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Some Results on Submodules Using (μ, ν, ω)-Single-Valued Neutrosophic Environment

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Abstract: The use of a single-valued neutrosophic set (svns) makes it much easier to manage situations in which one must deal with incorrect, unexpected, susceptible, faulty, vulnerable, and complicated information. This is a result of the fact that the specific forms of material being discussed here are more likely to include errors. This new theory has directly contributed to the expansion of both the concept of fuzzy sets and intuitionistic fuzzy sets, both of which have experienced additional development as a direct consequence of the creation of this new theory. In svns, indeterminacy is correctly assessed in a way that is both subtle and unambiguous. Furthermore, membership in the truth, indeterminacy, and falsity are all completely independent of one another. In the context of algebraic analysis, certain binary operations may be regarded as interacting with algebraic modules. These modules have pervasive and complicated designs. Modules may be put to use in a wide variety of different applications. Modules have applications in a diverse range of industries and market subsets due to their adaptability and versatility. Under the umbrella of the triplet (μ, ν, ω) structure, we investigate the concept of svns and establish a relationship between it and the single-valued neutrosophic module and the single-valued neutrosophic submodule, respectively. The purpose of this study is to gain an understanding of the algebraic structures of single-valued neutrosophic submodules under the triplet structure of a classical module and to improve the validity of this method by analyzing a variety of important facets. In this article, numerous symmetrical features of modules are also investigated, which demonstrates the usefulness and practicality of these qualities. The results of this research will allow for the successful completion of both of these objectives. The tactics that we have devised for use in this article are more applicable to a wide variety of situations than those that have been used in the past. Fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets are some of the tactics that fall under this category.

Keywords: (μ, ν, ω)-single-valued neutrosophic set; (μ, ν, ω)-single-valued neutrosophic module; (μ, ν, ω)-single-valued neutrosophic submodule; risk analysis; modeling; sensitivity analysis; efficiency analysis

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1. Introduction

The application of a newly suggested fuzzy algebraic structure has the effect of eliminating the limits that were previously imposed on previously developed fuzzy algebra structures. Due to the abundance of ambiguity and uncertainty in many parts of day-to-day life, the application of regular mathematics is not always practicable and may not even be possible at all in certain situations. In the process of resolving issues of this nature, the application of a wide range of fuzzy algebraic structures, such as fuzzy subgroups, fuzzy rings, fuzzy sub-fields, and fuzzy submodules, amongst others, has the potential to be of tremendous guidance. This is because these fuzzy algebraic structures are capable of representing a number of different types of information. The use of svns, which is a robust and all-encompassing formal framework, leads to the extension of both the fuzzy set and the intuitionistic fuzzy set, which are both categories of fuzzy sets.

1980 is the year in which Smarandache is credited with establishing neutrosophy as a distinct topic within the study of philosophy. It serves as the foundation upon which other academic disciplines such as philosophical logic, probability, set theory, and statistical analysis are constructed. As a consequence of this, he came up with the theory of neutrosophic logic and set, which provides an approximation of every statement of neutrosophic logic with the benefits of truth in the subcategory T, indeterminacy value in the subcategory I, and falsehood in the subcategory F. In light of the fact that the fuzzy set theory can only be used to depict situations in which there is uncertainty, the neutrosophic theory is the only viable option for describing scenarios in which there is indeterminacy. In [1], Smarandache provided an explanation of the neutrosophic idea, and in [2] Wang provided additional information on single-valued neutrosophic sets.

Researchers have already done extensive research on fuzzy and intuitionistic fuzzy sets [3–6], fuzzy logics [7–9], paraconsistent sets [10,11], fuzzy groups [12–15], complex fuzzy sets [16–18], fuzzy subrings and ideals [19–25], single-valued neutrosophic graphs and lattices [26–28], single-valued neutrosophic algebras [29,30] and many more interesting fields.

The neutrosophic theory ultimately led to the development of the algebraic neutrosophic structural principle. Kandasamy and Smarandache described shifts in the paradigm of algebraic structure theory in their paper, which may be found in [1,2]. The term "svns" is used to characterize them in addition to the terms "algebraic structures" and "topological structures" [31–33]. This concept was utilized by Çetkin, Aygün, and Çetkin in the context of neutrosophic subgroups [34], neutrosophic subrings [35], and neutrosophic submodules [36,37] of a certain classical group, ring, and module. Several recent research works on the process of group decision making with a variety of different characteristics are described in [38–41].

The motivation of the proposed concept is explained as follows: To present a more generalized concept, i.e., (1) (μ, ν, ω)-single-valued neutrosophic set; (2) (μ, ν, ω)-single-valued neutrosophic submodules; (3) Under triplet structure, the intersection of a finite number of svnsm is also (μ, ν, ω)-svnsm, but union may not be; (4) Several fundamental examples are provided for the superiority of this article.

Note that, clearly, $P^{\tilde{X}} = \tilde{P}, P^{\tilde{\emptyset}} = \tilde{\emptyset}$, which shows that our proposed definition can be converted into a single-valued neutrosophic set. The purpose of this paper is to present the study of single-valued neutrosophic submodules under triplet structure as a generalization of submodules, as a powerful extension of single-valued neutrosophic sets, as we know that modules are among the most basic and extensive algebraic structures that are researched in terms of a number of different binary operations.

Within the scope of this study, we analyze the idea of single-valued neutrosophic submodules under a triplet structure, as well as the noteworthy notions and characterizations offered in relation to this issue. In addition, we investigate the fundamental aspects of the ideas that are being presented.

We also demonstrate that svnsm must be (μ, ν, ω)-svnsm of module M, but (μ, ν, ω)-svnsm may not be a svnsm of module M. The article is organized as follows: in Section 2, we explain several basic ideas for svns. Section 3 explains the concept of (μ, ν, ω)-svnsm and some idealistic findings.

2. Preliminaries

This section covers basic definitions related to svns. In this section, we also present fundamental properties and relationships between svnss.

Definition 1 ([1])**.** *On the universe set X a svns P is defined as:*

$$P = \{\langle m,\ T_P(m),\ I_P(m),\ F_P(m)\rangle,\ m \in X\},$$

where $T, I, F : X \to [0,1]$, and $0 \le T_P(m) + I_P(m) + F_P(m) \le 3$, $\forall\ m \in X$, $T_P(m), I_P(m)$, $F_P(m) \in [0,1]$.

T_P, I_P and F_P represent the functions of truth, indeterminacy, and falsity-membership, respectively.

Definition 2 ([34])**.** *Let P be a svns on X and $\alpha \in [0,1]$. The α-level sets on P can be determined:*

$$\begin{aligned}(T_P)_\alpha &= \{m \in X \mid T_P(m) \ge \alpha\}, \\ (I_P)_\alpha &= \{m \in X \mid I_P(m) \ge \alpha\}, \text{ and} \\ (F_P)^\alpha &= \{m \in X \mid F_P(m) \le \alpha\}.\end{aligned}$$

Definition 3 ([2])**.** *Let P and Q be two single-valued neutrosophic sets (svnss) on X. Then*

1. *$P \subseteq Q$, if and only if $P(m) \le Q(m)$.*
   That is,
   $$T_P(m) \le T_Q(m),\ I_P(m) \le I_Q(m),\ \text{and } F_P(m) \ge F_Q(m).$$
   Also $P = Q$ if and only if $P \subseteq Q$ and $Q \subseteq P$.
2. *$P \cup Q = \{\langle \max\{T_P(m), T_Q(m)\}, \max\{I_P(m), I_Q(m)\}, \min\{F_P(m), F_Q(m)\}\rangle$, $\forall\ m \in X\}$.*
3. *$P \cap Q = \{\langle \min\{T_P(m), T_Q(m)\}, \min\{I_P(m), I_Q(m)\}, \max\{F_P(m), F_Q(m)\}\rangle$, $\forall\ m \in X\}$.*
4. *$(P \backslash Q) = \{\langle \min\{T_P(m), T_Q(m)\}, \min\{I_P(m), I_Q(m)\}, \max\{F_P(m), F_Q(m)\rangle, \forall\ m \in X\}$.*
5. *$c(P) = \{\langle F_P(m), 1 - I_P(m), T_P(m), \forall\ m \in X\rangle\}$. Here $c(c(P)) = P$.*

Definition 4 ([34])**.** *Let us define a function $g : X_1 \longrightarrow X_2$ and let P, Q be the svnss of X_1 and X_2, respectively. Then, the image of a svns P is also a svns of X_2 and as described below:*

$$\begin{aligned}g(P)(n) &= (T_{g(P)}(n), I_{g(P)}(n), F_{g(P)}(n) \\ &= (g(T_P)(n), g(I_P)(n), g(F_P)(n)),\ \ \forall\ n \in X_2.\end{aligned}$$

where

$$g(T_P)(n) = \begin{cases} \bigvee T_P(m), & \text{if } m \in g^{-1}(n), \\ 0, & \text{otherwise.} \end{cases}$$

$$g(I_P)(n) = \begin{cases} \bigvee I_P(m), & \text{if } m \in g^{-1}(n), \\ 0, & \text{otherwise.} \end{cases}$$

$$g(F_P)(n) = \begin{cases} \bigwedge F_P(m), & \text{if } m \in g^{-1}(n), \\ 1, & \text{otherwise.} \end{cases}$$

The preimage of a svns Q is a svns of X_1 and defined as:

$$\begin{aligned}g^{-1}(Q)(m) &= (T_{g^{-1}(Q)}(m), I_{g^{-1}(Q)}(m), F_{g^{-1}(Q)}(m) \\ &= (T_Q(g(m)), I_Q(g(m)), F_Q(g(m))) \\ &= B(g(m)),\ \ \forall\ m \in X_1.\end{aligned}$$

3. Single-Valued Neutrosophic Submodules under Triplet Structure

We define and investigate the basic properties and characterizations of a (μ, ν, ω)-svnm and (μ, ν, ω)-svnsm of a given classical module over a ring in this section. We typically start with some introductory (μ, ν, ω)-svns, the α-level set on (μ, ν, ω)-svns, operations and properties of (μ, ν, ω)-svns, and then study crucial results, propositions, theorems and several examples related to (μ, ν, ω)-svnm and (μ, ν, ω)-svnsm of a given classical module over a ring R. In addition, we present various homomorphism theorems for the validity of (μ, ν, ω)-svnsm.

