symmetry

MDPI

Article

Fixed Points of (α, β, F^*) and (α, β, F^{**})-Weak Geraghty Contractions with an Application

Khairul Habib Alam [1], Yumnam Rohen [1] and Naeem Saleem [2,*]

[1] Department of Mathematics, National Institute of Techmology Manipur, Imphal 795004, India
[2] Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan
* Correspondence: naeem.saleem2@gmail.com

Abstract: This study aims to provide some new classes of (α, β, F^*)-weak Geraghty contraction and (α, β, F^{**})-weak Geraghty contraction, which are self-generalized contractions on any metric space. Furthermore, we find that the mappings satisfying the definition of such contractions have a unique fixed point if the underlying space is complete. In addition, we provide an application showing the uniqueness of the solution of the two-point boundary value problem.

Keywords: fixed point; F-contraction; α-admissible mapping; (α, β, F^*) and (α, β, F^{**})-weak Geraghty contractions

Citation: Alam, K.H.; Rohen, Y.; Saleem, N. Fixed Points of (α, β, F^*) and (α, β, F^{**})-Weak Geraghty Contractions with an Application. *Symmetry* **2023**, *15*, 243. https://doi.org/10.3390/sym15010243

Academic Editor: Alexander Zaslavski

Received: 23 December 2022
Revised: 12 January 2023
Accepted: 13 January 2023
Published: 16 January 2023

1. Introduction and Preliminaries

The Banach contraction theorem [1] has numerous extensions and generalizations. In 1973, Micheal A. Geraghty [2] introduced an intriguing contraction. By taking this into account, he examined some auxiliary functions for the existence and uniqueness of mappings in any complete metric spaces. The idea of α-contractive and α-admissible mappings was first presented in 2012 by Samet et al. [3], who also produced a number of fixed-point results for mappings that satisfy such contraction conditions. Later in 2013, Karapinar et al. [4] introduced an idea of triangular α-admissible mapping, which extended the scope of the α-admissibile mappings. Cho et al. [5] introduced the idea of α-Geragthy contraction mappings, which generalizes the idea of α-admissible mappings. Chandok [6] state and proved some interesting fixed point results for (α, β)-admissible Geraghty contractive mappings in 2015. On the other hand, Wardowski [7] in 2012 introduced the concept of F-contraction, while Wardowski et al. [8] defined the F-weak contraction and demonstrated some fixed point results as a generalization of Banach's result in 2014. The outcomes of this deduction are presented in the publications [9–13] in the setting of generalized metric spaces. By altering the criteria of Wardowski [7], the authors in [14–16] developed a new class of functions and established numerous generalized contraction theorems. The findings of Alfaqih et al. [15], who introduced F^*-weak contraction, and Badre [16], who introduced F^{**}-weak contraction, are the ones that will have the biggest impact on our main findings.

Furthermore, symmetry is a potential property of a Banach space, which is closely connected to fixed point problems (as discussed in [17]). This enhances the practical application of the subject to different fields. Since symmetry is a self-mapping of object A such that the structure is preserved, Saleem et al. [18] and Sain [19] provided several possible ways that this mapping could occur. By using the concept of symmetry, Neugebauer [17] obtained several applications of a layered compression expansion fixed-point theorem in the existence of solutions of a 2nd ordered difference equation with Dirichlet boundary conditions.

This paper contains the following notations: $\mathbb{N}$ is the collection of natural numbers, $\mathbb{R}$ is the collection of real numbers, and the group of positive real numbers is designated as $\mathbb{R}_+$.

After the subsequent result of the Banach contraction theorem M. Edelstein [20] first presented in 1962, we will now review certain definitions, findings, and examples as follows:

Theorem 1 ([20]). *For all $\vartheta_1, \vartheta_2 \in V$ with $\vartheta_1 \neq \vartheta_2$, a self-mapping $H : V \longrightarrow V$ in a compact metric space (V, d) is such that $d(H\vartheta_1, H\vartheta_2) < d(\vartheta_1, \vartheta_2)$. Then H must have a unique fixed point in V.*

Definition 1 ([2]). *A self-mapping $H : V \longrightarrow V$, where (V, d) is any metric space, is called a Geraghty contraction if there is a function*

$$\beta : (0, +\infty) \longrightarrow [0, 1) \text{ with } \lim_{n \to +\infty} r_n = 0 \text{ whenever } \lim_{n \to +\infty} \beta(r_n) = 1 \tag{1}$$

such that for all $\vartheta_1, \vartheta_2 \in V$,

$$d(H\vartheta_1, H\vartheta_2) \leq \beta(d(\vartheta_1, \vartheta_2))d(\vartheta_1, \vartheta_2).$$

Consider the collection of this type of β function as $\mathfrak{B}$.

Example 1. *The functions $\frac{1}{1+r}$, e^{-r} belong to the collection $\mathfrak{B}$.*

Definition 2 ([3]). *Let $H : V \longrightarrow V$ and $\alpha : V \times V \longrightarrow [0, +\infty)$ are two mappings. Then, the self-mapping H is called α-admissible if for all $\vartheta_1, \vartheta_2 \in V$,*

$$\alpha(H\vartheta_1, H\vartheta_2) \geq 1, \text{ whenever } \alpha(\vartheta_1, \vartheta_2) \geq 1. \tag{2}$$

Definition 3 ([5]). *A self-mapping $H : V \longrightarrow V$, where (V, d) is any metric space, is called α-Geraghty generalized contraction mapping if there exist mappings $\alpha : V \times V \longrightarrow \mathbb{R}$ and $\beta \in \mathfrak{B}$ such that for all $\vartheta_1, \vartheta_2 \in V$,*

$$\alpha(\vartheta_1, \vartheta_2)d(H\vartheta_1, H\vartheta_2) \leq \beta(M(\vartheta_1, \vartheta_2))M(\vartheta_1, \vartheta_2), \tag{3}$$

where $M(\vartheta_1, \vartheta_2) = \max\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2)\}$.

Definition 4 ([7]). *A self-mapping $H : V \longrightarrow V$, where (V, d) is any metric space, is called an F-contraction if for some $\tau > 0$ and for all $\vartheta_1, \vartheta_2 \in V$, $d(H\vartheta_1, H\vartheta_2) > 0$, we have*

$$\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq F(d(\vartheta_1, \vartheta_2)). \tag{4}$$

where $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a mapping that satisfies:
(F1) $r < s \Rightarrow F(r) < F(s)$, for all $r, s \in \mathbb{R}_+$;
(F2) $\lim_{n \to \infty} r_n = 0$ if and only if $\lim_{n \to \infty} F(r_n) = -\infty$, where $\{r_n\}_{n \in \mathbb{N}}$ is any sequence in $\mathbb{R}_+$;
(F3) for some $\delta \in (0, 1)$, $\lim_{r \to 0^+} r^\delta F(r) = 0$.
Consider the collection of this type of function F as $\mathcal{F}$.

