\frac{1}{sup_{b\epsilon[T(p)]_{\frac{\beta}{3}}}M_F\left([S(p)]_{\frac{\alpha}{2}},b,t\right)}-1\right\}$$

$$=max\left\{\frac{1}{M_F\left(\frac{3p}{2},\left[\frac{p}{2},\frac{2p}{3}\right],t\right)}-1,\frac{1}{M_F\left(\left[\frac{p}{2},\frac{3p}{2}\right],\frac{2p}{3},t\right)}-1\right\}=max\left\{\frac{1}{M_F\left(\frac{3p}{2},\frac{p}{2},t\right)}-1,\frac{1}{M_F\left(\frac{p}{2},\frac{2p}{3},t\right)}-1\right\}.$$

$As\ \frac{1}{M_F(x,y,t)}-1=\frac{d(x,y)}{t},$

$$\frac{1}{H_{M_F}\left([S(p)]_{\frac{\alpha}{2}},[T(p)]_{\frac{\beta}{3}},t\right)}-1=max\left\{\frac{d\left(\frac{3p}{2},\frac{p}{2}\right)}{t},\frac{d\left(\frac{p}{2},\frac{2p}{3}\right)}{t}\right\}$$

$$=max\left\{\frac{\left|\frac{3p}{2}-\frac{p}{2}\right|}{t},\frac{\left|\frac{p}{2}-\frac{2p}{3}\right|}{t}\right\}=max\left\{\frac{p}{t},\frac{\left|\frac{p}{2}-\frac{2p}{3}\right|}{t}\right\}\frac{1}{H_{M_F}\left([S(p)]_{\frac{\alpha}{2}},[T(p)]_{\frac{\beta}{3}},t\right)}-1=\frac{p}{t},$$

and putting $p=1.5$ *and* $t=5$ *:*

$$\frac{1}{H_{M_F}\left([S(p)]_{\frac{\alpha}{2}},[T(p)]_{\frac{\beta}{3}},t\right)}-1=0.3.$$

Letting

$$\left(\frac{1}{M_F(x,y,t)}-1\right)+b\left(\frac{1}{M_F\left(x,[S(p)]_{\frac{\alpha}{2}},t\right)}-1+\frac{1}{M_F\left(y,[T(p)]_{\frac{\beta}{3}},t\right)}-1+\frac{1}{M_F\left(y,[S(p)]_{\frac{\alpha}{2}},t\right)}-1+\frac{1}{M_F\left(x,[T(p)]_{\frac{\beta}{3}},t\right)}-1\right)$$

$$+c\left(\frac{1}{\min\left\{M_F\left(x,[S(p)]_{\frac{\alpha}{2}},t\right),M_F\left(y,[T(p)]_{\frac{\beta}{3}},t\right),M_F\left(y,[S(p)]_{\frac{\alpha}{2}},t\right),M_F\left(x,[T(p)]_{\frac{\beta}{3}},t\right)\right\}}-1\right)$$