Definition 5. *If P is a single-valued neutrosophic subset of X then (μ, ν, ω)-single-valued neutrosophic subset P of X is categorize as:*

$$P^{(\mu,\nu,\omega)} = \Big\{ \langle m, T_P^{\mu}(m), I_P^{\nu}(m), F_P^{\omega}(m) \rangle \mid m \in X \Big\},$$

where

$$\begin{aligned} T_P^{\mu}(m) &= \vee\{T_P(m), \mu\}, \\ I_P^{\nu}(m) &= \vee\{I_P(m), \nu\}, \\ F_P^{\omega}(m) &= \wedge\{F_P(m), \omega\}, \end{aligned}$$

such that

$$0 \leq T_P^{\mu}(m) + I_P^{\nu}(m) + F_P^{\omega}(m) \leq 3.$$

where $\mu, \nu, \omega \in [0,1]$, also $T, I, F : X \rightarrow [0,1]$, such that T_P^{μ}, I_P^{ν} and F_P^{ω} represent the functions of truth, indeterminacy, and falsity-membership, respectively.

Definition 6. *Let X be a space of objects, with m denoting a generic entity belong to X. A (μ, ν, ω)-svns P on X is symbolized by truth T_P^{μ}, indeterminacy T_P^{μ} and falsity-membership function F_P^{ω}, respectively. For every m in X, $T_P^{\mu}(m), I_P^{\nu}(m), F_P^{\omega}(m) \in [0,1]$, write a (μ, ν, ω)-svns P accordingly as:*

$$P^{(\mu,\nu,\omega)} = \sum_{i}^{n} \langle T^{\mu}(m_i), I^{\nu}(m_i), F^{\omega}(m_i) \rangle / m_i,\ m_i \in X.$$

Definition 7. *Let P and Q be two (μ, ν, ω)-svnss on X. Then*

1. $P^{(\mu,\nu,\omega)} \subseteq Q^{(\mu,\nu,\omega)} \ \Leftrightarrow P^{(\mu,\nu,\omega)}(m) \leq Q^{(\mu,\nu,\omega)}(m)$.
   That is,

$$\begin{aligned} T_P^{\mu}(m) &\leq T_Q^{\mu}(m), \\ I_P^{\nu}(m) &\leq I_Q^{\nu}(m), \\ F_P^{\omega}(m) &\geq F_Q^{\omega}(m), \end{aligned}$$

   and

$$P^{(\mu,\nu,\omega)} \ = \ Q^{(\mu,\nu,\omega)} \Leftrightarrow P^{(\mu,\nu,\omega)} \subseteq Q^{(\mu,\nu,\omega)} \text{ and } Q^{(\mu,\nu,\omega)} \subseteq P^{(\mu,\nu,\omega)}.$$

2. *The union of $P^{(\mu,\nu,\omega)}$ and $Q^{(\mu,\nu,\omega)}$ is denoted by*

$$S^{(\mu,\nu,\omega)} = P^{(\mu,\nu,\omega)} \ \cup \ Q^{(\mu,\nu,\omega)},$$

   and defined as

$$S^{(\mu,\nu,\omega)}(m) = P^{(\mu,\nu,\omega)}(m) \ \vee \ Q^{(\mu,\nu,\omega)}(m),$$

   where

$$P^{(\mu,\nu,\omega)}(m) \vee Q^{(\mu,\nu,\omega)}(m) = \{ \langle T_P^{\mu}(m) \vee T_Q^{\mu}(m), I_P^{\nu}(m) \vee I_Q^{\nu}(m), F_P^{\omega}(m) \wedge F_Q^{\omega}(m) \rangle,\ \forall\, m \in X \}.$$

That is,

$$\begin{aligned} T_S^{\mu}(m) &= \max\{T_P^{\mu}(m),\ T_Q^{\mu}(m)\}, \\ I_S^{\nu}(m) &= \max\{I_P^{\nu}(m),\ I_Q^{\nu}(m)\}, \\ F_S^{\omega}(m) &= \min\{F_P^{\omega}(m),\ F_Q^{\omega}(m)\}. \end{aligned}$$

3. *The intersection of* $P^{(\mu,\nu,\omega)}$ *and* $Q^{(\mu,\nu,\omega)}$ *is denoted by*

$$S^{(\mu,\nu,\omega)} = P^{(\mu,\nu,\omega)} \cap Q^{(\mu,\nu,\omega)},$$

and defined as

$$S^{(\mu,\nu,\omega)}(m) = P^{(\mu,\nu,\omega)}(m) \wedge Q^{(\mu,\nu,\omega)}(m),$$

where

$$P^{(\mu,\nu,\omega)}(m) \wedge Q^{(\mu,\nu,\omega)}(m) = \{\langle T_P^{\mu}(m) \wedge T_Q^{\mu}(m),\ I_P^{\nu}(m) \wedge I_Q^{\nu}(m),\ F_P^{\omega}(m) \vee F_Q^{\omega}(m)\rangle,\ \forall\, m \in X\}.$$

That is,

$$\begin{aligned} T_S^{\mu}(m) &= \min\{T_P^{\mu}(m), T_Q^{\mu}(m)\}, \\ I_S^{\nu}(m) &= \min\{I_P^{\nu}(m), I_Q^{\nu}(m)\}, \\ F_S^{\omega}(m) &= \max\{F_P^{\omega}(m), F_Q^{\omega}(m)\}. \end{aligned}$$

4. $(P^{(\mu,\nu,\omega)} \backslash Q^{(\mu,\nu,\omega)}) = \{\langle \min\{T_P^{\mu}(m), T_Q^{\mu}(m)\}, \min\{I_P^{\nu}(m), I_Q^{\nu}(m)\}, \max\{F_P^{\omega}(m), F_Q^{\omega}(m)\rangle, \forall\, m \in X\}$.
5. $c(P^{(\mu,\nu,\omega)}) = \{\langle (F_P^{\omega}(m), 1 - I_P^{\nu}(m), T_P^{\mu}(m)), \rangle, \forall\, m \in X\}$. *Here,* $c(c(P^{(\mu,\nu,\omega)}) = P^{(\mu,\nu,\omega)}$.

Definition 8. *Let* P *be a* (μ,ν,ω)*-svns on* X *and* $\alpha \in [0,1]$*. The* α*-level sets on* P *can be determined as:*

$$\begin{aligned} (T_P^{\mu})_{\alpha} &= \{m \in X \mid T_P^{\mu}(m) \geq \alpha\}, \\ (I_P^{\nu})_{\alpha} &= \{m \in X \mid I_P^{\nu}(m) \geq \alpha\}, \\ (F_P^{\omega})^{\alpha} &= \{m \in X \mid F_P^{\omega}(m) \leq \alpha\}. \end{aligned}$$

Definition 9. *Suppose a function* $g : X_1 \longrightarrow X_2$ *and* $P,\ Q$ *are the two* (μ,ν,ω)*-svnss of* X_1 *and* X_2*, respectively. Then, the image of a* (μ,ν,ω)*-svns* $P^{(\mu,\nu,\omega)}$ *is a* (μ,ν,ω)*-svns of* X_2 *and it is defined as follows:*

$$\begin{aligned} g(P^{(\mu,\nu,\omega)})(n) &= (T_{g(P)}^{\mu}(n), I_{g(P)}^{\nu}(n), F_{g(P)}^{\omega}(n)) \\ &= (g(T_P^{\mu})(n), g(I_P^{\nu})(n), g(F_P^{\omega})(n)), \forall\, n \in X_2. \end{aligned}$$

where

$$g(T_P^{\mu})(n) = \begin{cases} \bigvee T_P^{\mu}(m), & \textit{if } m \in g^{-1}(n), \\ 0, & \textit{otherwise}. \end{cases}$$

$$g(I_P^{\nu})(n) = \begin{cases} \bigvee I_P^{\nu}(m), & \textit{if } m \in g^{-1}(n), \\ 0, & \textit{otherwise}. \end{cases}$$

$$g(F_P^{\omega})(n) = \begin{cases} \bigwedge F_P^{\omega}(m), & \textit{if } m \in g^{-1}(n), \\ 1, & \textit{otherwise}. \end{cases}$$

The preimage of a (μ,ν,ω)-svns Q is a (μ,ν,ω)-svns of X_1 and defined as follows:

$$\begin{aligned} g^{-1}(Q^{(\mu,\nu,\omega)})(m) &= (T^{\mu}_{g^{-1}(Q)}(m), I^{\nu}_{g^{-1}(Q)}(m), F^{\omega}_{g^{-1}(Q)}(m)) \\ &= (T^{\mu}_{Q}(g(m)), I^{\nu}_{Q}(g(m)), F^{\omega}_{Q}(g(m))) \\ &= Q^{(\mu,\nu,\omega)}(g(m)),\ \forall\, m \in X_1. \end{aligned}$$

Note: We define and explore the notion of a (μ,ν,ω)-svnsm of a given classical module M over a ring R. R is used throughout this article to represent a commutative ring with unity 1.