Example 2. *The functions $\ln r, \ln r + r, -\frac{1}{\sqrt{r}}, \ln(r + r^2)$ belong to the collection $\mathcal{F}$.*

Remark 1. *From the Inequality (4) and Condition (F1) of Definition 4, we have $d(H\vartheta_1, H\vartheta_2) < d(\vartheta_1, \vartheta_2)$ when $d(H\vartheta_1, H\vartheta_2) > 0$, and also when $d(\vartheta_1, \vartheta_2) = 0$ we have $d(H\vartheta_1, H\vartheta_2) = 0$; thus, F-contractions are continuous.*

Theorem 2 ([7]). *An F-contraction is a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space and must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n \in \mathbb{N}}$ converges to ϑ^* for every $\vartheta_0 \in V$.*

Proposition 1 ([21]). *Let $\{\vartheta_n\}$ be any sequence in any metric space (V, d) which is not Cauchy and*

$$\lim_{n\to+\infty} d(\vartheta_{n+1}, \vartheta_n) = 0,$$

then for any $\varepsilon > 0$, there will be subsequences $\{\vartheta_{p_n}\}, \{\vartheta_{q_n}\}$ of the sequence $\{\vartheta_n\}$ with

$$n \le q_n < p_n,\ d(\vartheta_{p_n-1}, \vartheta_{q_n}) < \varepsilon \le d(\vartheta_{p_n}, \vartheta_{q_n}), \forall n \in \mathbb{N}$$

and

$$\lim_{n\to+\infty} d(\vartheta_{p_n}, \vartheta_{q_n}) = \lim_{n\to+\infty} d(\vartheta_{p_n-1}, \vartheta_{q_n-1}) = \varepsilon.$$

Definition 5 ([8]). *A self-mapping $H : V \to V$, where (V, d) is any metric space, is called an F-weak contraction if for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$, there exists $\tau > 0$ and $F \in \mathcal{F}$ satisfies*

$$\begin{aligned} \tau + F(d(H\vartheta_1, H\vartheta_2)) \quad &\le \quad F\Big(\max\Big\{ d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \\ &\qquad \frac{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1)}{2} \Big\}\Big). \end{aligned}$$

Remark 2. *Since for every F-contraction H, we have*

$$\begin{aligned} \tau + F(d(H\vartheta_1, H\vartheta_2) \quad &\le \quad F(d(\vartheta_1, \vartheta_2)) \\ &\le \quad F\Big(\max\Big\{ d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \\ &\qquad \frac{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1)}{2} \Big\}\Big), \end{aligned}$$

This implies that every F-contraction is an F-weak contraction.

Example 3. *Let $H : [0, 1] \to [0, 1]$ be given by*

$$H(\vartheta) = \begin{cases} \frac{1}{2} & \text{if} \quad \vartheta \in [0, 1) \\ \frac{1}{4} & \text{if} \quad \vartheta = 1 \end{cases}$$

Clearly, H is not continuous and from Remark 1, H is not an F-contraction. For any $\vartheta_1 \in [0, 1)$ and $\vartheta_2 = 1$, we have

$$d(H\vartheta_1, H\vartheta_2) = d\left(\frac{1}{2}, \frac{1}{4}\right) = \left|\frac{1}{2} - \frac{1}{4}\right| = \frac{1}{4} > 0$$

and

$$\begin{aligned} \max\left\{ d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \frac{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1)}{2} \right\} \quad &\ge \\ d(\vartheta_2, H\vartheta_2) = \frac{3}{4}. \end{aligned}$$

So, if we take $F(r) = \ln r,\ r \in \mathbb{R}_+$, H becomes an F-weak contraction for $\tau = \ln 3$. This example proves that every F-weak contraction may not be an F-contraction.

Theorem 3 ([8]). *If an F-weak contraction $H : V \to V$, where (V, d) is a complete metric space is such that either H or F is continuous, then H must have unique a fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ will converge to ϑ^* for any choice $\vartheta_0 \in V$.*

Definition 6 ([14]). *A mapping $H : V \to V$, where (V, d) is any metric space, is called an F-Suzuki contraction if for all $\vartheta_1, \vartheta_2 \in V$, there exists $\tau > 0$ that satisfies*

$$\frac{1}{2}d(\vartheta_1, H\vartheta_1) < d(\vartheta_1, \vartheta_2) \Rightarrow \tau + F(d(H\vartheta_1, H\vartheta_2)) \leq F(d(\vartheta_1, \vartheta_2)).$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ be a mapping that satisfies:
(F1) For all $r, s \in \mathbb{R}_+$ with $r < s \Rightarrow F(r) < F(s)$.
(F2$'$) $\inf F = -\infty$.
(F3$'$) F is continuous on $\mathbb{R}_+$.
Consider the collection of this type of function F as $\mathfrak{F}$.

Example 4. *The functions $-\frac{1}{r}, -\frac{1}{r} + r, \frac{1}{1-e^r}, \frac{1}{e^r - e^{-r}}$ belong to $\mathfrak{F}$.*

Theorem 4 ([14]). *An F-Suzuki contraction self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ must converge to ϑ^* for every $\vartheta_0 \in V$.*

Theorem 5 ([14]). *Let $H : V \to V$ be a self map, where (V, d) is a complete metric space. If for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$, there exists $\tau > 0$ and $F \in \mathfrak{F}$ that satisfies*

$$\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq F(d(\vartheta_1, \vartheta_2)).$$

Then, H must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ will converge to ϑ^* for any $\vartheta_0 \in V$.

Remark 3. *The condition (F3) of Definition 4 and (F3$'$) of Definition 6 do not depend on each other. Indeed, for $p \geq 1$, $F(r) = -\frac{1}{r^p}$ does not satisfy (F3) but satisfies (F1), (F2) of Definition 4 and also the condition (F3$'$) of Definition 6. So, we have, $\mathfrak{F} \nsubseteq \mathcal{F}$. Again, $F(r) = -\frac{1}{(r+[r])^t}$ for $t \in (0, \frac{1}{a}), a > 1$ does not satisfy (F3$'$) but satisfies (F1), (F2) and also (F3) of Definition 4 for $k \in (\frac{1}{a}, 1)$. Consequently, $\mathcal{F} \nsubseteq \mathfrak{F}$. Additionally, if $F(r) = \ln r$, then $F \in \mathcal{F} \cap \mathfrak{F}$. Consequently, $\mathcal{F} \cap \mathfrak{F} \neq \phi$. As a result, some functions with fixed points do not satisfy the contraction condition by functions that belongs to $\mathcal{F}$.*

Alfaqih et al., in [15], introduced a new collection $\mathcal{F}'$ of functions $F : (0, \infty) \to \mathbb{R}$ satisfying only one side implication condition:
(F2$''$) $\lim_{n\to\infty} F(r_n) = -\infty \Rightarrow \lim_{n\to\infty} r_n = 0$, where $\{r_n\}_{n\in\mathbb{N}}$ is sequence in $\mathbb{R}_+$

Clearly, $\mathcal{F} \subset \mathcal{F}'$. The example given below demonstrates that in general, the converse inclusion is not true.

