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Post-Quantum Integral Inequalities for Three-Times (p,q)-Differentiable Functions

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Abstract: A new (p,q)-integral identity involving left and right post quantum derivatives, by using three times (p,q)-differentiable functions is established and then this identity is used to derive several new post-quantum Ostrowski type integral inequalities for three times (p,q)-differentiable functions. These results are generalizations of corresponding results in the area of integral inequalities.

Keywords: post quantum calculus; convex functions; Ostrowski's type inequalities

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1. Introduction

Convexity theory played an essential role in the development of the theory of mathematical inequalities [1]. Some inequalities have numerous applications and a growing role, both in different branches of pure mathematics and in a number of applied sciences. In the last few decades, many generalizations of convex functions have been formulated [2] and used, such as m-convex, (α,m)- convex, log-convex, h-convex, exponentially convex, harmonically convex, preinvex functions, generalized-convex functions and convex with respect to the pairs of functions. Inequalities and convexity play an important role in approximation and optimization theory.

Mathematical inequalities have also given interesting study perspectives in some economic fields, such as cost, income and profit [3], and also in welfare economics [4], wealth distribution, the study of utility and even prospect theory (e.g., comparing the satisfaction that the consumer gets if he makes one choice or another) [5].

Mathematical researchers have proposed many types of inequalities, such as the Hermite–Hadamard (H-H) type, the Ostrowski type, the Opial type, the H-H-Mercer type, the Simpson type, the Bullen type and other types by using convex functions and focusing on obtaining some new bounds for the left and right sides of these inequalities.

One of the most studied of such inequalities is the H-H one, also known as the "trapezoidal inequality". It was discovered independently by Hermite and Hadamard [6] and has attracted the interest of many researchers in the last few decades [7–16]. Compared with the initial perspective, this inequality has been and is being studied through a variety of new approaches. A possible explanation of the exceptional interest given to this type of inequality may be represented by the symmetry contained within. Even the new generalizations of this type of inequality preserve such symmetry, and sometimes they may contain a certain type of asymmetry.

Jackson analyzed the q-difference operator in [17]. The knowledge of q calculus [18] and difference equations was first used in physics and chemistry problems [17]. In [19], Siegel presented string theory involving q calculus.

The classical concept of quantum calculus introduced in [20] has been extended and generalized in several directions. In quantum calculus H-H, Ostrowski-, Gruss- and Montgomery-type inequalities have been studied in numerous articles [21–30]. This topic has a number of applications in many fields of mathematics (number theory, combinatorics, orthogonal polynomials and hypergeometric functions) and also in physics and mathematics (mechanics and relativity theory) [31–35].

Chakarabarti and Jajonnathan [36] introduced the concept of post-quantum calculus for the generalization of q-oscillator algebra, which is well known in physics, and in [37], Tunc and Gov introduced the concepts of (q_1, q_2) derivatives ${}_{\omega_1}D_{q_1,q_2}f(x)$ and (p,q) integrals over finite intervals.

The fundamental theorem of (p,q) calculus [38] and some (p,q) Taylor formulas were studied in [39], and as an example, the concept of (p,q) beta [38] and (p,q) gamma functions was introduced in [40]. (p,q) calculus is efficiently used in fields such as Lie groups, hypergeometric series and physics. In addition, (p,q) calculus is expected to be very important in the development of analysis and applications [40].

In [41], Kunt et al. found some (q_1, q_2) analogues of H-H-type inequalities and midpoints. Cortez et al. obtained the integral identities via the ϕ preinvex function in [42]. Using new ideas [41,43], in recent years, numerous authors have provided other refinements and extensions of the inequalities [44–46] of the H-H and Ostrowski types [18,38,44,47–53]. Studies relative to convexity and preinvex functions have applications such as maximizing the likelihood from multiple linear regression involving a Gauss–Laplace distribution [54–58].

Starting from the inequalities given in [48], our goal is to find a similar identity regarding the post-quantum left and right derivatives and the corresponding inequalities in the case of three-times differentiable functions in post-quantum calculus.

This article is organized into four main sections. After the introduction in Section 1, there follows a brief review of the fundamental notions of (p,q) calculus necessary for subsequent approaches (Section 2). Section 3 provides an identity that plays an essential role in developing the main results of this paper (i.e., new refinements to the Ostrowski-type integral inequality for three-times (p,q) differentiable functions). Section 4 presents the conclusions as well as future research perspectives.

2. Materials and Methods

The traditional H-H-type inequality says that "if $f : [a,b] \to \mathbb{R}$ is a convex function, then the following inequality takes place:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

and when f is a concave function, then previous inequality holds in the opposite direction" [6].

The Ostrowski integral inequality for differentiable functions was given in [59], being studied by many researchers:

Theorem 1. *"Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) whose derivative $f^{'} : [a,b] \to \mathbb{R}$ is bounded on (a,b) and*

$$||f^{'}||_\infty = sup_{t\in(a,b)}|f^{'}(t)| < \infty.$$

Then

$$|f(x) - \frac{1}{b-a}\int_a^b f(t)dt| \le \left[\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)||f^{'}||_\infty$$

for all $x \in [a,b]$" [48,59].