Definition 10. *Let M be a module over a ring R. A (μ,ν,ω)-svns P on M is called a (μ,ν,ω)-svnsm of M if the following conditions are satisfied:*

M1: $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$. *That is*

$$T^{\mu}_{P}(0) = 1,\ I^{\nu}_{P}(0) = 1,\ F^{\omega}_{P}(0) = 0.$$

M2:

$$P^{(\mu,\nu,\omega)}(m+n) \geq P^{(\mu,\nu,\omega)}(m) \wedge P^{(\mu,\nu,\omega)}(n), \forall\, m,n \in M.$$

That is,

$$\begin{aligned} T^{\mu}_{P}(m+n) &\geq T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n), \\ I^{\nu}_{P}(m+n) &\geq I^{\nu}_{P}(m) \wedge I^{\nu}_{P}(n), \\ F^{\omega}_{P}(m+n) &\leq F^{\omega}_{P}(m) \vee F^{\omega}_{P}(n). \end{aligned}$$

M3:

$$P^{(\mu,\nu,\omega)}(rm) \geq P^{(\mu,\nu,\omega)}(m), \forall\, m \in M, r \in R.$$

That is,

$$\begin{aligned} T^{\mu}_{P}(rm) &\geq T^{\mu}_{P}(m), \\ I^{\nu}_{P}(rm) &\geq I^{\nu}_{P}(m), \\ F^{\omega}_{P}(rm) &\leq F^{\omega}_{P}(m). \end{aligned}$$

(μ,ν,ω)-svnsm(M) denotes the set of all (μ,ν,ω)-single-valued neutrosophic submodules of M.

Definition 11. *Let P be a (μ,ν,ω)-svns on M, then $-P^{(\mu,\nu,\omega)}$ is a (μ,ν,ω)-svns on M, defined as follows:*

$$\begin{aligned} T^{\mu}_{-P}(m) &= T^{\mu}_{P}(-m), \\ I^{\nu}_{-P}(m) &= I^{\nu}_{P}(-m), \\ F^{\omega}_{-P}(m) &= F^{\omega}_{P}(-m),\ \ \forall\, m \in M. \end{aligned}$$

Proposition 1. *If P is a (μ,ν,ω)-svnsm of an R-module M, then $(-1)P^{(\mu,\nu,\omega)} = -P^{(\mu,\nu,\omega)}$.*

Example 1. *Take, for example, classical ring $R = Z_4 = \{\bar{0},\bar{1},\bar{2},\bar{3}\}$. Since each ring is a module in itself, we consider $M = Z_4$ as a classical module. Define svns P as follows:*

$$P = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.3,0.2,0.8\rangle\ /\bar{1} + \langle 0.8,0.5,0.4\rangle\ /\bar{2} + \langle 0.2,0.1,0.7\rangle\ /\bar{3}\}.$$

It is clear that the svns P is a not a svnsm of the module M.

Let $\mu = 0.6$, $\nu = 0.3$ and $\omega = 0.6$, So (μ,ν,ω)-svns become

$$P = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.6,0.3,0.6\rangle\ /\bar{1} + \langle 0.8,0.5,0.4\rangle\ /\bar{2} + \langle 0.6,0.3,0.6\rangle\ /\bar{3}\}.$$

It is clear that the (μ,ν,ω)-svns P is a (μ,ν,ω)-svnsm of the module $M = Z_4$.

Proof. Let $m \in M$ be an arbitrary element

$$\begin{aligned} T^{\mu}_{(-1)P}(m) &= \bigvee_{m=(-1)n} T^{\mu}_{P}(n) \\ &= \bigvee_{n=-m} T^{\mu}_{P}(m) = T^{\mu}_{P}(-m) \\ &= T^{\mu}_{-P}(m). \end{aligned}$$

$$\begin{aligned} I^{\nu}_{(-1)P}(m) &= \bigvee_{m=(-1)n} I^{\nu}_{P}(n) \\ &= \bigvee_{n=-m} I^{\nu}_{P}(m) = I^{\nu}_{P}(-m) \\ &= I^{\nu}_{-P}(m) \end{aligned}$$

$$\begin{aligned} F^{\omega}_{(-1)P}(m) &= \bigwedge_{m=(-1)n} F^{\omega}_{P}(n) \\ &= \bigwedge_{n=-m} F^{\omega}_{P}(m) = F^{\omega}_{P}(-m) \\ &= F^{\omega}_{-P}(m). \end{aligned}$$

This shows that $T^{\mu}_{(-1)P}(m) = T^{\mu}_{-P}(m)$, $I^{\nu}_{(-1)P}(m) = I^{\mu}_{-P}(m)$ and $F^{\omega}_{(-1)P}(m) = F^{\omega}_{-P}(m)$. Thus, this holds true for each $m \in M$,

$$(-1)P^{(\mu,\nu,\omega)} = (T^{\mu}_{(-1)P}, I^{\nu}_{(-1)P}, F^{\omega}_{(-1)P}) = (T^{\mu}_{-P}, I^{\nu}_{-P}, F^{\omega}_{-P}) = -P^{(\mu,\nu,\omega)}.$$

□

Definition 12. *Let P be a (μ,ν,ω)-svns on an R-module M with $r \in R$. Set rP as a neutrosophic set to M, define as:*

$$\begin{aligned} T^{\mu}_{rP}(m) &= \vee\{T^{\mu}_{P}(n) \mid n \in M,\ m = rn\}, \\ I^{\nu}_{rP}(m) &= \vee\{I^{\nu}_{P}(n) \mid n \in M,\ m = rn\}, \\ F^{\omega}_{rP}(m) &= \wedge\{F^{\omega}_{P}(n) \mid n \in M,\ m = rn\}. \end{aligned}$$

Definition 13. *Let $P,\ Q$ be (μ,ν,ω)-svnss on M. Then, their sum $P^{(\mu,\nu,\omega)} + Q^{(\mu,\nu,\omega)}$ is a (μ,ν,ω)-svns on M, defined as follows:*

$$\begin{aligned} T^{\mu}_{P+Q}(m) &= \vee\{T^{\mu}_{P}(n) \wedge T^{\mu}_{Q}(o) \mid m = n + o,\ n, o \in M\}, \\ I^{\nu}_{P+Q}(m) &= \vee\{I^{\nu}_{P}(n) \wedge I^{\nu}_{Q}(o) \mid m = n + o,\ n, o \in M\}, \\ F^{\omega}_{P+Q}(m) &= \wedge\{F^{\omega}_{P}(n) \vee F^{\omega}_{Q}(o) \mid m = n + o,\ n, o \in M\}. \end{aligned}$$

Proposition 2. *If P and Q are (μ,ν,ω)-svnss on M with $P^{(\mu,\nu,\omega)} \subseteq Q^{(\mu,\nu,\omega)}$, then $rP^{(\mu,\nu,\omega)} \subseteq rQ^{(\mu,\nu,\omega)}$ for each $r \in R$.*

Proof. By definition, it is obvious. □

Proposition 3. *If P is (μ,ν,ω)-svns on M, then $T^{\mu}_{rP}(rm) \geq T^{\mu}_{P}(m), I^{\nu}_{rP}(rm) \geq I^{\nu}_{P}(m)$ and $F^{\omega}_{rP}(rm) \leq F^{\omega}_{P}(m)$.*

Proof. By definition, it is obvious. □

Proposition 4. *If P is a (μ,ν,ω)-svns on M, then $r(sP^{(\mu,\nu,\omega)}) = (rs)P^{(\mu,\nu,\omega)}, \forall\, r,\ s \in R$.*

Proof. Consider $r, s \in R$ to be arbitrary, whereas $m \in M$.

$$\begin{aligned} T^{\mu}_{r(sP)}(m) &= \bigvee_{m=rn} T^{\mu}_{sP}(n) \\ &= \bigvee_{m=rn} \bigvee_{n=st} T^{\mu}_{P}(t) = \bigvee_{m=r(st)} T^{\mu}_{P}(t) \\ &= T^{\mu}_{(rs)P}(m). \end{aligned}$$

$$\begin{aligned} I^{\nu}_{r(sP)}(m) &= \bigvee_{m=rn} I^{\nu}_{sP}(n) \\ &= \bigvee_{m=rn} \bigvee_{n=st} I^{\nu}_{P}(t) = \bigvee_{m=r(st)} I^{\nu}_{P}(t) \\ &= I^{\nu}_{(rs)P}(m). \end{aligned}$$

$$\begin{aligned} F^{\omega}_{r(sP)}(m) &= \bigwedge_{m=rn} F^{\omega}_{sP}(n) \\ &= \bigwedge_{m=rn} \bigwedge_{n=st} F^{\omega}_{P}(t) = \bigwedge_{m=r(st)} F^{\omega}_{P}(t) \\ &= F^{\omega}_{(rs)P}(m). \end{aligned}$$

Therefore, we have the following equalities

$$\begin{aligned} T^{\mu}_{r(sP)}(m) &= T^{\mu}_{(rs)P}(m), \\ I^{\nu}_{r(sP)}(m) &= I^{\nu}_{(rs)P}(m), \\ F^{\omega}_{r(sP)}(m) &= F^{\omega}_{(rs)P}(m). \end{aligned}$$

Therefore,

$$r(sP^{(\mu,\nu,\omega)}) = (T^{\mu}_{r(sP)}, I^{\nu}_{r(sP)}, F^{\omega}_{r(sP)}),$$

$\Rightarrow r(sP^{(\mu,\nu,\omega)}) = (T^{\mu}_{(rs)P}, I^{\nu}_{(rs)P}, F^{\omega}_{(rs)P}) = (rs)P^{(\mu,\nu,\omega)}$. □

Proposition 5. *If P and Q are (μ,ν,ω)-svnss on M, then*

1. $T^{\mu}_{Q}(rm) \geq T^{\mu}_{P}(m)$, *for each $m \in M$, if and only if $T^{\mu}_{rP} \leq T^{\mu}_{Q}$.*
2. $I^{\nu}_{Q}(rm) \geq I^{\nu}_{P}(m)$, *for each $m \in M$, if and only if $I^{\nu}_{rP} \leq I^{\nu}_{Q}$.*
3. $F^{\omega}_{Q}(rm) \leq F^{\omega}_{P}(m)$, *for each $m \in M$, if and only if $F^{\omega}_{rP} \geq F^{\omega}_{Q}$.*