Example 5. *The function $\ln(\frac{r}{2} + \sin r) \in \mathcal{F}'$ does not satisfy (F1) of Definition 4. In addition, the functions $\cos r - \frac{1}{r}, \sin r - \frac{1}{r} \in \mathcal{F}'$ do not satisfy both (F1) and (F3) of Definition 4.*

Definition 7 ([15]). *A self-mapping $H : V \to V$, where (V, d) is a metric space, is called an F^*-weak contraction if for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$, there are some $\tau > 0$ and $F \in \mathcal{F}'$ that satisfy*

$$\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq F(m(\vartheta_1, \vartheta_2)), \tag{5}$$

where $m(\vartheta_1, \vartheta_2) = \max\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2)\}$.

Theorem 6 ([15]). *An F^*-weak contraction $H : V \to V$, where (V, d) is a complete metric space, is such that F is continuous; then, H must have a unique fixed point $\vartheta^* \in V$ and for all $\vartheta_0 \in V$, $\lim_{n\to\infty} H^n\vartheta_0 = \vartheta^*$. Moreover, $\lim_{n\to\infty} m(H^n\vartheta_0, \vartheta^*) = 0$ if and only if H is continuous at ϑ^*.*

The set $\mathfrak{F}'$ of mappings $F : \mathbb{R}_+ \to \mathbb{R}$ fulfilling (F1), (F2″) and (F3) was utilized by Sachin V. Bedre [16] to define a new contraction in his paper. We define set $\mathcal{F}''$ fulfilling (F1) and (F2″).

Definition 8 ([16]). *A self-mapping $H : V \to V$, where (V, d) is a metric space, is called an F^{**}-weak contraction if for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$, there are some $\tau > 0$ and $F \in \mathfrak{F}'$ which satisfies*

$$\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq F(m'(\vartheta_1, \vartheta_2)), \tag{6}$$

where

$$m'(\vartheta_1, \vartheta_2) = \max\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), d(\vartheta_1, H\vartheta_2), d(\vartheta_2, H\vartheta_1)\}.$$

Theorem 7 ([16]). *An F^{**}-weak contraction $H : V \to V$, where (V, d) is a complete metric space is such that F is continuous; then, H must have a unique fixed point $\vartheta^* \in V$ and for all $\vartheta_0 \in V$, $\lim\limits_{n\to\infty} H^n\vartheta_0 = \vartheta^*$. Moreover, $\lim\limits_{n\to\infty} m'(H^n\vartheta_0, \vartheta^*) = 0$ if and only if H is continuous at ϑ^*.*

2. Main Results

We use the definitions of F^* and F^{**}-weak contractions to define the (α, β, F^*)-weak Geragthy contraction and (α, β, F^{**})-weak Geraghty contraction as well as two new classes of contractions. In addition, we state and prove some results for the functions that will satisfy the definitions and some more extra conditions. There are also some examples satisfying our results and having a unique fixed point. There are some corollaries that are deduced from our main result.

Definition 9. *Let (V, d) be a metric space and $\alpha : V \times V \longrightarrow [0, +\infty)$. Then, a mapping $H : V \longrightarrow V$ is called an (α, β, F^*)-weak Geraghty generalized contraction if for some $F \in \mathcal{F}'$, $\tau > 0$, and for a $\beta \in \mathfrak{B}$, we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m(\vartheta_1, \vartheta_2))F(m(\vartheta_1, \vartheta_2)), \tag{7}$$

for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$, and $\alpha(\vartheta_1, \vartheta_2) \geq 1$,
where $m(\vartheta_1, \vartheta_2) = \max\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2)\}$.

Definition 10. *Let (V, d) be a metric space and $\alpha : V \times V \longrightarrow [0, +\infty)$. Then, a mapping $H : V \longrightarrow V$ will be called an (α, β, F^{**})-weak Geraghty generalized contraction if for some $F \in \mathcal{F}''$ and $\tau > 0$ and for a $\beta \in \mathfrak{B}$, we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m'(\vartheta_1, \vartheta_2))F(m'(\vartheta_1, \vartheta_2)), \tag{8}$$

for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$ and $\alpha(\vartheta_1, \vartheta_2) \geq 1$,
where

$$m'(\vartheta_1, \vartheta_2) = \max\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), d(\vartheta_1, H\vartheta_2), d(\vartheta_2, H\vartheta_1)\}.$$

Remark 4. *Note that $m(\vartheta_1, \vartheta_2) \leq m'(\vartheta_1, \vartheta_2)$ and $\mathcal{F}'' \subset \mathcal{F}'$. So, for $F \in \mathcal{F}''$, every (α, β, F^*)-weak Geraghty generalized contraction is again an (α, β, F^{**})-weak Geraghty generalized contraction. However, for $F \in \mathcal{F}'$, an (α, β, F^*)-weak Geraghty generalized contraction is not an (α, β, F^{**})-weak Geraghty generalized contraction.*

Theorem 8. *Suppose a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, is an (α, β, F^*)-weak Geraghty generalized contraction and F is continuous; then, the self-mapping H has a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ must converge to ϑ^* for every $\vartheta_0 \in V$. Moreover, H is a continuous at ϑ^* if and only if $\lim\limits_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.*

Proof. For arbitrary $\vartheta_0 \in V$, let us define a sequence $\{\vartheta_n\}$ by:

$$\begin{array}{rcl} \vartheta_1 & = & H\vartheta_0, \\ \vartheta_2 & = & H\vartheta_1, \\ & \vdots & \\ \vartheta_{n+1} & = & H\vartheta_n, \forall n \in \mathbb{N}. \end{array}$$

If there is $m \in \mathbb{N}$ with $\vartheta_{m+1} = \vartheta_m$, then we have proved it, since then ϑ_m is the required point. Suppose $\vartheta_n \neq \vartheta_{n+1}, \forall n \in \mathbb{N}$. Since $\alpha(\vartheta_n, \vartheta_{n-1}) \geq 1$, we have

$$\begin{array}{rcl} \tau + F(d(H\vartheta_n, H\vartheta_{n-1})) & \leq & \alpha(\vartheta_n, \vartheta_{n-1})(\tau + F(d(H\vartheta_n, H\vartheta_{n-1})) \\ & \leq & \beta(m(\vartheta_n, \vartheta_{n-1}))F(m(\vartheta_n, \vartheta_{n-1})) \\ \Rightarrow \quad \tau + F(d(\vartheta_{n+1}, \vartheta_n)) & \leq & \beta(m(\vartheta_n, \vartheta_{n-1}))F(m(\vartheta_n, \vartheta_{n-1})), \end{array}$$

where

$$\begin{array}{rcl} m(\vartheta_n, \vartheta_{n-1}) & = & \max\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, H\vartheta_n), d(\vartheta_{n-1}, H\vartheta_{n-1})\} \\ & = & \max\{d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_n, \vartheta_{n-1})\} \end{array}$$