$$= a\left(\frac{1}{M_F(x,y,t)}-1\right)+b\left(\frac{1}{M_F\left(x,\left[\frac{p}{2},\frac{3p}{2}\right],t\right)}-1+\frac{1}{M_F\left(y,\left[\frac{p}{2},\frac{2p}{3}\right],t\right)}-1+\frac{1}{M_F\left(y,\left[\frac{p}{2},\frac{3p}{2}\right],t\right)}-1+\frac{1}{M_F\left(x,\left[\frac{p}{2},\frac{2p}{3}\right],t\right)}-1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x,\left[\frac{p}{2},\frac{3p}{2}\right],t\right),M_F\left(y,\left[\frac{p}{2},\frac{2p}{3}\right],t\right),M_F\left(y,\left[\frac{p}{2},\frac{3p}{2}\right],t\right),M_F\left(x,\left[\frac{p}{2},\frac{2p}{3}\right],t\right)\right\}}-1\right)$$
$$= a\left(\frac{1}{M_F(x,y,t)}-1\right)+b\left(\frac{1}{M_F\left(x,\frac{p}{2},t\right)}-1+\frac{1}{M_F\left(y,\frac{p}{2},t\right)}-1+\frac{1}{M_F\left(y,\frac{p}{2},t\right)}-1+\frac{1}{M_F\left(x,\frac{p}{2},t\right)}-1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x,\frac{p}{2},t\right),M_F\left(y,\frac{p}{2},t\right),M_F\left(y,\frac{p}{2},t\right),M_F\left(x,\frac{p}{2},t\right)\right\}}-1\right)$$
$$= a\left(\frac{1}{M_F(x,y,t)}-1\right)+2b\left(\frac{1}{M_F\left(x,\frac{p}{2},t\right)}-1+\frac{1}{M_F\left(y,\frac{p}{2},t\right)}-1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x,\frac{p}{2},t\right),M_F\left(y,\frac{p}{2},t\right)\right\}}-1\right)$$
$$= a\left(\frac{d(x,y)}{t}\right)+2b\left(\frac{d\left(x,\frac{p}{2}\right)}{t}+\frac{d\left(y,\frac{p}{2}\right)}{t}\right)+c\left(\frac{1}{\min\left\{M_F\left(x,\frac{p}{2},t\right),M_F\left(y,\frac{p}{2},t\right)\right\}}-1\right)$$
$$= a\left(\frac{|x-y|}{t}\right)+2b\left(\frac{\left|x-\frac{p}{2}\right|}{t}+\frac{\left|y-\frac{p}{2}\right|}{t}\right)+c\left(\frac{1}{\min\left\{M_F\left(x,\frac{p}{2},t\right),M_F\left(y,\frac{p}{2},t\right)\right\}}-1\right),$$

Taking $x = 5,\ y = 7,\ p = 1.5,\ t = 5,\ a = \frac{1}{6},\ b = \frac{1}{11},$ *and* $c = \frac{1}{15}$, (19) gives

$$a\left(\frac{|x-y|}{t}\right)+2b\left(\frac{\left|x-\frac{p}{2}\right|}{t}+\frac{\left|y-\frac{p}{2}\right|}{t}\right)+c\left(\frac{1}{\min\left\{M_F\left(x,\frac{p}{2},t\right),M_F\left(y,\frac{p}{2},t\right)\right\}}-1\right)=0.531818$$

We obtain $0.3 \leq 0.531818$. *This satisfies all the requirements of Theorem 1. Thus, the contraction exists on the C-bàl. Now, we take points from the whole space instead of C-bàl. Now, taking* $x = 10,\ y = 12$ $,10, 12 \in \acute{X}$, *and* $\alpha(10, 12) \geq 1$, *and choosing* $p = 11,\ t = 8,\ a = \frac{1}{6}$, $b = \frac{1}{11}$, *and* $c = \frac{1}{15}$, *we obtain*

$$\frac{1}{H_{M_F}\left([S(p)]_{\frac{\alpha}{2}},[T(p)]_{\frac{\beta}{3}},t\right)}-1\leq a\left(\frac{1}{M_F(x,y,t)}-1\right)$$
$$+b\left(\frac{1}{M_F\left(x,[S(p)]_{\frac{\alpha}{2}},t\right)}-1+\frac{1}{M_F\left(y,[T(p)]_{\frac{\beta}{3}},t\right)}-1+\frac{1}{M_F\left(y,[S(p)]_{\frac{\alpha}{2}},t\right)}-1+\frac{1}{M_F\left(x,[T(p)]_{\frac{\beta}{3}},t\right)}-1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x,[S(p)]_{\frac{\alpha}{2}},t\right),M_F\left(y,[T(p)]_{\frac{\beta}{3}},t\right),M_F\left(y,[S(p)]_{\frac{\alpha}{2}},t\right),M_F\left(x,[T(p)]_{\frac{\beta}{3}},t\right)\right\}}-1\right)$$

.