Proof. (1) Suppose $T^{\mu}_{Q}(rm) \geq T^{\mu}_{P}(m)$, for each $m \in M$, then

$$T^{\mu}_{rP}(m) = \bigvee_{m=rn, n\in M} T^{\mu}_{P}(n).$$

Therefore,

$$T^{\mu}_{rP} \leq T^{\mu}_{Q}.$$

Conversely, suppose $T^{\mu}_{rP} \leq T^{\mu}_{Q}$. Then, $T^{\mu}_{rP}(m) = T^{\mu}_{Q}(m)$, for each $m \in M$. Hence,

$$T^{\mu}_{Q}(rm) \geq T^{\mu}_{rP}(rm) \geq T^{\mu}_{P}(m), \ \ \forall\, m \in M \ \ \text{(from Proposition 3)}.$$

(2) Suppose $I^{\nu}_{Q}(rm) \geq I^{\nu}_{P}(m)$, for each $m \in M$, then

$$I^{\nu}_{rP}(m) = \bigvee_{m=rn, n\in M} I^{\nu}_{P}(n).$$

Therefore,

$$I^{\nu}_{rP} \leq I^{\nu}_{Q}.$$

Conversely, suppose $I^{\nu}_{rP} \leq I^{\nu}_{Q}$. Then, $I^{\nu}_{rP}(m) = I^{\nu}_{Q}(m)$, for each $m \in M$.
Hence,

$$I^{\nu}_{Q}(rm) \geq I^{\nu}_{rP}(rm) \geq I^{\nu}_{P}(m), \ \ \forall\, m \in M \ \ \text{(from Proposition 3)}.$$

(3) Suppose $F^{\omega}_{Q}(rm) \leq I^{\omega}_{P}(m)$, for each $m \in M$, then

$$F^{\omega}_{rP}(m) = \bigwedge_{m=rn, n\in M} F^{\omega}_{P}(n).$$

Therefore,

$$F^{\omega}_{rP} \geq F^{\omega}_{Q}.$$

Conversely, suppose $F^{\omega}_{rP} \geq F^{\omega}_{Q}$. Then $F^{\omega}_{rP}(m) = F^{\omega}_{Q}(m)$, for each $m \in M$. Hence,

$$F^{\omega}_{Q}(rm) \leq F^{\omega}_{rP}(rm) \leq F^{\omega}_{P}(m), \ \forall\, m \in M \ \ \text{(using Proposition 3)}.$$

□

Proposition 6. *If P and Q are (μ,ν,ω)-svnss on M, then $r(P^{(\mu,\nu,\omega)} + Q^{(\mu,\nu,\omega)}) = rP^{(\mu,\nu,\omega)} + rQ^{(\mu,\nu,\omega)}$, $\forall\, r \in R$.*

Proof. Let P and Q be (μ,ν,ω)-svnss on M, $m \in M$ and $r \in R$.

$$\begin{aligned}
T^{\mu}_{r(P+Q)}(m) &= \bigvee_{m=rn} T^{\mu}_{(P+Q)}(n) \\
&= \bigvee_{m=rn} \bigvee_{n=t_1+t_2} (T^{\mu}_{P}(t_1) \wedge T^{\mu}_{Q}(t_2)) \\
&= \bigvee_{m=rt_1+rt_2} (T^{\mu}_{P}(t_1) \wedge T^{\mu}_{Q}(t_2)) \\
&= \bigvee_{m=m_1+m_2} (\bigvee_{m_1=rt_1} (T^{\mu}_{P}(t_1) \wedge \bigvee_{m_2=rt_2} T^{\mu}_{Q}(t_2))) \\
&= \bigvee_{m=m_1+m_2} (T^{\mu}_{rP}(m_1) \wedge T^{\mu}_{rQ}(m_2)) \\
&= T^{\mu}_{rP+rQ}(m).
\end{aligned}$$

$$\begin{aligned}
I^{\nu}_{r(P+Q)}(m) &= \bigvee_{m=rn} I^{\nu}_{(P+Q)}(n) \\
&= \bigvee_{m=rn} \bigvee_{n=t_1+t_2} (I^{\nu}_{P}(t_1) \wedge I^{\nu}_{Q}(t_2)) \\
&= \bigvee_{m=rt_1+rt_2} (I^{\nu}_{P}(t_1) \wedge I^{\nu}_{Q}(t_2)) \\
&= \bigvee_{m=m_1+m_2} (\bigvee_{m_1=rt_1} (I^{\nu}_{P}(t_1) \wedge \bigvee_{m_2=rt_2} I^{\nu}_{Q}(t_2))) \\
&= \bigvee_{m=m_1+m_2} (I^{\nu}_{rP}(m_1) \wedge I^{\nu}_{rQ}(m_2)) \\
&= I^{\nu}_{rP+rQ}(m).
\end{aligned}$$

$$\begin{aligned}
F^{\omega}_{r(P+Q)}(m) &= \bigwedge_{m=rn} F^{\omega}_{(P+Q)}(n) \\
&= \bigwedge_{m=rn} \bigwedge_{n=t_1+t_2} (F^{\omega}_{P}(t_1) \vee F^{\omega}_{Q}(t_2)) \\
&= \bigwedge_{m=rt_1+rt_2} (F^{\omega}_{P}(t_1) \vee F^{\omega}_{Q}(t_2)) \\
&= \bigwedge_{m=m_1+m_2} (\bigwedge_{m_1=rt_1} (F^{\omega}_{P}(t_1) \vee \bigwedge_{m_2=rt_2} F^{\omega}_{Q}(t_2))) \\
&= \bigwedge_{m=m_1+m_2} (F^{\omega}_{rP}(m_1) \vee F^{\omega}_{rQ}(m_2)) \\
&= F^{\omega}_{rP+rQ}(m).
\end{aligned}$$

Therefore, we have the equalities

$$\begin{aligned}
T^{\mu}_{r(P+Q)}(m) &= T^{\mu}_{rP+rQ}(m), \\
I^{\nu}_{r(P+Q)}(m) &= I^{\nu}_{rP+rQ}(m), \\
F^{\omega}_{r(P+Q)}(m) &= F^{\omega}_{rP+rQ}(m).
\end{aligned}$$

Hence,

$$\begin{aligned}
r(P^{(\mu,\nu,\omega)} + Q^{(\mu,\nu,\omega)}) &= (T^{\mu}_{r(P+Q)}, I^{\nu}_{r(P+Q)}, F^{\omega}_{r(P+Q)}) \\
&= (T^{\mu}_{rP+rQ}, I^{\nu}_{rP+rQ}, F^{\omega}_{rP+rQ}) \\
&= rP^{(\mu,\nu,\omega)} + rQ^{(\mu,\nu,\omega)}.
\end{aligned}$$

□

Proposition 7. *If P and Q are (μ, ν, ω)-svnss on M, then*
1. $T^{\mu}_{rP+sQ}(rm + sn) \geq T^{\mu}_{P}(m) \wedge T^{\mu}_{Q}(n)$,
2. $I^{\nu}_{rP+sQ}(rm + sn) \geq I^{\nu}_{P}(m) \wedge I^{\nu}_{Q}(n)$,
3. $F^{\omega}_{rP+sQ}(rm + sn) \leq F^{\omega}_{P}(m) \vee F^{\omega}_{Q}(n)$, *for each $m, n \in M$, $r, s \in R$.*

Proof. It is easy to prove with the help of Definitions 12 and 13 and Proposition 3. □

Proposition 8. *If P, Q, S are (μ, ν, ω)-svnss on M, then, for each $r, s \in R$, the following are satisfied;*

1. $T_S^{\mu}(rm + sn) \geq T_P^{\mu}(m) \wedge T_Q^{\mu}(n)$, *for all $m, n \in M$ if and only if $T_{rP+sQ}^{\mu} \leq T_S^{\mu}$.*
2. $I_S^{\nu}(rm + sn) \geq I_P^{\nu}(m) \wedge I_Q^{\nu}(n)$, *for all $m, n \in M$ if and only if $I_{rP+sQ}^{\nu} \leq I_S^{\nu}$.*
3. $F_S^{\omega}(rm + sn) \leq F_P^{\omega}(m) \vee F_Q^{\omega}(n)$, *for all $m, n \in M$ if and only if $F_{rP+sQ}^{\omega} \geq F_S^{\omega}$.*

Proof. It is easy to prove with the help of Proposition 7. □

Example 2. *Take an example for the above Proposition 7, classical ring $R = Z_2 = \{\bar{0}, \bar{1}\}$. Since each ring is a module in itself, we consider $M = Z_2$ as a classical module. Define svnss P and Q as follows:*

$$P = \{\langle 1, 1, 0\rangle\ /\bar{0} + \langle 0.6, 0.3, 0.6\rangle\ /\bar{1} \text{ and } Q = \{\langle 1, 1, 0\rangle\ /\bar{0} + \langle 0.8, 0.1, 0.4\rangle\ /\bar{1}\}.$$

Let $\mu = 0.6$, $\nu = 0.3$ and $\omega = 0.6$, So (μ, ν, ω)-svnss P and Q becomes

$$P = \{\langle 1, 1, 0\rangle\ /\bar{0} + \langle 0.6, 0.3, 0.6\rangle\ /\bar{1} \text{ and } Q = \{\langle 1, 1, 0\rangle\ /\bar{0} + \langle 0.8, 0.3, 0.4\rangle\ /\bar{1}\}.$$

We can examine that for truth-membership
$T_P^{\mu}(0) = 1$, $T_P^{\mu}(1) = 0.6$, $T_Q^{\mu}(0) = 1$, $T_Q^{\mu}(1) = 0.8$ *and*
$T_P^{\mu}(0) \wedge T_Q^{\mu}(0) = 1$, $T_P^{\mu}(0) \wedge T_Q^{\mu}(1) = 0.8$, $T_P^{\mu}(1) \wedge T_Q^{\mu}(0) = 0.6$, *and* $T_P^{\mu}(1) \wedge T_Q^{\mu}(1) = 0.6$.
Additionally, we can see that

$T_{rP}^{\mu}(0) = 1$, $T_{rP}^{\mu}(1) = 0.6$, $T_{sQ}^{\mu}(0) = 1$, $T_{sQ}^{\mu}(1) = 0.8$ *and* $T_{rP+sQ}^{\mu}(0) = 1$, $T_{rP+sQ}^{\mu}(1) = 0.8$.

