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On Rational Solutions of Dressing Chains of Even Periodicity

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Abstract: We develop a systematic approach to deriving rational solutions and obtaining classification of their parameters for dressing chains of even N periodicity or equivalent Painlevé equations invariant under $A^{(1)}_{N-1}$ symmetry. This formalism identifies rational solutions (as well as special function solutions) with points on orbits of fundamental shift operators of $A^{(1)}_{N-1}$ affine Weyl groups acting on seed configurations defined as first-order polynomial solutions of the underlying dressing chains. This approach clarifies the structure of rational solutions and establishes an explicit and systematic method towards their construction. For the special case of the $N = 4$ dressing chain equations, the method yields all the known rational (and special function) solutions of the Painlevé V equation. The formalism naturally extends to $N = 6$ and beyond as shown in the paper.

Keywords: Painlevé equations; affine Weyl symmetries; Bäcklund transformations; dressing chain equations; Hamilton equations

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1. Introduction and Background Information

Painlevé equations form a class of second-order nonlinear differential equations with solutions that have no movable critical singularities in the complex plane, see, e.g., [1]. Although this mathematical property motivated the discovery of Painlevé equations, these equations had an astonishing impact on several fields inside and outside mathematics in a relatively short time. A long and incomplete list of affected topics and models includes correlation functions of the Ising model, random matrix theory, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear and fiber optics, and Bose–Einstein condensation. Special solutions, such as rational solutions, turned out to be important in these applications, and various methods were applied in their study. To provide a systematic approach to the study of rational solutions, we here utilize the dressing chain and its connection to Painlevé equations. The dressing chain was derived by applying Darboux transformations to the spectral problem of second order differential equations [2]. Specifically, let us consider a sequence of second order differential operators L_n connected via first order Darboux transformations: $(\partial_z - j_n)L_n = (L_{n-1} + \alpha_n)(\partial_z - j_n)$, where α_n is a constant. Such symmetry is realized for

$$L_n = (\partial_z + j_n)(\partial_z - j_n) + \alpha_n = (\partial_z - j_{n+1})(\partial_z + j_{n+1}), \tag{1}$$

with L_n defined by products of two first order differential operators with their orders being interchanged when going from n to $n+1$. Comparing the two alternative expressions for L_n in Equation (1), we obtain the nonlinear lattice equations [2]:

$$(j_n + j_{n+1})_z = -j_n^2 + j_{n+1}^2 + \alpha_n, \; n = 1, \ldots, N, \quad j_{N+i} = j_i, \tag{2}$$

made finite by imposing the periodic boundary condition $j_{N+i} = j_i$. We refer to system (2) as a system of dressing chain equations of N-periodicity. Such a system possesses many important properties. For $N = 3$, it has been shown [2] that it passes the Kovalevskaya–Painlevé test, and its equivalence to the Painlevé IV equation has also been established [2,3].

For higher N, the system is equivalent to $A^{(1)}_{N-1}$ invariant Painlevé equations [3,4], and this equivalence will be utilized in this paper to construct and study rational solutions of Painlevé equations in the context of underlying periodic dressing chains. Quite recently the N cyclic dressing chain was also obtained in the self-similarity limit of the second flow of $sl(N)$ mKdV hierarchy [5].

As we will now show, the system (2) requires different treatments depending on whether N is odd or even. This becomes evident when we consider a regular sum $\sum_{n=1}^{N}(j_n + j_{n+1})_z$ and an alternating sum $\sum_{n=1}^{N}(-1)^n(j_n + j_{n+1})_z$ of derivatives of $j_n + j_{n+1}$. Calculating a regular sum using the dressing Equation (2) we obtain the same expression for both even and odd N

$$\sum_{n=1}^{N}(j_n + j_{n+1})_z = 2\sum_{n=1}^{N}(j_n)_z = \sum_{n=1}^{N}\alpha_n\,, \tag{3}$$

for the integration constant on the right hand side. As long as N is odd, calculating an alternating sum $\sum_{n=1}^{N}(-1)^n(j_n + j_{n+1})_z$ using the dressing Equation (2) will reproduce the same condition as in (3). For even N, the alternating sum $\sum_{n=1}^{N}(-1)^n(j_n + j_{n+1})_z$ is identically zero (positive and negative terms simply cancel). However the same expression calculated by plugging the right hand side of dressing Equation (2) yields for, e.g., $N = 4$, the expression $\left(j_1^2 + j_3^2 - j_2^2 - j_4^2\right) + \frac{1}{2}(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4)$. Thus, the dressing chains of even periodicity require imposition of a new quadratic constraint or modification of the dressing chain formulation. Such modification was proposed in [6], where the authors put forward a system of dressing chain equations of even $N = 4, 6, 8, \ldots$ periodicity defined as:

$$(j_i + j_{i+1})_z = -j_i^2 + j_{i+1}^2 + \alpha_i + (-1)^{i+1}\frac{(j_i + j_{i+1})\Psi}{\Phi}\,,\; i = 1, 2, \ldots, N, \quad j_{N+i} = j_i\,, \tag{4}$$

where

$$\Psi = \sum_{k=1}^{N}(-1)^{k+1}\left(j_k^2 - \frac{1}{2}\alpha_k\right), \qquad \Phi = \sum_{k=1}^{N} j_k\,. \tag{5}$$

This structure is such that both regular and alternating sums of derivatives of $j_i + j_{i+1}$ give consistent answers when applied to the system (4):

$$\sum_{i=1}^{N}(j_i + j_{i+1})_z = 2\Phi_z = \sum_{i=1}^{N}\alpha_i\,,$$
$$\sum_{i=1}^{N}(-1)^i(j_i + j_{i+1})_z = 2\sum_{k=1}^{N}(-1)^{k+1}j_k^2 + \sum_{k=1}^{N}(-1)^k\alpha_k - 2\frac{\Phi}{\Phi}\Psi = 0\,.$$

As shown in [6], such a system can be obtained by Dirac reduction from $N+1$ dressing chain (2) of odd periodicity.

The above equations as well as quantities Ψ and Φ are invariant under $A^{(1)}_{N-1}$ Bäcklund transformations s_i, $i = 1, \ldots, N$ [3]:

$$j_i \xrightarrow{s_i} j_i - \frac{\alpha_i}{j_i + j_{i+1}}, \quad j_{i+1} \xrightarrow{s_i} j_{i+1} + \frac{\alpha_i}{j_i + j_{i+1}}, \quad j_k \xrightarrow{s_i} j_k,\; k \neq i, k \neq i+1\,, \tag{6}$$

when transformations (6) are accompanied by transformations of coefficients

$$\alpha_i \to -\alpha_i, \quad \alpha_{i\pm1} \to \alpha_{i\pm1} + \alpha_i. \tag{7}$$

There are also two automorphisms π, ρ:

$$\begin{aligned} &\pi : j_i \to j_{i-1},\; \alpha_i \to \alpha_{i-1},\; \pi(\Phi) = \Phi,\; \pi(\Psi) = -\Psi \\ &\rho : z \to -z, j_i \to -j_{i+2},\;\; \alpha_i \to \alpha_{i+2},\;\; \rho(\Phi) = -\Phi,\; \rho(\Psi) = \Psi\,, \end{aligned} \tag{8}$$

that keep the dressing Equation (4) invariant.

For the redefined quantities

$$\bar{j}_n = j_n + \frac{(-1)^n}{2}\frac{\Psi}{\Phi}, \tag{9}$$

it holds that the corresponding sum $f_n = j_n + j_{n+1} = \bar{j}_n + \bar{j}_{n+1}$ is unchanged. Such redefinition leads to a formal absorption of Ψ terms so that they are no longer explicit in the dressing equations rewritten in terms of $\bar{j}_n$ that satisfy Equation (2) [6]. However, such a process introduces potential extra divergencies into an associated Sturm–Liouville problem. Throughout this paper we will work with (4) with a constant non-zero Ψ so that the polynomial seed solutions we will construct below will be free of divergencies.

We present the construction of rational and special function solutions for dressing chains of even periodicity. In this work, rational solutions are identified with points on the orbits of fundamental shift operators (sometimes also referred to in the literature as translations) of the extended affine Weyl group $A^{(1)}_{N-1}$ acting on the first-order polynomial seed solutions. In particular, for the seed solutions with all components being equal to each other, the construction yields rational solutions being ratios of Umemura polynomials [7]. The reduction procedure that yields special function solutions is outlined and is shown to reproduce rational solutions for appropriate values of the parameters of the underlying Riccati equations.

The presentation is organized as follows. In Section 2, we obtain the first-order polynomial solutions of the dressing chain Equation (4) with parameters $\alpha_i, i = 1, \ldots, N$ depending on one arbitrary variable and with a constant non-zero Ψ that ensures that the solution is polynomial.

In Section 3, we establish a connection between the dressing chain Equation (4) and Hamiltonian formalism for $N = 4, 6$ that can easily be generalized to arbitrary even N values. Essential for establishing this connection is the ability to cast the dressing chain Equation (4) as symmetric $A^{(1)}_{N-1}$-invariant Painlevé equations, such as as those given in Equations (18) and (A1) for $N = 4, 6$, respectively. We should point out that translating the system of equations depending on j_i into formalism that is expressed entirely in terms of $f_i = j_i + j_{j+1}$ is possible for even N thanks to the presence of Ψ terms on the right hand side of Equation (4). This is in contrast to odd N dressing chains where j_i and f_i are always fully interchangeable. For $N = 4$ the Hamiltonian formalism of Section 3 gives rise to the Painlevé V equation as briefly reviewed in Section 3.2. The first-order polynomial solutions in the setting of Hamiltonian formalism become the algebraic solutions of [8].

We are able to present power series expansions of Hamiltonian variables p and q in Section 3.4. We show how potential divergencies of power series solutions (that cannot be absorbed in Ψ) can be removed by appropriate Bäcklund transformations. After removing the eventual simple poles from rational solutions by acting with the Bäcklund transformations, we obtain rational solutions that are expandable in a series of positive powers of z and can be reproduced by actions of the shift operators as shown in the next section.

In Section 4, we derive rational solutions for $N = 4$ by acting with shift operators on the first-polynomial solutions (11) and (12) to obtain all known cases listed in ref. [9] that presented necessary and sufficient conditions for rational solutions of the Painlevé V equation. Ref. [10] showed how to act with shift operators on solutions (11) (expressed by tau functions) to obtain some of the cases of [9] (items I + II in Section (4.1)).

For the first-order polynomial seed solutions (11) (with all the components j_i equal to z/N), the action of shift operators yields rational solutions expressed by Umemura polynomials [7,11] and we use the shift operators to derive the recurrence relations that determine these polynomials. Extending structure of seed solutions to include solutions (12) (where $j_i + j_{i+1} = 0$ for some i) requires exclusion of those shift operators that are ill-defined when acting on such solutions as discussed in Section 4.5. Those of the shift operators that are well-defined generate the remaining rational solutions from solutions (12), see item III in Section 4.1. This new approach leads to a systematic and unified way to derive all rational Painlevé V solutions. Based on results for $N = 4$ we conjecture that for all even N values all rational solutions are obtainable through actions of shift operators on first-order polynomial solutions.

In Section 5, we provide explicit construction of special function solutions and rational solutions for $N = 6$. The rational solutions are always identified with orbits of the fundamental shift operators. For the seed solution with all components being equal or only one of the components being negative, we are able to express the corresponding rational solutions by Umemura-type polynomials. Existence of special function solutions is established for the remaining cases, with a sufficient number of constraints imposed on α_i parameters to insure reduction of Hamiltonian equations to one single Riccati equation. For $N = 6$ this happens for three independent constraints. However we also encounter hybrid situations with one single Riccati equation and one coupled quadratic (in q_i, p_i) equations for some cases with two constraints. In such cases there exists a special function solution for only one of the variables. Interestingly, when α_i parameters are associated with orbits of the shift operators, we obtain closed expressions in terms of Whittaker functions that describe rational solutions for all underlying variables of the reduced system.

2. Preliminaries. The Seed Solutions as the First-Order Polynomial Solutions of Even Chains

For simplicity, we first carry out the discussion for $N = 4$ before proceeding to the case of $N = 6$ and making general comments about higher N cases.

We are looking for the first-order polynomial solutions to Equation (4) of the type

$$j_i = c_i z, \quad \sum_{i=1}^{4} c_i = 1\,,$$

that satisfy the $\Phi = z$ condition. With such ansatz, the quantity Ψ defined in (5) can only contain terms with z^2 or a constant. The terms quadratic in z can be absorbed in j_i via (9) transformation. Thus, without losing any generality we can assume that

$$\Psi = \frac{1}{2}(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4) = 1 - \alpha_1 - \alpha_3\,, \tag{10}$$

where we used that $\sum_{i=1}^{4} \alpha_i = 2$.

One can easily see that the condition for Ψ not to contain z^2 for the polynomial solutions of the first-order amounts to $j_{n+1}^2 - j_n^2 = 0$ on the right hand side of the dressing equations. Thus, the solution must be $j_i = zc(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ with $\epsilon_i = \pm 1$ and c a non-zero constant. Since $\Phi = z \neq 0$ we must also have $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 \neq 0$. This argument eliminates the case of two epsilons being negative, $\epsilon_i = -1, \epsilon_j = -1, i \neq j$, as this would violate $\Phi \neq 0$. Therefore the only two independent (up to π) polynomial solutions are:

$$j_i = \frac{z}{4}(1,1,1,1) \qquad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a}) \qquad \Phi = z, \Psi = 1 - 2\mathsf{a}\,, \tag{11}$$

$$j_i = \frac{z}{2}(1,1,-1,1) \qquad (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\mathsf{a}, 0, 0, 2-\mathsf{a}) \qquad \Phi = z, \Psi = 1 - \mathsf{a}\,. \tag{12}$$

Both solutions depend on only one free parameter a. The remaining first order polynomial solutions can be obtained by acting with π, π^2, and π^3 on solution (12) (recall that $\pi^4 = 1$ for $N = 4$ cyclicity and so $\pi^3 = \pi^{-1}$). Note that in case of solution (12), the action of automorphism π is such that it simply moves the -1 term in expression for j_i and zeros in expression for α_i to the right. It is important to point out that there could be other potential solutions of the first-order polynomial type like for example $j_i = (z/2)(1,0,1,0)$. However, such solutions would involve z^2 terms in Ψ and could be transformed by the transformation (9) involving the z^2 part of Ψ to the solution (12) or its π variants.

One can easily extend this analysis to higher N with Ψ and Φ defined in the definition (5). For the $N = 6$ first-order polynomial solutions we take:

$$\Psi = \frac{1}{2}(-\alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5 + \alpha_6) = 1 - \alpha_1 - \alpha_3 - \alpha_5 \,,$$

and obtain five different first-order polynomial solutions:

$$j_i = \frac{z}{6}(1,1,1,1,1,1), \quad \alpha_i = (\mathsf{a}, \frac{2}{3} - \mathsf{a}, \mathsf{a}, \frac{2}{3} - \mathsf{a}, \mathsf{a}, \frac{2}{3} - \mathsf{a})\,, \tag{13}$$

$$j_i = \frac{z}{4}(1,1,1,1,1,-1), \quad \alpha_i = (\mathsf{a}, 1 - \mathsf{a}, \mathsf{a}, 1 - \mathsf{a}, 0, 0)\,, \tag{14}$$

$$j_i = \frac{z}{2}(1,1,1,1,-1,-1), \quad \alpha_i = (\mathsf{a}, 2 - \mathsf{a}, \mathsf{a}, 0, -\mathsf{a}, 0)\,, \tag{15}$$

$$j_i = \frac{z}{2}(1,1,1,-1,1,-1), \quad \alpha_i = (2 - \mathsf{a}, \mathsf{a}, 0, 0, 0, 0)\,, \tag{16}$$

$$j_i = \frac{z}{2}(1,1,-1,1,1,-1), \quad \alpha_i = (2 - \mathsf{a}, 0, 0, \mathsf{a}, 0, 0)\,, \tag{17}$$

since all these configurations seems to be distinct and can not be connected by permutation generated by π or multiples of π's. All the above solutions depend on one arbitrary parameter a. Note that $j_i = z(1,1,1,-1,-1,-1)$ is not a solution because it would violate the $\Phi \neq 0$ condition. Thus the number of configurations is equal to $1 + 1 + p(6-2,2) = 5$, where $p(6-2,2) = p(4,2) = 3$ is a number of partitions of 4 in two parts (of positive integers and zero): $4 = 4 + 0 = 3 + 1 = 2 + 2$. For $N = 8$ we find a number of the first-order polynomial solutions to be $1 + 1 + p(8-2,2) + p(8-3,3)$ with $p(8-2,2) = p(6,2) = 4$ and $p(8-3,3) = p(5,3) = 5$. Generally a number of the first-order polynomial solutions is given by $1 + 1 + \sum_{k=2}^{N/2-1} p(N-k,k)$, where $p(N-k,k)$ is a number of distinct partitions of $N-k$ in k parts consisting of positive integers and zero.

For arbitrary even N with $\Phi = z$, $\Psi = 1 - \sum_{k=1}^{N/2} \alpha_{2k-1}$ and an arbitrary variable a, there will always be a fully symmetric solution:

$$j_i = \frac{z}{N},\ i = 1,\dots,N, \quad \alpha_{2j-1} = \mathsf{a},\ \alpha_{2j} = \frac{4}{N} - \mathsf{a},\ j = 1,\dots,N/2\,,$$

which is a fixed point of π^2 automorphism. The remaining solutions will have one and up to $N/2 - 1$ negative components $j_i = -\frac{z}{N}$ with varying distance between the negative components. For example, for only one negative component in the last position we get

$$j_i = \frac{z}{N-2},\ i = 1,\dots,N-2,\ \alpha_{2j-1} = \mathsf{a},\ \alpha_{2j} = \frac{4}{N-2} - \mathsf{a},\ j = 1,\dots,N/2-1\,,$$

with $j_k = 0, \alpha_k = 0$ for $k = N-1, N$, and so on for solutions with more negative components.

One needs to point out that the first-order solutions (13)–(17) appeared also as simple rational solutions expressed in terms of $f_i = j_i + j_{i+1}$ that give rise to other rational solutions via Bäcklund transformations in the framework of $A_5^{(1)}$ Painlevé equations (equivalent to $N = 6$ dressing chain equations) in ref. [12].

3. Hamiltonian Formalism and Polynomial Solutions

3.1. Hamilton Equations and Their Algebraic Solutions

For $N = 4$, we will show how the first-order polynomial solutions (11) and (12) are equivalent to all algebraic solutions found for the Painlevé V equation in [8]. These solutions will then serve as the seeds of all rational solutions [9] of the Painlevé V equation via shift transformations.

Thanks to the presence of Ψ in the dressing Equation (4) they can be rewritten in terms $f_i = j_i + j_{i+1}, i = 1,2,3,4$ as

$$\begin{aligned} z\frac{df_1}{dz} &= f_1f_3(f_2 - f_4) + (1-\alpha_3)f_1 + \alpha_1 f_3\,,\; z\frac{df_2}{dz} = f_2f_4(f_3 - f_1) + (1-\alpha_4)f_2 + \alpha_2 f_4, \\ z\frac{df_3}{dz} &= f_1f_3(f_4 - f_2) + (1-\alpha_1)f_3 + \alpha_3 f_1\,,\; z\frac{df_4}{dz} = f_4f_2(f_1 - f_3) + (1-\alpha_2)f_4 + \alpha_4 f_2\,, \end{aligned} \tag{18}$$

after multiplication by $\Phi = f_1 + f_3 = f_2 + f_4 = z$ and use of definition of Ψ from (5). Recall that it follows from relation (3) that $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$.