Finally, we obtain $1.375 \leq 0.3458$. *This is not true. Hence, the contraction exists only on C-bàl. Thus, all requisites of Theorem 1 are fulfilled.*

Corollary 1. *Let* $(\acute{X}, M_F, *)$ *be a complete FMS. Let* $x_0 \in B_F(\overleftarrow{x_0}, r, t) \subseteq X'$, $\alpha : \underleftarrow{\acute{X}} \times \acute{X} \to [0, \infty)$ *and* $S, T : \acute{X} \to W(\acute{X})$ *be two fuzzy-dominated mappings on* $\{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t)$ *satisfying:*

$$\frac{1}{H_{M_F}\left([S(x)]_{\alpha(x)},[T(y)]_{\beta(y)},t\right)}-1\leq b\left(\frac{1}{M_F\left(x,[S(x)]_{\alpha(x)},t\right)}-1+\frac{1}{M_F\left(y,[T(y)]_{\beta(y)},t\right)}-1+\frac{1}{M_F\left(y,[S(x)]_{\alpha(x)},t\right)}-1+\frac{1}{M_F\left(x,[T(y)]_{\beta(y)},t\right)}-1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x,[S(x)]_{\alpha(x)},t\right),M_F\left(y,[T(y)]_{\beta(y)},t\right),M_F\left(y,[S(x)]_{\alpha(x)},t\right),M_F\left(x,[T(y)]_{\beta(y)},t\right)\right\}}-1\right), \quad (19)$$

where $\alpha, \beta \in (0, 1)$ *and* $x, y \in \{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t), \alpha(x, y) \geq 1$. *Moreover,*

$$\frac{1}{M_F(x_0, x_1, t)}-1\leq(1-\beta)r\leq r \quad (20)$$

whenever $b \in \left(0, \frac{1}{9}\right)$, $c \in \left(0, \frac{1}{13}\right)$, *and* $\beta = max\left(\frac{2b+c}{1-2b}, \frac{2b}{1-2b-c}, \frac{2b+c}{1-2b-c}\right) < 1$. *Then,* $\{TS(x_n)\}$ *is a sequence in* $B_F(\overleftarrow{x_0}, r, t)$ *and* $\{TS(x_n)\} \to x \in B_F(\overleftarrow{x_0}, r, t)$. *Again, if (19) holds for* x, *then* S *and* T *have a common fuzzy FP in* $B_F(x_0, r, t)$.

If we put $b = 0$ *in Theorem 1, we obtain the above result.*

Corollary 2. *Let* $(\acute{X}, M_F, *)$ *be a complete FMS. Let* $x_0 \in B_F(\overleftarrow{x_0}, r, t) \subseteq X'$, $\alpha : \overleftarrow{X'} \times X' \to [0, \infty)$ *and* $S, T : \acute{X} \to W(\acute{X})$ *be two fuzzy-dominated mappings on* $\{TS(x_n)\} \cap B_F(x_0, r, t)$ *satisfying:*

$$\frac{1}{H_{M_F}\left([S(x)]_{\alpha(x)}, [T(y)]_{\beta(y)}, t\right)} - 1 \leq a\left(\frac{1}{M_F(x,y,t)} - 1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x, [S(x)]_{\alpha(x)}, t\right), M_F\left(y, [T(y)]_{\beta(y)}, t\right), M_F\left(y, [S(x)]_{\alpha(x)}, t\right), M_F\left(x, [T(y)]_{\beta(y)}, t\right)\right\}} - 1\right), \tag{21}$$

where $\alpha, \beta \in (0,1)$ *and* $x, y \in \{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t), \alpha(x, y) \geq 1$. *Moreover,*

$$\frac{1}{M_F(x_0, x_1, t)} - 1 \leq (1-\beta)r \leq r, \tag{22}$$

whenever $a \in (0, \frac{1}{4})$, $c \in \left(0, \frac{1}{13}\right)$, *and* $\beta = max\left(a + c, \frac{a}{1-c}, \frac{a+c}{1-c}\right) < 1$. *Then,* $\{TS(x_n)\}$ *is a sequence in* $B_F(\overleftarrow{x_0}, r, t)$ *and* $\{TS(x_n)\} \to x \in B_F(\overleftarrow{x_0}, r, t)$. *Again, if (21) holds for* x, *then* S *and* T *have a common fuzzy FP in* $B_F(x_0, r, t)$.