Case 1: *Let $m = 0$, $n = 0$ and $r, s \in R = Z_2$, clearly* $T_{rP+sQ}^{\mu}(r0 + s0) = 1 \geq T_P^{\mu}(0) \wedge T_Q^{\mu}(0) = 1$.

Case 2: *Let $m = 0$, $n = 1$ and $r, s \in R = Z_2$, clearly* $T_{rP+sQ}^{\mu}(r0 + s1) = 1$ *or* $0.8 \geq T_P^{\mu}(0) \wedge T_Q^{\mu}(1) = 0.8$.

Case 3: *Let $m = 1$, $n = 0$ and $r, s \in R = Z_2$, clearly* $T_{rP+sQ}^{\mu}(r1 + s0) = 1$ *or* $0.8 \geq T_P^{\mu}(1) \wedge T_Q^{\mu}(0) = 0.6$.

Case 4: *Let $m = 1$, $n = 1$ and $r, s \in R = Z_2$, clearly* $T_{rP+sQ}^{\mu}(r1 + s1) = 1$ *or* $0.8 \geq T_P^{\mu}(1) \wedge T_Q^{\mu}(0) = 0.6$.

$\Rightarrow$ *(μ, ν, ω)-svnss P and Q satisfy the condition*
(1) $T_{rP+sQ}^{\mu}(rm + sn) \geq T_P^{\mu}(m) \wedge T_Q^{\mu}(n)$,
Similarly, we can show that for indeterminacy membership

(2) $I_{rP+sQ}^{\nu}(rm + sn) \geq I_P^{\nu}(m) \wedge I_Q^{\nu}(n)$,
Now, we prove for the falsity membership

$F_P^{\mu}(0) = 0$, $F_P^{\mu}(1) = 0.6$, $F_Q^{\mu}(0) = 0$, $F_Q^{\mu}(1) = 0.4$ *and*
$F_P^{\mu}(0) \vee F_Q^{\mu}(0) = 0$, $F_P^{\mu}(0) \vee F_Q^{\mu}(1) = 0.4$, $F_P^{\mu}(1) \vee F_Q^{\mu}(0) = 0.6$, *and* $F_P^{\mu}(1) \vee F_Q^{\mu}(1) = 0.6$.
Additionally, we can see that

$F_{rP}^{\mu}(0) = 0$, $F_{rP}^{\mu}(1) = 0.6$, $F_{sQ}^{\mu}(0) = 0$, $F_{sQ}^{\mu}(1) = 0.4$ *and* $F_{rP+sQ}^{\mu}(0) = 0$, $F_{rP+sQ}^{\mu}(1) = 0$.

Case 1: *Let $m = 0$, $n = 0$ and $r, s \in R = Z_2$, clearly* $F_{rP+sQ}^{\mu}(r0 + s0) = 0 \leq F_P^{\mu}(0) \vee F_Q^{\mu}(0) = 0$.

Case 2: *Let $m = 0$, $n = 1$ and $r, s \in R = Z_2$, clearly* $F_{rP+sQ}^{\mu}(r0 + s1) = 0 \leq F_P^{\mu}(0) \vee F_Q^{\mu}(1) = 0.4$.

Case 3: *Let $m = 1$, $n = 0$ and $r, s \in R = Z_2$, clearly* $F_{rP+sQ}^{\mu}(r1 + s0) = 0 \leq F_P^{\mu}(1) \vee F_Q^{\mu}(0) = 0.6$.

Case 4: *Let $m = 1$, $n = 1$ and $r, s \in R = Z_2$, clearly* $F_{rP+sQ}^{\mu}(r1 + s1) = 0 \leq F_P^{\mu}(1) \vee F_Q^{\mu}(0) = 0.6$.

$\Rightarrow$ (μ,ν,ω)-*svnss P and Q satisfy the condition*
(3) $F^{\omega}_{rP+sQ}(rm+sn) \leq F^{\omega}_{P}(m) \vee F^{\omega}_{Q}(n)$, *for each* $m,n \in M$, $r,s \in R$.

Example 3. *Take an example for the above Proposition 8. Let us take the classical ring* $R = Z_2 = \{\bar{0},\bar{1}\}$. *Since each ring is a module in itself, we consider* $M = Z_2$ *as a classical module. Define svnss* P, Q *and* S *as follows:*

$$P = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.3,0.2,0.8\rangle\ /\bar{1},\ Q = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.4,0.5,0.4\rangle\ /\bar{1}\} \text{ and } S = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.2,0.1,0.7\rangle\ /\bar{1}\}.$$

Let $\mu = 0.6$, $\nu = 0.3$ *and* $\omega = 0.6$. *Therefore,* (μ,ν,ω)-*svnss* P, Q *and* S *become*

$$P = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.6,0.3,0.6\rangle\ /\bar{1}, Q = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.6,0.5,0.4\rangle\ /\bar{1}\} \text{ and } S = \{\langle 1,1,0\rangle\ /\bar{0} + \langle 0.6,0.3,0.6\rangle\ /\bar{1}\}.$$

We can see that for truth-membership
$T^{\mu}_{P}(0) = 1$, $T^{\mu}_{P}(1) = 0.6$, $T^{\mu}_{Q}(0) = 1$, $T^{\mu}_{Q}(1) = 0.8$, $T^{\mu}_{S}(0) = 1$, $T^{\mu}_{S}(1) = 0.6$ *and*
$T^{\mu}_{P}(0) \wedge T^{\mu}_{Q}(0) = 1$, $T^{\mu}_{P}(0) \wedge T^{\mu}_{Q}(1) = 0.6$, $T^{\mu}_{P}(1) \wedge T^{\mu}_{Q}(0) = 0.6$, *and* $T^{\mu}_{P}(1) \wedge T^{\mu}_{Q}(1) = 0.6$.
Additionally, we can see that
$T^{\mu}_{rP}(0) = 1$, $T^{\mu}_{rP}(1) = 0.6$, $T^{\mu}_{sQ}(0) = 1$, $T^{\mu}_{sQ}(1) = 0.6$ *and* $T^{\mu}_{rP+sQ}(0) = 1$, $T^{\mu}_{rP+sQ}(1) = 0.6$.

Case 1: *Let* $m = 0$, $n = 0$ *and* $r,s \in R = Z_2$, *clearly* $T^{\mu}_{S}(r0+s0) = 1 \geq T^{\mu}_{P}(0) \wedge T^{\mu}_{Q}(0) = 1$.

Case 2: *Let* $m = 0$, $n = 1$ *and* $r,s \in R = Z_2$, *clearly* $T^{\mu}_{S}(r0+s1) = 1$ *or* $0.6 \geq T^{\mu}_{P}(0) \wedge T^{\mu}_{Q}(1) = 0.6$.

Case 3: *Let* $m = 1$, $n = 0$ *and* $r,s \in R = Z_2$, *clearly* $T^{\mu}_{S}(r1+s0) = 1$ *or* $0.6 \geq T^{\mu}_{P}(1) \wedge T^{\mu}_{Q}(0) = 0.6$.

Case 4: *Let* $m = 1$, $n = 1$ *and* $r,s \in R = Z_2$, *clearly* $T^{\mu}_{S}(r1+s1) = 1$ *or* $0.6 \geq T^{\mu}_{P}(1) \wedge T^{\mu}_{Q}(0) = 0.6$.

In all cases, we can see that $T^{\mu}_{S}(rm+sn) \geq T^{\mu}_{P}(m) \wedge T^{\mu}_{Q}(n)$, $\forall\ m,n \in M$
$\Leftrightarrow T^{\mu}_{rP+sQ}(0) = 1 \leq T^{\mu}_{S}(0) = 1$, *and* $T^{\mu}_{rP+sQ}(1) = 0.6 \leq T^{\mu}_{S}(1) = 0.6$.
$\Rightarrow$ (μ,ν,ω)-*svnss* P, Q *and* S *satisfy the condition*
(1) $T^{\mu}_{S}(rm+sn) \geq T^{\mu}_{P}(m) \wedge T^{\mu}_{Q}(n)$, *for all* $m,n \in M$ *if and only if* $T^{\mu}_{rP+sQ} \leq T^{\mu}_{S}$.
Similarly, we can show for the other clauses, i.e., indeterminacy membership as well as falsity membership.

Theorem 1. *Let* P *be a* (μ,ν,ω)-*svns on* M *and* $r,s \in R$. *Then, the following conditions must hold;*

1. $T^{\mu}_{rP} \leq T^{\mu}_{P} \Leftrightarrow T^{\mu}_{P}(rm) \geq T^{\mu}_{P}(m)$,
   $I^{\nu}_{rP} \leq I^{\nu}_{P} \Leftrightarrow I^{\nu}_{P}(rm) \geq I^{\nu}_{P}(m)$ *and*
   $F^{\omega}_{rP} \geq F^{\omega}_{P} \Leftrightarrow F^{\omega}_{P}(rm) \leq F^{\omega}_{P}(m)$, *for each* $m \in M$.
2. $T^{\mu}_{rP+sP} \leq T^{\mu}_{P} \Leftrightarrow T^{\mu}_{P}(rm+sn) \geq T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n)$,
   $I^{\nu}_{rP+sP} \leq I^{\nu}_{P} \Leftrightarrow I^{\nu}_{P}(rm+sn) \geq I^{\nu}_{P}(m) \wedge I^{\nu}_{P}(n)$,
   $F^{\omega}_{rP+sP} \geq F^{\omega}_{P} \Leftrightarrow F^{\omega}_{P}(rm+sn) \leq F^{\omega}_{P}(m) \vee F^{\omega}_{P}(n)$.