Let $m(\vartheta_n, \vartheta_{n-1}) = d(\vartheta_n, \vartheta_{n+1})$, then

$$\begin{array}{rcl} \tau + F(d(\vartheta_{n+1}, \vartheta_n)) & \leq & \beta(m(\vartheta_n, \vartheta_{n-1}))F(m(\vartheta_n, \vartheta_{n-1})) \\ \Rightarrow \quad \tau + F(d(\vartheta_{n+1}, \vartheta_n)) & \leq & F(m(\vartheta_n, \vartheta_{n-1})), \text{Since } \beta(m(\vartheta_n, \vartheta_{n+1})) \leq 1. \end{array}$$

which implies $\tau < 0$, which is a contradiction.

Hence $m(\vartheta_n, \vartheta_{n-1}) = d(\vartheta_n, \vartheta_{n-1})$; thus, we have

$$\begin{array}{rcl} F(d(\vartheta_{n+1}, \vartheta_n)) & \leq & F(d(\vartheta_n, \vartheta_{n-1})) - \tau \\ & \leq & F(d(\vartheta_{n-1}, \vartheta_{n-2})) - 2\tau \\ & \vdots & \\ & \leq & F(d(\vartheta_0, \vartheta_1)) - n\tau, \end{array}$$

$$\lim_{n \to +\infty} F(d(\vartheta_{n+1}, \vartheta_n)) = -\infty.$$

which implies

$$\lim_{n \to +\infty} d(\vartheta_{n+1}, \vartheta_n) = 0. \tag{9}$$

Now, we will demonstrate that sequence $\{\vartheta_n\}$ is a Cauchy sequence. On the other hand, suppose that $\{\vartheta_n\}$ is not a Cauchy sequence. Then, according to Proposition 1, for any $\varepsilon > 0$, there will be two subsequences $\{\vartheta_{p_n}\}, \{\vartheta_{q_n}\}$ of the sequence $\{\vartheta_n\}$ which satisfies

$$n \leq q_n < p_n, \ d(\vartheta_{p_n - 1}, \vartheta_{q_n}) < \varepsilon \leq d(\vartheta_{p_n}, \vartheta_{q_n}), \forall n \in \mathbb{N} \tag{10}$$

and

$$\lim_{n \to +\infty} d(\vartheta_{p_n}, \vartheta_{q_n}) = \lim_{n \to +\infty} d(\vartheta_{p_n - 1}, \vartheta_{q_n - 1}) = \varepsilon. \tag{11}$$

Then, we will find $N \in \mathbb{N}$ such that $d(\vartheta_{p_n}, \vartheta_{q_n}) > 0, \forall n \geq N$. Putting $\vartheta_1 = \vartheta_{p_n - 1}$ and $\vartheta_2 = \vartheta_{q_n - 1}$ in the contraction condition of the definition, we have

$$\begin{array}{rcl} \tau + F(d(H\vartheta_{p_n-1}, H\vartheta_{q_n-1})) & \leq & \alpha(\vartheta_{p_n-1}, \vartheta_{q_n-1})(\tau + F(d(H\vartheta_{p_n-1}, H\vartheta_{q_n-1})) \\ & \leq & \beta(m(\vartheta_{p_n-1}, \vartheta_{q_n-1}))F(m(\vartheta_{p_n-1}, \vartheta_{q_n-1})) \end{array}$$

$$\Rightarrow \tau + F(d(\vartheta_{p_n}, \vartheta_{q_n})) \leq \beta(m(\vartheta_{p_n-1}, \vartheta_{q_n-1}))F(m(\vartheta_{p_n-1}, \vartheta_{q_n-1})),$$

where

$$\begin{aligned} m(\vartheta_{p_n-1}, \vartheta_{q_n-1}) &= \max\{d(\vartheta_{p_n-1}, \vartheta_{q_n-1}), d(\vartheta_{p_n-1}, H\vartheta_{p_n-1}), d(\vartheta_{q_n-1}, H\vartheta_{q_n-1}\} \\ &= \max\{d(\vartheta_{p_n-1}, \vartheta_{q_n-1}), d(\vartheta_{p_n-1}, \vartheta_{p_n}), d(\vartheta_{q_n-1}, \vartheta_{q_n}\}. \end{aligned}$$

Thus, using (9) and (11), we have

$$\lim_{n\to+\infty} m(\vartheta_{p_n-1}, \vartheta_{q_n-1}) = \varepsilon.$$

Since F is a continuous, limiting as $n \to +\infty$, from the inequality,

$$\begin{aligned} \tau + F(d(\vartheta_{p_n}, \vartheta_{q_n})) &\leq \beta(m(\vartheta_{p_n-1}, \vartheta_{q_n-1}))F(m(\vartheta_{p_n-1}, \vartheta_{q_n-1})) \\ &\leq F(m(\vartheta_{p_n-1}, \vartheta_{q_n-1})), \end{aligned}$$

we have $\tau \leq 0$, i.e., we arrived at a contradiction. So, the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Being (V, d) a complete metric space, the sequence $\{\vartheta_n\}$ is convergent. Then, there must be a $\vartheta^* \in V$ at which $\{\vartheta_n\}$ converges.

To prove $H\vartheta^* = \vartheta^*$, if there is a subsequence $\{\vartheta_{p_n}\}$ of $\{\vartheta_n\}$ satisfying $\lim_{n\to+\infty} \vartheta_{p_n} = H\vartheta^*$, then $H\vartheta^* = \vartheta^*$. Suppose there is no such subsequence; then, $H\vartheta_{p_n} \neq H\vartheta^*$ for any subsequence $\{\vartheta_{p_n}\}$. In this case, $H\vartheta^* \neq \vartheta^*$, i.e., $d(H\vartheta_{p_n}, H\vartheta^*) > 0$ and $d(H\vartheta^*, \vartheta^*) > 0$. Which gives

$$\tau + F(d(H\vartheta_{p_n}, H\vartheta^*)) \leq \beta(m(\vartheta_{p_n}, \vartheta^*))F(m(\vartheta_{p_n}, \vartheta^*)) \leq F(m(\vartheta_{p_n}, \vartheta^*)),$$

where

$$\begin{aligned} m(\vartheta_{p_n}, \vartheta^*) &= \max\{d(\vartheta_{p_n}, \vartheta^*), d(\vartheta_{p_n}, H\vartheta_{p_n}), d(\vartheta^*, H\vartheta^*)\} \\ &= \max\{d(\vartheta_{p_n}, \vartheta^*), d(\vartheta_{p_n}, \vartheta_{p_n+1}), d(\vartheta^*, H\vartheta^*)\} \\ \Rightarrow \lim_{n\to+\infty} m(\vartheta_{p_n}, \vartheta^*) &= d(\vartheta^*, H\vartheta^*). \end{aligned}$$

By limiting, the last inequality becomes $\tau + F(d(\vartheta^*, H\vartheta^*)) \leq F(d(\vartheta^*, H\vartheta^*))$. Again, we arrived at a contradiction. Hence, $H\vartheta^* = \vartheta^*$.