If we put $c = 0$, *in Theorem 1, we obtain the above result.*

Corollary 3. *Let* $(\acute{X}, M_F, *)$ *be a complete FMS. Let* $x_0 \in B_F(\overleftarrow{x_0}, r, t) \subseteq X'$, $\alpha : \overleftarrow{X'} \times X' \to [0, \infty)$ *and* $S, T : \acute{X} \to W(\acute{X})$ *be two fuzzy-dominated mappings on* $\{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t)$ *satisfying:*

$$\frac{1}{H_{M_F}\left([S(x)]_{\alpha(x)}, [T(y)]_{\beta(y)}, t\right)} - 1 \leq a\left(\frac{1}{M_F(x,y,t)} - 1\right)$$
$$+b\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(y, [T(y)]_{\beta(y)}, t\right)} - 1 + \frac{1}{M_F\left(y, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(x, [T(y)]_{\beta(y)}, t\right)} - 1\right), \tag{23}$$

where $\alpha, \beta \in (0,1)$ *and* $x, y \in \{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t), \alpha(x, y) \geq 1$. *Moreover,*

$$\frac{1}{M_F(x_0, x_1, t)} - 1 \leq (1-\beta)r \leq r \tag{24}$$

whenever $a \in (0, \frac{1}{4})$, $b \in \left(0, \frac{1}{9}\right)$, *and* $\beta = max\left(\frac{a+2b}{1-2b}, \frac{a+2b}{1-2b}, \frac{a+2b}{1-2b}\right) < 1$. *Then,* $\{TS(x_n)\}$ *is a sequence in* $B_F(\overleftarrow{x_0}, r, t)$ *and* $\{TS(x_n)\} \to x \in B_F(\overleftarrow{x_0}, r, t)$ *Again, if (23) holds for* x, *then* S *and* T *have a common fuzzy FP in* $B_F(x_0, r, t)$.

3. Fixed-Point Result for Graph Contractions

Here, we prove an important application of Theorem 1 in graph theory. Jachymski [22] established the comparable result in metric spaces endowed with a graph that initiates the notion of graphic contractions in metric FP theory. Hussain et al. [23] gave FP results for graphic contraction including an application to a system of integral equations. If there exists a distance between any two different vertices, then the graph Q is said to be a connected graph [24–29].

Definition 11. Let *P be a nonempty set and* $Q = (V(Q), L(Q))$ *be a graph with* $B = V(Q)$. *A fuzzy map G from B to* $W(B)$ *is named fuzzy-graph-dominated on B if* $(e, c) \in L(Q)$, *for* $e \in B$, $c \in [Ge]_\beta$, *and* $0 < \beta \leq 1$.

Theorem 2. Let $(\acute{X}, M_F, *)$ *be a complete FMS equipped with graph Q,* $x_0, \in \acute{X}$ *so that*

(i) $S, T : \acute{X} \to W(\acute{X})$ *are fuzzy-graph-dominated mappings on* $\{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t)$.

$$(ii)\quad \frac{1}{H_{M_F}\left([S(x)]_{\alpha(x)}, [T(y)]_{\beta(y)}, t\right)} - 1 \leq a\left(\frac{1}{M_F(x,y,t)} - 1\right)$$
$$+b\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(y, [T(y)]_{\beta(y)}, t\right)} - 1 + \frac{1}{M_F\left(y, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(x, [T(y)]_{\beta(y)}, t\right)} - 1\right) \tag{25}$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x, [S(x)]_{\alpha(x)}, t\right), M_F\left(y, [T(y)]_{\beta(y)}, t\right), M_F\left(y, [S(x)]_{\alpha(x)}, t\right), M_F\left(x, [T(y)]_{\beta(y)}, t\right)\right\}} - 1\right),$$

where $\alpha, \beta \in (0, 1)$ *and* $x, y \in \{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t), \alpha(x, y) \geq 1$. *Moreover,*