The above system of equations can be cast into a Hamiltonian system with

$$H = -q\,(q-z)\,p\,(p-z) + (1-\alpha_1-\alpha_3)\,pq + \alpha_1 zp - \alpha_2 zq\,, \tag{19}$$

with Hamilton equations

$$\begin{aligned} zq_z &= -q(q-z)(2p-z) + (1-\alpha_1-\alpha_3)q + \alpha_1 z\,, \\ zp_z &= p(p-z)(2q-z) - (1-\alpha_1-\alpha_3)p + \alpha_2 z\,, \end{aligned} \tag{20}$$

derived from

$$zq_z = \frac{dH}{dp}, \qquad zp_z = -\frac{dH}{dq}\,.$$

The Hamilton Equation (20) reproduces the $N = 4$ system of Equation (18) after substitution $(q,p) \to (f_1,f_2,f_3,f_4)$, such that

$$q = f_1 = z - f_3, \qquad p = f_2 = z - f_4\,.$$

The Bäcklund transformations (6) and automorphisms (8) are given in the setting of Hamilton Equation (20) by

$$\begin{aligned} & s_1 : q \to q,\; p \to p + \frac{\alpha_1}{q},\; \alpha_1 \to -\alpha_1,\; \alpha_2 \to \alpha_2 + \alpha_1, \alpha_3 \to \alpha_3 \\ & s_2 : q \to q - \frac{\alpha_2}{p},\; p \to p,\; \alpha_2 \to -\alpha_2,\; \alpha_1 \to \alpha_2 + \alpha_1, \alpha_3 \to \alpha_3 + \alpha_2 \\ & s_3 : q \to q,\; p \to p - \frac{\alpha_3}{z-q},\; \alpha_3 \to -\alpha_3,\; \alpha_2 \to \alpha_2 + \alpha_3, \alpha_4 \to \alpha_3 + \alpha_4 \\ & s_4 : q \to q + \frac{\alpha_4}{z-p},\; p \to p,\; \alpha_4 \to -\alpha_4,\; \alpha_1 \to \alpha_4 + \alpha_1, \alpha_3 \to \alpha_3 + \alpha_4 \\ & \pi : q \to z - p,\; p \to q,\; \alpha_i \to \alpha_{i-1}, \\ & \rho : z \to -z,\; q \to z - q,\; p \to p - z,\; \alpha_1 \leftrightarrow \alpha_3,\; \alpha_2 \leftrightarrow \alpha_4, \end{aligned} \tag{21}$$

where α_4 is understood as $2 - \alpha_1 - \alpha_2 - \alpha_3$ in terms of $\alpha_i, i = 1,2,3$ appearing in the Hamiltonian formalism.

Solutions (11) and (12) as well solutions that can be obtained from (12) by an automorphism $\pi : j_i \to j_{i-1}, \alpha_i \to \alpha_{i-1}$ are given in terms of q, p by

$$q = z/2,\; p = z/2,\; (\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a})\,, \tag{22}$$

$$q = z,\; p = 0,\; (\mathsf{a}, 0, 0, 2-\mathsf{a})\,, \tag{23}$$

$$q = z,\; p = z,\; (\mathsf{a}, 2-\mathsf{a}, 0, 0)\,, \tag{24}$$

$$q = 0,\; p = z,\; (0, \mathsf{a}, 2-\mathsf{a}, 0)\,, \tag{25}$$

$$q = 0,\; p = 0,\; (0, 0, \mathsf{a}, 2-\mathsf{a})\,, \tag{26}$$

where (22) is derived from (11) while the remaining solutions are obtained from (12) and its π variants. Solution (22) is a fixed point of π^2 while all the remaining solutions can be connected to each other by the π automorphism. All these solutions coincide with a complete set of algebraic solutions found by Watanabe [8].

For $N = 6$ we define the Hamiltonian formalism in terms of quantities:

$$q_1 = j_1 + j_2, \quad p_1 = j_2 + j_3, \quad q_2 = j_1 + j_2 + j_3 + j_4, \quad p_2 = j_4 + j_5\,, \tag{27}$$

which satisfy equations

$$\begin{aligned} zq_{1,z} &= q_1(q_2 - q_1)(2p_1 - z) + q_1(z - q_2)(2p_1 + 2p_2 - z) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_3 - \alpha_5) \\ zq_{2,z} &= (q_2 - q_1)(z - q_2)(2p_2 - z) + q_1(z - q_2)(2p_1 + 2p_2 - z) + z(\alpha_1 + \alpha_3) + q_2(1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \\ zp_{1,z} &= p_1 p_2(2q_2 - 2q_1 - z) + p_1(z - p_1 - p_2)(z - 2q_1) + z\alpha_2 - p_1(1 - \alpha_1 - \alpha_3 - \alpha_5) \\ zp_{2,z} &= p_1 p_2(2q_1 - 2q_2 + z) + p_2(z - p_1 - p_2)(z - 2q_2) + z\alpha_4 - p_2(1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \end{aligned} \tag{28}$$

that can be derived from $N = 6$ dressing chain (4) (explicitly given for $N = 6$ in the appendix in Equation (A1)). Equation (28) can be realized as Hamilton equations $zq_{i\,z} = \partial H/\partial p_i$ and $zp_{i\,z} = -\partial H/\partial q_i$ for $i = 1, 2$ with the Hamiltonian:

$$\begin{aligned} H = &-\sum_{i=1}^{2} p_i(p_i - z)q_i(q_i - z) - 2p_1 q_1 p_2(q_2 - z) + \sum_{i=1}^{2} p_i z \sum_{j=1}^{i} \alpha_{2i-1} - \sum_{i=1}^{2} q_i z \alpha_{2i} \\ &+ \sum_{i=1}^{2} q_i p_i(1 - \alpha_1 - \alpha_3 - \alpha_5)\,. \end{aligned}$$

One advantage of variables $q_i, p_i, i = 1, 2$ is that they make expressions for Bäcklund transformations (6) more transparent. The actions of Bäcklund transformations on these variables are given by

$$\begin{aligned} &s_1(p_1) = p_1 + \frac{\alpha_1}{q_1},\ s_2(q_1) = q_1 - \frac{\alpha_2}{p_1},\ s_3(p_1) = p_1 - \frac{\alpha_3}{q_2 - q_1},\ s_3(p_2) = p_2 + \frac{\alpha_3}{q_2 - q_1}, \\ &s_4(q_2) = q_2 - \frac{\alpha_4}{p_2},\ s_5(p_2) = p_2 - \frac{\alpha_5}{z - q_2},\ s_6(q_1) = q_1 + \frac{\alpha_6}{z - p_1 - p_2},\ s_6(q_2) = q_2 + \frac{\alpha_6}{z - p_1 - p_2}, \end{aligned} \tag{29}$$

where we only listed those transformations that are not identities and each s_i is accompanied by transformation (7) of α_i. The automorphism π acts in this setting as follows:

$$\pi : q_1 \to z - p_1 - p_2,\ p_1 \to q_1,\ q_2 \to z - p_2,\ p_2 \to q_2 - q_1,\ \alpha_i \to \alpha_{i-1}\,.$$

The first-order polynomial solutions (13)–(17) are expressed in terms of variables defined in relation (27) as the following solutions to Hamilton Equation (28):

$$q_1 = p_1 = p_2 = \frac{z}{3},\ q_2 = \frac{2z}{3}, \qquad \alpha_i = (\mathsf{a}, \frac{2}{3} - \mathsf{a}, \mathsf{a}, \frac{2}{3} - \mathsf{a}, \mathsf{a}, \frac{2}{3} - \mathsf{a})\,, \tag{30}$$

$$q_1 = p_1 = p_2 = \frac{z}{2},\ q_2 = z, \qquad \alpha_i = (\mathsf{a}, 1 - \mathsf{a}, \mathsf{a}, 1 - \mathsf{a}, 0, 0)\,, \tag{31}$$

$$q_1 = p_1 = z,\ q_2 = 2z,\ p_2 = 0 \qquad \alpha_i = (\mathsf{a}, 2 - \mathsf{a}, \mathsf{a}, 0, -\mathsf{a}, 0)\,, \tag{32}$$

$$q_1 = q_2 = p_1 = z,\ p_2 = 0 \qquad \alpha_i = (2 - \mathsf{a}, \mathsf{a}, 0, 0, 0, 0)\,, \tag{33}$$

$$q_1 = q_2 = p_2 = z,\ p_1 = 0 \qquad \alpha_i = (2 - \mathsf{a}, 0, 0, \mathsf{a}, 0, 0)\,, \tag{34}$$

We notice that the solution (30) is a fixed point of π^2 automorphism, as it is obvious comparing with its form in expression (13).

3.2. Connection of $N = 4$ Formalism to Painlevé V Equation

It is well-known that Equations (18) or (20) lead to the Painlevé V equation. We will here establish this relation explicitly in order to relate the parameters of both theories. We first define $w = q/z$. Taking a derivative of the top equation in (20) and eliminating p_z and p, we obtain the second order equation

$$w_{zz} = -\frac{w_z}{z} + \left(\frac{1}{2w} + \frac{1}{2(w-1)}\right)w_z^2 + \frac{\alpha w}{z^2(w-1)} + \frac{\beta(w-1)}{z^2 w} + \gamma w(w-1) + \delta z^2 w(w-1)(2w-1)\,, \tag{35}$$

with

$$\alpha = -\frac{1}{2}\alpha_3^2,\ \beta = \frac{1}{2}\alpha_1^2,\ \gamma = 2 - 2\alpha_2 - \alpha_1 - \alpha_3 = \alpha_4 - \alpha_2,\ \delta = \frac{1}{2}\,. \tag{36}$$

We need two additional steps to cast Equation (35) into a standard form of Painlevé V equation.

First we perform a change of variables $z \to t$ where $t = \epsilon z^2/2$ then followed by a transformation $y = w/(w-1)$.

In terms of y, Equation (35) takes a form of standard Painlevé V equation

$$y_{tt} = -\frac{y_t}{t} + \left(\frac{1}{2y} + \frac{1}{y-1}\right)y_t^2 + \frac{(y-1)^2}{t^2}\left(\bar{\alpha}y + \bar{\beta}\frac{1}{y}\right) + \frac{\bar{\gamma}}{x}y + \bar{\delta}\frac{y(y+1)}{y-1}\,, \tag{37}$$

where

$$\bar{\alpha} = \frac{1}{8}\alpha_3^2,\ \ \bar{\beta} = -\frac{1}{8}\alpha_1^2,\ \ \bar{\gamma} = -\frac{1}{2\epsilon}(2 - 2\alpha_2 - \alpha_1 - \alpha_3) = \frac{\alpha_2 - \alpha_4}{2\epsilon},\ \ \bar{\delta} = -\frac{1}{2}\frac{1}{\epsilon^2}\,. \tag{38}$$

For $\bar{\delta}$ to take a conventional value of $-\frac{1}{2}$ we need $\epsilon^2 = 1$.

3.3. Riccati Solutions of Equation (18)

Let us reduce Equation (18) by setting either $\alpha_2 = 0,\ f_2 = 0,\ f_4 = z$ or $\alpha_3 = 0,\ f_3 = 0,\ f_1 = z$. Using that $f_3 = z - f_1$ in the first case and $f_4 = z - f_2$ in the second case we can rewrite the remaining equations for $F_i = f_i/z, i = 1,2$ as

$$\frac{d}{dz}F_i = -zF_i(1-F_i) - \frac{\alpha_i + \alpha_{i+2}}{z}F_i + \frac{\alpha_i}{z}, \quad i = 1,2\,, \tag{39}$$

in which we recognize Riccati equations [13]. Without losing generality we will discuss the solution for the case of $i = 1$ with the principal solution given in terms of Whittaker functions as

$$F_1(z) = -\alpha_1 \frac{\mathrm{WhittakerM}(-\frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_1 + 1, -\frac{1}{2} + \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_3, \frac{1}{2}z^2)}{z^2\mathrm{WhittakerM}(-\frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_1, -\frac{1}{2} + \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_3, \frac{1}{2}z^2)} + \frac{\alpha_1}{z^2}\,. \tag{40}$$

The above expression becomes a rational function for at least one of the two parameters α_1, α_3 being equal to a negative even integer, and the other equal to an arbitrary integer but not equal to the opposite of that negative even integer ($\alpha_1 + \alpha_3 \neq 0$):

$$\alpha_i = -2n, \qquad \alpha_{i+2} = m \neq 2n,\ i = 1,3 \quad n \in \mathbb{Z}_+, 0,\ m \in \mathbb{Z}\,.$$

For the special case $\alpha_1 = 0 = \alpha_3$, it holds that $F_1 = 0$. With the above conditions being satisfied, the rational solutions occur for Painlevé parameters:

$$\bar{\alpha} = \frac{1}{2}n^2,\ \bar{\beta} = -\frac{1}{2}(\frac{m}{2})^2, \quad \text{or} \quad \bar{\alpha} = \frac{1}{2}(\frac{m}{2})^2,\ \bar{\beta} = -\frac{1}{2}n^2\,.$$

Let us recall that since $\alpha_2 = 0$ then $\epsilon\bar{\gamma} = -\alpha_4/2 = -(2 - \alpha_1 - \alpha_3)/2$. Thus, if $\alpha_1 = -2n, n \in \mathbb{Z}_+$, then we can rewrite α_3 as $\alpha_3 = 2(1 + n + \epsilon\bar{\gamma})$. If $\alpha_3 = -2n, n \in \mathbb{Z}_+$ then $\alpha_1 = 2(1 + n + \epsilon\,\bar{\gamma})$.

Riccati Equation (39) takes a more familiar look when we rewrite it in terms of a variable $x = -z^2/2$:

$$\frac{d}{dx}F_i = F_i(1-F_i) - \frac{\alpha_i + \alpha_{i+2}}{2x}F_i + \frac{\alpha_1}{2x}\,.$$

To linearize this equation we set $F_i = w_{i\,x}/w_i$ and for brevity introduce coefficients $b_i = (\alpha_i + \alpha_{i+2})/2$ and $a_i = \alpha_i/2$. In this way we obtain the second-order Kummer's equation:

$$x w_{i\,xx} + (b_i - x) w_{i\,x} - a_i w_i = 0\,. \tag{41}$$

We look for solutions of Kummer's equation denoted as $U(a, b, x)$ that are polynomials in x of a finite, let us say n, degree. This occurs for $a = -n$ and for $a - b = -n - 1$ for $n = 0, 1, 2, 3 \ldots$ and in the latter case it holds that [14]:

$$U(a, a+n+1, x) = x^{-a} \sum_{r=0}^{n} \binom{n}{r} (a)_r\, x^{-r}, \tag{42}$$

where $(a)_r$ is a Porchhammer symbol.

We will connect this polynomial with the case of $\alpha_3 = 0$ and $a = \alpha_2/2, b = (\alpha_2 + \alpha_4)/2$ for $\alpha_i = (\alpha_1 + 2n, -2n, 0, 2 - \alpha_1)$, which we will revisit later in Equation (111) in Section 4.5, where it will be obtained by an action of T_2^{-n} shift operator on polynomial solutions (12). For such values of a and b we will need to calculate $U(-n, 1 - n - \frac{\alpha_1}{2}, x)$. Thanks to Kummer's transformation $U(a, b, x) = x^{1-b} U(a - b + 1, 2 - b, x)$ [14] we obtain a relation

$$U(-n, 1 - n - \alpha_1/2, x) = x^{n + \alpha_1/2}\, U(\frac{\alpha_1}{2}, +\frac{\alpha_1}{2} + n + 1, x)\,, \tag{43}$$

which is a polynomial of degree n according to Equation (42).

For the case of $\alpha_2 = 0$ we have $a = \alpha_1/2$ and $b = \alpha_1/2 + \alpha_3/2$. We will consider $\alpha_i = (\alpha_1, 0, -2n, 2 - \alpha_1 + 2n)$, which as shown in Section 4.5 are obtained by action of T_4^n shift operator on the polynomial solution (12). Accordingly, we are dealing with the Kummer function $U(\alpha_1/2, \alpha_1/2 - n, x)$. This expression is not a polynomial, as we can verify by explicitly calculating this function for $n = 1$ obtaining $U(\alpha_1/2, \alpha_1/2 - 1, x) = (2x + \alpha_1 - 2)e^x$ with U_x/U being however a rational function. In Section 4.5 we will prove that the action of T_4^n shift operator on the polynomial solution (12) generates solutions of the Riccati Equation (39) for $\alpha_i = (\alpha_1, 0, -2n, 2 - \alpha_1 + 2n)$.

3.4. Power Series Representation of p and q Variables

For $N = 4$ we will show that $q = j_1 + j_2, p = j_2 + j_3$ can be represented by power series in odd powers of z and the results are (up to an action with π automorphism and its powers)

$$q = \sum_{i=1}^{2} (c_i z + e_i z^3 + \ldots), \qquad p = \sum_{i=1}^{2} (c_i z + e_i z^3 + \ldots)\,,$$

or

$$q = \sum_{i=1}^{2} (c_i z + e_i z^3 + \ldots), \qquad p = \frac{\alpha_3 - \alpha_1}{z} + \sum_{i=1}^{2} (c_i z + e_i z^3 + \ldots)\,.$$

The second case can be transformed by s_1 Bäcklund transformation to the previous case.

Consider power series expansion $j_i = k_i z^{-m} + \ldots$ with the first term being lowest power in z. Comparing both sides of Equation (4), we notice that the lowest terms on the left and the right sides will be of the order

$$z^{-m-1} \sim z^{-2m} + z^{-m-1} (\Psi_{(-2m)} z^{-2m} + \ldots + \Psi_{(0)})\,, \tag{44}$$

where we use the expansion of Ψ in (5) in powers of z:

$$\Psi = \ldots + \frac{\Psi_{(-2)}}{z^2} + \frac{\Psi_{(-1)}}{z} + \Psi_{(0)} + \Psi_{(+1)} z^1 + \ldots.$$

For the terms on both sides of (44) to match and cancel each other we need to take $m = 1$ and set all $\Psi_{(k)} = 0, k < 0$. In such case only Ψ_0 contributes to the above equation.

Without losing generality we therefore adopt the expansion

$$j_i(z) = \frac{a_i}{z} + b_i + c_i z + d_i z^2 + e_i z^3 + \dots . \tag{45}$$

For expansion in (45), it follows that

$$\begin{aligned}\Psi_{(-2)} &= a_1^2 + a_3^2 - a_2^2 - a_4^2 = -2(a_1 + a_2)(a_2 + a_3)\,,\\ \Psi_{(-1)} &= 2(a_1 b_1 + a_3 b_3 - a_2 b_2 - a_4 b_4) = -2((a_1 + a_2)(b_2 + b_3) + (a_2 + a_3)(b_1 + b_2))\,,\end{aligned} \tag{46}$$

after we used that $a_4 = -a_1 - a_2 - a_3$ and $b_4 = -b_1 - b_2 - b_3$.

Next, we will effectively work with the dressing Equation (2) without Ψ to see whether solutions for $j_i = a_i/z + b_i + c_i z$ will be such that the divergent terms can be absorbed in Ψ of Equation (4) via transformation (9):

$$j_i \;\to j_i + (-1)^i \frac{1}{2z}\Psi = j_i + (-1)^i \frac{1}{2z}\Psi_{(0)} + (-1)^i \frac{1}{2z}\Psi_{(1)} z + \dots .$$

On the z^{-2} level of such dressing equations one finds the following expressions:

$$-(a_i + a_{i+1}) = a_{i+1}^2 - a_i^2 = (a_{i+1} + a_i)(a_{i+1} - a_i),\ i = 1, \dots, N\,, \tag{47}$$

which imposes that

$$a_i + a_{i+1} = 0 \ \text{ or } \ a_{i+1} - a_i = -1\,,$$

for each $i = 1, 2, 3, 4$. There are two independent solutions of the above equations:

$$a_i = (1, -1, 1, -1)\, a\,, \tag{48}$$

$$a_i = (a, -a, -1 - a, 1 + a)\,, \tag{49}$$

that all satisfy $\sum_i a_i = 0$. There are other similar solutions that one can obtain from (49) by acting with π, π^2, π^3 transformations to obtain other solutions, such as $a_i = (a, a-1, 1-a, -a)$ and $a_i = (-1+a, 1-a, -a, a)$. It therefore suffices to use the solution (49). The top Equation (48) is such that $a_i + a_{i+1} = 0$ for every $i = 1, 2, 3, 4$. Such divergence can be absorbed by the transformation (9) with $\Psi = 2a$. In addition, the divergent terms will be absent from expressions for p and q.