Now, we will prove the uniqueness of the point ϑ^*. On the contrary, suppose that there are two distinct points $\vartheta^*, \vartheta^{**} \in V$ such that $\{\vartheta_n\}$ converges to both $\vartheta^*, \vartheta^{**}$ and $H\vartheta^* = \vartheta^*, H\vartheta^{**} = \vartheta^{**}$. Since $\vartheta^* \neq \vartheta^{**}$, we have $d(H\vartheta^*, H\vartheta^{**}) = d(\vartheta^*, \vartheta^{**}) > 0$ and this implies

$$\tau + F(d(H\vartheta^*, H\vartheta^{**})) \leq \beta(m(\vartheta^*, \vartheta^{**}))F(m(\vartheta^*, \vartheta^{**})) \leq F(m(\vartheta^*, \vartheta^{**})),$$

where

$$\begin{aligned} m(\vartheta^*, \vartheta^{**}) &= \max\{d(\vartheta^*, \vartheta^{**}), d(\vartheta^*, H\vartheta^*), d(\vartheta^{**}, H\vartheta^{**})\} \\ &= d(\vartheta^*, \vartheta^{**}). \end{aligned}$$

So, from the last inequality, $\tau + F(d(\vartheta^*, \vartheta^{**})) \leq F(d(\vartheta^*, \vartheta^{**}))$, which is a contradiction. So, we have $\vartheta^* = \vartheta^{**}$, which implies ϑ^* is unique.
For the last part, let $\vartheta_n \to \vartheta^* = H\vartheta^*$ and suppose H is continuous at ϑ^*. Then, $H\vartheta_n \to H\vartheta^* = \vartheta^*$, i.e., $\lim_{n\to+\infty} d(\vartheta_n, \vartheta^*) = 0 \Rightarrow \lim_{n\to+\infty} d(H\vartheta_n, H\vartheta^*) = 0$, which gives

$$\begin{aligned} \lim_{n\to+\infty} m(\vartheta_n, \vartheta^*) &= \lim_{n\to+\infty} (\max\{d(\vartheta_n, \vartheta^*), d(\vartheta_n, H\vartheta_n), d(H\vartheta^*, \vartheta^*)\}) \\ &= \max\{d(\vartheta^*, \vartheta^*), d(\vartheta^*, H\vartheta^*)\} \\ &= 0. \end{aligned}$$

Conversely, let

$$
\begin{aligned}
& \lim_{n\to+\infty} m(\vartheta_n, \vartheta^*) = 0 \\
\Rightarrow\ & \lim_{n\to+\infty} (\max\{d(\vartheta_n, \vartheta^*), d(\vartheta_n, H\vartheta_n), d(H\vartheta^*, \vartheta^*)\}) = 0 \\
\Rightarrow\ & \lim_{n\to+\infty} d(\vartheta_n, H\vartheta_n) = 0 \\
\Rightarrow\ & \lim_{n\to+\infty} H\vartheta_n = \lim_{n\to+\infty} \vartheta_n = \vartheta^* = H\vartheta^*
\end{aligned}
$$

Thus, H is continuous at ϑ^*. $\square$

Example 6. *Let $V = [0,1]$ and d_u is a usual metric of $\mathbb{R}$. Then, (V, d_u) is a complete metric space. Let $H : V \longrightarrow V$ given by*

$$H(\vartheta) = \begin{cases} \frac{3}{4} & \text{if} \quad \vartheta \in [0,1) \\ \frac{1}{3} & \text{if} \quad \vartheta = 1 \end{cases},$$

$\alpha : V \times V \longrightarrow [0, +\infty)$ given by

$$\alpha(\vartheta_1, \vartheta_2) = \begin{cases} 1 & \text{if} \quad \vartheta_1 \in [0,1) \\ 0 & \text{otherwise} \end{cases},$$

$\beta \in \mathfrak{B}$ given by

$$\beta(r) = \frac{1}{1+r}, \forall r \in [0, +\infty),$$

and $F \in \mathcal{F}'$ given by

$$F(r) = \cos r - \frac{1}{r}, \text{ for all } r \in \mathbb{R}_+.$$

Then, H becomes a (α, β, F^*)-weak Geragthy contraction for $\tau = 1$, so, by Theorem 8, it has a unique fixed point. Clearly, $\frac{3}{4} \in [0,1]$ is the only one point such that $H(\frac{3}{4}) = \frac{3}{4}$.

Theorem 9. *Suppose a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, is an (α, β, F^{**})-weak Geraghty generalized contraction and F is continuous; then, the self-mapping H must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ must converge to ϑ^* for every $\vartheta_0 \in V$. Moreover, H is continuous at ϑ^* if and only if $\lim_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.*

Proof. Moving forward, we shall have a sequence $\{\vartheta_n\}$, in a manner identical to the demonstration of Theorem 8, such that

$$\tau + F(d(\vartheta_{n+1}, \vartheta_n)) \leq \beta(m'(\vartheta_n, \vartheta_{n-1}))F(m'(\vartheta_n, \vartheta_{n-1})),$$

where

$$
\begin{aligned}
m'(\vartheta_n, \vartheta_{n-1}) &= \max\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, H\vartheta_n), d(\vartheta_{n-1}, H\vartheta_{n-1}), d(\vartheta_n, H\vartheta_{n-1}), \\
&\quad d(\vartheta_{n-1}, H\vartheta_n)\} \\
&\leq \max\{d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_{n-1}, \vartheta_n) + d(\vartheta_n, \vartheta_{n+1})\} \\
&= d(\vartheta_{n-1}, \vartheta_n) + d(\vartheta_n, \vartheta_{n+1}).
\end{aligned}
$$

So, we have

$$
\begin{aligned}
F(d(\vartheta_{n+1}, \vartheta_n)) &\leq F(d(\vartheta_n, \vartheta_{n-1}) + d(\vartheta_n, \vartheta_{n+1})) - \tau \\
&\leq F(d(\vartheta_{n-1}, \vartheta_{n-2}) + d(\vartheta_{n-1}, \vartheta_n)) - 2\tau \\
&\ \ \vdots \\
&\leq F(d(\vartheta_0, \vartheta_1) + d(\vartheta_1, \vartheta_2)) - n\tau
\end{aligned}
$$

which implies

$$\lim_{n\to+\infty} F(d(\vartheta_{n+1},\vartheta_n)) = -\infty.$$

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_{n+1},\vartheta_n) = 0. \tag{12}$$