$$(iii)\ \frac{1}{M_F(x_0, x_1, t)} - 1 \leq (1 - \beta)r \leq r,$$

whenever $a \in (0, \frac{1}{4}), b \in \left(0, \frac{1}{9}\right), c \in \left(0, \frac{1}{13}\right)$, *and* $\beta = max\left(\frac{a+2b+c}{1-2b}, \frac{a+2b}{1-2b-c}, \frac{a+2b+c}{1-2b-c}\right) < 1$. *Then,* $\{TS(x_n)\}$ *is a sequence in* $B_F(\overleftarrow{x_0}, r, t)$, $\{TS(x_n)\} \to x \in B_F(\overleftarrow{x_0}, r, t)$, *and* $x, y \in \{TS(x_n)\}$, $(x, y) \in L(Q)$. *Then,* $(x_n, x_{n+1}) \in L(Q)$ *and* $\{TS(x_n)\} \to k^*$. *In addition, if (25) holds for* $(x_n, k^*) \in L(Q)$, $(k^*, x_n) \in L(Q)$ *for each* $n \in N$, k^* *belongs to both* $[T(k^*)]_{\beta(k^*)}$ *and* $k^* \in [S(k^*)]_{\alpha(k^*)}$.

Proof. Define $\alpha : \acute{X} \times \acute{X} \to [0, \infty)$ by $\alpha(x, y) = 1$ if $x, \in B_F(\overleftarrow{x_0}, r, t)$ and $(x, y) \in L(Q)$. Otherwise, take $\alpha(x, y) = 0$. The graph-dominated notion on $B_F(\overleftarrow{x_0}, r, t)$ is that $(x, y) \in L(Q)$ for all $y \in [S(x)]_{\gamma(x)}$ and $(x, y) \in L(Q)$ for each $y \in [T(y)]_{\beta(y)}$. Thus, $\alpha(x, y) = 1$ for each $y \in [S(x))]_{\alpha(x)}$ and $\alpha(x, y) = 1$ for all $y \in [T(y)]_{\beta(y)}$. This signifies that

$$inf\left\{\alpha(x, y) : y \in [S(x)]_{\alpha(x)} = 1 \text{ and } \inf\left\{\alpha(x, y) : y \in [Ty]_{\beta(y)}\right\} = 1\right.$$

Hence, $\alpha_*\left(x, \left[S(x)]_{\alpha(x)}\right)\right) = 1$, $\alpha_*\left(x, \left[T(y)]_{\beta(y)}\right)\right) = 1$ for every $x \in B_F(\overleftarrow{x_0}, r, t)$, as both the maps are α_* -dominated on $B_F(\overleftarrow{x_0}, r, t)$ Further, (24) can be defined by

$$\frac{1}{H_{M_F}\left([S(x)]_{\alpha(x)}, [T(y)]_{\beta(y)}, t\right)} - 1 \leq a\left(\frac{1}{M_F(x,y,t)} - 1\right)$$
$$+b\left(\frac{1}{M_F\left(x, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(y, [T(y)]_{\beta(y)}, t\right)} - 1 + \frac{1}{M_F\left(y, [S(x)]_{\alpha(x)}, t\right)} - 1 + \frac{1}{M_F\left(x, [T(y)]_{\beta(y)}, t\right)} - 1\right)$$
$$+c\left(\frac{1}{\min\left\{M_F\left(x, [S(x)]_{\alpha(x)}, t\right), M_F\left(y, [T(y)]_{\beta(y)}, t\right), M_F\left(y, [S(x)]_{\alpha(x)}, t\right), M_F\left(x, [T(y)]_{\beta(y)}, t\right)\right\}} - 1\right),$$

where $\alpha, \beta \in (0, 1)$, $a \in (0, \frac{1}{4})$, $b \in \left(0, \frac{1}{9}\right)$, $c \in \left(0, \frac{1}{13}\right)$, and $x, y \in \{TS(x_n)\} \cap B_F(\overleftarrow{x_0}, r, t)$, $\alpha(x, y) \geq 1$. In addition, (ii) holds. Using Theorem 1, $\{TS(x_n)\}$ is the sequence in $B_F(\overleftarrow{x_0}, r, t)$ and $\{TS(x_n)\} \to k^* \in B_F(\overleftarrow{x_0}, r, t)$. Now, $x_n, k^* \in B_F(\overleftarrow{x_0}, r, t)$ and either $(x_n, k^*) \in L(Q)$ or $(k^*, x_n) \in L(Q)$ signifies that either $\alpha(x_n, k^*) \geq 1$ or $\alpha(k^*, x_n) \geq 1$. Hence, all specifications of Theorem 1 are proven. Thus, k^* belongs to both $[T(k^*)]_{\beta(k^*)}$ and $k^* \in [S(k^*)]_{\alpha(k^*)}$. □

4. Application to Fredhlom-Type Integral Equations

Clearly, many authors have proven many different types of linear and nonlinear-Volterra and Fredhlom integral equations (FIEs) by applying the generalized contractions principle. Aydi at al. [30], Hussain et al. [31], Nashine et al. [32], Rasham et al. [17], and Rehman et al. [11] proved significant FP results for the existence of a solution of linear and nonlinear integral equations. For further FP results with applications to the system of integral equations, see [33–35].