Throughout this paper, it is assumed that $0 < q < p \le 1$, and $[a,b]$ is a real interval with $a < b$. In addition, the (p,q) number of n is $[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + \ldots + pq^{n-2} + q^{n-1}, \quad n \in \mathbb{N}$:

Definition 1 ([46]). *"For a continuous function $f : [a, b] \to \mathbb{R}$, the $(p, q)_a$ derivative of function f at t on $[a, b]$ is defined by*

$$ {}_aD_{p,q}f(t) = \frac{f(pt + (1-p)a) - f(qt + (1-q)a)}{(p-q)(t-a)}, \qquad t \neq a, \tag{2} $$

$$ {}_aD_{p,q}f(a) = \lim_{t \to a} {}_aD_{p,q}f(t)." $$

Definition 2 ([47]). *"For a continuous function $f : [a, b] \to \mathbb{R}$, the $(p, q)^b$ derivative of a function f at t on $[a, b]$ is defined by*

$$ {}^bD_{p,q}f(t) = \frac{f(qt + (1-q)b) - f(pt + (1-p)b)}{(p-q)(b-t)}, \qquad t \neq b, \tag{3} $$

$$ {}^bD_{p,q}f(a) = \lim_{t \to b} {}^bD_{p,q}f(t)." $$

Example 1. *We define the function $f : [a, b] \to \mathbb{R}$ by $f(x) = Ax^2 + Bx + C$, where A, B and C are constants with $0 < q < p \leq 1$. Then, for $x \neq a$, we obtain*

$$ \begin{aligned} {}_aD_{p,q}(Ax^2 + Bx + C) &= \frac{A(px + (1-p)a)^2 + B(px + (1-p)a) + C}{(p-q)(x-a)} \\ &- \frac{A(qx + (1-q)a)^2 + B(qx + (1-q)a) + C}{(p-q)(x-a)} \\ &= \frac{A(x-a)(p-q)[(p+q)x + a(2-(p+q))] + B(p-q)(x-a)}{(p-q)(x-a)} \\ &= A([2]_{p,q}(x-a) + 2a) + B. \end{aligned} $$

Definition 3 ([46,48]). *"For a continuous function $f : [a, b] \to \mathbb{R}$, the $(p, q)_a$ integral of a function f at t on $[a, b]$ is defined by*

$$ \int_a^x f(t)_a d_{p,q}t = (p-q)(x-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + (1 - \frac{q^n}{p^{n+1}})a \right) \tag{4} $$

for $x \in [a, b]$. The function f is called a $(p, q)_a$ integrable function on $[a, b]$ if $\int_a^b f(t)_a d_{p,q}t$ exists".

Definition 4 ([47,48]). *"For a continuous function $f : [a, b] \to \mathbb{R}$, the $(p, q)^b$ integral of a function f at t on $[a, b]$ is defined by*

$$ \int_x^b f(t)^b d_{p,q}t = (p-q)(b-x) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} x + (1 - \frac{q^n}{p^{n+1}})b \right) \tag{5} $$

for $x \in [a, b]$. The function f is called a $(p, q)^b$ integrable function on $[a, b]$ if $\int_a^b f(t)^b d_{p,q}t$ exists".

Example 2. *We define the function $f : [a, b] \to \mathbb{R}$ with $f(t) = At^2 + B$, where A and B are constants. Then, we have*

$$ \begin{aligned} \int_a^b (At^2 + B)_a d_{p,q}t &= A(p-q)(b-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(\frac{q^n}{p^{n+1}} b + \right)^2 + B(p-q)(b-a) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \\ &= A(b-a)[\frac{b^2}{[3]_{p,q}} + a^2 \left(1 - \frac{2}{[2]_{p,q}} + \frac{1}{[3]_{p,q}} \right) + 2ab \left(\frac{1}{[2]_{p,q}} - \frac{1}{[3]_{p,q}} \right)] \\ &+ B(b-a). \end{aligned} $$

Lemma 1 ([46]). *"For $\alpha \in \mathbb{R} - \{-1\}$, the following equality holds:*

$$\int_a^b (t-a)^\alpha {}_a d_{p,q} t = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_{p,q}}." \tag{6}$$

Theorem 2 ([37]). *"Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions and $r > 1$ with $\frac{1}{s} + \frac{1}{r} = 1$. Then,*

$$\int_a^b |f(t)g(t)| {}_a d_{p,q} t \le \left(\int_a^b |f(t)|^s {}_a d_{p,q} t \right)^{\frac{1}{s}} \left(\int_a^b |f(t)|^r {}_a d_{p,q} t \right)^{\frac{1}{r}}." \tag{7}$$

The basic properties of these derivatives and integrals can be found in [18,38].