The other solution (49) is such that either $a_1 + a_2 = 0$ or $a_2 + a_3 = 0$, ensuring $\Psi_{-2} = 0$ according to relation (46). However the divergent terms are such that they cannot be removed the transformation (9) and the divergent terms will be present in expressions for p. Let us illustrate this by applying the transformation (9) with $\Psi = -2(1 + a)$. This results in $a_i = (1 + 2a, -(1 + 2a), 0, 0)$. As we will show below, such divergent terms can be removed by a Bäcklund transformation. The calculations done for $N = 4$ and $N = 6$ suggest that this is a general feature for all N values.

Now for solution (48) we obtain that the condition (46) for $\Psi_{(-1)} = -2((a_1 + a_2)(b_2 + b_3) + (a_2 + a_3)(b_1 + b_2)) = 0$ is satisfied automatically and accordingly b_i can be chosen arbitrarily. For (49) and the other configurations that can be obtained from (49) by π, we obtain conditions $(-1 - 2a)(b_1 + b_2) = 0$, $(-1 + 2a)(b_2 + b_3) = 0$ and $(1 - 2a)(b_1 + b_2) = 0$. Accordingly, b_i can be chosen arbitrarily if $a = \pm 1/2$ or we will have a $b_2 = -b_3$ or $b_2 = -b_1$ condition imposing one condition on b_i.

Now consider the z^{-1} level of Equation (4) without Ψ. With such a redefined system one obtains on the z^{-1} level $0 = a_{i+1} b_{i+1} - a_i b_i$. For the solutions in (48) and (49) we find that we can write $b_i = b(1, -1, 1, -1)$ and we can set $b = 0$ without losing any generality as the terms can be added or removed by the transformation (9). A similar conclusion can be obtained for other coefficients of terms with z to the even power: z^{2k}. Such terms will not contribute to $q = j_1 + j_2$, $p = j_2 + j_3$ and we don't need to consider them in what follows.

Now consider z^0 levels of the Equation (2):

$$(1+2a_i)c_i + (1-2a_{i+1})c_{i+1} = \alpha_i, \quad i = 1, \ldots, N\,, \tag{50}$$

using $b_i^2 = b_{i+1}^2$.

We first enter values for a_i from (48) into the above equation to obtain

$$a = \frac{1}{2}(-1+\alpha_1+\alpha_3)\,,$$

using that $\sum_i c_i = 1$. For a_i given in (49) we find

$$a = \frac{1}{2}(-1+\alpha_1-\alpha_3)\,, \tag{51}$$

and

$$c_1 + c_2 = \frac{\alpha_1}{\alpha_1 - \alpha_3}\,. \tag{52}$$

We will now apply our results to $q = j_1 + j_2, p = j_2 + j_3$ variables. For $a_i = (a, -a, -1-a, 1+a)$ and a given in (51) it holds that $-(1+2a) = \alpha_3 - \alpha_1$ and

$$q = c_{12} z + e_{12} z^3 + \ldots, \qquad p = \frac{\alpha_3 - \alpha_1}{z} + c_{23} z + e_{23} z^3 + \ldots. \tag{53}$$

Here, for brevity, we introduced $c_{12} = c_1 + c_2$ given in Equation (52). Explicit calculation gives

$$c_{23} = \frac{\alpha_3^2 + \alpha_3\alpha_2 - 2\alpha_3 + \alpha_1^2 + \alpha_1^2 + \alpha_2\alpha_1 - 2\alpha_2 - 2\alpha_1}{\alpha_3^2 - 2\alpha_3\alpha - 1 - 4 + \alpha_1^2}\,.$$

It follows that the singular term in p in (53) can be removed by s_1 transformation: $q \to q$, $p \to p + \alpha_1/q$ with

$$\begin{aligned} \frac{\alpha_1}{q} &= \frac{1}{\frac{1}{\alpha_1}(-\frac{\alpha_1}{\alpha_3-\alpha_1} z + e_{12} z^3 + \ldots)} = \frac{1}{\frac{-z}{\alpha_3-\alpha_1}(1 - \frac{z^2 e_{12}(\alpha_3-\alpha_1)}{\alpha_1} + \ldots)}\,, \\ &= -\frac{\alpha_3-\alpha_1}{z}(1 + \frac{z^2 e_{12}(\alpha_3-\alpha_1)}{\alpha_1} + z^4 \ldots) = -\frac{\alpha_3-\alpha_1}{z} - z\frac{e_{12}(\alpha_3-\alpha_1)}{\alpha_1} + z^3 \ldots)\,, \end{aligned} \tag{54}$$

which shows that the transformed p given by $p + \alpha_1/q$ will no longer contain a singular term. Its power expansion will start with the term proportional to z and will only contain odd powers of z.

The initial position of the pole can be obviously moved from p to q by the π automorphism. This will lead to s_1 being transformed by π to other s_i, which will remove the divergent terms. With this understanding we continue to consider the above configuration without any loss of generality. One can therefore effectively only consider the case of $a_i = a(1, -1, 1, -1)$ from (48) with

$$q = c_{12} z + e_{12} z^3 + \ldots, \quad p = c_{23} z + e_{23} z^3 + \ldots,$$

with

$$c_{12} = \frac{\alpha_1}{\alpha_1 + \alpha_3}, \; c_{23} = \frac{\alpha_2}{2 - \alpha_1 - \alpha_3}\,.$$

Amazingly, the first terms of a general expression for q, p agree with a general formula

$$q = \frac{\alpha_1}{\alpha_1 + \alpha_3} z, \quad p = \frac{\alpha_2}{2 - \alpha_1 - \alpha_3} z, \tag{55}$$

that reproduces all the cases of (22)–(26) for the corresponding values of α_i.

Let us illustrate all this by the following example.

Example 1. *The solution*

$$q = \frac{z(-468+z^4)}{2(z^4-324)},\ \ p = \frac{z^2-18}{2z},\ \alpha_1 = \frac{13}{2}, \alpha_2 = -1, \alpha_3 = -\frac{5}{2}, \tag{56}$$

is taken from ref. [15], where it was obtained using Maya diagram techniques. Clearly $p = z/2 - 9/z$ *contains a singularity. Note that indeed* $-9/z = (\alpha_3 - \alpha_1)/z$ *in agreement with relation (53). Applying* s_1*, we obtain:*

$$q = \frac{z(-468+z^4)}{2(z^4-324)},\ \ p = \frac{z(z^4+8z^2-468)}{2(z^4-468)},\ \alpha_1 = -\frac{13}{2}, \alpha_2 = \frac{11}{2}, \alpha_3 = -\frac{5}{2}, \tag{57}$$

with polynomial expansions:

$$q(z) = \frac{13}{18}z + \frac{1}{1458}z^5 + \frac{1}{472{,}392}z^9 + \dots,\ \ p(z) = \frac{1}{2}z - \frac{1}{117}z^3 - \frac{1}{54{,}756}z^7 + \dots.$$

Note that $13/18 = \alpha_1/(\alpha_1+\alpha_3)$*. We will show below how to derive the rational solution (57) from the seed solutions (11)–(12) (or (22)–(26)) by the shift operators.*

Applying Equations (47) and (50) to $N = 6$, we find that the number of solutions increased from two to three (up to an action of π automorphism) and they are given by:

$$a_i = a\,(1,-1,1,-1,1,-1), \qquad a = -\frac{1}{2}(1-\alpha_1-\alpha_3-\alpha_5), \tag{58}$$

$$a_i = (a,-a,-1-a,1+a,-1-a,1+a), \quad a = -\frac{1}{2}(1+\alpha_1-\alpha_3-\alpha_5), \tag{59}$$

$$a_i = (a,-a,-1-a,1+a,a,-a), \qquad a = -\frac{1}{2}(1+\alpha_1-\alpha_3+\alpha_5), \tag{60}$$

for expansions $j_i(z) = a_i/z + c_i z + \dots, i = 1,\dots,6$. For solutions (59) and (60) there will be poles in expansions of $p_i, i = 1,2$.

Note that from Equation (50) we find $c_1 + c_2 = \alpha_1/(1+2a)$ and $c_3 + c_4 = -\alpha_3/(1+2a)$, where a is given in relations (59) and (60), respectively.

In the case of solution (59) the expansion of p_1 starts with a pole $p_1 = -(\alpha_1 - \alpha_3 - \alpha_5)/z + \dots$ while the expansion of q_1 is $q_1 = (c_1+c_2)z + \dots = \alpha_1 z/(\alpha_1 - \alpha_3 - \alpha_5) + \dots$. Consequently, the action of s_1 on p_1 removes the pole similarly to what we have seen for the $N = 4$ case in expression (54).

In the case of solution (60), both expansions of $p_i, i = 1,2$ will start with divergent terms: $p_1 = -(\alpha_1 - \alpha_3 + \alpha_5)/z + \dots$ and $p_2 = (\alpha_1 - \alpha_3 + \alpha_5)/z + \dots$. Since $q_1 = (c_1 + c_2)z + \dots$ and $q_2 = (c_1 + c_2 + c_3 + c_4)z + \dots$, we easily find that $q_1 - q_2 = \alpha_3 z/(\alpha_1 - \alpha_3 + \alpha_5) + \dots$. Consequently, the action of s_3 from Equation (29) on p_1 and p_2 will remove these divergencies. For those solutions that are obtained from solutions (59) or (60) by acting with automorphism π or its powers, the divergencies will be removed by appropriate Bäcklund transformations that are conjugations of s_1, s_3, e.g., $\pi s_1 \pi^{-1}$, $\pi s_3 \pi^{-1}$, etc.

4. Construction of Rational Solutions

In this section, we will describe a method to derive all rational solutions that are obtainable from the first-order polynomial solutions of dressing Equation (4) via the combined actions of fundamental shift operators $T_i, i = 1,\dots,N$ from (68).

4.1. Summary of the Results for $N = 4$

For $N = 4$ the seeds solutions (11) and (12) of dressing Equation (4) are equivalent to Watanabe's algebraic solutions (22)–(26) in the setting of Hamilton Equation (20). It is convenient to give the classification of solutions in terms of parameters $\alpha_1, \alpha_3, (\alpha_2 - \alpha_4)/2$ of the dressing chain equations that define the Painlevé V parameters $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ via relations (38) with $\bar{\delta}$ parameter being non-zero and here equal to $\bar{\delta} = -1/2$ (for $\epsilon^2 = 1$).

The rational solutions obtained by acting with the shift operators fall into three classes of parameters $\alpha_1, \alpha_3, (\alpha_2 - \alpha_4)/2$, and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ depending on whether the fundamental shift operators act on solutions

- $j_i = (z/4)(1,1,1,1)$ from (11) (items (Ia,Ib) and item (II)). In case of item (II) an intermediary step of acting with s_1 in addition to the shift operators is involved, see f.i. Equation (53).
- $j_i = (z/2)(1,1,-1,1)$ from (12) (items (IIIa,IIIb)).

These three cases are as follows:

(I)

$$\alpha_1 = A + 2n_1 - 2n_2,\ \alpha_3 = A + 2n_3 - 2n_4,\ \frac{\alpha_2 - \alpha_4}{2} = n_2 - n_3 - n_4 + n_1,$$

with $n_i \in \mathbb{Z}, i = 1, \ldots, 4$ and A arbitrary. The above implies either (Ia) or (Ib):

(Ia) $\bar{\alpha} = \frac{1}{2}(a)^2,\ \ \bar{\beta} = -\frac{1}{2}(a+n)^2$ and $\bar{\gamma} = \epsilon m$ where $m + n$ is even and equal to $2(n_1 - n_3)$ and $a = A/2 + n_3 - n_4$ arbitrary,

(Ib) $\bar{\alpha} = \frac{1}{2}(b+n)^2,\ \ \bar{\beta} = -\frac{1}{2}(b)^2$ and $\bar{\gamma} = \epsilon m$ where $m + n$ is even and equal to $2(n_2 - n_4)$ and $b = A/2 + n_1 - n_2$ arbitrary

(II)

$$\begin{aligned} \alpha_1 &= 1 + 2n_1 - 2n_2,\ \alpha_2 = -A + 2n_2 - 2n_3, \\ \alpha_3 &= 1 + 2n_3 - 2n_4, \quad \alpha_4 = A + 2n_4 - 2n_1\,, \end{aligned}$$

which imply

$$\bar{\alpha} = \frac{1}{2}\left(\frac{1}{2} + m\right)^2,\ \bar{\beta} = -\frac{1}{2}\left(\frac{1}{2} + n\right)^2,\ \bar{\gamma} = (-A + n + m)\epsilon\,,$$

where A is arbitrary and n, m are integers.

(IIIa)

$$\alpha_1 = A + 2n_1 + 2n_2,\ \alpha_3 = -2n_4,\ \frac{\alpha_2 - \alpha_4}{2} = -\frac{\alpha_4}{2} = \frac{A}{2} - 1 - n_4 + n_1 - n_2, \quad n_2, n_4 \in \mathbb{Z}_+,\ n_1 \in \mathbb{Z},$$

with A arbitrary and $\mathbb{Z}_+$ that includes positive integers and zero. Accordingly, eliminating the arbitrary number A from the above equations, we can write

$$\bar{\alpha} = \frac{1}{8}\alpha_3^2 = \frac{1}{2}(n)^2,\ \ \bar{\beta} = -\frac{1}{8}\alpha_1^2 = -\frac{1}{2}(\epsilon\bar{\gamma} + 1 + m)^2,$$

where $n = n_4, m = n_4 + 2n_2 \in \mathbb{Z}_+$ and with $n + m$ being an even integer.

(IIIb)

$$\alpha_1 = -2n_2,\ \alpha_3 = A + 2n_3 + 2n_4, \quad \frac{\alpha_2 - \alpha_4}{2} = 1 - \frac{A}{2} + n_2 - n_3 + n_4,\ \ n_2, n_4 \in \mathbb{Z}_+,\ n_3 \in \mathbb{Z},$$

with A arbitrary. $\mathbb{Z}_+$ includes positive integers and zero. Accordingly, eliminating the arbitrary number A from the above equations, we can write

$$\bar{\alpha} = \frac{1}{8}\alpha_3^2 = \frac{1}{2}(-\epsilon\bar{\gamma} + 1 + m)^2,\ \ \bar{\beta} = -\frac{1}{8}\alpha_1^2 = -\frac{1}{2}(n)^2,$$

where $n = n_2, m = n_2 + 2n_4 \in \mathbb{Z}_+$, and with $n + m$ being an even integer.

Comments: Integers n, m in (IIIa) and (IIIb) have been derived as positive integers. However they both enter quadratic expressions in which their overall sign can be reversed.

4.2. Applying the Shift Operators to Obtain Rational Solutions

For $N = 4$ we will show how to reproduce items (I)–(III) listed in Section (4.1) in the setting of Painlevé V equation using the following construction:

- The seeds of all rational solutions are the first-order polynomial solutions (11), (12) and its π variants. Note that these seed solutions all depend on an arbitrary real parameter, customarily chosen here as a.
- A class of rational solutions that can be obtained by successive operation by shift operators T_i, defined in the next Section 4.3, of the form:

$$T_1^{n_1} T_2^{n_2} T_3^{n_3} T_4^{n_4}, \qquad n_i \in \mathbb{Z}, \tag{61}$$

on polynomial solutions, (11) can be expanded in positive power series in z and does not contain a pole singularity and, if necessary (as in the case of Equation (53)), having this singularity removed by s_1 Bäcklund transformation. These two cases are described by the parameters presented in the above items I and II, respectively.
- A class of rational solutions obtained from the seed polynomial solutions (12) will be derived by successive operation with shift operators T_i of the type

$$T_i^{n_i} T_j^{n_j} T_k^{-n_k}, \qquad n_j, n_k \in \mathbb{Z}_+,\ n_i \in \mathbb{Z}, \tag{62}$$

for distinct i, j, k and $\mathbb{Z}_+$ containing positive integers and zero as only actions with shift operators given in Equation (62) that are not causing divergencies. The results are summarized in item III in Section (4.1).

We conclude that the well known fundamental results on classification of rational solutions of the Painlevé V equation first presented in [9] are here obtained by acting with the operators (61) on the first-order polynomial solutions (11) and (12). In the latter case, we will encounter restrictions on those values of n_i for which the operators (61) are well defined, as indicated in Equation (62). See also [10], which derived the rational solutions described above in items (Ia,Ib) and (II) via shift operators acting on solutions expressed by τ functions and corresponding to (11). The results of ref. [9] were summarized succinctly in [1].

4.3. The Fundamental Shift Operators for $A^{(1)}_{N-1}$

To analyze transformations under the shift operators which we will introduce in this subsection it is convenient to first introduce the following representation of α_i parameters for the $N = 4$ case:

$$\alpha_1 = 2(v_2 - v_1), \quad \alpha_2 = 2(v_3 - v_2), \quad \alpha_3 = 2(v_4 - v_3), \quad \alpha_4 = 2 + 2(v_1 - v_4)\,. \tag{63}$$

One checks that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2 + 2(v_2 - v_1 + v_3 - v_2 + v_4 - v_3 + v_1 - v_4) = 2\,,$$

is satisfied automatically without imposing any condition on v's.

Obviously, adding a constant term to all v_i will not change the final result in (63) and thus we have an equivalence:

$$(v_1, v_2, v_3, v_4) \sim (v_1 + c, v_2 + c, v_3 + c, v_4 + c). \tag{64}$$

The Bäcklund transformations $s_i, i = 1, 2, 3$ act in terms of v_i simply as permutations between v_i and v_{i+1}: $s_i : v_i \leftrightarrow v_{i+1}$, while $s_i(v_j) = v_j, j \neq i, i+1$. The automorphism π acts as follows: $\pi(v_i) = v_{i-1}, i = 2, 3, 4$ and $\pi(v_1) = v_4 - 1$.

Next, we introduce the shift operators

$$T_1 = \pi s_3 s_2 s_1, \quad T_2 = s_1 \pi s_3 s_2, \quad T_3 = s_2 s_1 \pi s_3, \quad T_4 = s_3 s_2 s_1 \pi, \tag{65}$$

that act as simple translations on the v_i variables: $T_i(v_j) = v_j - \delta_{i,j}$ leading to:

$$T_i(v_i) = v_i - 1, T_i(v_j) = v_j \longrightarrow \begin{cases} T_i(\alpha_i) = \alpha_i + 2, \\ T_i(\alpha_{i-1}) = \alpha_{i-1} - 2 \end{cases} , \tag{66}$$

or

$$\begin{aligned} T_1(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_1 + 2, \alpha_2, \alpha_3, \alpha_4 - 2)\,, \\ T_2(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_1 - 2, \alpha_2 + 2, \alpha_3, \alpha_4)\,, \\ T_3(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_1, \alpha_2 - 2, \alpha_3 + 2, \alpha_4)\,, \\ T_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (\alpha_1, \alpha_2, \alpha_3 - 2, \alpha_4 + 2)\,. \end{aligned} \tag{67}$$

Comparing expressions (67) and (66) we see that in the v_i representation it is very convenient to study how the parameter space of solutions of the dressing equation is being formed under actions of the shift operators. Generally the orbit of $v_i = (v_1, v_2, v_3, v_4)$ under an action with $T_1^{n_1} T_2^{n_2} T_3^{n_3} T_4^{n_4}$ from Equation (61) will be described by $v_i = (v_1 - n_1, v_2 - n_2, v_3 - n_3, v_4 - n_4)$. We are then able to associate a rational solution to each point of the orbit following the approach of Section 4.2.