Let $\acute{X} = C([0,c], R)$ be the set consisting of all continuous real-valued functions on $[0,\varepsilon]$ where $o < \varepsilon \in R$. Now, we prove a special case of FIEs for the second type given by:

$$x(t) = \int_0^{\varepsilon} k_1(\tau, q, x(q))dq \tag{26}$$

$$y(t) = \int_0^{\varepsilon} k_2(\tau, q, y(q))dq \tag{27}$$

where $\tau \in [0,\varepsilon]$ and $k_1, k_2 : [0,\varepsilon] \times [0,\varepsilon] \times R \to R$.

The metric space $d : \acute{X} \times \acute{X} \to R$ is induced and defined by

$$d(x,y) = \| \frac{x-y}{2} \| \tag{28}$$

The continuous t-norm of the binary operator $*$ is defined by $\alpha * \beta = \alpha\beta \; \forall \, \alpha, \beta \in [0,\varepsilon]$. We can express FMS $M_F : \acute{X} \times \acute{X} \times (0,\infty) \to [0,1]$ as

$$M_F(x,y,t) = \frac{t}{t + d(x,y)} \; \forall \, x, y \in \mathrm{X} \text{ and } t > 0 \tag{29}$$

Theorem 3. *Assume for* $\rho \in (0,1)$:

$$\| \, S(x) \, , T(y) \, \| \leq \rho G((S,T),x,y) \forall x,y \in X \tag{30}$$

$$\begin{gathered} G((S,T),x,y) = max(\| \tfrac{x-y}{2} \|, \| \tfrac{S(x)-x}{2} \|, \| \tfrac{T(y)-y}{2} \|, \\ \| \tfrac{S(x)-y}{2} \|, \| \tfrac{T(y)-x}{2} \|, \| \tfrac{S(x)-x}{2} \| + \| \tfrac{T(y)-y}{2} \| + \| \tfrac{S(x)-y}{2} \| + \| \tfrac{T(y)-x}{2} \|) \end{gathered} \tag{31}$$

Then, the FIEs (26) and (27) have unique solutions.

Proof. Define mappings $S, T : \acute{X} \to \acute{X}$ by

$$S(x(\tau))_{=} \int_0^{\varepsilon} k_1(\tau, q, x(q))dq \tag{32}$$

$$T(y(\tau)) = \int_0^{\varepsilon} k_2(\tau, q, y(q))dq \tag{33}$$

S and T are well defined and (26) and (27) have unique solutions if and only if S and T have unique FPs in $\acute{X}$. Now, we want to prove that Theorem 1 is workable for integral operator S and T; thus, we have the following six cases $\forall \, x, y \in \acute{X}$.□

Case-I. Let the maximum term in (31) be $\| \frac{x-y}{2} \|$. Then, $G((S,T),x,y) = \| \frac{x-y}{2} \|$; therefore, in the outlook of (28) and (29), we obtain

$$\frac{1}{M_F(S(x) \, ,T(y),t)} - 1 = d\left(\frac{S(x) \, ,T(y)}{t}\right) \leq \frac{\rho G((S,T),x,y)}{t} = \rho \| \frac{x-y}{2} \| = \rho \frac{1}{M_F(x,y,t)} - 1.$$

$$\text{This means that } \frac{1}{M_F(S(x) \, , T_y, t)} - 1 \leq \rho \frac{1}{M_F(x,y,t)} - 1), \text{ for } t > 0, \tag{34}$$

$\forall\ x, y \in \acute{X}$ such that $S(x)\ \neq T(y)$. The inequality (34) holds if $S(x) = T(y)$. Thus, the integral operators S and T satisfy all the conditions of Theorem 1. Then, the integral operators S and T have unique solutions.