Proof. It is easy to prove with the help of Propositions 5 and 8. □

Theorem 2. *Let* P *be a* (μ,ν,ω)-*svns on* M. *Then,* P *is a svnsm of* $M \Leftrightarrow P$ *is a single-valued neutrosophic subgroup of the additive group* M, *in the notion of [34], and meets the requirements* $T^{\mu}_{rP} \leq T^{\mu}_{P}$, $I^{\nu}_{rP} \leq I^{\nu}_{P}$ *and* $F^{\omega}_{rP} \geq F^{\omega}_{P}$ *for every* $r \in R$.

Proof. From the description of a single-valued neutrosophic subgroup in [34], also using Theorem 1, it is easy to prove. □

Theorem 3. *Assume that* P *is a* (μ,ν,ω)-*svns on* M. *Then,* $P \in svnsm(M) \Leftrightarrow$ *the characteristics below hold:*

1. $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$.
2. $P^{(\mu,\nu,\omega)}(rm+sn) \geq P^{(\mu,\nu,\omega)}(m) \wedge P^{(\mu,\nu,\omega)}(n)$, *for every* $m,n \in M$, $r,s \in R$.

Proof. Assume that P is a (μ, ν, ω)-svnsm of M and $e, f \in M$. It is clearly shown that $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$ by using the condition (M1) of Definition 10. The foregoing statements are also correct based on (M2) and (M3).

$$\begin{aligned} T_P^{\mu}(rm+sn) &\geq T_P^{\mu}(rm) \wedge T_P^{\mu}(sn) \geq T_P^{\mu}(m) \wedge T_P^{\mu}(n), \\ I_P^{\nu}(rm+sn) &\geq I_P^{\nu}(rm) \wedge I_P^{\nu}(sn) \geq I_P^{\nu}(m) \wedge I_P^{\nu}(n), \\ F_P^{\omega}(rm+sn) &\leq T_P^{\mu}(rm) \vee F_P^{\omega}(sn) \leq F_P^{\omega}(m) \vee F_P^{\omega}(n), \ \forall\, m,n \in M, r,s \in R. \end{aligned}$$

Hence,

$$\begin{aligned} P^{(\mu,\nu,\omega)}(rm+sn) &= (T_P^{\mu}(rm+sn), I_P^{\nu}(rm+sn), F_P^{\omega}(rm+sn)) \\ &\geq (T_P^{\mu}(m) \wedge T_P^{\mu}(n), I_P^{\nu}(m) \wedge I_A(n), F_P^{\omega}(m) \vee F_P^{\omega}(n)) \\ &= (T_P^{\mu}(m), I_P^{\nu}(m), F_P^{\omega}(m)) \wedge (T_P^{\mu}(n), I_P^{\nu}(n), F_P^{\omega}(n)) \\ &= P^{(\mu,\nu,\omega)}(m) \wedge P^{(\mu,\nu,\omega)}(n). \end{aligned}$$

$\Rightarrow \ P^{(\mu,\nu,\omega)}(rm+sn) \geq P^{(\mu,\nu,\omega)}(m) \wedge P^{(\mu,\nu,\omega)}(n).$

Conversely, assume $P^{(\mu,\nu,\omega)}$ meets the conditions (i) and (ii). Therefore, the assumption is evident that $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$.

$$\begin{aligned} T_P^{\mu}(m+n) &= T_P^{\mu}(1.m+1.n) \geq T_P^{\mu}(m) \wedge T_P^{\mu}(n), \\ I_P^{\nu}(m+n) &= I_P^{\nu}(1.m+1.n) \geq I_P^{\nu}(m) \wedge I_P^{\nu}(n), \\ F_P^{\omega}(m+n) &= F_P^{\omega}(1.m+1.m) \leq F_P^{\omega}(m) \vee F_P^{\omega}(n). \end{aligned}$$

Therefore, $P^{(\mu,\nu,\omega)}(m+n) \geq P^{(\mu,\nu,\omega)}(m) \wedge P^{(\mu,\nu,\omega)}(n)$.

Furthermore, the requirement (M2) of Definition 10 is fulfilled. Let us now demonstrate the condition's legitimacy (M3). According to the hypothesis,

$$\begin{aligned} T_P^{\mu}(rm) &= T_P^{\mu}(rm+r0) \geq T_P^{\mu}(m) \wedge T_P^{\mu}(0) = T_P^{\mu}(m), \\ I_P^{\nu}(rm) &= I_P^{\nu}(rm+r0) \geq I_P^{\nu}(m) \wedge I_P^{\nu}(0) = I_P^{\nu}(m), \\ F_P^{\omega}(rm) &= F_P^{\omega}(rm+r0) \leq F_P^{\omega}(m) \vee F_P^{\omega}(0) = F_P^{\omega}(m), \ \ \forall\, m,n \in M, r \in R. \end{aligned}$$

As a result, (M3) of Definition 10 is achieved. □

Theorem 4. *Assume P and Q are (μ, ν, ω)-svnsm of a classical module M, then $P \cap Q$ is also a (μ, ν, ω)-svnsm of M.*

Proof. Since $P, Q \in (\mu, \nu, \omega)$-svnsm(M), we have $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$, and $Q^{(\mu,\nu,\omega)}(0) = \tilde{X}$.

$$\begin{aligned} T_{P \cap Q}^{\mu}(0) &= T_P^{\mu}(0) \wedge T_Q^{\mu}(0) = 1, \\ I_{P \cap Q}^{\nu}(0) &= I_P^{\nu}(0) \wedge I_Q^{\nu}(0) = 1, \\ F_{P \cap Q}^{\omega}(0) &= F_P^{\omega}(0) \vee F_Q^{\omega}(0) = 0. \end{aligned}$$

Hence, $(P^{(\mu,\nu,\omega)} \cap Q^{(\mu,\nu,\omega)})(0) = \tilde{X}$ and we find that the condition (M1) of Definition 10 is met. Let $m, n \in M, r, s \in R$. According to Theorem 3, it is sufficient to demonstrate that

$$(P^{(\mu,\nu,\omega)} \cap Q^{(\mu,\nu,\omega)})(rm+sn) \geq (P^{(\mu,\nu,\omega)} \cap Q^{(\mu,\nu,\omega)})(m) \ \wedge \ (P^{(\mu,\nu,\omega)} \cap Q^{(\mu,\nu,\omega)})(n).$$

That is,

$$\begin{aligned} T_{P \cap Q}^{\mu}(rm+sn) &\geq T_{P \cap Q}^{\mu}(m) \ \wedge \ T_{P \cap Q}^{\mu}(n), \\ I_{P \cap Q}^{\nu}(rm+sn) &\geq I_{P \cap Q}^{\nu}(m) \ \wedge \ I_{P \cap Q}^{\nu}(n), \\ F_{P \cap Q}^{\omega}(rm+sn) &\leq F_{P \cap Q}^{\omega}(m) \ \vee \ F_{P \cap Q}^{\omega}(n). \end{aligned}$$

Now, consider the truth, indeterminacy and falsity membership degree of the intersection,

$$\begin{aligned}
T^{\mu}_{P\,\cap\,Q}(rm+sn) &= T^{\mu}_{P}(rm+sn) \wedge T^{\mu}_{Q}(rm+sn) \\
&\geq (T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n)) \wedge (T^{\mu}_{Q}(m) \wedge T^{\mu}_{Q}(n)) \\
&= (T^{\mu}_{P}(m) \wedge T^{\mu}_{Q}(m)) \wedge (T^{\mu}_{P}(n) \wedge T^{\mu}_{Q}(n)) \\
&= T^{\mu}_{P\,\cap\,Q}(m) \wedge T^{\mu}_{P\,\cap\,Q}(n). \\
\Rightarrow\ T^{\mu}_{P\,\cap\,Q}(rm+sn) &\geq T^{\mu}_{P\,\cap\,Q}(m) \wedge T^{\mu}_{P\,\cap\,Q}(n) \\
I^{\nu}_{P\,\cap\,Q}(rm+sn) &= I^{\mu}_{P}(rm+sn) \wedge T^{\nu}_{Q}(rm+sn) \\
&\geq (I^{\nu}_{P}(m) \wedge I^{\nu}_{P}(n)) \wedge (T^{\nu}_{Q}(m) \wedge I^{\nu}_{Q}(n)) \\
&= (I^{\mu}_{P}(m) \wedge T^{\nu}_{Q}(m)) \wedge (I^{\mu}_{P}(n) \wedge I^{\mu}_{Q}(n)) \\
&= I^{\nu}_{P\,\cap\,Q}(m) \wedge T^{\mu}_{P\,\cap\,Q}(n). \\
\Rightarrow\ I^{\nu}_{P\,\cap\,Q}(rm+sn) &\geq I^{\nu}_{P\,\cap\,Q}(m) \wedge I^{\nu}_{P\,\cap\,Q}(n) \\
F^{\omega}_{P\,\cap\,Q}(rm+sn) &= F^{\omega}_{P}(rm+sn) \vee I^{\omega}_{Q}(rm+sn) \\
&\leq (F^{\omega}_{P}(m) \vee F^{\omega}_{P}(n)) \vee (F^{\omega}_{Q}(m) \vee F^{\omega}_{Q}(n)) \\
&= (F^{\omega}_{P}(m) \vee F^{\omega}_{Q}(m)) \vee (F^{\omega}_{P}(n) \vee F^{\omega}_{Q}(n)) \\
&= F^{\omega}_{P\,\cap\,Q}(m) \vee F^{\omega}_{P\,\cap\,Q}(n). \\
\Rightarrow F^{\omega}_{P\,\cap\,Q}(rm+sn) &\leq F^{\omega}_{P\,\cap\,Q}(m) \vee F^{\omega}_{P\,\cap\,Q}(n).
\end{aligned}$$

Hence, $P\,\cap\,Q \in (\mu,\nu,\omega)$-*svnsm*$(M)$. □

Note: Let N be a nonempty subset of M, which is a submodule of $M \Leftrightarrow rm+sn \in N,\ \forall\, m,n \in M,\ r,\ s \in R$.