It is easy to extend the definition of the fundamental shift operators to arbitrary N [4,10,16]:

$$T_1 = \pi s_{N-1} \cdots s_2 s_1,\ T_2 = s_1 \pi s_{N-1} \cdots s_2, \ldots, T_N = s_{N-1} \cdots s_2 s_1 \pi\,, \tag{68}$$

that for every N, the weight lattice of $A_{N-1}^{(1)}$ is generated. The shift operators commute with each other

$$T_i T_j = T_j T_i\,,$$

and satisfy $T_1 T_2 \cdots T_N = 1$, where we used that $\pi^N = 1$ and that $\pi s_i = s_{i+1} \pi$. These operators act on parameters α_i as

$$T_i(\alpha_{i-1}) = \alpha_{i-1} - 2, \quad T_i(\alpha_i) = \alpha_i + 2, \quad T_i(\alpha_j) = \alpha_j \ (j \neq i-1, i), \tag{69}$$

and further satisfy $\pi T_i = T_{i+1} \pi$, $T_i(\Phi) = \Phi$, $T_i(\Psi) = -\Psi$. The inverse shift operators for $N = 4$ are:

$$T_1^{-1} = s_1 s_2 s_3 \pi^3, \quad T_2^{-1} = s_2 s_3 \pi^3 s_1, \quad T_3^{-1} = s_3 \pi^3 s_1 s_2, \quad T_4^{-1} = \pi^3 s_1 s_2 s_3\,. \tag{70}$$

For convenience, we also list the shift operators for $N = 6$:

$$\begin{aligned} &T_1 = \pi s_5 s_4 s_3 s_2 s_1,\ T_2 = s_1 \pi s_5 s_4 s_3 s_2,\ T_3 = s_2 s_1 \pi s_5 s_4 s_3,\ T_4 = s_3 s_2 s_1 \pi s_5 s_4 \\ &T_5 = s_4 s_3 s_2 s_1 \pi s_5,\ T_6 = s_5 s_4 s_3 s_2 s_1 \pi\,, \end{aligned} \tag{71}$$

and their inverse

$$\begin{aligned} &T_1^{-1} = s_1 s_2 s_3 s_4 s_5 \pi^{-1},\ T_2^{-1} = s_2 s_3 s_4 s_5 \pi^{-1} s_1,\ T_3^{-1} = s_3 s_4 s_5 \pi^{-1} s_1 s_2,\ T_4^{-1} = s_4 s_5 \pi^{-1} s_1 s_2 s_3, \\ &T_5^{-1} = s_5 \pi^{-1} s_1 s_2 s_3 s_4,\ T_6^{-1} = \pi^{-1} s_1 s_2 s_3 s_4 s_5 \end{aligned} . \tag{72}$$

Within the framework of dressing chain equations with Bäcklund transformations (6) it is actually possible to establish general transformation rules for the shift operator T_i acting on $j_{i+1}, j_{i+2}, \ldots$ for $i = 1., \ldots, N$, which applies to $N = 4, 6$ and the initial configurations (11), (13):

$$
\begin{aligned}
j_{i+1,n+1} &= T_i(j_{i+1,n}) = j_{i\,n} - \frac{\mathsf{a}+2n}{j_{i,n}+j_{i+1,n}} \\
j_{i+2,n+1} &= T_i(j_{i+2,n}) = j_{i+1,n} + \frac{\mathsf{a}+2n}{j_{i,n}+j_{i+1,n}} \\
&\quad - \frac{4/N+2n}{j_{i+1,n}+j_{i+2,n}+j_{i+3,n}-T_i(j_{i+1,n})} \\
j_{i+3,n+1} &= T_i(j_{i+3,n}) = j_{i+2,n} + \frac{4/N+2n}{j_{i,n}+j_{i+1,n}+j_{i+2,n}-T_i(j_{i+1,n})} \\
&\quad - \frac{4/N+a+2n}{j_{i,n}+j_{i+1,n}+j_{i+2,n}+j_{i+3,n}-T_i(j_{i+1,n}+j_{i+2,n})}
\end{aligned}
\tag{73}
$$

etc., where $j_{i+k,n} = T_i^n(j_{i+k,0})$ with $j_{i+k,0} = z/N$ and $k = 1, 2 \ldots$. The above equations lead to

$$
\begin{aligned}
j_{i+1,n+1} + j_{i+2,n+1} = T_i(j_{i+1,n}+j_{i+2,n}) &= j_{i\,n} + j_{i+1,n} \\
&- (1+\frac{N}{2}n)\frac{4}{N}\frac{j_{i,n}+j_{i+1,n}}{(j_{i,n}+j_{i+1,n})(j_{i+1,n}+j_{i+2,n})+\mathsf{a}+2n}
\end{aligned}
\tag{74}
$$

which for $i = 1$ will lead to recurrence relations for $p = j_2 + j_3$ in case of $N = 4$ and for $p_1 = j_2 + j_3$ in case of $N = 6$. These recurrence relations will establish Umemura polynomial solutions as will be shown below.

4.4. Shift Operators Acting on the Solution $j_i = \frac{z}{4}(1,1,1,1)$ in Equation (11)

4.4.1. Parameters of the Solutions Obtained from the Seed Solution $j_i = \frac{z}{4}(1,1,1,1)$ by Action of the Shift Operators

Consider solution (11) such that $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a})$ with an arbitrary parameter a and $q = p = z/2$. According to relation (67), these solutions under action of (61) will have the following final parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:

$$
\alpha_1 = \mathsf{a} + 2n_1 - 2n_2,\ \alpha_3 = \mathsf{a} + 2n_3 - 2n_4,\ \alpha_2 = 1 - \mathsf{a} + 2n_2 - 2n_3,\ \alpha_4 = 1 - \mathsf{a} + 2n_4 - 2n_1\,. \tag{75}
$$

Thus, in agreement with item I in Section 4.1 we find

$$
\begin{aligned}
\frac{\alpha_1-\alpha_3}{2} &= n_1 - n_2 - n_3 + n_4 = k_1 - k_2 = 2k_- \\
\frac{\alpha_2-\alpha_4}{2} &= n_1 - n_3 + n_2 - n_4 = k_1 + k_2 = 2k_+\,,
\end{aligned}
\tag{76}
$$

where we introduced

$$
k_1 = n_1 - n_3, \quad k_2 = n_2 - n_4, \quad k_\pm = \frac{1}{2}(k_1 \pm k_2)\,. \tag{77}
$$

In terms of these parameters, we can decompose $T_1^{n_1}T_2^{n_2}T_3^{n_3}T_4^{n_4}$ into a product of different factors

$$
T_1^{n_1}T_2^{n_2}T_3^{n_3}T_4^{n_4} = T_1^{k_1}T_2^{k_2}(T_1T_3)^{n_3}(T_2T_4)^{n_4} = (T_1T_2)^{k_+}(T_1T_2^{-1})^{k_-}(T_1T_3)^{n_3}(T_2T_4)^{n_4}\,, \tag{78}
$$

with each factor acting independently of the others on parameters in Equation (76). Their action on expression (11) with $(\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a})$ induces the following transformations:

1. $(T_1T_3)^{n_3}$ increases arbitrary parameter a: $\mathsf{a} \to \mathsf{a} + 2n_3$ but leaves $q = p = z/2$ of equation (22) unchanged.
2. $(T_2T_4)^{n_4}$ decreases arbitrary parameter a: $\mathsf{a} \to \mathsf{a} - 2n_4$ but leaves $q = p = z/2$ of Equation (22) unchanged.

3. $(T_1T_2)^{k_+}$ increases $\frac{1}{2}(\alpha_2-\alpha_4) \to \frac{1}{2}(\alpha_2-\alpha_4)+2k_+$
4. $(T_1T_2^{-1})^{k_-}$ increases $\frac{1}{2}(\alpha_1-\alpha_3) \to \frac{1}{2}(\alpha_1-\alpha_3)+2k_-$

The conclusion in point 1 follows easily from the transformation rule:

$$(T_1T_3)^k(j_i) = (T_2T_4)^{-k}(j_i) = \frac{z}{4} + (-1)^{i+1}\frac{2k}{z} \tag{79}$$

where $j_i = z/4$ is one of the components of solution (11). A similar argument applies to point 2 since $T_1T_2T_3T_4 = 1$. The first two top transformations in points 1 and 2 do not induce any change in $\frac{1}{2}(\alpha_2-\alpha_4)$ nor in $\frac{1}{2}(\alpha_1-\alpha_3)$, thus the shift operators $(T_1T_3)^{n_3}$ and $(T_2T_4)^{n_4}$ equally increase Painlevé V parameters $\bar{\alpha}$ and $\bar{\beta}$ and are not changing the $\epsilon\gamma$ parameter. The above discussion shows that the two seed configurations $(\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a})$ and $(\mathsf{b}, 1-\mathsf{b}, \mathsf{b}, 1-\mathsf{b})$, both corresponding to the solution (22) with parameters a and b such that $\mathsf{b} = \mathsf{a} + 2m$, with m being an integer, can be connected by the transformation $(T_1T_3)^{n_3}(T_2T_4)^{n_4}$ with $m = n_3 - n_4$, leaving $q = p = z/2$ of equation (22) unchanged. Thus, they both can give rise to an identical solution $y, (\alpha, \beta, \gamma, \delta)$ of the Painlevé V equation via actions of different fundamental shift operators. However, this ambiguity disappears when the two seed solutions are considered as solutions (11) of the dressing chain since their $j_i(z)$ components will transform non-trivially under $(T_1T_3)^{n_3}(T_2T_4)^{n_4}$ according to relation (79) as long as $n_3 \neq n_4$.

The shift operator $(T_1T_2)^{k_+}$ increases $\epsilon\gamma$ by $2k_+$, while $(T_1T_2^{-1})^{k_-}$ changes a difference between $\bar{\alpha}$ and $\bar{\beta}$ of Painlevé V parameters. To illustrate how the Painlevé V parameters $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ transform under the above combinations of shift operators, we recall expressions (38) and take into account expressions (75) to obtain:

$$\bar{\alpha} = \frac{(\mathsf{a}/2 + n_3 - n_4)^2}{2},\ \bar{\beta} = -\frac{(\mathsf{a}/2 + n_1 - n_2)^2}{2},\ \bar{\gamma} = \epsilon(n_2 - n_3 - n_4 + n_1)\,. \tag{80}$$

In terms of integers $k_\pm$, the above expressions can be rewritten succinctly as:

$$\bar{\alpha} = \frac{1}{2}\left(\sqrt{-2\bar{\beta}} + n_2 - n_1 + n_3 - n_4\right)^2 = \frac{1}{2}\left(\sqrt{-2\bar{\beta}} - 2k_-\right)^2,\ \bar{\gamma} = 2\epsilon k_+\,.$$

Sometimes one encounters a pole in an initial expression for p as was the case in solution (56), where s_1 was used to remove the pole from p. To cover such a case, we apply s_1 Bäcklund transformation to obtain a configuration $(-\mathsf{a}, 1, \mathsf{a}, 1)$. Then, applying π automorphism we arrive at

$$(1, -\mathsf{a}, 1, \mathsf{a})\,.$$

Acting with $T_1^{n_1}T_2^{n_2}T_3^{n_3}T_4^{n_4}$ from (61) will yield:

$$\begin{aligned} \alpha_1 &= 1 + 2n_1 - 2n_2, & \alpha_2 &= -\mathsf{a} + 2n_2 - 2n_3, \\ \alpha_3 &= 1 + 2n_3 - 2n_4, & \bar{\alpha}_4 &= +\mathsf{a} + 2n_4 - 2n_1, \end{aligned} \tag{81}$$

with

$$\bar{\alpha} = \frac{1}{2}\left(\frac{1}{2} + n_3 - n_4\right)^2,\ \ \bar{\beta} = -\frac{1}{2}\left(\frac{1}{2} + n_1 - n_2\right)^2,\ \ \bar{\gamma} = \epsilon(-\mathsf{a} + n_2 - n_3 - n_4 + n_1),$$

setting $\mathsf{a} = A, n_3 = 0, n_4 = -m, n_1 = n, n_2 = 0$ we get item (II) in Section (4.1), in agreement with [9], see also [10].

Example 2. *Consider again the case of solution* (56) *with*

$$\alpha_1 = \frac{13}{2},\quad \alpha_2 = -1,\quad \alpha_3 = -\frac{5}{2},\quad \alpha_4 = -1\,,$$

and $p = z/2 - 9/z$ that contains a pole that can be removed by s_1. Fitting the above α's into relation (75) does not work since the method works for p being expandable in a positive series in z. We therefore try to fit it into a structure obtained from T_i's acting on configuration $(-\mathsf{a}, 1, \mathsf{a}, 1)$:

$$\begin{aligned} \bar{\alpha}_1 &= -\mathsf{a} + 2n_1 - 2n_2, \ \bar{\alpha}_2 = 1 + 2n_2 - 2n_3, \\ \bar{\alpha}_3 &= \mathsf{a} + 2n_3 - 2n_4, \quad \bar{\alpha}_4 = 1 + 2n_4 - 2n_1, \end{aligned} \tag{82}$$

For $\bar{\alpha}_1 = \frac{13}{2}, \bar{\alpha}_2 = -1, \bar{\alpha}_3 = -\frac{5}{2}, \bar{\alpha}_4 = -1$, it is now easy to find a class of solutions

$$n_2 = -1 + n_3, \ n_1 = 1 + n_4, \ \mathsf{a} = -\frac{5}{2} - 2n_3 + 2n_4$$

with n_3, n_4 being arbitrary integers. If we set f.i. $n_3 = n_4 = 1$, then $n_2 = 0$ and $\mathsf{a} = -5/2$ from the solution.

Note that relations (82) are equivalent with

$$\bar{\alpha} = \frac{1}{2}(\mathsf{a}/2 + n_3 - n_4)^2, \ \bar{\beta} = -\frac{1}{2}(-\mathsf{a}/2 + n_1 - n_2)^2, \ \bar{\gamma} = \epsilon\,(n_2 - n_3 - n_4 + n_1), \tag{83}$$

Setting $a = -(\mathsf{a}/2 + n_3 - n_4) = \mathsf{a}/2 - n_1 + n_2$, we can rewrite the above as

$$\bar{\alpha} = \frac{1}{2}(a+m)^2, \ \bar{\beta} = -\frac{1}{2}(a)^2, \ \bar{\gamma} = \epsilon\,k\,,$$

with $m = n_1 - n_2 + n_3 - n_4, k = n_2 - n_3 - n_4 + n_1$ and $k + m = 2n_1 - 2n_4$ being an even number (see also [9] or (I) in Section 4.1.

Example 3. *In this example, instead of connecting the solution (56) to the seed solution with $(-\alpha_1, 1, \alpha_1, 1)$ we will rather take the polynomial solution (57) with $\alpha_1 = -\frac{13}{2}, \alpha_2 = \frac{11}{2}, \alpha_3 = -\frac{5}{2}, \alpha_4 = \frac{11}{2}$ obtained by acting with s_1 on solution (56) from [15] and show that it can be obtained from polynomial solution (11) with*

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\mathsf{a}, 1 - \mathsf{a}, \mathsf{a}, 1 - \mathsf{a})\,,$$

by successive operations of translation operations T_i, each acting n_i times. Recalling the actions of T_i (67), we obtain the following 4 conditions for the solution (57) to be obtained from the solution (11) by T_i's each acting n_i times:

$$\begin{aligned} \mathsf{a} + 2n_1 - 2n_2 &= -\frac{13}{2}, \quad (1 - \mathsf{a}) + 2n_2 - 2n_3 = \frac{11}{2} \\ \mathsf{a} + 2n_3 - 2n_4 &= -\frac{5}{2}, \quad (1 - \mathsf{a}) + 2n_4 - 2n_1 = \frac{11}{2}\,, \end{aligned}$$

with a general solution given in terms of arbitrary n_3, n_4:

$$n_1 = n_3 - 1, \ n_2 = n_4 + 1, \ \mathsf{a} = -\frac{5}{2} + 2n_4 - 2n_3,$$

that involves action by the shift operators equal to

$$T_1^{-1+n_3} T_2^{1+n_4} T_3^{n_3} T_4^{n_4} = (T_1 T_3)^{-1+n_3} (T_2 T_4)^{1+n_4} T_3 T_4^{-1}\,.$$

The above expression shows that there is no ambiguity related to the choice of n_3 and n_4 as $(T_1T_3)^{-1+n_3}$ and $(T_2T_4)^{1+n_4}$ do not change the form of the solution. Therefore, for simplicity we eliminate the first two factors of the above expression by choosing:

$$n_3 - 1 = n_4 + 1 = 0 \ \rightarrow \ n_1 = n_2 = 0, \ \mathsf{a} = -\frac{13}{2},$$

and thus the action of shift operators (61) becomes that of $T_3T_4^{-1}$. The action of the inverse operator $T_4^{-1} = \pi^{-1}s_1s_2s_3$ on $p = q = z/2, (\mathsf{a}, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a})$ is well defined and yields

$$q = \frac{1}{2}z\frac{(z^2-4\mathsf{a})}{(z^2-4\mathsf{a}-4)},\ p = \frac{1}{2}z\frac{(z^2-4\mathsf{a}+4)}{(z^2-4\mathsf{a})},\ (\mathsf{a}, 1-\mathsf{a}, 2+\mathsf{a}, -1-\mathsf{a})\,.$$

Applying T_3 on the above expressions we get:

$$q = \frac{z(16\mathsf{a}^2+32\mathsf{a}-z^4)}{2(4\mathsf{a}+8+z^2)(-z^2+4\mathsf{a}+8)},\quad p = \frac{z(16\mathsf{a}^2+32\mathsf{a}-8z^2-z^4)}{2(16\mathsf{a}^2+32\mathsf{a}-z^4)}$$
$$(\mathsf{a}, -1-\mathsf{a}, 4+\mathsf{a}, -1-\mathsf{a})\,,$$

which for $\mathsf{a} = -\frac{13}{2}$ reproduces expression (57).

4.4.2. Umemura Polynomial Solutions Obtained from $j_i = \frac{z}{4}(1,1,1,1)$ Seed Solution through Action of the Shift Operators

As follows from relations (73) applied to the $N = 4$ case, we have the following recurrence relations

$$\begin{aligned} j_{4,n+1} &= T_3(j_{4,n}) = j_{3\,n} - \frac{\mathsf{a}+2n}{j_{3,n}+j_{4,n}} \\ j_{1,n+1} &= T_3(j_{1,n}) = j_{4,n} + \frac{\mathsf{a}+2n}{j_{3,n}+j_{4,n}} - (1+2n)\frac{j_{3,n}+j_{4,n}}{(j_{3,n}+j_{4,n})(j_{4,n}+j_{1,n})+\mathsf{a}+2n}\,, \end{aligned} \tag{84}$$

for transformations induced by T_3.