In recent years, new estimations of post-quantum Ostrowski-type integral inequalities for twice (p,q) differentiable functions were presented in [48]:

Theorem 3 ([48]). *"Let $f : a,b \to \mathbb{R}$ be a twice (p,q) differentiable function such that ${}^bD^2_{p,q}f$ and ${}_aD^2_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD^2_{p,q}f|$ and $|{}_aD^2_{p,q}f|$ are convex functions, then*

$$|{}^b_aL_{p,q}(x)| \le (x-a)^2(b-x)^2[(x-a)\left(\frac{1}{[4]_{p.q}}|{}_aD^2_{p,q}f(x)| + \frac{[4]_{p,q}-[3]_{p.q}}{[3]_{p,q}[4]_{p,q}}|{}_aD^2_{p,q}f(a)|\right)$$
$$+(b-x)\left(\frac{1}{[4]_{p.q}}|{}^bD^2_{p,q}f(x)| + \frac{[4]_{p,q}-[3]_{p.q}}{[3]_{p,q}[4]_{p,q}}|{}^bD^2_{p,q}f(b)|\right)]."$$

Theorem 4 ([48]). *"Let $f : a,b \to \mathbb{R}$ be a twice (p,q) differentiable function such that ${}^bD^2_{p,q}f$ and ${}_aD^2_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD^2_{p,q}f|^r$ and $|{}_aD^2_{p,q}f|^r$ are convex functions for $r \ge 1$, then*

$$\begin{aligned} |{}^b_aL_{p,q}(x)| &\le (x-a)^2(b-x)^2\left(\frac{1}{[3]_{p,q}}\right)^{1-\frac{1}{r}} \\ &\times [(x-a)\left(\frac{1}{[4]_{p.q}}|{}_aD^2_{p,q}f(x)|^r + \frac{[4]_{p,q}-[3]_{p.q}}{[3]_{p,q}[4]_{p,q}}|{}_aD^2_{p,q}f(a)|^r\right)^{\frac{1}{r}} \\ &+ (b-x)\left(\frac{1}{[4]_{p.q}}|{}^bD^2_{p,q}f(x)|^r + \frac{[4]_{p,q}-[3]_{p.q}}{[3]_{p,q}[4]_{p,q}}|{}^bD^2_{p,q}f(b)|^r\right)^{\frac{1}{r}}]." \end{aligned}$$

Theorem 5 ([48]). *"Let $f : a,b \to \mathbb{R}$ be a twice (p,q) differentiable function such that ${}^bD^2_{p,q}f$ and ${}_aD^2_{p,q}f$ are continuous and integrable functions on J_1 and J_2, respectively. If $|{}^bD^2_{p,q}f|^r$ and $|{}_aD^2_{p,q}f|^r$ are convex functions for $r > 1$, $\frac{1}{s} + \frac{1}{r} = 1$, then*

$$\begin{aligned} |{}^b_aL_{p,q}(x)| &\le (x-a)^2(b-x)^2\left(\frac{1}{[2s+1]_{p,q}}\right)^{\frac{1}{s}} \\ &\times [(x-a)\left(\frac{|{}_aD^2_{p,q}f(x)|^r + (p+q-)|{}_aD^2_{p,q}f(a)|^r}{[2]_{p,q}}\right)^{\frac{1}{r}} \\ &+ (b-x)\left(\frac{|{}^bD^2_{p,q}f(x)|^r + |{}^bD^2_{p,q}f(b)|^r}{[2]_{p,q}}\right)^{\frac{1}{r}}]." \end{aligned}$$

3. Results

Several new estimates of post-quantum H-H-type integral inequalities for three-times (p,q) differentiable functions are presented below, starting from the results presented in [48]. Let us define $J_1 = [b - p(b-x), b]$ and $J_2 = [a, a + p(x-a)]$. At first, a basic result (a new identity) for the following inequalities is established:

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be a three-times (p,q) differentiable function with ${}_aD^3_{p,q}f$ and ${}^bD^3_{p,q}f$ continuous and integrable functions on J_2 and J_1, respectively. Then, we have*

$$\begin{aligned} {}_a^bM_{p,q}(x) &= (x-a)^3(b-x)^3[(a-x)\int_0^1 t^3\,{}_aD^3_{p,q}f(tx+(1-t)a)d_{p,q}t \\ &+ (b-x)\int_0^1 t^3\,{}^bD^3_{p,q}f(tx+(1-t)b)d_{p,q}t], \end{aligned} \tag{8}$$

where

$$\begin{aligned} {}_a^bM_{p.q}(x) &= \frac{[2]_{p,q}[3]_{p,q}}{p^6q^6}[(b-x)^3\int_a^{p^3x+(1-p^3)a} f(t)\,{}_ad_{p.q}t + (x-a)^3\int_{p^3x+(1-p^3)b}^{b} f(t)\,{}^bd_{p,q}t] \\ &- \frac{1}{(p-q)^2pq}\{\frac{p^5-[2]_{p,q}[3]_{p,q}q(p-q)^2}{p^2q^5}[(x-a)(b-x)^3f(p^2x+(1-p^2)a) \\ &+ (x-a)^3(b-x)f(p^2x+(1-p^2)b)] + \frac{p}{q^3}[(x-a)(b-x)^3f(q^2x+(1-q^2)a) \\ &+ (x-a)^3(b-x)f(q^2x+(1-q^2)b)] + \frac{p^3-q[3]_{p,q}}{pq^4}[(x-a)(b-x)^3 \\ &\times f(pqx+(1-pq)a) + (x-a)^3(b-x)f(pqx+(1-pq)b)]\}. \end{aligned}$$