Proposition 9. *Suppose M is a module over R. $P \in (\mu,\nu,\omega)$-svnsm(M) $\Leftrightarrow \forall\, \alpha \in [0,1]$, α-level sets of $P^{(\mu,\nu,\omega)}$, $(T^{\mu}_{P})_{\alpha}$, $(I^{\nu}_{P})_{\alpha}$ and $(F^{\omega}_{P})^{\alpha}$ are classical submodules of M where $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$.*

Proof. Let $P \in (\mu,\nu,\omega)$-svnsm(M), $\alpha \in [0,1]$, $m,n \in (T^{\mu}_{P})_{\alpha}$ and $r,s \in R$ can represent a certain element. Then,

$$T^{\mu}_{P}(m) \geq \alpha,\ T^{\mu}_{P}(n) \geq \alpha \text{ and } T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n) \geq \alpha.$$

By using Theorem 3, we have

$$T^{\mu}_{P}(rm+sn) \geq T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n) \geq \alpha.$$

Hence,

$$rm+sn \in (T^{\mu}_{P})_{\alpha}.$$

As a result, with each $\alpha \in [0,1]$, $(T^{\mu}_{P})_{\alpha}$ is a classical submodule of M. Similarly, for $m,n \in (I^{\nu}_{P})_{\alpha}$, $(F^{\omega}_{P})^{\alpha}$, we obtain $rm+sn \in (I^{\nu}_{P})_{\alpha}$, $(F^{\omega}_{P})^{\alpha}$ for each $\alpha \in [0,1]$. Consequently, $(I^{\nu}_{P})_{\alpha}$, $(F^{\omega}_{P})^{\alpha}$ with each $\alpha \in [0,1]$ are classical submodules of M.

Conversely, let $(T^{\mu}_{P})_{\alpha}$ with each $\alpha \in [0,1]$ be a classical submodule of M.

Let $m,n \in M$, $\alpha = T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n)$. Then, $T^{\mu}_{P}(m) = \alpha$ and $T^{\mu}_{P}(n) = \alpha$. Thus, m, $n \in (T^{\mu}_{P})_{\alpha}$.

Since $(T^{\mu}_{P})_{\alpha}$ is a classical submodule of M, we have $rm+sn \in (T^{\mu}_{P})_{\alpha}$ for all $r,s \in R$.

$$\Rightarrow\ (T^{\mu}_{P})(rm+sn) \geq \alpha = T^{\mu}_{P}(m) \wedge T^{\mu}_{P}(n).$$

Similarly, $(I^{\nu}_{P})_{\alpha}$ with each $\alpha \in [0,1]$ is a classical submodule of M.

Let $m,n \in M$, $\alpha = I^{\nu}_{P}(m) \wedge I^{\nu}_{P}(n)$. Then, $I^{\nu}_{P}(m) = \alpha$ and $I^{\nu}_{P}(n) = \alpha$. Thus, $m,n \in (I^{\nu}_{P})_{\alpha}$.

Since $(I_P^\nu)_\alpha$ is a classical submodule of M, we have $rm + sn \in (I_P^\nu)_\alpha$ for all $r, s \in R$.

$$\Rightarrow \ (I_P^\nu)(rm + sn) \geq \alpha = I_P^\nu(m) \wedge I_P^\nu(n).$$

Now, we consider $(F_P^\omega)^\alpha$. Let $m, n \in M$, $\alpha = F_P^\omega(m) \vee F_P^\omega(n)$. Then, $F_P^\omega(m) = \alpha$, $F_P^\omega(n) = \alpha$.

Thus, $m, n \in (F_P^\omega)^\alpha$. Since $(F_P^\omega)^\alpha$ is a submodule of M, we have $rm + sn \in (F_P^\omega)^\alpha$ for all $r, s \in R$.

Thus, $(F_P^\omega)(rm + sn) \leq \alpha = F_P^\omega(m) \vee F_P^\omega(n)$. It is also obvious that $P^{(\mu,\nu,\omega)}(0) = \tilde{X}$.

As a result, the conditions of Theorem 3 are fulfilled. □

Proposition 10. *Assume that P and Q are two (μ, ν, ω)-svnss on X and Y, respectively. Then, for the α- levels, the following equalities hold.*

$$\begin{aligned} (T^\mu_{P\times Q})_\alpha &= (T^\mu_P)_\alpha \times (T^\mu_Q)_\alpha, \\ (I^\nu_{P\times Q})_\alpha &= (I^\nu_P)_\alpha \times (I^\nu_Q)_\alpha, \\ (F^\omega_{P\times Q})^\alpha &= (F^\omega_P)^\alpha \times (F^\omega_Q)^\alpha. \end{aligned}$$

Proof. Let $(m, n) \in (T^\mu_{P\times Q})_\alpha$ be arbitrary.

Therefore,

$$\begin{aligned} &T^\mu_{P\times Q}(m,n) \geq \alpha \Leftrightarrow T^\mu_P(m) \wedge T^\mu_Q(n) \geq \alpha, \\ &\Leftrightarrow T^\mu_P(m) \geq \alpha,\ T^\mu_P(n) \geq \alpha \Leftrightarrow (m,n) \in (T^\mu_P)_\alpha \times (T^\mu_Q)_\alpha. \end{aligned}$$

Now, let $(m, n) \in (I^\nu_{P\times Q})_\alpha$ be arbitrary.

Therefore,

$$\begin{aligned} &I^\nu_{P\times Q}(m,n) \geq \alpha \Leftrightarrow I^\nu_P(m) \wedge I^\nu_Q(n) \geq \alpha, \\ &\Leftrightarrow I^\nu_P(m) \geq \alpha,\ I^\nu_P(n) \geq \alpha, \Leftrightarrow (m,n) \in (I^\nu_P)_\alpha \times (T^\nu_Q)_\alpha. \end{aligned}$$

Similarly, let $(m, n) \in (F^\omega_{P\times Q})^\alpha$ be arbitrary.

Therefore,

$$\begin{aligned} &F^\omega_{P\times Q}(m,n) \leq \alpha \Leftrightarrow F^\omega_P(m) \vee F^\omega_Q(n) \leq \alpha, \\ &\Leftrightarrow F^\omega_P(m) \leq \alpha,\ F^\omega_P(n) \leq \alpha \Leftrightarrow (m,n) \in (F^\omega_P)^\alpha \times (F^\omega_Q)^\alpha. \end{aligned}$$

□

Proposition 11. *Let P and Q be two (μ, ν, ω)-svnss on X and Y, respectively, and let $g : X \to Y$ be a mapping. Therefore, the preceding must be applicable:*

1.

$$g((T^\mu_P))_\alpha \subseteq (T^\mu_{g(P)})_\alpha,$$

$$g((I^\nu_P)_\alpha) \subseteq (I^\nu_{g(P)})_\alpha,$$

$$g((F^\omega_P)^\alpha) \supseteq (F^\omega_{g(P)})^\alpha.$$

2.

$$g^{-1}((T^\mu_Q)_\alpha) = (T^\mu_{g^{-1}(Q)})_\alpha,$$

$$g^{-1}((I^\nu_Q)_\alpha) = (I^\nu_{g^{-1}(Q)})_\alpha,$$

$$g^{-1}((F^\omega_Q)^\alpha) = (F^\omega_{g^{-1}(Q)})^\alpha.$$

Proof. (1) Let $n \in g((T^\mu_P)_\alpha)$. Then, $\exists\ m \in (T^\mu_P)_\alpha$ such that $g(m) = n$. Hence, $T^\mu_P(m) \geq \alpha$.

Therefore, $\bigvee_{m\in g^{-1}(n)} T^{\mu}_{P}(m) \geq \alpha$. That is, $T^{\mu}_{g(P)}(n) \geq \alpha$ and $n \in (T^{\mu}_{g(P)})_{\alpha}$. Hence, $g((T^{\mu}_{P})_{\alpha}) \subseteq (T^{\mu}_{g(P)})_{\alpha}$.

Similarly, $n \in g((I^{\nu}_{P})_{\alpha})$. Then, $\exists\, m \in (I^{\nu}_{P})_{\alpha}$ such that $g(m) = n$. Thus, $I^{\nu}_{P}(m) \geq \alpha$.

Therefore, $\bigvee_{m\in g^{-1}(n)} I^{\nu}_{P}(m) \geq \alpha$. That is, $I^{\nu}_{g(P)}(n) \geq \alpha$ and $n \in (I^{\nu}_{g(P)})_{\alpha}$. Therefore, $g((I^{\nu}_{P})_{\alpha}) \subseteq (I^{\nu}_{g(P)})_{\alpha}$.

Additionally, $n \in g((F^{\omega}_{P})^{\alpha})$. Then, $\exists\, m \in (F^{\omega}_{P})^{\alpha}$ such that $g(m) = n$. This implies $F^{\omega}_{P}(m) \leq \alpha$.

Therefore, $\bigwedge_{m\in g^{-1}(n)} F^{\omega}_{P}(m) \leq \alpha$. That is, $F^{\omega}_{g(P)}(n) \leq \alpha$ and $n \in (F^{\omega}_{g(P)})^{\alpha}$. Hence, $g((F^{\omega}_{P})^{\alpha}) \supseteq (F^{\omega}_{g(P)})_{\alpha}$.