We will now demonstrate that $\{\vartheta_n\}$ is Cauchy. On the other hand, suppose that $\{\vartheta_n\}$ is not Cauchy. Then, according to Proposition 1 for any $\varepsilon > 0$, there will be two subsequences $\{\vartheta_{p_n}\}, \{\vartheta_{q_n}\}$ of the sequence $\{\vartheta_n\}$ that satisfies

$$n \le q_n < p_n,\ d(\vartheta_{p_n-1},\vartheta_{q_n}) < \varepsilon \le d(\vartheta_{p_n},\vartheta_{q_n}), \forall n \in \mathbb{N}$$

and

$$\lim_{n\to+\infty} d(\vartheta_{p_n},\vartheta_{q_n}) = \lim_{n\to+\infty} d(\vartheta_{p_n-1},\vartheta_{q_n-1}) = \varepsilon. \tag{13}$$

Now,

$$\begin{aligned} \varepsilon \le d(\vartheta_{p_n},\vartheta_{q_n}) &\le d(\vartheta_{p_n},\vartheta_{q_n+1}) + d(\vartheta_{q_n},\vartheta_{q_n+1}) \\ \Rightarrow \varepsilon &\le d(\vartheta_{p_n},\vartheta_{q_n+1}) \le d(\vartheta_{p_n},\vartheta_{q_n}) + d(\vartheta_{q_n},\vartheta_{q_n+1}), \\ &\quad \text{by limiting } n \to +\infty \text{ and triangle inequality.} \end{aligned}$$

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_{p_n},\vartheta_{q_n+1}) = \varepsilon,\ \textit{by } (12)\ \textit{and } (13). \tag{14}$$

Similarly, we have

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_{p_n+1},\vartheta_{q_n}) = \varepsilon. \tag{15}$$

Using the inequalities

$$d(\vartheta_{p_n+1},\vartheta_{q_n+1}) \le d(\vartheta_{p_n+1},\vartheta_{q_n}) + d(\vartheta_{p_n},\vartheta_{q_n}) + d(\vartheta_{q_n},\vartheta_{q_n+1}),$$

$$\varepsilon \le d(\vartheta_{p_n},\vartheta_{q_n}) \le d(\vartheta_{p_n},\vartheta_{p_n+1}) + d(\vartheta_{p_n+1},\vartheta_{q_n+1}) + d(\vartheta_{q_n},\vartheta_{q_n+1}),$$

and (12)–(15), we have

$$\Rightarrow \lim_{n\to+\infty} d(\vartheta_{p_n+1},\vartheta_{q_n+1}) = \varepsilon. \tag{16}$$

Again

$$\begin{aligned} m'(\vartheta_{p_n-1},\vartheta_{q_n-1}) &= \max\{d(\vartheta_{p_n-1},\vartheta_{q_n-1}), d(\vartheta_{p_n-1},H\vartheta_{p_n-1}), d(\vartheta_{q_n-1},H\vartheta_{q_n-1}), \\ &\quad d(\vartheta_{p_n-1},H\vartheta_{q_n-1}), d(\vartheta_{q_n-1},H\vartheta_{p_n-1})\} \\ &= \max\{d(\vartheta_{p_n-1},\vartheta_{q_n-1}), d(\vartheta_{p_n-1},\vartheta_{p_n}), d(\vartheta_{q_n-1},\vartheta_{q_n}), \\ &\quad d(\vartheta_{p_n-1},\vartheta_{q_n}), d(\vartheta_{q_n-1},\vartheta_{p_n})\}. \end{aligned}$$

Thus using (12)–(16) we have

$$\lim_{n\to+\infty} m'(\vartheta_{p_n-1},\vartheta_{q_n-1}) = \varepsilon.$$

Being F continuous, limiting as $n \to +\infty$, from the inequality,

$$\begin{aligned} \tau + F(d(\vartheta_{p_n},\vartheta_{q_n})) &\le \beta(m'(\vartheta_{p_n-1},\vartheta_{q_n-1}))F(m'(\vartheta_{p_n-1},\vartheta_{q_n-1})) \\ &\le F(m'(\vartheta_{p_n-1},\vartheta_{q_n-1})), \end{aligned}$$

we have $\tau \le 0$; i.e., we arrived at a contradiction. So, the sequence $\{\vartheta_n\}_{n\in\mathbb{N}}$ is Cauchy. Since (V,d) is complete, the sequence $\{\vartheta_n\}$ is convergent. Then, there must be a $\vartheta^* \in V$ at which $\{\vartheta_n\}$ converges.

To prove $H\vartheta^* = \vartheta^*$, if there is a subsequence $\{\vartheta_{p_n}\}$ of $\{\vartheta_n\}$ satisfying $\lim_{n\to+\infty} \vartheta_{p_n} = H\vartheta^*$, then $H\vartheta^* = \vartheta^*$. Suppose there is no such subsequence; then, $H\vartheta_{p_n} \neq H\vartheta^*$ for any subsequence $\{\vartheta_{p_n}\}$. In this case, $H\vartheta^* \neq \vartheta^*$, i.e., $d(H\vartheta_{p_n}, H\vartheta^*) > 0$ and $d(H\vartheta^*, \vartheta^*) > 0$, which gives

$$\tau + F(d(H\vartheta_{p_n}, H\vartheta^*)) \leq \beta(m'(\vartheta_{p_n}, \vartheta^*))F(m'(\vartheta_{p_n}, \vartheta^*)) \leq F(m'(\vartheta_{p_n}, \vartheta^*)),$$

where

$$\begin{aligned} m'(\vartheta_{p_n}, \vartheta^*) &= \max\{d(\vartheta_{p_n}, \vartheta^*), d(\vartheta_{p_n}, H\vartheta_{p_n}), d(\vartheta^*, H\vartheta^*), d(\vartheta_{p_n}, H\vartheta^*), \\ &\quad d(H\vartheta_{p_n}, \vartheta^*)\} \\ &= \max\{d(\vartheta_{p_n}, \vartheta^*), d(\vartheta_{p_n}, \vartheta_{p_n+1}), d(\vartheta^*, H\vartheta^*), d(\vartheta_{p_n}, H\vartheta^*), \\ &\quad d(\vartheta_{p_n+1}, \vartheta^*)\} \\ \Rightarrow \lim_{n\to+\infty} m'(\vartheta_{p_n}, \vartheta^*) &= d(\vartheta^*, H\vartheta^*). \end{aligned}$$

By limiting, the last inequality becomes $\tau + F(d(\vartheta^*, H\vartheta^*)) \leq F(d(\vartheta^*, H\vartheta^*))$, which is a contradiction. $H\vartheta^* = \vartheta^*$.