Proof. By using Definition 1, we have

$$\begin{aligned} &{}_aD^3_{p,q}f(tb+(1-t)a) = {}_aD_{p,q}({}_aD^2_{p,q}f(ta+(1-t)b)) = \frac{1}{(p-q)^3(b-a)^3t^3pq} \\ &\times[\frac{qf(p^3tb+a(1-p^3t))+pf(pq^2tb+a(1-q^2pt))-[2]_{p,q}f(p^2qtb+a(1-p^2tq))}{p^2} \\ &-\frac{qf(p^2qtb+a(1-p^2qt))+pf(q^3tb+a(1-q^3t))-[2]_{p,q}f(pq^2bt+a(1-pq^2t))}{q^2}]. \end{aligned} \tag{9}$$

Now, by applying the relation in Equation (9) and Definition 3, we obtain

$$\begin{aligned} &I = \int_0^1 t^3\,{}_aD^3_{p,q}f(tx+(1-t)a)d_{p,q}t = \frac{1}{(p-q)^3(b-a)^3pq} \\ &\times\int_0^1[\frac{qf(p^3tx+a(1-p^3t))+pf(pq^2tx+a(1-q^2pt))-[2]_{p,q}f(p^2qtx+a(1-p^2tq))}{p^2} \\ &-\frac{qf(p^2qtx+a(1-p^2qt))+pf(q^3tx+a(1-q^3t))-[2]_{p,q}f(pq^2xt+a(1-pq^2t))}{q^2}]d_{p,q}t, \end{aligned}$$

or we can obtain

$$
\begin{aligned}
I &= \frac{1}{(p-q)^3(x-a)^3pq}\int_0^1\{\frac{q}{p^2}f(p^3tx+a(1-p^3t))-\frac{p}{q^2}f(q^3tx+a(1-q^3t))\\
&+ \left(\frac{p+q}{q^2}+\frac{1}{p}\right)f(pq^2tx+a(1-pq^2t))\\
&- \left(\frac{1}{q}+\frac{p+q}{p^2}\right)f(p^2qtx+a(1-p^2qt))\}d_{p,q}t\\
&= \frac{1}{(p-q)^3(x-a)^4pq}\{(x-a)(p-q)\frac{q}{p^2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(p^3\frac{q^n}{p^{n+1}}x+a(1-p^3\frac{q^n}{p^{n+1}}))\\
&- \frac{p}{q^2}(x-a)(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(q^3\frac{q^n}{p^{n+1}}x+a(1-q^3\frac{q^n}{p^{n+1}}))\\
&+ \frac{p^2+pq+q^2}{pq^2}(x-a)(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(pq^2\frac{q^n}{p^{n+1}}x+a(1-pq^2\frac{q^n}{p^{n+1}}))\\
&- \frac{p^2+pq+q^2}{p^2q}(x-a)(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(p^2q\frac{q^n}{p^{n+1}}x+a(1-p^2q\frac{q^n}{p^{n+1}}))\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(x-a)(p-q)}{(x-a)^4(p-q)^3pq}\{\frac{q}{p^2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(p^3\frac{q^n}{p^{n+1}}x+a(1-p^3\frac{q^n}{p^{n+1}}))\\
&- \frac{p^4}{q^5}\sum_{n=0}^{\infty}\frac{q^{n+3}}{p^{n+4}}f(p^3\frac{q^{n+3}}{p^{n+4}}x+a(1-p^3\frac{q^{n+3}}{p^{n+4}}))\\
&+ \frac{p^2+pq+q^2}{q^4}p\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f(p^3\frac{q^{n+2}}{p^{n+3}}x+a(1-p^3\frac{q^{n+2}}{p^{n+3}}))\\
&- \frac{p^2+pq+q^2}{pq^2}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f(p^3\frac{q^{n+1}}{p^{n+2}}x+a(1-p^3\frac{q^{n+1}}{p^{n+2}}))\}\\
&= \frac{1}{(p-q)^3(x-a)^4p^4q}\{-\frac{[2]_{p,q}[3]_{p,q}(p-q)^3}{p^2q^4}\int_a^{p^3x+(1-p^3)a}f(t)_a d_{p,q}t\\
&+ p^3[(x-a)(p-q)[\frac{p^5-[2]_{p,q}[3]_{p,q}q(p-q)}{p^2q^5}f(p^2x+a(1-p^2))\\
&+ \frac{p}{q^3}f(q^2x+a(1-q^2))+\frac{p^3-q[3]_{p,q}}{pq^4}f(pqx+a(1-pq))]]\}.
\end{aligned}
\tag{10}
$$