Since $\sum_{i=1}^4 j_{4,n} = z$ and $\sum_{i=1}^4 j_{4,n+1} = z$, we find for $T_3(j_{2\,n}+j_{3,n}) = z - T_3(j_{1\,n}+j_{4,n})$

$$\begin{aligned} T_3(j_{2\,n}+j_{3,n}) &= j_{1\,n}+j_{2,n}+(1+2n)\frac{j_{3,n}+j_{4,n}}{(j_{3,n}+j_{4,n})(j_{4,n}+j_{1,n})+\mathsf{a}+2n} \\ &= j_{1\,n}+j_{2,n}+(1+2n)\frac{z-(j_{1,n}+j_{2,n})}{(z-j_{1,n}-j_{2,n})(z-(j_{2,n}+j_{3,n}))+\mathsf{a}+2n}\,, \end{aligned}$$

which can be rewritten as

$$p_{n+1} = q_n + (2n+1)\frac{(z-q_n)}{d_n}\,, \tag{85}$$

where for $q = j_1 + j_2, p = j_2 + j_3$ we introduced the following notation

$$q_n(z;\mathsf{a}) = T_3^n(q_0),\quad p_n(z;\mathsf{a}) = T_3^n(p_0),\quad q_0 = p_0 = \frac{z}{2}\,, \tag{86}$$

and

$$d_n(z;\mathsf{a}) = (z-q_n)(z-p_n) + 2n + \mathsf{a}\,. \tag{87}$$

Similarly from Equation (73) we find

$$T_3(j_{1\,n}+j_{2,n}) = j_{1\,n}+j_{4,n}+\frac{(\mathsf{a}+2n)}{(z-j_{1,n}-j_{2,n})} - \frac{(\mathsf{a}+2n+1)}{z-T_3(z-j_{2,n}-j_{3,n})}\,,$$

that can be rewritten as

$$q_{n+1} = z - p_n + \frac{\mathsf{a}+2n}{z-q_n} - \frac{\mathsf{a}+2n+1}{p_{n+1}} = \frac{d_n}{z-q_n} - \frac{\mathsf{a}+2n+1}{p_{n+1}}\,, \tag{88}$$

and together with Equation (85) form two recurrence relations for the canonical quantities q_n, p_n. One finds from relations (85) and (88) that

$$\frac{q_n}{z - q_n} d_n = q_{n+1} p_{n+1} + \mathsf{a} \,,$$

which shows that the quantity d_n is useful in describing transition from p_n, q_n to p_{n+1}, q_{n+1}. Indeed, we will be able below to formulate the recurrence relation for Umemura polynomials based on the existence of alternative expressions (94) for d_n.

It is convenient to introduce the polynomials $U_n(z;\mathsf{a})$ to which we will refer as Umemura polynomials [7,11] defined for $n = 0, 1, 2, 3$ by

$$U_0(z;\mathsf{a}) = 1, \quad U_1(z;\mathsf{a}) = 1 \,, \tag{89}$$

$$U_2(z;\mathsf{a}) = z^2 + 4\mathsf{a} \,, \tag{90}$$

$$U_3(z;\mathsf{a}) = z^6 + 12z^4\mathsf{a} + 12z^4 + 48z^2\mathsf{a}^2 + 96z^2\mathsf{a} + 192\mathsf{a}^2 + 128\mathsf{a} + 64\mathsf{a}^3 \,. \tag{91}$$

Note that $U_n(z;\mathsf{a}) = z^{n(n-1)} + \ldots$ is a polynomial of the $n(n-1)$-th order. In terms of the above polynomials, we can express q_1, p_1, d_1 in the following way

$$\begin{aligned} q_1(z;\mathsf{a}) &= \frac{z}{2} \frac{U_2(z;\mathsf{a}) U_1(z;\mathsf{a}+3)}{U_2(z;\mathsf{a}+1) U_1(z;\mathsf{a}+2)}, \quad p_1(z;\mathsf{a}) = \frac{z}{2} \frac{U_2(z;\mathsf{a}+1) U_1(z;\mathsf{a})}{U_2(z;\mathsf{a}) U_1(z;a+1)}, \\ d_1(z;\mathsf{a}) &= \frac{z^2}{4} \frac{U_2(z;\mathsf{a}+2)\, U_2(z;\mathsf{a}-1)}{U_2(z;\mathsf{a}+1)\, U_2(z;\mathsf{a})} + 2 + \mathsf{a} = \frac{1}{4} \frac{U_3(z;\mathsf{a}) U_1(z;\mathsf{a}+1)}{U_2(z;\mathsf{a}) U_2(z;a+1)} \,. \end{aligned} \tag{92}$$

The repeating action of T_3 operator on expressions (92) gives rise to:

$$\begin{aligned} q_n(z;\mathsf{a}) &= \frac{z}{2} \frac{U_n(z;\mathsf{a}+3)\, U_{n+1}(z;\mathsf{a})}{U_n(z;\mathsf{a}+2)\, U_{n+1}(z;\mathsf{a}+1)}, \quad p_n(z;\mathsf{a}) = \frac{z}{2} \frac{U_n(z;\mathsf{a})\, U_{n+1}(z;\mathsf{a}+1)}{U_n(z;\mathsf{a}+1)\, U_{n+1}(z;\mathsf{a})} \\ d_n(z;\mathsf{a}) &= \frac{z^2}{4} \frac{U_{n+1}(z;\mathsf{a}+2)\, U_{n+1}(z;\mathsf{a}-1)}{U_{n+1}(z;\mathsf{a}+1)\, U_{n+1}(z;\mathsf{a})} + 2n + \mathsf{a} \,. \end{aligned} \tag{93}$$

Using the recurrence relations (85), (88) one can alternatively express the quantity $d_n = (z - q_n)(z - p_n) + 2n + \mathsf{a}$ as

$$\begin{aligned} d_n(z;\mathsf{a}) &= \frac{1}{4} \frac{U_n(z;\mathsf{a}+1) U_{n+2}(z;\mathsf{a})}{U_{n+1}(z;\mathsf{a}) U_{n+1}(z;\mathsf{a}+1)} \\ &= \frac{1}{4} \frac{U_n(z;\mathsf{a}+2) U_{n+2}(z;\mathsf{a}-1)}{U_{n+1}(z;\mathsf{a}) U_{n+1}(z;\mathsf{a}+1)} + (2n+1) \,. \end{aligned} \tag{94}$$

Comparing the bottom of expressions (93) with the two expressions in Equation (94), we obtain two alternative recurrence relations for the Umemura polynomials which independently can be used to generate higher level Umemura polynomials.

It is convenient at this point to introduce the variable $x = \frac{z^2}{4}$ and polynomials

$$W_n(x;\mathsf{a}) = 2^{-n(n-1)} U_n(z;\mathsf{a}) \,, \tag{95}$$

which satisfy two recurrence relations that follow from comparing expressions (93) with (94):

$$\begin{aligned} W_{n-1}(x;\mathsf{a}+1)W_{n+1}(x;\mathsf{a}) &= xW_n(x;\mathsf{a}+2)W_n(x;\mathsf{a}-1) \\ &\quad + (2n-2+\mathsf{a})W_n(x;\mathsf{a})W_n(x;\mathsf{a}+1) \end{aligned} \tag{96}$$

$$W_{n-1}(x;\mathsf{a}+3)W_{n+1}(x;\mathsf{a}) = xW_n(x;\mathsf{a}+3)W_n(x;\mathsf{a}) + \mathsf{a}W_n(x;\mathsf{a}+2)W_n(x;\mathsf{a}+1)\,. \tag{97}$$

Such redefined Umemura polynomials $W_n(x;\mathsf{a})$ are given for $n = 0,1,2,3,5$ by

$$W_0(x;\mathsf{a}) = 1, \quad W_1(x;\mathsf{a}) = 1\,, \tag{98}$$

$$W_2(x;\mathsf{a}) = x + \mathsf{a}\,, \tag{99}$$

$$\begin{aligned} W_3(x;\mathsf{a}) &= x^3 + 3x^2\mathsf{a} + 3x^2 + 3x\mathsf{a}^2 + 6x\mathsf{a} + 3\mathsf{a}^2 + 2\mathsf{a} + \mathsf{a}^3 \\ &= (x+\mathsf{a})^3 + 3(x+\mathsf{a})^2 + 2\mathsf{a}\,, \end{aligned} \tag{100}$$

$$\begin{aligned} W_4(x;\mathsf{a}) &= 48\,\mathsf{a} + 60\,x^3 + x^6 + 12\,x^5 + 45\,x^4 + 144\,x\,\mathsf{a} + 240\,x^2\,\mathsf{a} + 300\,x\,\mathsf{a}^2 + 6\,x^5\,\mathsf{a} + 15\,x^4\,\mathsf{a}^2 \\ &\quad + 60\,x^4\,\mathsf{a} + 20\,x^3\,\mathsf{a}^3 + 120\,x^3\,\mathsf{a}^2 + 190\,x^3\,\mathsf{a} + 15\,x^2\,\mathsf{a}^4 + 120\,x^2\,\mathsf{a}^3 + 300\,x^2\,\mathsf{a}^2 + 60\,x\,\mathsf{a}^4 \\ &\quad + 210\,x\,\mathsf{a}^3 + 6\,x\,\mathsf{a}^5 + 124\,\mathsf{a}^2 + 120\,\mathsf{a}^3 + 12\,\mathsf{a}^5 + \mathsf{a}^6 + 55\,\mathsf{a}^4\,, \end{aligned} \tag{101}$$

from which higher polynomials can be obtained using recurrence relations (96) or (97). In addition, the polynomials $W_n(x;\mathsf{a})$ satisfy the identity

$$2W_{n+1}(x,\mathsf{a})W_n(x;\mathsf{a}+1) - W_{n+1}(x,\mathsf{a}+1)W_n(x;\mathsf{a}) = W_{n+1}(x;\mathsf{a}-1)W_n(x;\mathsf{a}+2)\,, \tag{102}$$

established on the basis of consistency of the shift operator approach with various operators T_i connected via π. Although we have chosen arbitrarily to generate the recurrence relations by acting with T_3, we could take any other shift operator as a starting point and be able to transfer from one formalism to another by applying the automorphism π through relation $\pi T_i = T_{i+1}\pi$. The identity (102) ensures that acting with any of the shift operators $T_i, i = 1,2,3,4$ on expressions (92) will give rise to solutions that are still expressible in terms of Umemura polynomials $U_n(z;\mathsf{a})$. For example, the repeating action of T_1 operator on expressions (92) yields:

$$q_n^{(1)}(z;\mathsf{a}) = \frac{z}{2}\frac{U_n(z,\mathsf{a}+1)U_{n+1}(z,\mathsf{a}+2)}{U_n(z,\mathsf{a}+2)U_{n+1}(z,\mathsf{a}+1)}, \tag{103}$$

$$p_n^{(1)}(z;\mathsf{a}) = \frac{z}{2}\frac{U_n(z,\mathsf{a}+2)U_{n+1}(z,\mathsf{a}-1)}{U_n(z,\mathsf{a}+1)U_{n+1}(z,\mathsf{a})}. \tag{104}$$

Consider again equation (93) for $q_n(z;\mathsf{a})$ and plug q_n into expression $y = (q/z)(q/z-1)^{-1}$ for solution of the Painlevé V equation derived in Section 3.2. After some simple algebra we find:

$$y = \frac{W_n(x;\mathsf{a}+3)\,W_{n+1}(x;\mathsf{a})}{W_n(x;\mathsf{a}+3)\,W_{n+1}(x;\mathsf{a}) - 2W_n(x;\mathsf{a}+2)\,W_{n+1}(x;\mathsf{a}+1)}\,.$$

Using the identity (102) to rewrite the denominator, we obtain

$$y = -\frac{W_n(x;\mathsf{a}+3)\,W_{n+1}(x;\mathsf{a})}{W_n(x;\mathsf{a}+1)\,W_{n+1}(x;\mathsf{a}+2)}\,, \tag{105}$$

for $(\mathsf{a}, 1-\mathsf{a}-2n, \mathsf{a}+2n, 1-\mathsf{a})$ with the Painlevé parameters:

$$\bar{\alpha} = \frac{1}{2}(\frac{\mathsf{a}}{2}+n)^2, \;\; \bar{\beta} = -\frac{1}{2}(\frac{\mathsf{a}}{2})^2, \;\; \bar{\gamma} = -\frac{1}{\epsilon}n\,,$$

agreeing with the solution (Ib) given at the beginning of Section 4.

Consider now solution (103), generated by acting n times with the shift operator T_1. The parameters α_i for this solution are equal to $(\mathsf{a}+2n, 1-\mathsf{a}, \mathsf{a}, 1-\mathsf{a}-2n)$. Plugging the above $q(z)$ into expression $y = (q/z)(q/z-1)^{-1}$ and using the identity (102) we get

$$y = -\frac{W_n(x;\mathsf{a}+1)\, W_{n+1}(x;\mathsf{a}+2)}{W_n(x;\mathsf{a}+3)\, W_{n+1}(x;\mathsf{a})}\,, \tag{106}$$

with the Painlevé V parameters

$$\bar{\alpha} = \frac{1}{2}\left(\frac{\mathsf{a}}{2}\right)^2, \;\; \bar{\beta} = -\frac{1}{2}\left(\frac{\mathsf{a}}{2}+n\right)^2, \;\; \bar{\gamma} = n, \; \bar{\delta} = -\frac{1}{2}\,,$$

that agree with the solution (Ia) given at the beginning of section 4 for the Painlevé V variable $t = 2x$.

The fact that the above y satisfies the Painlevé V equation is equivalent to the Umemura polynomials $W_n(x,\mathsf{a})$ satisfying the σ-type of relation, which can be given a form of a Toda-like equation:

$$\frac{W_{n-1}(x;\mathsf{a})W_{n+1}(x;\mathsf{a})}{W_n^2(x;\mathsf{a})} = x+\mathsf{a}+3(n-1)+2\frac{d}{dx}x\frac{d}{dx}\ln W_n(x;\mathsf{a})\,.$$

Next we define quantity:

$$\omega_a = \frac{W_n(x;\mathsf{a})\, W_{n+1}(x;\mathsf{a}+1)}{W_n(x;\mathsf{a}+1)\, W_{n+1}(x;\mathsf{a})} - 1, \tag{107}$$

where we suppressed dependence on n on the left hand side. It is interesting to notice that, as follows from applications of all three identities (96), (97) and (102), ω_a satisfies a discrete Painlevé II equation [11]:

$$\omega_{a-1} + \omega_{a+1} = \frac{2}{x}\frac{1}{1-\omega_a^2}(n+(1-n-a)\omega_a)\,. \tag{108}$$

See [17] for an early observation that Bäcklund transformations of continuous models can give rise to a discrete structure.

4.5. Action of the Shift Operators on $j_i = \frac{z}{2}(1,1,-1,1)$ Solution in (12)

By acting with $T_1^{n_1}T_2^{n_2}T_3^{n_3}T_4^{n_4}$ on $j_i = (z/2)(1,1,-1,1)$ from Equation (12) with $\alpha_i = (\mathsf{a},0,0,2-\mathsf{a})$ we will arrive, in principle, at the following parameters of the final configuration

$$\begin{aligned}\alpha_1 &= \mathsf{a}+2n_1-2n_2,\; \alpha_2 = 2n_2-2n_3,\\ \alpha_3 &= 2n_3-2n_4, \quad \alpha_4 = 2-\mathsf{a}+2n_4-2n_1\,,\end{aligned}$$

or

$$\alpha_1 = \mathsf{a}+2n_1-2n_2,\; \alpha_3 = 2n_3-2n_4,\; \frac{\alpha_2-\alpha_4}{2} = \frac{\mathsf{a}}{2}-1+n_2-n_3-n_4+n_1,\; n_i\in\mathbb{Z}\,. \tag{109}$$

However not all of the shift transformations are well defined when acting on $j_i = (z/2)(1,1,-1,1)$. Since $j_2+j_3 = 0$ and $j_3+j_4 = 0$ we see from the definition (6) that actions of s_2, s_3 involve divisions by zero and therefore are not allowed. Recalling the definitions (65) and (70), we accordingly need to exclude T_2, T_3 and T_3^{-1}, T_4^{-1}, as these operators contain s_3 and s_2 transformations at the positions to the right. Because the shift operators in (65) and (70) contain ordered products of neighboring Bäcklund transformations of the type $s_{i+1}s_i$ the divergence is only generated by the s_i located to the right. If the result of acting by s_i is not divergent, then acting with s_{i+1} would not be divergent, as follows from the definition (6).

Accordingly, to avoid divergencies, we will only consider the operators $T_1^{n_1} T_4^{n_4} T_2^{-n_2}$ with $n_2, n_4 \in \mathbb{Z}_+$ and $n_1 \in \mathbb{Z}$.

Indeed, one can verify that $T_2^{-1} = s_2 s_3 \pi^{-1} s_1$ is permissible and generates

$$
\begin{aligned}
T_2^{-1} &: q = z, p = 0 \to q = z, p = \frac{2z}{\mathsf{a} - z^2}, \, (2 + \mathsf{a}, -2, 0, 2 - \mathsf{a}) \\
T_2^{-n} &: q = z, p = 0 \to q_n = z, p_n = \frac{2nzR_{n-1}(\mathsf{a}; z)}{R_n(\mathsf{a}; z)}, \, (2n + \mathsf{a}, -2n, 0, 2 - \mathsf{a}),
\end{aligned}
\tag{110}
$$

where $R_n(\mathsf{a}; z)$ is found to satisfy the recurrence relation:

$$R_{n+1}(\mathsf{a}; z) = 2nz^2 R_{n-1}(\mathsf{a}; z) + (-z^2 + 2n + \mathsf{a}) R_n(\mathsf{a}; z), \quad n = 1, 2, \ldots,$$

with $R_0(\mathsf{a}; z) = 1$. The solution to this recurrence relation is given by

$$R_n(\alpha_1; z) = \sum_{r=0}^{n} \binom{n}{r} (\mathsf{a})_{r,2} \, (-z^2)^{n-r}, \quad n = 0, 1, 2, \ldots, \tag{111}$$

where we used the Pochhammer k-symbol $(x)_{n,k}$ defined as $(x)_{n,k} = x(x+2)(x+2k)\cdots(x+(n-1)k)$. We notice that $R_n(\mathsf{a}; z)$ can be expressed as a function of $x = -z^2/2$ and in terms of x it holds that $dR_n(\mathsf{a}; x)/dx = 2nR_{n-1}(\mathsf{a}; x)$. Thus we find that p_n from Equation (110) satisfies $p_n/z = f_2/z = d(\ln R_n(\mathsf{a}; x))/dx$. Based on discussion around Equation (43) from Section 3.3, we expect that $R_n(\mathsf{a}; x)$ is related to Kummer's polynomial $U(-n, 1 - n - \mathsf{a}/2, x)$. Indeed an explicit calculation of expression (111) yields $R_n(\mathsf{a}; x) = 2^n x^{n+\mathsf{a}/2} U(\frac{\mathsf{a}}{2}, \frac{\mathsf{a}}{2} + n + 1, -\frac{z^2}{2})$, which according to relation (43) is equal (up to an overall constant) to $U(-n, 1 - n - \mathsf{a}/2, x)$, a solution to the Kummer's Equation (41) with $a = \alpha_2/2 = -n$, $b = (\alpha_2 + \alpha_4)/2 = -n + 1 - \mathsf{a}/2$. Here, we obtained this solution through acting n-th times with T_2^{-1} on the first-order solution (12). Since the Kummer's functions found many applications in, e.g., solvable quantum mechanics, atomic physics, and critical phenomena, among other fields, the fact that, as shown above, their form can be reproduced by action of the shift operators should be of potential interest for these applications and efforts to expand them.

The shift operator T_1 essentially acts as an identity

$$T_1 : q = z, \; p = 0, \; \alpha_i = (\mathsf{a}, 0, 0, 2 - \mathsf{a}) \; \longrightarrow \; q = z, \; p = 0, \; \alpha_i = (2 + \mathsf{a}, 0, 0, -\mathsf{a}),$$

its only action is to increase $\mathsf{a} \to \mathsf{a} + 2$.