Now, we will prove the uniqueness of the point ϑ^*. On the contrary, suppose there are two distinct points $\vartheta^*, \vartheta^{**} \in V$ such that $\{\vartheta_n\}$ converges to both $\vartheta^*, \vartheta^{**}$ and $H\vartheta^* = \vartheta^*, H\vartheta^{**} = \vartheta^{**}$. Since $\vartheta^* \neq \vartheta^{**}$, we have $d(H\vartheta^*, H\vartheta^{**}) = d(\vartheta^*, \vartheta^{**}) > 0$, and this implies

$$\tau + F(d(H\vartheta^*, H\vartheta^{**})) \leq \beta(m'(\vartheta^*, \vartheta^{**}))F(m'(\vartheta^*, \vartheta^{**})) \leq F(m'(\vartheta^*, \vartheta^{**})),$$

where

$$\begin{aligned} m'(\vartheta^*, \vartheta^{**}) &= \max\{d(\vartheta^*, \vartheta^{**}), d(\vartheta^*, H\vartheta^*), d(\vartheta^{**}, H\vartheta^{**}), d(\vartheta^*, H\vartheta^{**}), d(\vartheta^{**}, H\vartheta^*)\} \\ &= d(\vartheta^*, \vartheta^{**}). \end{aligned}$$

By limiting, the last inequality becomes $\tau + F(d(\vartheta^*, \vartheta^{**})) \leq F(d(\vartheta^*, \vartheta^{**}))$, which is a contradiction. So, we have $\vartheta^* = \vartheta^{**}$, which implies ϑ^* is unique.

For the last part, let $\vartheta_n \to \vartheta^* = H\vartheta^*$ and suppose H is continuous at ϑ^*. Then, $H\vartheta_n \to H\vartheta^* = \vartheta^*$, i.e., $\lim_{n\to+\infty} d(\vartheta_n, \vartheta^*) = 0 \Rightarrow \lim_{n\to+\infty} d(H\vartheta_n, H\vartheta^*) = 0$, which gives

$$\begin{aligned} \lim_{n\to+\infty} m'(\vartheta_n, \vartheta^*) &= \lim_{n\to+\infty}(\max\{d(\vartheta_n, \vartheta^*), d(\vartheta_n, H\vartheta_n), d(H\vartheta^*, \vartheta^*), d(\vartheta_n, H\vartheta^*), \\ &\quad d(\vartheta^*, H\vartheta_n)\}) \\ &= \max\{d(\vartheta^*, \vartheta^*), d(\vartheta^*, H\vartheta^*)\} \\ &= 0. \end{aligned}$$

Conversely, let

$$\begin{aligned} &\lim_{n\to+\infty} m'(\vartheta_n, \vartheta^*) = 0 \\ \Rightarrow\ &\lim_{n\to+\infty}(\max\{d(\vartheta_n, \vartheta^*), d(\vartheta_n, H\vartheta_n), d(H\vartheta^*, \vartheta^*), d(\vartheta_n, H\vartheta^*), d(\vartheta^*, H\vartheta_n)\}) = 0 \\ \Rightarrow\ &\lim_{n\to+\infty} d(\vartheta_n, H\vartheta_n) = 0 \\ \Rightarrow\ &\lim_{n\to+\infty} H\vartheta_n = \lim_{n\to+\infty} \vartheta_n = \vartheta^* = H\vartheta^* \end{aligned}$$

Thus, H is continuous at ϑ^*. $\square$

Example 7. *Let* $V = V_1 \cup V_2$, *where*

$$V_1 = \left\{\frac{3^n}{3k+1} : n, k \in \{0, 1, 2 \ldots\}\right\} \cup \left\{0\right\},$$

$$V_2 = \left\{\frac{3^n}{3k+2} : n, k \in \{0, 1, 2 \ldots\}\right\}$$

and d_u is a usual metric of $\mathbb{R}$. Then, (V, d_u) is a complete metric space. Let $H : V \longrightarrow V$ be given by

$$H(\vartheta) = \begin{cases} \frac{3x}{5} & if \quad x \in V_1 \\ \frac{x}{8} & if \quad x \in V_2 \end{cases},$$

$\alpha : V \times V \longrightarrow [0, +\infty)$ be given by

$$\alpha(\vartheta_1, \vartheta_2) = \begin{cases} 1 & \text{if} \quad \vartheta_1 \in V \\ 0 & \text{otherwise} \end{cases},$$

$\beta \in \mathfrak{B}$ be given by

$$\beta(r) = \frac{1}{1+r}, \forall r \in [0, +\infty),$$

and $F \in \mathcal{F}''$ be given by

$$F(r) = -\frac{1}{r^2}, \text{ for all } r \in \mathbb{R}_+.$$

Then, H becomes a (α, β, F^{**})-weak Geragthy contraction for $\tau = \frac{5}{2}$, so, by Theorem 9, it has a unique fixed point. Clearly, $0 \in V$ is the only one point such that $H(0) = 0$.

Corollary 1. *Suppose a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, is such that for some $F \in \mathcal{F}'$ and $\tau > 0$ and for any mapping $\beta \in \mathfrak{B}$, we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m_1(\vartheta_1, \vartheta_2))F(m_1(\vartheta_1, \vartheta_2)),$$

for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$ and $\alpha(\vartheta_1, \vartheta_2) \geq 1$.
Where,

$$m_1(\vartheta_1, \vartheta_2) = \max\left\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \frac{d(\vartheta_1, H\vartheta_1)d(\vartheta_2, H\vartheta_2)}{d(\vartheta_1, \vartheta_2)}\right\}.$$

If F is continuous, then the self-mapping $H : V \longrightarrow V$ must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ must converge to ϑ^* for every $\vartheta_0 \in V$. Moreover, H is continuous at u^* if and only if $\lim_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.