By using Definition 2, we have

$$
{}^bD^3_{p,q}f(ta+(1-t)b) = {}^bD_{p,q}({}^bD_{p,q}f(ta+(1-t)b)) = \frac{1}{(p-q)^3(b-a)^3t^3pq}
$$
$$
\times[\frac{pf(q^3ta+b(1-q^3t))+qf(p^2qta+b(1-p^2qt))-[2]_{p,q}f(pq^2ta+b(1-pq^2t))}{q^2}
$$
$$
-\frac{pf(q^2pta+b(1-q^2pt))+qf(p^3ta+b(1-p^3t))-[2]_{p,q}f(p^2qta+b(1-p^2qt))}{p^2}]. \tag{11}
$$

Now, by applying the relation in Equation (11) and Definition 4, we obtain

$$
J = \int_0^1 t^3\, {}^bD^3_{p,q}f(tx+(1-t)b)d_{p,q}t = \frac{1}{(p-q)^3(b-x)^3pq}
$$

$$\times \int_0^1 [\frac{pf(q^3tx+b(1-q^3t))+qf(p^2qtx+b(1-p^2qt))-[2]_{p,q}f(pq^2tx+b(1-pq^2t))}{q^2}$$
$$-\frac{pf(q^2ptx+b(1-q^2pt))+qf(p^3tx+b(1-p^3t))-[2]_{p,q}f(p^2qtx+b(1-p^2qt))}{p^2}]d_{p,q}t,$$

or we can obtain

$$\begin{aligned} J &= \frac{1}{(p-q)^3(b-x)^3pq}\int_0^1\{\frac{p}{q^2}f(q^3tx+b(1-q^3t))-\frac{q}{p^2}f(p^3tx+b(1-p^3t)) \\ &- \left(\frac{p+q}{q^2}+\frac{1}{p}\right)f(pq^2tx+b(1-pq^2t))+\left(\frac{1}{q}+\frac{p+q}{p^2}\right)f(p^2qtx+b(1-p^2qt))\}d_{p,q}t. \end{aligned}$$

In the same way, we find that

$$\begin{aligned} J &= \frac{(p-q)(b-x)}{(b-x)^4(p-q)^3pq}\{\frac{p}{q^2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(q^3\frac{q^n}{p^{n+1}}x+b(1-q^3\frac{q^n}{p^{n+1}})) \\ &- \frac{q}{p^2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(p^3\frac{q^n}{p^{n+1}}x+b(1-p^3\frac{q^n}{p^{n+1}})) \\ &- \frac{p^2+pq+q^2}{pq^2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(pq^2\frac{q^n}{p^{n+1}}x+b(1-pq^2\frac{q^n}{p^{n+1}})) \\ &+ \frac{p^2+pq+q^2}{p^2q}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(p^2q\frac{q^n}{p^{n+1}}x+b(1-p^2q\frac{q^n}{p^{n+1}}))\} \\ \\ &= \frac{(p-q)(b-x)}{(b-x)^4(p-q)^3pq}\{\frac{p^4}{q^5}\sum_{n=0}^{\infty}\frac{q^{n+3}}{p^{n+4}}f(p^3\frac{q^{n+3}}{p^{n+4}}x+b(1-p^3\frac{q^{n+3}}{p^{n+4}})) \\ &- \frac{q}{p^2}\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f(p^3\frac{q^n}{p^{n+1}}x+b(1-p^3\frac{q^n}{p^{n+1}})) \\ &- \frac{p^2+pq+q^2}{q^4}p\sum_{n=0}^{\infty}\frac{q^{n+2}}{p^{n+3}}f(p^3\frac{q^{n+2}}{p^{n+3}}x+b(1-p^3\frac{q^{n+3}}{p^{n+3}})) \\ &+ \frac{p^2+pq+q^2}{pq^2}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}f(p^3\frac{q^{n+1}}{p^{n+2}}x+b(1-p^3\frac{q^{n+1}}{p^{n+2}}))\}, \end{aligned}$$

or

$$\begin{aligned} J &= \frac{1}{(b-x)^4(p-q)^3p^4q}\{\frac{[2]_{p,q}[3]_{p,q}(p-q)^3}{p^2q^5}\int_{p^3x+b(1-p^3)}^{b}f(t)d_{p,q}t \\ &- p^3[\frac{p^5-[2]_{p,q}[3]_{p,q}(p-q)q}{p^2q^5}(b-x)(p-q)f(p^2x+b(1-p^2)) \\ &+ \frac{p}{q^3}(b-x)(p-q)f(q^2x+b(1-q^2)) \\ \\ &+\frac{p^3-q[3]_{p,q}}{pq^4}(b-x)(p-q)f(pqx+b(1-pq))]\}. \end{aligned} \quad (12)$$

Now, by multiplying Equation (10) by $-(x-a)^4(b-x)^3$ and Equation (12) by $(b-x)^4(x-a)^3$, as well as adding the resultant equality, we obtain the desired equality. □