Let us now take a closer look at the action of T_4 on $q = z, p = 0$. Acting once with T_4 yields:

$$q_1 = z - \frac{2z}{z^2 - \mathsf{a} + 2} = T_4(q_0), \quad p_1 = 0, \quad (\mathsf{a}, 0, -2, 4 - \mathsf{a}), \tag{112}$$

Acting n times with T_4 on $q_0 = z, p = 0$ we get $q_n = T_4^n(q_0)$ that satisfies the recurrence relation

$$q_n = z - \frac{2nz}{zq_{n-1} + 2n - \mathsf{a}}, \qquad (\mathsf{a}, 0, -2n, 2(n+1) - \mathsf{a}), \tag{113}$$

the corresponding expression for p_n is

$$p_n = q_{n-1} + \frac{2n - \mathsf{a}}{z} - \frac{2n}{z - q_n} = 0,$$

where the zero on the right hand side follows from the recurrence relation (113) connecting q_n, q_{n-1}.

It we assume that $F_{n-1} = q_{n-1}/z$ satisfies the Riccati Equation (39) for $i = 1$ and $\alpha_3 = -2(n-1)$, then it follows that $F_n = q_n/z$ with q_n determined through the recurrence relation (113) will satisfy the same Riccati Equation (39) for $\alpha_3 = -2n$. Since for $q_0 = z$ the function $F_0 = q_0/z = 1$ satisfies the Riccati Equation (39) for $\alpha_3 = 0$ this concludes the induction proof for q_n being equal to $zF_{\mathsf{a},\alpha_3=-2n}$, where F_{a,α_3} is given by expression (40) in terms of Whittaker functions.

Based on the above discussion, we can rewrite Equation (109) as

$$\alpha_1 = \mathsf{a} + 2n_1 + 2n_2,\ \alpha_3 = -2n_4,\ \frac{\alpha_2 - \alpha_4}{2} = \frac{\mathsf{a}}{2} - 1 - n_2 - n_4 + n_1,\ n_2, n_4 \in \mathbb{Z}_+, n_1 \in \mathbb{Z}, \quad (114)$$

after making a transformation $n_2 \to -n_2$.

Accordingly, Equation (114) gives rise to

$$\bar{\alpha} = \frac{1}{2}(-n_4)^2,\ \bar{\beta} = -\frac{1}{2}(\mathsf{a}/2 + n_1 + n_2)^2,\ \bar{\gamma} = \epsilon(\frac{\mathsf{a}}{2} - 1 - n_4 - n_2 + n_1), \quad (115)$$

or after elimination of an arbitrary constant a:

$$\bar{\alpha} = \frac{1}{2}(-n_4)^2,\ \bar{\beta} = -\frac{1}{2}(\epsilon\bar{\gamma} + 1 + n_4 + 2n_2)^2, \quad (116)$$

After learning how solution (12) transforms under a product of fundamental shift operators we turn our attention to the action of these operators on solutions that can be obtained from (12) by an automorphism π. Acting with π and π^2 on (12) we obtain, respectively, $j_i = (z/2)(1,1,1,-1)$ with $(2-\mathsf{a},\mathsf{a},0,0)$ and $j_i = (z/2)(-1,1,1,1)$ with $(0,2-\mathsf{a},\mathsf{a},0)$ as seeds configurations.

For $j_i = (z/2)(1,1,1,-1)$, we see that $j_3 + j_4 = 0$ and $j_4 + j_1 = 0$. Thus, comparing with relations (6) we recognize that the Bäcklund transformations $s_3, s_3\pi^{-1}, s_4, s_1\pi$ would involve divisions by zero. Accordingly, among the eight shift operators listed in (65) and (70), we need to discard $T_4, T_3, T_4^{-1}, T_1^{-1}$ that contain the above-mentioned Bäcklund transformations in the positions to the right. Accordingly, we will only act with $T_1^{n_1}T_3^{-n_3}T_2^{n_2}$ with $n_1, n_3 \in \mathbb{Z}_+, n_2 \in \mathbb{Z}$, generating the following transformations of $(2-\mathsf{a},\mathsf{a},0,0)$:

$$\begin{aligned} &\alpha_1 = 2 - \mathsf{a} + 2n_1 - 2n_2,\ \alpha_2 = \mathsf{a} + 2n_2 + 2n_3, \\ &\alpha_3 = -2n_3, \quad \alpha_4 = -2n_1, \quad n_1, n_3 \in \mathbb{Z}_+,\ n_2 \in \mathbb{Z}, \end{aligned} \quad (117)$$

The Painlevé parameters corresponding to (117) are:

$$\bar{\alpha} = \frac{1}{2}(n_3)^2,\ \bar{\beta} = -\frac{1}{2}(1 - \mathsf{a}/2 + n_1 - n_2)^2,\ \bar{\gamma} = \epsilon(\frac{\mathsf{a}}{2} + n_2 + n_1 - n_3),$$

or

$$\bar{\alpha} = \frac{1}{2}(n_3)^2,\ \bar{\beta} = -\frac{1}{2}(\epsilon\bar{\gamma} - 1 - 2n_1 + n_3)^2,$$

with n_1, n_3 being positive integers or zero. The above equation is similar to relation (116).

For $j_i = (z/2)(-1,1,1,1)$ we see that $j_1 + j_4 = 0$ and $j_1 + j_2 = 0$. We conclude from relations (6) that the Bäcklund transformations $s_1, s_3\pi^{-1}, s_4, s_1\pi, s_2\pi$ would involve divisions by zero. We therefore need to exclude $T_4, T_1, T_2^{-1}, T_1^{-1}$ among the eight shift operators listed in (65) and (70). The action with the remaining shift operators $T_2^{n_2}T_4^{-n_4}T_3^{n_3}$ with $n_2, n_4 \in \mathbb{Z}_+, n_3 \in \mathbb{Z}$ generates the following transformation of $(0,2-\mathsf{a},\mathsf{a},0)$:

$$\begin{aligned} &\alpha_1 = -2n_2,\ \alpha_2 = 2 - \mathsf{a} + 2n_2 - 2n_3, \\ &\alpha_3 = \mathsf{a} + 2n_3 + 2n_4, \quad \alpha_4 = -2n_4, \quad n_2, n_4 \in \mathbb{Z}_+,\ n_3 \in \mathbb{Z}, \end{aligned} \quad (118)$$

The Painlevé parameters corresponding to (118) are:

$$\bar{\alpha} = \frac{1}{2}\Big(\frac{\mathsf{a}}{2} + n_3 + n_4\Big)^2,\ \bar{\beta} = -\frac{1}{2}(-n_2)^2,\ \bar{\gamma} = \epsilon\,(1 - \frac{\mathsf{a}}{2} + n_2 - n_3 + n_4),$$

or

$$\bar{\alpha} = \frac{1}{2}(\epsilon\,\bar{\gamma} - 1 - n_2 - 2n_4)^2,\ \bar{\beta} = -\frac{1}{2}(n_2)^2, \quad (119)$$

with n_2, n_4 being positive integers or zero. Relations (116) and (119) constitute item (III) in Section (4.1).

Example 4. *Let us now consider the following example with solution taken from [15]:*

$$q = \frac{z(z^4 - 14z^2 + 63)}{z^4 - 18z^2 + 99}, \quad p = \frac{z^6 - 21z^4 + 189z^2 - 693}{z(z^4 - 14z^2 + 63)}, \ \alpha_1 = 7, \alpha_2 = 6, \alpha_3 = -4 \tag{120}$$

Expression for p has a pole which can be removed by applying s_1. Applying s_1 we get

$$q = \frac{z(z^4 - 14z^2 + 63)}{z^4 - 18z^2 + 99}, \quad p = z, \quad \alpha_1 = -7, \alpha_2 = 13, \alpha_3 = -4, \ \alpha_4 = 0\,. \tag{121}$$

We will match it with the initial configuration of (24) with $p = z, q = z$ and $(2 - \mathsf{a}, \mathsf{a}, 0, 0)$ on which we can act with $T_2^{n_2}, T_3^{-n_3}, T_1^{n_1}$ (but not T_3^{+1}) to get:

$$\begin{aligned} \alpha_1 &= -7 = 2 - \mathsf{a} + 2n_1 - 2n_2, \ \alpha_2 = 13 = \mathsf{a} + 2n_2 + 2n_3, \\ \alpha_3 &= -4 = -2n_3, \quad \alpha_4 = 0 = -2n_1, \ n_1, n_3 \in \mathbb{Z}_+, \ n_2 \in \mathbb{Z}\,. \end{aligned}$$

We choose $\mathsf{a} = 9, n_1 = n_2 = 0, n_3 = 2$ to get the desired result. One can show for the corresponding combination of shift operators that $T_3^{-2} = \pi^2 s_1 s_2 s_3 s_4 s_1 s_2$ and acting with such operator on $p = z, q = z$ and $\alpha_i = (-7, 9, 0, 0)$ one easily reproduces the solution (121). Alternatively, we can obtain this solution as a special function solution when we recognize that for the condition $\alpha_4 = 0$ from Equation (121), the Hamilton Equation (20) is solved by $p = z$, which when inserted in the first equation in (20) reduces this equation to the Riccati equation $zq_z = -zq(q - z) + (1 - \alpha_1 - \alpha_3)q + \alpha_1 z$, solved by

$$q = \frac{\alpha_3}{z} \frac{\mathrm{WhittakerM}(\frac{\alpha_3}{4} - \frac{\alpha_1}{4} + 1, -\frac{1}{2} + \frac{\alpha_1}{4} + \frac{\alpha_3}{4}, \frac{z^2}{2})}{\mathrm{WhittakerM}(\frac{\alpha_3}{4} - \frac{\alpha_1}{4}, -\frac{1}{2} + \frac{\alpha_1}{4} + \frac{\alpha_3}{4}, \frac{z^2}{2})} + \frac{z^2 - \alpha_3}{z}\,.$$

Inserting $\alpha_1 = -7, \alpha_3 = -4$, we recover from the above expression the rational solution (121).

By comparing with results in [9], we conclude that acting with shift operators on the first-order polynomial solutions of $N = 4$ dressing chain produces all rational solutions of the associated Painlevé system. We therefore conjecture that the same technique will produce all rational solutions for higher even N values and discuss realization of this statement for $N = 6$ in the next section.

5. Special Function and Rational Solutions of $N = 6$ Equations

5.1. Reductions of $N = 6$ Hamilton Equation (28)

Recall that in Section 3.3 we considered $N = 4$ solutions (22) with $\alpha_i = (\mathsf{a}, 0, 0, 2 - \mathsf{a})$. Having the parameters α_2 or α_3 set to zero resulted in the $N = 4$ Hamilton Equation (20) being reduced to a single Riccati equation. For example, for $\alpha_3 = 0$ the Hamilton Equation (20) is solved by $q = z$ and a solution of the Riccati equation $zp_z = zp(p - z) - (1 - \alpha_1)p + \alpha_2 z$. Similarly for $\alpha_2 = 0$ the Hamilton Equation (20) is solved by $p = 0$ and a solution of the Riccati equation $zq_z = zq(q - z) + (1 - \alpha_1 - \alpha_3)q + \alpha_1 z$. Accordingly, we determined a class of special function solutions to the Painlevé V equation that became rational solutions when the α_i parameters coincided with orbits of $(\mathsf{a}, 0, 0, 2 - \mathsf{a})$ obtained by an action of appropriate shift operators.

In this subsection we will carry out a similar discussion for the $N = 6$ case, investigating conditions for the presence of special function solutions to the Hamilton Equation (28). The Hamilton Equation (28) represents four coupled non-linear third-order differential equations. Setting various components of α_i to zero introduces connections between $q_i, p_i, i = 1, 2$ and accordingly reduces a number of coupled non-linear equations. Imposing three constraints on parameters of an $N = 6$ Hamilton system (28) reduces the system to only one solvable second-order Riccati equation with a special function solution. The three constraints emerge when the two of j_i are negative, as in solutions (14)–(17).

When the reduced systems are realized on orbits of shift operators $T_i^{n_i}$ acting on seed solutions (14)–(17) all these Riccati solutions become rational solutions parameterized by integers n_i.

5.1.1. One-Constraint Reductions of $N = 6$ Hamilton Equations

We will proceed by listing possible conditions on α_i parameters together with expressions for those $q_i, p_i, i = 1, 2$ that solve the reduced Equation (28) obtained as a result of imposing constraints. For example, the formula:

$$\alpha_6 = 0 \quad \longrightarrow \quad p_2 = z - p_1 \,, \tag{122}$$

means that inserting the condition $\alpha_6 = 0$ into the last two equations for p_1, p_2 in (28) causes each of them to reduce to one identical equation for p_1:

$$zp_{1,z} = p_1(z - p_1)(2q_2 - 2q_1 - z) + z\alpha_2 - p_1(1 - \alpha_1 - \alpha_3 - \alpha_5) \,,$$

with $p_2 = z - p_1$. The reduced system of the remaining three Hamilton equations only depends on three variables q_1, q_2, p_1 after imposition of one single constraint.

We list below other single constraints and the corresponding simple solutions for quantities entering Equation (28):

$$\alpha_5 = 0 \quad \longrightarrow \quad q_2 = z \,, \tag{123}$$
$$\alpha_4 = 0 \quad \longrightarrow \quad p_2 = 0 \,, \tag{124}$$
$$\alpha_3 = 0 \quad \longrightarrow \quad q_1 = q_2 \,, \tag{125}$$
$$\alpha_2 = 0 \quad \longrightarrow \quad p_1 = 0 \,, \tag{126}$$
$$\alpha_1 = 0 \quad \longrightarrow \quad q_1 = 0 \,. \tag{127}$$

5.1.2. Multi-Constraint Reductions of $N = 6$ Hamilton Equations

One can combine the above single constraints of α_i parameters into a set of two and more constraints. As we will see below, the set of three constraints leads to the constrained system described by a single Riccati equation.

Imposing two constraints leads as a rule to two coupled non-linear equations but not always equations that are quadratic in their underlying variables.

Let us first consider the following example of two constraints:

$$\alpha_6 = 0 \;\&\; \alpha_5 = 0 \quad \longrightarrow \quad p_1 + p_2 = z, \; q_2 = z \,, \tag{128}$$

that combines $p_1 + p_2 = z$ that follows from $\alpha_6 = 0$ and relation $q_2 = z$ that follows from $\alpha_5 = 0$. Imposing these two relations, we can rewrite the Hamiltonian equations only in terms of, e.g., p_2, q_1 entering cubic non-linear equations:

$$\begin{aligned} zp_{2,z} &= (z - p_2)p_2(2q_1 - z) + z\alpha_4 - p_2(1 - \alpha_1 - \alpha_3) \,, \\ zq_{1,z} &= q_1(z - q_1)(z - 2p_2) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_3) \,. \end{aligned} \tag{129}$$

For the two constraints:

$$\alpha_4 = 0 \;\&\; \alpha_3 + \alpha_5 = 0 \quad \longrightarrow \quad p_2 = 0, \; q_1 = z \,, \tag{130}$$

the remaining variables p_1, q_2 enter two coupled second-order equations:

$$\begin{aligned} zp_{1\,z} &= -zp_1(z - p_1) + z\alpha_2 - p_1(1 - \alpha_1) \,, \\ zq_{2\,z} &= z(z - q_2)(2p_1 - q_2) + z(\alpha_1 + \alpha_3) + q_2(1 - \alpha_1) \,. \end{aligned} \tag{131}$$

Only the first equation is a Riccati equation solvable in terms of Kummer/ Whittaker functions.

Next consider the two constraints

$$\alpha_6 = 0 \;\&\; \alpha_1 + \alpha_5 = 0 \quad \longrightarrow \quad p_2 = z - p_1,\; q_2 = q_1 + z\,. \tag{132}$$

The two remaining equations for q_1, p_1 are found to be

$$\begin{aligned} zq_{1,z} &= zq_1(2p_1 - z) - zq_1^2 + z\alpha_1 + q_1(1 - \alpha_3)\,, \\ zp_{1,z} &= zp_1(z - p_1) + z\alpha_2 - p_1(1 - \alpha_3)\,. \end{aligned} \tag{133}$$

The second equation among Equation (133) is a regular Riccati equation but the first one is a coupled Riccati equation. We will see below in Example 7 that the coupled Equations (131) and (133) become fully solvable on orbits of the shift operators.

Combining together conditions into three conditions yields one single second-order Riccati equation emerging from such a reduction process.

$$\alpha_2 = 0 \;\&\; \alpha_5 = 0 \;\&\; \alpha_6 = 0 \quad \longrightarrow \quad q_2 = z,\; p_1 = 0, p_2 = z\,. \tag{134}$$

In this case there only remains one Riccati equation for the remaining variable q_1:

$$zq_{1\,z} = -zq_1(z - q_1) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_3)\,. \tag{135}$$

Replacing α_2 with α_4 in (134) yields

$$\alpha_2 = 0 \;\&\; \alpha_5 = 0 \;\&\; \alpha_6 = 0 \quad \longrightarrow \quad q_2 = z,\; p_1 = z, p_2 = 0\,, \tag{136}$$

with a Riccati equation for q_1

$$zq_{1\,z} = zq_1(z - q_1) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_3)\,. \tag{137}$$

Similarly, the three constraints

$$\alpha_6 = 0 \;\&\; \alpha_4 = 0 \;\&\; \alpha_3 + \alpha_5 = 0 \quad \longrightarrow \quad p_1 = z,\; p_2 = 0,\; q_1 = z\,, \tag{138}$$

leave only one Riccati equation for q_2: $zq_{2,z} = z(z - q_2)^2 + z^2(z - q_2) + z(\alpha_1 + \alpha_3) + q_2(1 - \alpha_1)$. Entering $f_3 = q_2 - z$, we get a simple-looking Riccati equation for f_3:

$$zf_{3\,z} = -z^2 f_3 + zf_3^2 + z\alpha_3 + f_3(1 - \alpha_1)\,. \tag{139}$$

A similar case is that of three constraints with α_3 replaced by α_1:

$$\alpha_6 = 0 \;\&\; \alpha_4 = 0 \;\&\; \alpha_1 + \alpha_5 = 0 \quad \longrightarrow \quad p_1 = z,\; p_2 = 0,\; q_2 = q_1 + z\,, \tag{140}$$

which leaves only one Riccati equation for q_1: $zq_{1\,z} = zq_1(z - q_1) + z\alpha_1 + q_1(1 - \alpha_3)$.

Further we also list the three constraints:

$$\alpha_6 = 0 \;\&\; \alpha_5 = 0 \;\&\; \alpha_3 = 0 \quad \longrightarrow \quad q_1 = z,\; p_2 + p_1 = z,\; q_2 = z\,, \tag{141}$$

As seen before, $\alpha_6 = 0$ leads to $p_2 = z - p_1$ and $\alpha_5 = 0$ leads to $q_2 = z$. One of the remaining Hamilton equations is $zq_{1,z} = q_1(z - q_1)(2p_1 - z) + z\alpha_1 + q_1(1 - \alpha_1)$ with the solution $q_1 = z$, which, when inserted in the equation for p_1, gives Riccati equation: $zp_{1,z} = -zp_1(z - p_1) + z\alpha_2 - p_1(1 - \alpha_1)$.