(2)

$$\begin{aligned}
(T^{\mu}_{g^{-1}(Q)})_{\alpha} &= \{m \in X : T^{\mu}_{g^{-1}(Q)}(m) \geq \alpha\} \\
&= \{m \in X : T^{\mu}_{Q}(g(m)) \geq \alpha\} \\
&= \{m \in X : g(m) \in (T^{\mu}_{Q})_{\alpha}\} \\
&= \{m \in X : m \in g^{-1}((T^{\mu}_{Q})_{\alpha})\} \\
&= g^{-1}((T^{\mu}_{Q})_{\alpha}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(I^{\nu}_{g^{-1}(Q)})_{\alpha} &= \{m \in X : I^{\nu}_{g^{-1}(Q)}(m) \geq \alpha\} \\
&= \{m \in X : I^{\nu}_{Q}(g(m)) \geq \alpha\} \\
&= \{m \in X : g(m) \in (I^{\nu}_{Q})_{\alpha}\} \\
&= \{m \in X : m \in g^{-1}((I^{\nu}_{Q})_{\alpha})\} \\
&= g^{-1}((I^{\nu}_{Q})_{\alpha}).
\end{aligned}$$

Additionally,

$$\begin{aligned}
(F^{\omega}_{g^{-1}(Q)})^{\alpha} &= \{m \in X : F^{\omega}_{g^{-1}(Q)}(m) \leq \alpha\} \\
&= \{m \in X : F^{\omega}_{Q}(g(m)) \leq \alpha\} \\
&= \{m \in X : g(m) \in (F^{\omega}_{Q})^{\alpha}\} \\
&= \{m \in X : m \in g^{-1}((F^{\omega}_{Q})^{\alpha})\} \\
&= g^{-1}((F^{\omega}_{Q})^{\alpha}).
\end{aligned}$$

□

Theorem 5. *Assume $g : M \to N$ to be a homomorphism of modules, whereas M, N are the classical modules. If P is a (μ,ν,ω)-svnsm of M, then the image $g(P)$ is a (μ,ν,ω)-svnsm of N.*

Proof. It is sufficient to prove by Proposition 9 that

$$(T^{\mu}_{g(P)})_{\alpha},\ (I^{\nu}_{g(P)})_{\alpha}, (F^{\omega}_{g(P)})^{\alpha}$$

are (μ,ν,ω)-svnsm of N, $\forall\, \alpha \in [0,1]$.

Let $n_1, n_2 \in (T^{\mu}_{g(P)})_{\alpha}$. Then, $T^{\mu}_{g(P)}(n_1) \geq \alpha$ and $T^{\mu}_{g(P)}(n_2) \geq \alpha$. There exist $m_1,\ m_2 \in M$ such that

$$T^{\mu}_{P}(m_1) \geq T^{\mu}_{g(P)}(n_1) \geq \alpha \text{ and } T^{\mu}_{P}(m_2) \geq T^{\mu}_{g(P)}(n_2) \geq \alpha.$$

Therefore,
$$T_P^{\mu}(m_1) \geq \alpha,\ T_P^{\mu}(m_2) \geq \alpha \text{ and } T_P^{\mu}(m_1) \wedge T_P^{\mu}(m_2) \geq \alpha.$$
Since P is a (μ, ν, ω)-svnsm of M, for any $r, s \in R$ we have
$$T_P^{\mu}(rm_1 + sm_2) \geq T_P^{\mu}(m_1) \wedge T_P^{\mu}(m_2) \geq \alpha.$$
Hence,
$$rm_1 + sm_2 \in (T_P^{\mu})_\alpha.$$
$$\begin{aligned} &\Rightarrow \ g(rm_1 + sm_2) \in g((T_P^{\mu})_\alpha) \subseteq (T_{g(P)})_\alpha \\ &\Rightarrow \ rg(m_1) + sg(m_2) \in (T_{g(P)})_\alpha \\ &\Rightarrow \ rn_1 + sn_2 \in (T^{\mu}_{g(P)})_\alpha. \end{aligned}$$
Therefore, $(T^{\mu}_{g(P)})_\alpha$ is a submodule of N.

Similarly, $\forall\, \alpha \in [0,1]$, consider $n_1, n_2 \in (I^{\nu}_{g(P)})_\alpha$. Then, $I^{\nu}_{g(P)}(n_1) \geq \alpha$ and $I^{\nu}_{g(P)}(n_2) \geq \alpha$. There exist $m_1,\ m_2 \in M$, such that
$$I_P^{\nu}(m_1) \geq I^{\nu}_{g(P)}(n_1) \geq \alpha \text{ and } I_P^{\nu}(m_2) \geq I^{\nu}_{g(P)}(n_2) \geq \alpha.$$
Therefore,
$$I_P^{\nu}(m_1) \geq \alpha,\ I_P^{\nu}(m_2) \geq \alpha \text{ and } I_P^{\nu}(m_1) \wedge I_P^{\nu}(m_2) \geq \alpha.$$
Since P is a (μ, ν, ω)-svnsm of M, for any $r, s \in R$ we have
$$I_P^{\nu}(rm_1 + sm_2) \geq I_P^{\nu}(m_1) \wedge I_P^{\nu}(m_2) \geq \alpha.$$
Hence,
$$rm_1 + sm_2 \in (I_P^{\nu})_\alpha).$$
$$\begin{aligned} &\Rightarrow \ g(rm_1 + sm_2) \in g((I_P^{\nu})_\alpha) \subseteq (I_{g(P)})_\alpha \\ &\Rightarrow \ rg(m_1) + sg(m_2) \in (I_{g(P)})_\alpha \\ &\Rightarrow \ rn_1 + sn_2 \in (I^{\nu}_{g(P)})_\alpha. \end{aligned}$$
Therefore, $(I^{\nu}_{g(P)})_\alpha$ is a submodule of N.

Similarly, for all $\alpha \in [0,1]$, consider $n_1, n_2 \in (n^{\omega}_{g(P)})^\alpha$. Then, $n^{\omega}_{g(P)}(n_1) \leq \alpha$ and $n^{\omega}_{g(P)}(n_2) \leq \alpha$. There exist $m_1,\ m_2 \in M$, such that
$$F_P^{\omega}(m_1) \leq F^{\omega}_{g(P)}(n_1) \leq \alpha$$
and
$$F_P^{\omega}(m_2) \leq F^{\omega}_{g(P)}(n_2) \leq \alpha.$$
Therefore, $F_P^{\omega}(m_1) \leq \alpha$, $F_P^{\omega}(m_2) \leq \alpha$ and $F_P^{\omega}(m_1) \vee F_P^{\omega}(m_2) \leq \alpha$. Since P is a (μ, ν, ω)-svnsm of M, for any $r, s \in R$ we have $F_P^{\omega}(rm_1 + sm_2) \leq F_P^{\omega}(m_1) \vee F_P^{\omega}(m_2) \leq \alpha$.

Hence,
$$rm_1 + sm_2 \in (F_P^{\omega})_\alpha).$$
$$\begin{aligned} &\Rightarrow \ g(rm_1 + sm_2) \in g((F_P^{\omega})^\alpha) \supseteq (F_{g(P)})^\alpha \\ &\Rightarrow \ rg(m_1) + sg(m_2) \in (F_{g(P)})^\alpha \\ &\Rightarrow \ rn_1 + sn_2 \in (F^{\omega}_{g(P)})^\alpha. \end{aligned}$$

Therefore, $(F^{\omega}_{g(P)})^{\alpha}$ is a submodule of N. Consequently, for every $\alpha \in [0,1]$, $(T^{\mu}_{g(P)})_{\alpha}$ $(I^{\nu}_{g(P)})_{\alpha}$, $(F^{\omega}_{g(P)})^{\alpha}$ are classical submodules of N. Thus, $g(P)$ is a (μ,ν,ω)-svnsm of N via the use of Proposition 9. □

Theorem 6. *Assume $g : M \rightarrow N$ to be a homomorphism of modules, whereas M, N are the classical modules. If Q is a (μ,ν,ω)-svnsm of N, then the preimage $g^{-1}(Q)$ is a (μ,ν,ω)-svnsm of M.*

Proof. Using Proposition 11 (2), we have

$$\begin{aligned} g^{-1}((T^{\mu}_{Q})_{\alpha}) &= (T^{\mu}_{g^{-1}(Q)})_{\alpha}, \\ g^{-1}((I^{\nu}_{Q})_{\alpha}) &= (I^{\nu}_{g^{-1}(Q)})_{\alpha}, \\ g^{-1}((F^{\omega}_{Q})^{\alpha}) &= (F^{\omega}_{g^{-1}(Q)})^{\alpha}. \end{aligned}$$

Since preimage of a (μ,ν,ω)-svnsm is a (μ,ν,ω)-svnsm, by Proposition 9 we arrive at a conclusion. □

Corollary 1. *If $g : M \rightarrow N$ is a surjective module homomorphism and $\{P_i : i \in I\}$ is a family of (μ,ν,ω)-svnsm of M, then $g(\cap P_i)$ is a (μ,ν,ω)-svnsm of N.*

Corollary 2. *If $g : M \rightarrow N$ is a homomorphism of modules and $\{Q_j : j \in I\}$ is a family of (μ,ν,ω)-svnsm of N, then $g^{-1}(\cap Q_j)$ is a (μ,ν,ω)-svnsm of M.*

4. Conclusions

A svns under triplet structure is a type of svns that can be employed to identify significant problems in the fields of research, engineering, denoising, clustering, segmentation, and a variety of medical image-processing applications. Therefore, the study of svns under triplet structure and their characteristics has a massive impact, both in terms of attaining a knowledge of the basic principles of vulnerability and the applications that can benefit from this knowledge. This is because the study focuses on the characteristics of the svns under triplet structure rather than the structure of the svns. In this article, we defined svns and svnsm in terms of triplet structure and provided a number of essential conclusions related to these concepts. As a result, the objective of this work is to use a number of various ideas in order to produce some major findings on svnsm under the triplet structure. Since the study analyzes a wide range of symmetrical aspects of modules, it provides a compelling illustration of the importance of the work being carried out. In the field of algebraic structure theory, it contains an innovative concept with the potential to be used in the future to solve a range of algebraic problems.

- This approach is frequently extended to the generators of arbitrary nonempty families of neutrosophic submodules, as well as structure maintaining features such as the isomorphism of neutrosophic submodules. Neutrosophic submodules give a solid mathematical framework for clarifying related scientific issues in image processing, control theory, and economics.
- This notion can be expanded to soft neutrosophic modules, weak soft neutrosophic modules, strong soft neutrosophic modules, soft neutrosophic module homomorphism, and soft neutrosophic module isomorphism. Furthermore, scholars might explore the homological properties of these modules.
- This study can be broadened to include the cyclic fuzzy neutrosophic normal soft group, neutrosophic rings, and ideals.
- In the future, researchers may extend this concept to topological spaces, fields, and vector spaces.

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