Case-II. If $\| \frac{S(x)-x}{2} \|$ is the maximum term of (31), then $G((S,T),x,y) = \| \frac{S(x)-x}{2} \|$. Therefore, using (28) and (29), we have

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 = d\left(\frac{S(x)\ ,T(y)}{t}\right) \leq \frac{\rho\ G((S,T),x,y)}{t} = \rho\| \frac{S(x)-x}{2t} \| = \rho\frac{1}{M_F(S(x)\ ,x,t)} - 1.$$

It yields that

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 \leq \rho\frac{1}{M_F(S(x)\ ,x,t)} - 1, \text{ for } t > 0 \tag{35}$$

$\forall\ x, y \in \acute{X}$ such that $S(x)\ \neq T(y)$.

Case-III. If $\| \frac{T(y)-y}{2} \|$ is the maximum term in (31), then

$$G((S,T),x,y) = \| \frac{T(y)-y}{2} \|$$

Therefore, using (28) and (29), we have

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 = d\left(\frac{S(x)\ ,T(y)}{t}\right) \leq \frac{\rho\ G(S,x,y)}{t} = \rho\| \frac{T(y)-y}{2t} \| = \rho\frac{1}{M_F(T(y),y,t)} - 1.$$

That is,

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 \leq\ \rho\frac{1}{M_F(T(y),y,t)} - 1,\ for\ t > 0, \tag{36}$$

$\forall\ x, y \in \acute{X}$ such that $S(x)\ \neq T(y)$.

Case-IV. If $\| \frac{S(x)-y}{2} \|$ is the maximum term in (31), then

$$G((S,T),x,y) = \| \frac{S(x)-y}{2} \|$$

Therefore, using (28) and (29), we have

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 = d\left(\frac{S(x)\ ,T(y)}{t}\right) \leq \frac{\rho\ G((S,T),x,y)}{t} = \rho\| \frac{S(x)-y}{2t} \| = \rho\frac{1}{M_F(S(x)\ ,y,t)} - 1.$$

Hence,

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 \leq \rho\frac{1}{M_F(S(x)\ ,y,t)} - 1,\ \ for\ t > 0, \tag{37}$$

$\forall\ x, y \in \acute{X}$ such that $S(x)\ \neq T(y)$.

Case-V. If $\| \frac{T(y)-x}{2} \|$ is the maximum term in (31), then

$$G((S,T),x,y) = \| \frac{T(y)-x}{2} \|$$

Using (28) and (29), we have

$$\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 = d\left(\frac{S(x)\ ,T(y)}{t}\right) \leq \frac{\rho G((S,T),x,y)}{t} = \rho\| \frac{T(y)-x}{2t} \| = \rho\frac{1}{M_F(T(y),x,t)} - 1$$

It implies that

$$\frac{1}{M_F\big(S(x)\ ,T_y,t\big)} - 1 \leq \rho\frac{1}{M_F(T(y),x,t)} - 1,\ \ for\ t > 0, \tag{38}$$

$\forall\ x, y \in \acute{X}$ such that $S(x)\ \neq T_y$.

The inequalities (34), (36), (38) and (41) hold if $S(x) = T(y)$. Thus, the integral operators S and T fulfill all requirements of Theorem 1 with $\rho = c$ and $a = b = 0$. The integral operators S and T have unique solutions.

Case-VI. If $\parallel \frac{S(x)-x}{2} \parallel + \parallel \frac{T(y)-y}{2} \parallel + \parallel \frac{S(x)-y}{2} \parallel + \parallel \frac{T(y)-x}{2} \parallel$ is the maximum term in (31), then

$$G((S,T),x,y) = \parallel \frac{S(x)-x}{2} \parallel + \parallel \frac{T(y)-y}{2} \parallel + \parallel \frac{S(x)-y}{2} \parallel + \parallel \frac{T(y)-x}{2} \parallel$$

Therefore, from (28) and (29), we have

$$\begin{gathered}\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 = d\left(\frac{S(x)\ ,T(y)}{t}\right) \leq \frac{\rho\ G((S,T),x,y)}{t} \\ = \rho\left(\parallel \frac{S(x)-x}{2t} \parallel + \parallel \frac{T(y)-y}{2t} \parallel + \parallel \frac{S(x)-y}{2t} \parallel + \parallel \frac{T(y)-x}{2t} \parallel\right).\end{gathered}$$