Another example of three independent constraints:

$$\alpha_6 = 0 \;\&\; \alpha_3 = 0 \;\&\; \alpha_2 = 0 \quad \longrightarrow \quad p_1 = 0,\; p_2 = z\,. \tag{142}$$

For the remaining quantities q_1, q_2, the $N = 6$ Hamilton Equation (28) then gives:

$$zq_{1\,z} = -zq_1(q_2 - q_1) + zq_1(z - q_2) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_5)\,,$$
$$zq_{2\,z} = z(z - q_2)(q_2 - q_1) + zq_1(z - q_2) + z\alpha_1 + q_2(1 - \alpha_1 - \alpha_5)\,.$$

Taking the difference of the above two equations yields an equation for $q_2 - q_1$ which is solved for $q_2 = q_1$. Thus, we are left with one Riccati equation for q_1: $zq_{1\,z} = zq_1(z - q_1) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_5)$.

Another case of three constraints

$$\alpha_5 = 0 \;\&\; \alpha_4 = 0 \;\&\; \alpha_3 = 0 \quad \longrightarrow \quad q_1 = z,\; p_2 = 0,\; q_2 = z\,, \tag{143}$$

lead to one single Riccati equation for the remaining quantity p_1:

$$zp_{1,z} = -zp_1(z - p_1) + z\alpha_2 - p_1(1 - \alpha_1)\,. \tag{144}$$

As seen above, the three constraints reduce the four Hamiltonian equations in (28) to one Riccati equation for the remaining variables. As expected, imposing all four constraints applied on the four Hamiltonian equations in (28) leads only to trivial solutions:

$$\alpha_6 = 0 \;\&\; \alpha_5 = 0 \;\&\; \alpha_4 = 0 \;\&\; \alpha_3 = 0 \quad \longrightarrow \quad p_1 = z,\; p_2 = 0,\; q_1 = q_2 = z\,. \tag{145}$$

As we will see below in example 7 there are cases of two constraints with two remaining Riccati equations that decouple under special circumstances when the parameters are chosen to coincide with the orbits of the shift operators.

5.2. the Orbit Construction of Rational Solutions for $N = 6$

In this section we apply the technique introduced in previous sections to the case of $N = 6$, for which we already found the first-order polynomial solutions in Equations (13)–(17).

As found in Section 3.4 for the $N = 6$ case after the appropriate actions by s_1 and s_3, the variables $p_i, i = 1, 2$ can be expanded in positive power series that do not contain pole singularities. Such rational solutions can then be reproduced by actions of the shift operators on solutions (13)–(17) or (30)–(34).

5.2.1. Umemura Polynomial Solutions for $N = 6$

In this subsection we will apply the fundamental shift operator techniques to

(I) The seed solution (13) with all components $j_i = z/N = z/6$;

(II) The seed solution (14) with one of the components being negative and equal to $-z/(N-2) = -z/4$.

The case (I) will require a new class of Umemura polynomials (146) with the leading order term being $z^{n(n-p)}, p = 1, n/2 - 1, n/2$ with the last two cases being new. In case (II) we will be able to essentially reduce the problem to that of $N = 4$ and express the solutions in terms of regular Umemura polynomials with the leading order term being $z^{n(n-1)}$.

Case (I). Recall the relevant $N = 6$ shift operators from definitions (71) and (72). For solution (13) with all $j_i = z/6, i = 1, 2, 3, 4, 5, 6$ it holds that $j_i + j_{i+1} \neq 0$ for all $i = 1, 2, 3, 4, 5, 6$. Thus all s_i transformations acting via relation (6) are well defined and action by

$$T_1^{n_1} T_2^{n_2} T_3^{n_3} T_4^{n_4} T_5^{n_5} T_6^{n_6}, \qquad n_i \in \mathbb{Z}, i = 1, 2, 3, 4, 5, 6\,,$$

produces rational solutions with the transformed $\bar{\alpha}_i$:

$$\left(\mathsf{a} + 2n_1 - 2n_2, \frac{2}{3} - \mathsf{a} + 2n_2 - 2n_3, \mathsf{a} + 2n_3 - 2n_4, \frac{2}{3} - \mathsf{a} + 2n_4 - 2n_5, \mathsf{a} + 2n_5 - 2n_6, \frac{2}{3} - \mathsf{a} + 2n_6 - 2n_1\right).$$

We can rewrite the above action of the shift operators as follows

$$
\begin{aligned}
T_1^{n_1}T_2^{n_2}T_3^{n_3}T_4^{n_4}T_5^{n_5}T_6^{n_6} &= (T_1T_3T_5)^{n_1}(T_2T_4T_6)^{n_2}T_3^{n_3-n_1}T_5^{n_5-n_1}T_4^{n_4-n_2}T_6^{n_6-n_2}\\
&= (T_1T_3T_5)^{n_1}(T_2T_4T_6)^{n_2}(T_3T_4)^{k_+}(T_3T_4^{-1})^{k_-}(T_5T_6)^{m_+}(T_5T_6^{-1})^{m_-}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
k_+ &= \frac{1}{2}(n_3-n_1+n_4-n_2), \qquad & k_- &= \frac{1}{2}(n_3-n_1-n_4+n_2)\,,\\
m_+ &= \frac{1}{2}(n_5-n_1+n_6-n_2), \qquad & m_- &= \frac{1}{2}(n_5-n_1-n_6+n_2)\,.
\end{aligned}
$$

One can easily prove that $(T_1T_3T_5)^n$ only shifts the parameter a: $\mathsf{a} \to \mathsf{a}-n$ without changing the functional form of the solution (30). Similarly, $(T_2T_4T_6)^n$ only shifts the parameter a: $\mathsf{a} \to \mathsf{a}+n$ leaving the solutions (30) unchanged.

For $(T_3T_4)^{k_+}$ we find that it results in $(\alpha_2-\alpha_4)/2=-2k_+, (\alpha_3-\alpha_5)/2=0$, while for $(T_3T_4^{-1})^{k_-}$ we obtain $(\alpha_2-\alpha_4)/2=0, (\alpha_3-\alpha_5)/2=2k_-$.

For $(T_5T_6)^{m_+}$ we find that it results in $(\alpha_6-\alpha_4)/2=2m_+, (\alpha_3-\alpha_5)/2=0$, while for $(T_5T_6^{-1})^{m_-}$ we obtain $(\alpha_6-\alpha_4)/2=0, (\alpha_3-\alpha_5)/2=-2m_-$.

For $N=6$ we introduce the following notation:

$$
U_{k,n}(z;\mathsf{a}) = z^{(n+k)(n)}+\ldots, \quad k>0,\, n>0\,, \tag{146}
$$

which generalizes Umemura polynomials of the type $U_{1,n-1}(z;a)=z^{n(n-1)}+\ldots$ seen in the previous section for $N=4$. These new Umemura polynomials take the following special values for $n=0,1,2$:

$$
\begin{aligned}
&U_{-1,1}(z;\mathsf{a})=1\,,\; U_{-1,0}(z;\mathsf{a})=U_{0,0}(z;\mathsf{a})=1\,,\\
&U_{1,-1}(z;\mathsf{a})=1, \quad U_{1,0}(z;\mathsf{a})=1\,, && (147)\\
&U_{1,1}(z;\mathsf{a})=z^2+9\mathsf{a}\,, && (148)\\
&U_{0,2}(z;\mathsf{a})=U_{3,1}(z;\mathsf{a})=z^4+18\,z^2\,\mathsf{a}+24\,z^2+162\,\mathsf{a}+81\,\mathsf{a}^2+72 && (149)\\
&U_{1,2}(z;\mathsf{a})=648+2106\,a+540\,z^2\,\mathsf{a}+2187\,\mathsf{a}^2+252\,z^2+33\,z^4\\
&\qquad +z^6+27\,z^4\,\mathsf{a}+243\,z^2\,\mathsf{a}^2+729\,\mathsf{a}^3 && (150)
\end{aligned}
$$

and enter the following expressions for solutions we obtained by acting once with the shift operator T_1 on the $n=0$ configuration (13) with $q_1=p_1=p_2=z/3$ and $q_2=2z/3$ and $\alpha_i=(\mathsf{a},2/3-\mathsf{a},\mathsf{a},2/3-\mathsf{a},\mathsf{a},2/3-\mathsf{a})$:

$$
q_1^{(1)} = \frac{z}{3}\frac{U_{-1,1}(z;\mathsf{a})\,U_{0,2}(z;\mathsf{a}+2/3)}{U_{-1,1}(z;\mathsf{a}+2/3)\,U_{0,2}(z;\mathsf{a})} = \frac{1}{3}\frac{z\big(z^4+18\,z^2\mathsf{a}+36\,z^2+270\,\mathsf{a}+81\,\mathsf{a}^2+216\big)}{z^4+18\,z^2\mathsf{a}+24\,z^2+162\,\mathsf{a}+81\,\mathsf{a}^2+72} \tag{151}
$$

$$
p_1^{(1)} = \frac{z}{3}\frac{U_{-1,1}(z;\mathsf{a}+2/3)\,U_{1,1}(z;\mathsf{a}-2/3)}{U_{-1,1}(z;\mathsf{a})\,U_{1,1}(z;\mathsf{a})} = \frac{1}{3}\frac{z\big(z^2+9\,\mathsf{a}-6\big)}{z^2+9\,\mathsf{a}} \tag{152}
$$

$$
\begin{aligned}
q_2^{(1)} &= \frac{2z}{3}\,\frac{U_{1,2}(z;\mathsf{a})}{U_{0,2}(z;\mathsf{a})\,U_{1,1}(z;\mathsf{a}+2/3)}\\
&= \frac{2}{3}\frac{z\big(648+2106\,\mathsf{a}+540\,z^2a+2187\,a^2+252\,z^2+33\,z^4+z^6+27\,z^4\mathsf{a}+243\,z^2\mathsf{a}^2+729\,\mathsf{a}^3\big)}{(z^2+9\,\mathsf{a}+6)(z^4+18\,z^2\mathsf{a}+24\,z^2+162\,\mathsf{a}+81\,\mathsf{a}^2+72)}
\end{aligned} \tag{153}
$$

$$
p_2^{(1)} = \frac{z}{3}\frac{U_{1,1}(z;\mathsf{a}+2/3)\,U_{0,2}(z;\mathsf{a}-4/3)}{U_{1,1}(z;\mathsf{a})\,U_{0,2}(z;\mathsf{a}-2/3)} = \frac{1}{3}\frac{z(z^2+9\mathsf{a}+6)(z^4+18z^2\mathsf{a}-54\mathsf{a}+81\mathsf{a}^2)}{(z^4+18\,z^2\mathsf{a}+12\,z^2+54\,\mathsf{a}+81\,\mathsf{a}^2)(z^2+9\,\mathsf{a})} \tag{154}
$$

The repeated action n-th times with the shift operator T_1 on (13) with $q_1 = p_1 = p_2 = z/3$ and $q_2 = 2z/3$ can be described by generalization of (151)–(154) given by:

$$q_1^{(n)} = \frac{z}{3}\frac{U_{n-2,n}(z;\mathsf{a})\,U_{n-1,n+1}(z;\mathsf{a}+2/3)}{U_{n-2,n}(z;\mathsf{a}+2/3)\,U_{n-1,n+1}(z;\mathsf{a})} \tag{155}$$

$$p_1^{(n)} = \frac{z}{3}\frac{U_{n-2,n}(z;\mathsf{a}+2/3)\,U_{n,n}(z;\mathsf{a}-2/3)}{U_{n-2,n}(z;\mathsf{a})\,U_{n,n}(z;\mathsf{a})} \tag{156}$$

$$q_2^{(n)} = \frac{2z}{3}\frac{U_{1,2n}(z;\mathsf{a})}{U_{n-1,n+1}(z;\mathsf{a})\,U_{n,n}(z;\mathsf{a}+2/3)} \tag{157}$$

$$p_2^{(n)} = \frac{z}{3}\frac{U_{n,n}(z;\mathsf{a}+2/3)\,U_{n-1,n+1}(z;\mathsf{a}-4/3)}{U_{n,n}(z;\mathsf{a})\,U_{n-1,n+1}(z;\mathsf{a}-2/3)}\,, \tag{158}$$

where $U_{k,n}(z;\mathsf{a})$ polynomials of the type shown in Equation (146).

Case (II). For the solution (14) with $j_i = \frac{z}{4}(1,1,1,1,1,-1)$, it holds that $j_5 + j_6 = 0$, $j_6 + j_1 = 0$ and that makes $s_i, s_{i-1}\pi^{-1}, s_{i+1}\pi$ with $i = 5,6$ ill-defined. Accordingly, $T_5, T_6, T_1^{-1}, T_6^{-1}$ are ill-defined. Rational solutions will be produced from the seed solution (14) by action of

$$T_1^{n_1}T_2^{n_2}T_3^{n_3}T_4^{n_4}T_5^{-n_5},\;\; n_1, n_5 \in \mathbb{Z}_+,\;\; n_2, n_3, n_4 \in \mathbb{Z}\,,$$

that yields the orbit parameters:

$$(\mathsf{a}+2n_1-2n_2, 1-\mathsf{a}+2n_2-2n_3, \mathsf{a}+2n_3-2n_4, 1-\mathsf{a}+2n_4+2n_5, -2n_5, -2n_1)\,. \tag{159}$$

When we set $n_5 = 0$ in expression (159) we obtain the condition (123) with $\alpha_5 = 0$, $q_2 = z$. Inserting these conditions into $N = 6$ Hamilton Equation (28), we find that q_1, p_1 satisfy separately the $N = 4$ Hamilton Equation (20) although they still couple to p_2, but only in the last of Equation (28). Explicitly, we find the solutions in terms of Umemura polynomials from Section 4.4:

$$\begin{aligned} q_1^{(n)}(z;\mathsf{a}) &= T_1^n(q_1^{(0)}) = \frac{z}{2}\frac{U_n(z;-\mathsf{a}+2)\,U_{n+1}(z;-\mathsf{a}+3)}{U_n(z;-\mathsf{a}+3)\,U_{n+1}(z;-\mathsf{a}+2)},\\ p_1^{(n)}(z;\mathsf{a}) &= T_1^n(p_1^{(0)}) = \frac{z}{2}\frac{U_n(z;-\mathsf{a}+3)\,U_{n+1}(z;-\mathsf{a})}{U_n(z;-\mathsf{a}+2)\,U_{n+1}(z;-\mathsf{a}+1)}\,, \end{aligned} \tag{160}$$

with $q_1^{(0)} = p_1^{(0)} = z/2$. Given these two solutions one finds the expression for $p_2^{(n)}$ by solving the corresponding equation of p_2 among the $N = 6$ Hamilton Equation (28). Since $p_2^{(n)}$ can be obtained by repeated actions of the shift operator, it is given by a ratio of polynomials as illustrated in the Example given below.

Example 5. *Let us consider an orbit generated by T_2^n. Entering $n_2 = n$ and $n_1 = n_3 = n_4 = n_5 = 0$ into expression (159), we find that $\alpha_6 = \alpha_5 = 0$ as in (128). The expressions for p_2 found by applying T_2^n on solutions (14) with $p_2 = z/2, q_1 = z/2$ are:*

$$\begin{aligned} p_2(n=1,\mathsf{a},z) &= \frac{z}{2}\frac{4+z^2-4\mathsf{a}}{z^2-4\mathsf{a}+8},\;\; p_2(n=-1,\mathsf{a},z) = \frac{z}{2}\frac{z^2-4\mathsf{a}+4}{z^2-4\mathsf{a}}\,,\\ q_1(n=1,\mathsf{a},z) &= \frac{z}{2}\frac{z^2-4\mathsf{a}+8}{z^2-4\mathsf{a}+4},\;\; q_1(n=-1,\mathsf{a},z) = \frac{z}{2}\frac{-8+z^2-4\mathsf{a}}{-4+z^2-4\mathsf{a}}\,, \end{aligned}$$

and one can verify that they satisfy the relevant reduced Hamilton Equation (129) for p_2, q_1 with $\alpha_1 = \mathsf{a}-2n, \alpha_2 = 1-\mathsf{a}+2n, \alpha_3 = \mathsf{a}, \alpha_4 = 1-\mathsf{a}$.

5.2.2. Riccati Solutions for $N = 6$

In this subsubsection we will consider solutions constructed out of the seed solutions with two negative j_i components. First consider the solution given in (15) with

$j_i = \frac{z}{2}(1,1,1,1,-1,-1)$ and $j_4 + j_5 = 0, j_6 + j_1 = 0$. These conditions render $s_i, s_{i-1}\pi^{-1}$, $s_{i+1}\pi, i = 4,6$ ill-defined. Using these arguments, we find that $T_4, T_6, T_1^{-1}, T_5^{-1}$ are ill-defined. Also, by inspection we find that $T_3, T_5, T_2^{-1}, T_6^{-1}$ are ill-defined as well. Rational solutions will be produced from the seed solution (15) by action of

$$T_1^{n_1} T_2^{n_2} T_3^{-n_3} T_4^{-n_4}, \quad n_1, n_2, n_3, n_4 \in \mathbb{Z}_+,$$

that yields

$$(\mathsf{a}_3 + 2n_1 - 2n_2, 2 - \mathsf{a}_3 - 2n_2 + 2n_3, \mathsf{a}_3 - 2n_3 + n_4, -2n_4, -\mathsf{a}_3, -2n_1)\,. \tag{161}$$

Example 6. *Consider an orbit generated by T_2^n obtained by inserting $n_2 = n$ and $n_1 = n_3 = n_4 = 0$ into the above expression (161). This results in $\alpha_4 = 0, \alpha_6 = 0, \alpha_3 + \alpha_5 = 0$, which are the three constraints shown in (138). The corresponding Riccati Equation (139) becomes:*

$$z q_{2\,z} = -3z^2 q_2 + z q_2^2 + 2z^3 + z(2\mathsf{a}_3 - 2n) + q_2(1 - \mathsf{a}_3 + 2n)\,,$$

after inserting $\alpha_1 = -2n + \mathsf{a}_3$. Solving this equation for q_2 we get:

$$q_2 = z + \frac{\mathsf{a}_3}{z} + \frac{2}{z} \frac{\mathrm{WhittakerW}(\frac{n}{2} + \frac{\mathsf{a}_3}{4} + 1, -\frac{1}{2} - \frac{n}{2} + \frac{\mathsf{a}_3}{4}, \frac{z^2}{2})}{\mathrm{WhittakerW}(\frac{n}{2} + \frac{\mathsf{a}_3}{4}, -\frac{1}{2} - \frac{n}{2} + \frac{\mathsf{a}_3}{4}, \frac{z^2}{2})}\,.$$

For $n = 0$ we obtain $q_2 = 2z$, and next

$$n = 1, \quad q_2 = 2z \frac{(-\mathsf{a}_3 + 1 + z^2)}{(-\mathsf{a}_3 + 2 + z^2)}\,,$$

$$n = 2, \quad q_2 = 2z \frac{(-4\mathsf{a}_3 + 4 + \mathsf{a}_3^2 - 2\mathsf{a}_3 z^2 + 2z^2 + z^4)}{(-6\mathsf{a}_3 + 8 + \mathsf{a}_3^2 - 2\mathsf{a}_3 z^2 + 4z^2 + z^4)}\,.$$

This is in agreement with results obtained by acting explicitly by $T_2^n, n = 0,1,2$ on solution (32)