Some new post-quantum Ostrowski-type integral inequalities for three-times (p,q) differentiable functions are given below:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a three-times (p,q) differentiable function such that ${}_aD^3_{p,q}f$ and ${}^bD^3_{p,q}f$ are continuous and integrable functions on J_2 and J_1, respectively. If $|{}_aD^3_{p,q}f|$ and $|{}^bD^3_{p,q}f|$ are convex functions, then the following inequality takes place:*

$$|{}^b_aM_{p,q}(x)| \le (x-a)^3(b-x)^3\{(x-a)[|{}_aD^3_{p,q}f(x)|\frac{1}{[5]_{p,q}} + |{}_aD^3_{p,q}f(a)|\frac{[5]_{p,q}-[4]_{p,q}}{[4]_{p,q}[5]_{p,q}}]$$
$$+(b-x)[|{}^bD^3_{p,q}f(x)|\frac{1}{[5]_{p,q}} + |{}^bD^3_{p,q}f(b)|\frac{[5]_{p,q}-[4]_{p,q}}{[4]_{p,q}[5]_{p,q}}]\}. \tag{13}$$

Proof. Taking into account the modulus of Equation (8) and the convexity of the functions $|{}_aD^3_{p,q}f|$ and $|{}^bD^3_{p,q}f|$, we have

$$\begin{aligned}|{}^b_aM_{p,q}(x)| \quad &\le \quad (x-a)^3(b-x)^3[(x-a)\int_0^1 t^3|{}_aD^3_{p,q}f(tx+(1-t)a)|d_{p,q}t \\ &+ \quad (b-x)\int_0^1 t^3|{}^bD^3_{p,q}f(tx+(1-t)b)|d_{p,q}t\end{aligned}$$

$$\begin{aligned}&\le \quad (x-a)^3(b-x)^3\{(x-a)\int_0^1 t^3\Big(t|{}_aD^3_{p,q}f(x)| + (1-t)|{}_aD^3_{p,q}f(a)|\Big)d_{p,q}t \\ &+ \quad (b-x)\int_0^1 t^3\Big(t|{}^bD^3_{p,q}f(x)| + (1-t)|{}^bD^3_{p,q}f(b)|\Big)d_{p,q}t\}\end{aligned}$$

$$\begin{aligned}&= \quad (x-a)^3(b-x)^3\{(x-a)[|{}_aD^3_{p,q}f(x)|\int_0^1 t^4 d_{p,q}t + |{}_aD^3_{p,q}f(a)|\int_0^1 (t^3-t^4)d_{p,q}t] \\ &+ \quad (b-x)[|{}^bD^3_{p,q}f(x)|\int_0^1 t^4 d_{p,q}t + |{}^bD^3_{p,q}f(b)|\int_0^1 (t^3-t^4)d_{p,q}t]\},\end{aligned}$$

This completes the proof. □

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a three-times (p,q) differentiable function such that ${}_aD^3_{p,q}f$ and ${}^bD^3_{p,q}f$ are continuous and integrable functions on J_2 and J_1, respectively. If $|{}_aD^3_{p,q}f|^r$ and $|{}^bD^3_{p,q}f|^r$ are convex functions for $r \ge 1$, then the following inequality takes place:*

$$\begin{aligned}|{}^b_aM_{p,q}(x)| \quad &\le \quad (x-a)^3(b-x)^3\frac{1}{[4]^{1-\frac{1}{r}}_{p,q}} \\ &\times \quad [(x-a)\left(\frac{1}{[5]_{p,q}}|{}_aD^3_{p,q}f(x)|^r + \frac{[5]_{p,q}-[4]_{p,q}}{[4]_{p,q}[5]_{p,q}}|{}_aD^3_{p,q}f(a)|^r\right)^{\frac{1}{r}} \\ &+ \quad (b-x)\left(\frac{1}{[5]_{p,q}}|{}^bD^3_{p,q}f(x)|^r + \frac{[5]_{p,q}-[4]_{p,q}}{[4]_{p,q}[5]_{p,q}}|{}^bD^3_{p,q}f(b)|^r\right)^{\frac{1}{r}}]. \end{aligned} \tag{14}$$

Proof. By using Lemma 2 and the power mean inequality, we obtain

$$\begin{aligned}|{}^b_aM_{p,q}(x)| \quad &\le \quad (x-a)^3(b-x)^3[(x-a)\int_0^1 t^3|{}_aD^3_{p,q}f(tx+(1-t)a)|d_{p,q}t \\ &+ \quad (b-x)\int_0^1 t^3|{}^bD^3_{p,q}f(tx+(1-t)b)|d_{p,q}t\end{aligned}$$

$$\leq (x-a)^3(b-x)^3[(x-a)\left(\int_0^1 t^3 d_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^3|_aD_{p,q}^3f(tx+(1-t)a)|^r d_{p,q}t\right)^{\frac{1}{r}}$$
$$+ (b-x)\left(\int_0^1 t^3 d_{p,q}t\right)^{1-\frac{1}{r}}\left(\int_0^1 t^3|^bD_{p,q}^3f(tx+(1-t)b)|^r d_{p,q}t\right)^{\frac{1}{r}}].$$