It implies that

$$\begin{gathered}\frac{1}{M_F(S(x)\ ,T(y),t)} - 1 \\ \leq \rho\left(\frac{1}{M_F(x,S(x)\ ,t)} - 1 + \frac{1}{M_F(y,T(y),t)} - 1 + \frac{1}{M_F(y,S(x)\ ,t)} - 1 + \frac{1}{M_F(x,T(y),t)} - 1\right), \\ for\ t > 0,\end{gathered} \tag{39}$$

$\forall\ x, y \in \acute{X}$ such that $S(x)\ \neq T(y)$. Inequality (39) holds if $S(x) = T(y)$. Thus, the integral operator S fulfills all conditions of Theorem 1 with $\rho = b$ and $a = c = 0$. The integral operators S and T have unique FPs. Now, we look at a specific type of example for an instance of an FIE of the second kind.

Example 2. *Take $\acute{X} = [0,1]$ and put $\varepsilon = 1$ in (26) and (27).*

$$x(t) = \int_0^1 k_1(\tau, q, x(q))dq, \tag{40}$$

$$y(t) = \int_0^1 k_2(\tau, q, y(q))dq, \tag{41}$$

where $k_1(\tau, q, x(q)) = \frac{4}{7(\tau+1+x(q))}$ *and* $k_2(\tau, q, y(q)) = \frac{4}{7(\tau+1+y(q))}$.

Equations (40) and (41) are the special kinds of integral equations where $\tau \in [0,1]$. Then

$$\begin{gathered}\parallel k_1(\tau, q, x(q) - k_2(\tau, q, y(q)) \parallel = \parallel \frac{4}{7(\tau+1+x(q)} - \frac{4}{7(\tau+1+y(q))} \parallel \\ = \frac{4}{7} \parallel \frac{x(q)-y(q)}{(\tau+1+x(q))(\tau+1+y(q))} \parallel \leq \frac{4}{7} \parallel x(q) - y(q) \parallel = \frac{4}{7} G((S,T),x,y),\end{gathered}$$

where

$$G((S,T),x,y) = \parallel x(q) - y(q) \parallel.$$

Now, we have to show that

$$\parallel Sx(\tau) - Ty(\tau) \parallel \leq \rho\ G(S,T),x,y).$$

From (32) and (33), we have

$$\begin{gathered}\parallel Sx(\tau) - Ty(\tau) \parallel = \parallel \int_0^1 k_1(\tau, q, x(q))dq - \int_0^1 k_2(\tau, q, y(q))dq \parallel \\ = \int_0^1 \parallel k_1(\tau, q, x(q)) - k_2(\tau, q, y(q)) \parallel dq \leq \int_0^1 \frac{4}{7} G((S,T),x,y)dq \\ = \frac{4}{7} G((S,T),x,y) \int_0^1 1dq = \frac{4}{7} G((S,T),x,y)\end{gathered}$$

As a result, all requirements of Theorem 3 hold with $\rho = \frac{4}{7} < 1$. The integral Equations (40) and (41) have unique solutions.

5. Conclusions

In this paper, we prove the existence of some new symmetrical fuzzy FP results for α_*-dominated mappings satisfying a new generalized advanced contraction on C-bàl in complete FMSs. In addition, some new definitions and examples are introduced. Furthermore, the notion of fuzzy-graph-dominated mappings is established in FMSs and some common fuzzy FP point theorems are proven for graphic contraction. Some illustrative examples are presented to show the validity of our new obtained results. To demonstrate the originality of our work, we give an application to an FIE that investigates the unique solution under a certain generalized contraction. Our results generalize many latest results [11,18,20,26,28,34] and many classical results in the literature. The obtained results improve and refine the corresponding results in the ordered metric space, ordered dislocated metric space, and partial metric spaces. The research work performed in this paper, in the future, will set a direction to work on families of fuzzy mappings, bipolar fuzzy mappings, L-fuzzy mappings, and intuitionistic fuzzy mappings.

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