Similar considerations are involved in a study of an orbit generated by T_3^{-n} obtained by inserting $n_3 = n$ and $n_1 = n_2 = n_4 = 0$ into expression (161). This results in $\alpha_4 = 0$, $\alpha_6 = 0, \alpha_1 + \alpha_5 = 0$, which are the three constraints shown in (140). Entering $\alpha_1 = \mathsf{a}_3$ and or $\alpha_3 = \mathsf{a}_3 - 2n$ into the Riccati equation for q_1 shown below Equation (140) we find a solution:

$$q_1(n, \mathsf{a}_3, z) = -2n \frac{\mathrm{WhittakerM}(-\frac{\mathsf{a}_3}{4} - \frac{n}{2} + 1, -\frac{1}{2} + \frac{\mathsf{a}_3}{4} - \frac{n}{2}, \frac{z^2}{2})}{z\mathrm{WhittakerM}(-\frac{\mathsf{a}_3}{4} - \frac{n}{2}, -\frac{1}{2} + \frac{\mathsf{a}_3}{4} - \frac{n}{2}, \frac{z^2}{2})} + \frac{(z^2 + 2n)}{z}\,, \tag{162}$$

with $q_1(0, \mathsf{a}_3, z) = z$ and

$$q_1(1, \mathsf{a}_3, z) = z \frac{\mathsf{a}_3 + z^2}{\mathsf{a}_3 - 2 + z^2}, \; q_1(2, \mathsf{a}_3, z) = z \frac{-2\mathsf{a}_3 + \mathsf{a}_3^2 + 2\mathsf{a}_3 z^2 + z^4}{\mathsf{a}_3^2 - 6\mathsf{a}_3 + 8 + 2\mathsf{a}_3 z^2 - 4z^2 + z^4}\,.$$

Example 7. *The two examples we will here consider involve systems that are characterized by two conditions imposed on the parameters α_i. Such a situation leads to a system of reduced Hamilton equations quadratic in canonical variables. In examples shown here, the reduced Hamilton equations consist of one simple Riccati equation and one quadratic equation with coupled underlying canonical variables. However, when α_i parameters are those of an orbit (161), the coupled Hamilton equations system separates into two independent and solvable Riccati equations.*

First, we consider an T_1^n orbit which is obtained by inserting $n_1 = n$ and $n_2 = n_3 = n_4 = 0$ into the above expression (161). The orbit configuration agrees with the two constraints of (130) and with the corresponding coupled Hamilton equations (131), of which only the first equation is a Riccati equation, which after inserting $\alpha_1 = \mathsf{a}_3 + 2n$ yields

$$z p_{1\,z} = -z p_1(z - p_1) + z(2 - \mathsf{a}_3) - p_1(1 - \mathsf{a}_3 - 2n)\,,$$

with solution:

$$p_1(n, \mathsf{a}_3, z) = \frac{n\mathsf{a}_3}{z} \frac{\mathrm{WhittakerW}(-\frac{1}{2} + \frac{n}{2} - \frac{\mathsf{a}_3}{4}, \frac{\mathsf{a}_3}{4} + \frac{n}{2}, \frac{z^2}{2})}{\mathrm{WhittakerW}(\frac{1}{2} + \frac{n}{2} - \frac{\mathsf{a}_3}{4}, \frac{\mathsf{a}_3}{4} + \frac{n}{2}, \frac{z^2}{2})} + \frac{(z^2 - 2n)}{z}, \tag{163}$$

for which we find for $n = 0, 1, 2$:

$$\begin{aligned} p_1(n = 0, \mathsf{a}_3, z) &= z, \quad p_1(n = 1, \mathsf{a}_3, z) = z\frac{\mathsf{a}_3 + z^2 - 2}{\mathsf{a}_3 + z^2} \\ p_1(n = 2, \mathsf{a}_3, z) &= z\frac{-2\mathsf{a}_3 + \mathsf{a}_3^2 + 2z^2\mathsf{a}_3 + z^4 - 4z^2}{2\mathsf{a}_3 + \mathsf{a}_3^2 + 2z^2\mathsf{a}_3 + z^4}. \end{aligned} \tag{164}$$

However, for $\alpha_1 = \mathsf{a}_3 + 2n$ it appears that Equation (131) effectively decouples. We can namely define q_1 such that

$$q_1(n, \mathsf{a}_3, z) = q_2(n, \mathsf{a}_3, z) - z\,,$$

that satisfies the Riccati equation:

$$zq_{1\,z} = -zq_1(z - q_1) + z(-\mathsf{a}_3) + q_1(1 + \mathsf{a}_3 + 2n)\,,$$

with solution:

$$q_1 = \frac{2}{z} \frac{\mathrm{WhittakerW}(\frac{n}{2} - \frac{\mathsf{a}_3}{4} + 1, \frac{1}{2} + \frac{\mathsf{a}_3}{4} + \frac{n}{2}, \frac{z^2}{2})}{\mathrm{WhittakerW}(\frac{n}{2} - \frac{\mathsf{a}_3}{4}, \frac{1}{2} + \frac{\mathsf{a}_3}{4} + \frac{n}{2}, \frac{z^2}{2})} - \frac{\mathsf{a}_3}{z},$$

which explicitly gives the values

$$\begin{aligned} q_1(n = 0, \mathsf{a}_3, z) &= z, \quad q_1(n = 1, \mathsf{a}_3, z) = z\frac{\mathsf{a}_3 + z^2}{\mathsf{a}_3 + z^2 + 2} \\ q_1(n = 2, \mathsf{a}_3, z) &= z\frac{2\mathsf{a}_3 + \mathsf{a}_3^2 + 2z^2\mathsf{a}_3 + z^4}{6\mathsf{a}_3 + 8 + \mathsf{a}_3^2 + 2z^2\mathsf{a}_3 + 4z^2 + z^4}, \end{aligned}$$

that reproduces $q_2(n, \mathsf{a}_3, z)$ after adding z.

Quite similar behavior will take place for an orbit T_4^{-n} obtained by inserting $n_4 = n$ and $n_1 = n_2 = n_3 = 0$ into the above expression (161). Here, α_i parameters satisfy two conditions: $\alpha_6 = 0$ and $\alpha_1 + \alpha_5 = 0$, which coincide with expression (132). The two $N = 6$ Hamilton Equation (28) for remaining variables q_1, p_1 shown in (133) are such that the first equation contains a coupling between these two variables. Although the second equation is a regular Riccati equation. We consider the case of $\alpha_3 = \mathsf{a}_3 + 2n$ and $\alpha_2 = 2 - \mathsf{a}_3, \alpha_1 = \mathsf{a}_3$. The solution to the second equation in (133) is:

$$p_1(n, \mathsf{a}_3, z) = -\frac{n\mathsf{a}_3}{z} \frac{\mathrm{WhittakerW}(-\frac{1}{2} + \frac{n}{2} - \frac{\mathsf{a}_3}{4}, \frac{\mathsf{a}_3}{4} + \frac{n}{2}, -\frac{z^2}{2})}{\mathrm{WhittakerW}(\frac{1}{2} + \frac{n}{2} - \frac{\mathsf{a}_3}{4}, \frac{\mathsf{a}_3}{4} + \frac{n}{2}, -\frac{z^2}{2})} + \frac{(z^2 + 2n)}{z}, \tag{165}$$

Entering $n = 0, 1, 2$ into (165) we get:

$$\begin{aligned} p_1(n = 0, \mathsf{a}_3, z) &= z, \; p_1(n = 1, \mathsf{a}_3, z) = z\frac{-\mathsf{a}_3 + z^2 + 2}{-\mathsf{a}_3 + z^2} \\ p_1(n = 2, \mathsf{a}_3, z) &= z\frac{-2\mathsf{a}_3 + \mathsf{a}_3^2 - 2z^2\mathsf{a}_3 + z^4 - 4z^2}{2\mathsf{a}_3 + \mathsf{a}_3^2 - 2z^2\mathsf{a}_3 + z^4}, \end{aligned} \tag{166}$$

It further holds for the particular values $\alpha_3 = \mathsf{a}_3 + 2n$ and $\alpha_2 = 2 - \mathsf{a}_3, \alpha_1 = \mathsf{a}_3$ that characterize the orbit, that q_1 from Equation (28) solves the Riccati equation

$$zq_{1\,z} = zq_1(z - q_1) - z\mathsf{a}_3 + q_1(1 + \mathsf{a}_3 + 2n)\,, \tag{167}$$

and the solution is

$$q_1(n, \mathsf{a}_3, z) = -2n \frac{\text{WhittakerM}(\frac{\mathsf{a}_3}{4} - \frac{n}{2} + 1, -\frac{1}{2} - \frac{\mathsf{a}_3}{4} - \frac{n}{2}, \frac{z^2}{2})}{z\text{WhittakerM}(\frac{\mathsf{a}_3}{4} - \frac{n}{2}, -\frac{1}{2} - \frac{\mathsf{a}_3}{4} - \frac{n}{2}, \frac{z^2}{2})} + \frac{(z^2 + 2n)}{z} .$$

For $n = 0, 1, 2$ the above $q_1(n, \mathsf{a}_3, z)$ is equal to

$$q_1(n = 0, \mathsf{a}_3, z) = z, \quad q_1(n = 1, \mathsf{a}_3, z) = z \frac{-\mathsf{a}_3 + z^2}{-\mathsf{a}_3 - 2 + z^2},$$
$$q_1(n = 2, \mathsf{a}_3, z) = z \frac{2\mathsf{a}_3 + \mathsf{a}_3^2 - 2\mathsf{a}_3 z^2 + z^4}{\mathsf{a}_3^2 + 6\mathsf{a}_3 + 8 - 2\mathsf{a}_3 z^2 - 4z^2 + z^4},$$

which agrees with the separate calculation involving the relevant shift operator.

Next, consider the solution given in (16) $j_i = \frac{z}{2}(1, 1, 1, -1, 1, -1)$ with the corresponding parameters $\alpha_i = (2 - \mathsf{a}_2, \mathsf{a}_2, 0, 0, 0, 0)$ for which $j_3 + j_4 = 0$, $j_4 + j_5 = 0$, $j_5 + j_6 = 0$, $j_6 + j_1 = 0$. With these quantities being zero, we are not permitted to act with $s_i, s_{i-1}\pi^{-1}$, $s_{i+1}\pi$ with $i = 3, 4, 5, 6$ on j_i in (16) in order to avoid division by zero. For these reasons we can not act with the shift operators $T_i, T_{i+1}^{-1}, i = 3, 4, 5, 6$ on the solution given in (16). We can therefore only act with

$$T_1^{n_1} T_2^{n_2} T_3^{-n_3}, \; n_1, n_3 \in \mathbb{Z}_+, \; n_2 \in \mathbb{Z} ,$$

that yields

$$(2 - \mathsf{a}_2 + 2n_1 - 2n_2, \mathsf{a}_2 - n_2 + 2n_3, 0 - 2n_3, 0, 0, 0) . \tag{168}$$

Example 8. *The action with the shift operator T_1^n is implemented by setting $n_1 = n, n_2 = n_3 = 0$. Then the parameters α_i automatically satisfy the three conditions $\alpha_5 = 0, \alpha_3 = 0, \alpha_4 = 0$ as in Equation (143). The single Riccati equation for the remaining quantity p_1 is given in Equation (144). Inserting $\alpha_1 = 2n + 2 - \mathsf{a}_2$ into Equation (144) leads to rational solution given by:*

$$p_1(n, \mathsf{a}_2, z) = \frac{2}{z} \frac{\text{WhittakerW}(\frac{\mathsf{a}_2}{4} + \frac{n}{2} + 1, \frac{1}{2} + \frac{n}{2} - \frac{\mathsf{a}_2}{4}, \frac{z^2}{2})}{\text{WhittakerW}(\frac{\mathsf{a}_2}{4} + \frac{n}{2}, \frac{1}{2} + \frac{n}{2} - \frac{\mathsf{a}_2}{4}, \frac{z^2}{2})} + \frac{\mathsf{a}_2}{z} ,$$

for which we find for $n = 0, 1, 2$:

$$p_1(n = 0, \mathsf{a}_2, z) = z, \quad p_1(n = 1, \mathsf{a}_2, z) = z \frac{-\mathsf{a}_2 + z^2}{-\mathsf{a}_2 + 2 + z^2}$$
$$p_1(n = 2, \mathsf{a}_2, z) = z \frac{-2\mathsf{a}_2 + \mathsf{a}_2^2 - 2z^2\mathsf{a}_2 + z^4}{8 - 6\mathsf{a}_2 + \mathsf{a}_2^2 + 4z^2 - 2z^2\mathsf{a}_2 + z^4} . \tag{169}$$

They agree with expressions obtained directly by acting with T_1 on the solution given in (16).

Finally, we consider the solution given in (17)

$$j_i = \frac{z}{2}(1, 1, -1, 1, 1, -1), \quad \alpha_i = (2 - \mathsf{a}_4, 0, 0, \mathsf{a}_4, 0, 0) ,$$

for which $j_2 + j_3 = 0$, $j_3 + j_4 = 0$, $j_5 + j_6 = 0, j_6 + j_1 = 0$. Accordingly $s_i, s_{i-1}\pi^{-1}, s_{i+1}\pi$ with $i = 2, 3, 5, 6$ will involve division with zero. This observation excludes $T_2, T_3, T_5, T_6, T_1^{-1}$, $T_3^{-1}, T_4^{-1}, T_6^{-1}$. Thus we generate rational solutions by acting with

$$T_1^{n_1} T_4^{n_4} T_2^{-n_2} T_5^{-n_5}, n_1, n_2, n_4, n_5 \in \mathbb{Z}_+ ,$$

that produces the parameter change

$$(2 - \mathsf{a}_4 + 2n_1 + 2n_2, -2n_2, -2n_4, \mathsf{a}_4 + 2n_4 + 2n_5, -2n_5, -2n_1)\,. \tag{170}$$

Example 9. *Here, we discuss the case of T_5^{-n}, the parameters α_i are those in expression (170) which one obtains after setting $n_5 = n, n_1 = n_2 = n_4 = 0$ and which satisfy $\alpha_2 = 0, \alpha_3 = 0, \alpha_6 = 0$ as in Equation (142). As shown below (142), we are left with one Riccati equation for q_1: $zq_{1\,z} = zq_1(z - q_2) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_5)$. Substituting $\alpha_1 = 2 - \mathsf{a}_4$ and $\alpha_5 = -2n$, we obtain*

$$zq_{1\,z} = zq_1(z - q_1) + z\alpha_1 + q_1(1 - \alpha_1 - \alpha_5) = zq_1(z - q_1) + z(2 - \mathsf{a}_4) + q_1(1 - 2 + \mathsf{a}_4 + 2n)\,. \tag{171}$$

The solution is

$$q_1(n, \mathsf{a}_4, z) = -\frac{n\mathsf{a}_4}{z}\frac{\mathrm{WhittakerW}(-\frac{1}{2} - \frac{\mathsf{a}_4}{4} + \frac{n}{2}, \frac{n}{2} + \frac{\mathsf{a}_4}{4}, -\frac{z^2}{2}))}{\mathrm{WhittakerW}(\frac{1}{2} - \frac{\mathsf{a}_4}{4} + \frac{n}{2}, \frac{n}{2} + \frac{\mathsf{a}_4}{4}, -\frac{z^2}{2})} + \frac{z^2 + 2n}{z}\,, \tag{172}$$

for which we find for $n = 0, 1, 2$:

$$\begin{aligned} &q_1(n = 0, \mathsf{a}_4, z) = z, \quad q_1(n = 1, \alpha_3, z) = z\frac{-\mathsf{a}_4 + z^2 + 2}{-\mathsf{a}_4 + z^2} \\ &p_1(n = 2, \mathsf{a}_4, z) = z\frac{-2\mathsf{a}_4 + \mathsf{a}_4^2 - 2z^2\mathsf{a}_4 + z^4 + 4z^2}{2\mathsf{a}_4 + \mathsf{a}_4^2 - 2z^2\mathsf{a}_4 + z^4}\,, \end{aligned} \tag{173}$$

which are in agreement with the results of acting with T_5 on the solution given in (17).

6. Summary and Comments

We identified rational solutions of the dressing chain equations of even periodicity with points of an orbit generated by the fundamental shift operators acting on all first-order polynomial solutions. It was described how additional Bäcklund transformation was needed to regularize those solutions that initially contained a simple pole.

For those first-order polynomial solutions which contain neighboring j_i and j_{i+1} such that: $j_i + j_{i+1} = 0$ for some $1 \leq i \leq N$ the action of some shift operators is not well-defined. Accordingly, those shift operators needed to be excluded in such cases and we have described the exclusion procedure in the paper. For orbits of the remaining well defined shift operators, we showed how this structure for $N = 4$ is responsible for a separate class of corresponding rational solutions (item III in Section 4.1) of the Painlevé V equation. We also showed how the rational solutions generated by a single shift operator T_i^n are expressed by Kummer/Whittaker polynomials with arguments depending on integer n.

The advantage of the formalism we presented is that it is universal, meaning that the derivation applies to all even-cyclic dressing chain systems or equivalent $A_{N-1}^{(1)}$ Painlevé equations as illustrated for the case of $N = 6$ in addition to the $N = 4$ case.

It is interesting to compare the derivation of elementary seed solutions for even-cyclic dressing chains with those encountered for odd-cyclic dressing chains. There are fundamental differences as the α_i parameters are fixed and do not depend on arbitrary variables. Also in contrast to the even-cyclic dressing chains, the fundamental variables $j_i, i = 1, ..., N$ of the odd-cyclic dressing chains that satisfy Equation (2) and the Painlevé variables $f_i, i = 1, ..., N$ are fully equivalent, as the relation $f_i = j_i + j_{i+1}$ is reversible through expression $j_i = \frac{1}{2}\sum_{k=0}^{N-1} f_{i+k}$ for odd values of N. For example for $N = 3$ one finds two elementary seed solutions that can be written as $j_i = (z/2)(1, 1, 1)$, $\alpha_i = (1, 1, 1)$, and $j_i = (3z)/2)(-1, 1, 1)$, $\alpha_i = 3(0, 1, 0)$. It is well known that the rational solutions of the Painlevé IV equation can all be obtained by Bäcklund transformations from the above two seed solutions [18], whether expressee in terms of J_i or f_i.

The natural next step, which we plan to pursue in the future, is to apply this framework to obtain closed determinant or special function expressions for rational solutions of all dressing chain equations of even periodicity generated by combined shift operators.

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Appendix A. Derivation of $A_5^{(1)}$ Painlevé Equations for $N = 6$ Dressing Chain

The dressing chain Equation (4) can be rewritten entirely in terms of f_i without any references to j_i after inserting the value for Ψ. It needs to be emphasized that such elimination of j_i variables while expressing the dressing chain equations in terms of f_i requires inserting the definition of Ψ from (5) into Equation (4). Such substitution of j_i by f_i would not work with Equation (2) for even values of N.

For $N = 6$, such a procedure yields $A_5^{(1)}$ Painlevé equations:

$$\begin{aligned}
z f_{1\,z} &= f_1 f_3 (f_2 - f_4 - f_6) + f_1 f_5 (f_4 + f_2 - f_6) + z\alpha_1 + f_1 (1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \\
z f_{2\,z} &= f_2 f_4 (f_3 - f_1 - f_5) + f_2 f_6 (f_3 - f_1 + f_5) + z\alpha_2 - f_2 (1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \\
z f_{3\,z} &= f_1 f_3 (f_4 - f_2 + f_6) + f_3 f_5 (f_4 - f_2 - f_6) + z\alpha_3 + f_3 (1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \\
z f_{4\,z} &= f_2 f_4 (f_1 - f_3 + f_5) + f_4 f_6 (f_5 - f_1 - f_3) + z\alpha_4 - f_4 (1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \\
z f_{5\,z} &= f_1 f_5 (f_6 - f_2 - f_4) + f_3 f_5 (f_6 + f_2 - f_4) + z\alpha_5 + f_5 (1 - \alpha_1 - \alpha_3 - \alpha_5)\,, \\
z f_{6\,z} &= f_2 f_6 (f_1 - f_3 - f_5) + f_4 f_6 (f_1 + f_3 - f_5) + z\alpha_6 - f_6 (1 - \alpha_1 - \alpha_3 - \alpha_5)\,,
\end{aligned} \tag{A1}$$

with $z = f_1 + f_3 + f_5 = f_2 + f_4 + f_6$.

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