With the convexity of $|_aD_{p,q}^3f|^r$ and $|^bD_{p,q}^3f|^r$, we have

$$\begin{aligned} |{}_a^bM_{p,q}(x)| &\leq (x-a)^3(b-x)^3 \\ &\times [(x-a)\frac{1}{[4]_{p,q}^{1-\frac{1}{r}}}\left(\frac{1}{[5]_{p,q}}|_aD_{p,q}^3f(x)|^r+(\frac{1}{[4]_{p,q}}-\frac{1}{[5]_{p,q}})|_aD_{p,q}^3f(a)|^r\right)^{\frac{1}{r}} \\ &+ (b-x)\frac{1}{[4]_{p,q}^{1-\frac{1}{r}}}\left(\frac{1}{[5]_{p,q}}|^bD_{p,q}^3f(x)|^r+(\frac{1}{[4]_{p,q}}-\frac{1}{[5]_{p,q}})|^bD_{p,q}^3f(b)|^r\right)^{\frac{1}{r}}]. \end{aligned}$$

□

Theorem 8. *Let $f:[a,b]\to\mathbb{R}$ be a three-times (p,q) differentiable function such that ${}_aD_{p,q}^3f$ and ${}^bD_{p,q}^3f$ are continuous and integrable functions on J_2 and J_1, respectively. If $|_aD_{p,q}^3f|^r$ and $|^bD_{p,q}^3f|^r$ are convex functions for $r\geq 1$ and $\frac{1}{r}+\frac{1}{s}=1$, then the following inequality holds:*

$$\begin{aligned} |{}_a^bM_{p,q}(x)| &\leq (x-a)^3(b-x)^3\left(\frac{1}{[3s+1]_{p,q}}\right)^{\frac{1}{s}} \\ &\times [(x-a)\left(\frac{1}{[2]_{p,q}}|_aD_{p,q}^3f(x)|^r+(1-\frac{1}{[2]_{p,q}})|_aD_{p,q}^3f(a)|^r\right)^{\frac{1}{r}} \\ &+ (b-x)\left(\frac{1}{[2]_{p,q}}|^bD_{p,q}^3f(x)|^r+(1-\frac{1}{[2]_{p,q}})|^bD_{p,q}^3f(b)|^r\right)^{\frac{1}{r}}]. \end{aligned} \tag{15}$$

Proof. By using Holder's inequality and applying Lemma 2, we obtain

$$\begin{aligned} |{}_a^bM_{p,q}(x)| &\leq (x-a)^3(b-x)^3[(x-a)\int_0^1 t^3|_aD_{p,q}^3f(tx+(1-t)a)|d_{p,q}t \\ &+ (b-x)\int_0^1 t^3|^bD_{p,q}^3f(tx+(1-t)b)|d_{p,q}t]. \end{aligned}$$

$$\begin{aligned} |{}_a^bM_{p,q}(x)| &\leq (x-a)^3(b-x)^3 \\ &\times [(x-a)\left(\int_0^1 t^{3s}d_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1 |_aD_{p,q}^3f(tx+(1-t)a)|^r d_{p,q}t\right)^{\frac{1}{r}} \\ &+ (b-x)\left(\int_0^1 t^{3s}d_{p,q}t\right)^{\frac{1}{s}}\left(\int_0^1 |^bD_{p,q}^3f(tx+(1-t)b)|^r d_{p,q}t\right)^{\frac{1}{r}}]. \end{aligned}$$

Now, by taking into account Lemma 1 and the convexity of $|_aD_{p,q}^3f|^r$ and $|^bD_{p,q}^3f|^r$ from the hypothesis, we obtain

$$
\begin{aligned}
|{}_a^b M_{p,q}(x)| \quad &\leq \quad (x-a)^3(b-x)^3 \\
&\times \quad [(x-a)\left(\frac{1}{[3s+1]_{p,q}}\right)^{\frac{1}{s}}\left(\int_0^1 (t|{}_aD_{p,q}^3 f(x)|^r + (1-t)|{}_aD_{p,q}^3 f(a)|^r)d_{p,q}t\right)^{\frac{1}{r}} \\
&+ \quad (b-x)\left(\frac{1}{[3s+1]_{p,q}}\right)^{\frac{1}{s}}\left(\int_0^1 (t|{}^bD_{p,q}^3 f(x)|^r + (1-t)|{}^bD_{p,q}^3 f(b)|^r)d_{p,q}t\right)^{\frac{1}{r}}].
\end{aligned}
$$

From here, we have

$$
|{}_a^b M_{p,q}(x)| \leq (x-a)^3(b-x)^3
$$

$$
\times[(x-a)\left(\frac{1}{[3s+1]_{p,q}}\right)^{\frac{1}{s}}\left(\frac{1}{[2]_{p,q}}|{}_aD_{p,q}^3 f(x)|^r + (1-\frac{1}{[2]_{p,q}})|{}_aD_{p,q}^3 f(a)|^r\right)^{\frac{1}{r}}
$$

$$
+(b-x)\left(\frac{1}{[3s+1]_{p,q}}\right)^{\frac{1}{s}}\left(\frac{1}{[2]_{p,q}}|{}^bD_{p,q}^3 f(x)|^r + (1-\frac{1}{[2]_{p,q}})|{}^bD_{p,q}^3 f(b)|^r\right)^{\frac{1}{r}}],
$$

The proof is finished. □

Remark 1. *Under conditions of Theorem 6, if $|{}_aD_{p,q}^3 f| \leq M$ and $|{}^bD_{p,q}^3 f| \leq M$, then we have*

$$
|{}_a^b M_{p,q}(x)| \leq M(x-a)^3(b-x)^3(b-a)\frac{1}{[4]_{p,q}}, \tag{16}
$$

where M is a positive constant.

Remark 2. *Under the conditions in Theorem 8, if $|{}_aD_{p,q}^3 f| \leq M$ and $|{}^bD_{p,q}^3 f| \leq M$, then we have*

$$
|{}_a^b M_{p,q}(x)| \leq M(x-a)^3(b-x)^3(b-a)\frac{1}{[3s+1]_{p,q}^{\frac{1}{s}}}, \tag{17}
$$

where M is a positive constant.

Remark 3. *All previous results can also be particularized for quantum calculus when we have $p = 1$. This means the identity in Equation (8) and the inequalities in Equations (13)–(15) can be rewritten for quantum calculus if we set $p = 1$.*

Therefore, the following can be obtained:

Remark 4. *(a) If $p = 1$ in Theorem 6, then the following inequality holds:*

$$
|{}_a^b M_q(x)| \leq (x-a)^3(b-x)^3\{(x-a)[|{}_aD_q^3 f(x)|\frac{1}{[5]_q} + |{}_aD_q^3 f(a)|\frac{[5]_q-[4]_q}{[4]_q[5]_q}]
$$

$$
+(b-x)[|{}^bD_q^3 f(x)|\frac{1}{[5]_q} + |{}^bD_q^3 f(b)|\frac{[5]_q-[4]_q}{[4]_q[5]_q}]\}.
$$

(b) If we have $p = 1$ in Theorem 7, then the following inequality holds:

$$
\begin{aligned}
|{}_a^b M_q(x)| \quad &\leq \quad (x-a)^3(b-x)^3 \\
&\times \quad \frac{1}{[4]_q^{1-\frac{1}{r}}}[(x-a)\left(\frac{1}{[5]_q}|{}_aD_q^3 f(x)|^r + \frac{[5]_q-[4]_q}{[4]_q[5]_q}|{}_aD_q^3 f(a)|^r\right)^{\frac{1}{r}} \\
&+ \quad (b-x)\left(\frac{1}{[5]_q}|{}^bD_q^3 f(x)|^r + \frac{[5]_q-[4]_q}{[4]_q[5]_q}|{}^bD_q^3 f(b)|^r\right)^{\frac{1}{r}}].
\end{aligned}
$$

(c) If $p = 1$ in Theorem 8, then the following inequality holds:

$$
\begin{aligned}
|{}_a^bM_q(x)| \quad &\leq \quad (x-a)^3(b-x)^3 \\
&\times \quad \left(\frac{1}{[3s+1]_q}\right)^{\frac{1}{s}}[(x-a)\left(\frac{1}{[2]_q}|{}_aD_q^3f(x)|^r + (1-\frac{1}{[2]_q})|{}_aD_q^3f(a)|^r\right)^{\frac{1}{r}} \\
&+ \quad (b-x)\left(\frac{1}{[2]_q}|{}^bD_q^3f(x)|^r + (1-\frac{1}{[2]_q})|{}^bD_q^3f(b)|^r\right)^{\frac{1}{r}}].
\end{aligned}
$$

Analogue results similar to the inequalities in Equations (16) and (17) are given in [48,52] for twice (p,q) differentiable functions and in [22].

4. Discussion and Conclusions

Starting from the identity and inequalities given in [48], our goal was to find a similar identity involving the left and right post-quantum derivatives and the corresponding inequalities in the case of the three-times (p,q) differentiable functions in the frame of post-quantum calculus.

In this paper, we presented a new (p,q) integral identity by using the third $(p,q)_a$ and $(p,q)^b$ derivatives. This identity was used to establish new post-quantum Ostrowski-type integral inequalities for three-times (p,q)-differentiable convex functions in Theorems 6–8. Several examples and consequences were presented in Examples 1 and 2 and Remark 4 to illustrate the investigated results.

It was also proven that these newly established integral inequalities can be turned into q Ostrowski-type integral inequalities for convex functions or classical Ostrowski-type inequalities for convex functions. These results contribute to the enrichment of the specialized literature in the field of Ostrowski-type inequalities. We hope that our results can be applied in interpolation theory, hypergeometric series, quantum algebra and applied mathematics.

It is a new and interesting problem that these results can be used to obtain similar inequalities for the new concepts of convexity, which have been extended and generalized in different dirrections.

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