Proof. Putting $\vartheta_1 = \vartheta_n, \vartheta_2 = \vartheta_{n-1}$, m_1 becomes

$$\begin{aligned} m_1(\vartheta_n, \vartheta_{n-1}) &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, H\vartheta_n), d(\vartheta_{n-1}, H\vartheta_{n-1}), \\ &\quad \frac{d(\vartheta_n, H\vartheta_n)d(\vartheta_{n-1}, H\vartheta_{n-1})}{d(\vartheta_n, \vartheta_{n-1})}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_{n-1}, \vartheta_n), \\ &\quad \frac{d(\vartheta_n, \vartheta_{n+1})d(\vartheta_{n-1}, \vartheta_n)}{d(\vartheta_n, \vartheta_{n-1})}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1})\Big\}. \end{aligned}$$

which shows that the proof is now similar to Theorem 8. □

Corollary 2. *Suppose a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, is such that for some $F \in \mathcal{F}'$ and $\tau > 0$ and for any mapping $\beta \in \mathfrak{B}$, we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m_2(\vartheta_1, \vartheta_2))F(m_2(\vartheta_1, \vartheta_2)),$$

for all $\vartheta_1, \vartheta_2 \in V$ *with* $d(H\vartheta_1, H\vartheta_2) > 0$ *and* $\alpha(\vartheta_1, \vartheta_2) \geq 1$.
Where,

$$m_2(\vartheta_1, \vartheta_2) = \max\Big\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \frac{d(\vartheta_1, H\vartheta_2)d(\vartheta_2, H\vartheta_1)}{d(\vartheta_1, \vartheta_2)}\Big\}.$$

If F is continuous, then the self-mapping $H : V \longrightarrow V$ *must have a unique fixed point* $\vartheta^* \in V$ *and sequence* $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ *must converge to* ϑ^* *for every* $\vartheta_0 \in V$. *Moreover, H is continuous at* ϑ^* *if and only if* $\lim_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.

Proof. Putting $\vartheta_1 = \vartheta_n, \vartheta_2 = \vartheta_{n-1}$, m_2 becomes

$$\begin{aligned} m_2(\vartheta_n, \vartheta_{n-1}) &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, H\vartheta_n), d(\vartheta_{n-1}, H\vartheta_{n-1}), \\ &\quad \frac{d(\vartheta_n, H\vartheta_{n-1})d(\vartheta_{n-1}, H\vartheta_n)}{d(\vartheta_n, \vartheta_{n-1})}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_{n-1}, \vartheta_n), \\ &\quad \frac{d(\vartheta_n, \vartheta_n)d(\vartheta_{n-1}, \vartheta_{n+1})}{d(\vartheta_n, \vartheta_{n-1})}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1})\Big\}. \end{aligned}$$

which shows that the proof is now similar to Theorem 8. □

Corollary 3. *Suppose a self-mapping* $H : V \longrightarrow V$, *where* (V, d) *is a complete metric space, is such that for some* $F \in \mathcal{F}'$ *and* $\tau > 0$ *and for any mapping* $\beta \in \mathfrak{B}$, *we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m_3(\vartheta_1, \vartheta_2))F(m_3(\vartheta_1, \vartheta_2)),$$

for all $\vartheta_1, \vartheta_2 \in V$ *with* $d(H\vartheta_1, H\vartheta_2) > 0$ *and* $\alpha(\vartheta_1, \vartheta_2) \geq 1$.
Where,

$$m_3(\vartheta_1, \vartheta_2) = \max\Big\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \frac{d(\vartheta_1, H\vartheta_1)d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_2)d(\vartheta_2, H\vartheta_1)}{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1)}\Big\}.$$

If F is continuous, then the self-mapping $H : V \longrightarrow V$ *must have a unique fixed point* $\vartheta^* \in V$ *and sequence* $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ *must converge to* ϑ^* *for every* $\vartheta_0 \in V$. *Moreover, H is continuous at* ϑ^* *if and only if* $\lim_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.

Proof. Putting $\vartheta_1 = \vartheta_n, \vartheta_2 = \vartheta_{n-1}$, m_3 becomes

$$\begin{aligned} m_3(\vartheta_n, \vartheta_{n-1}) &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, H\vartheta_n), d(\vartheta_{n-1}, H\vartheta_{n-1}), \\ &\quad \frac{d(\vartheta_n, H\vartheta_n)d(\vartheta_n, H\vartheta_{n-1}) + d(\vartheta_{n-1}, H\vartheta_{n-1})d(\vartheta_{n-1}, H\vartheta_n)}{d(\vartheta_n, H\vartheta_{n-1}) + d(\vartheta_{n-1}, H\vartheta_n)}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_{n-1}, \vartheta_n), \\ &\quad \frac{d(\vartheta_n, \vartheta_{n+1})d(\vartheta_n, \vartheta_n) + d(\vartheta_{n-1}, \vartheta_n)d(\vartheta_{n-1}, \vartheta_{n+1})}{d(\vartheta_n, \vartheta_n) + d(\vartheta_{n-1}, \vartheta_{n+1})}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1})\Big\}. \end{aligned}$$

which shows that the proof is now similar to Theorem 8. □

Corollary 4. *Suppose a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, is such that for some $F \in \mathcal{F}''$ and $\tau > 0$ and for any mapping $\beta \in \mathfrak{B}$, we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m_4(\vartheta_1, \vartheta_2))F(m_4(\vartheta_1, \vartheta_2)),$$

for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$ and $\alpha(\vartheta_1, \vartheta_2) \geq 1$.
Where,

$$m_4(\vartheta_1, \vartheta_2) = \max\left\{d(\vartheta_1, \vartheta_2), \frac{d(\vartheta_1, H\vartheta_1) + d(\vartheta_2, H\vartheta_2)}{2}\right\}.$$

If F is continuous, then the self-mapping $H : V \longrightarrow V$ must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ must converge to ϑ^* for every $\vartheta_0 \in V$. Moreover, H is continuous at ϑ^* if and only if $\lim_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.

Proof. Putting $\vartheta_1 = \vartheta_n, \vartheta_2 = \vartheta_{n-1}$, m_4 becomes

$$\begin{aligned} m_4(\vartheta_n, \vartheta_{n-1}) &= \max\left\{d(\vartheta_n, \vartheta_{n-1}), \frac{d(\vartheta_n, H\vartheta_n) + d(\vartheta_{n-1}, H\vartheta_{n-1})}{2}\right\} \\ &= \max\left\{d(\vartheta_n, \vartheta_{n-1}), \frac{d(\vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n-1}, \vartheta_n)}{2}\right\} \\ &\leq d(\vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n-1}, \vartheta_n). \end{aligned}$$

which shows that the proof is now similar to Theorem 9. □

Corollary 5. *Suppose a self-mapping $H : V \longrightarrow V$, where (V, d) is a complete metric space, is such that for some $F \in \mathcal{F}''$ and $\tau > 0$ and for any mapping $\beta \in \mathfrak{B}$, we have*

$$\alpha(\vartheta_1, \vartheta_2)(\tau + F(d(H\vartheta_1, H\vartheta_2)) \leq \beta(m_5(\vartheta_1, \vartheta_2))F(m_5(\vartheta_1, \vartheta_2)),$$

for all $\vartheta_1, \vartheta_2 \in V$ with $d(H\vartheta_1, H\vartheta_2) > 0$ and $\alpha(\vartheta_1, \vartheta_2) \geq 1$.
Where,

$$m_5(\vartheta_1, \vartheta_2) = \max\left\{d(\vartheta_1, \vartheta_2), d(\vartheta_1, H\vartheta_1), d(\vartheta_2, H\vartheta_2), \frac{d(\vartheta_1, H\vartheta_2) + d(\vartheta_2, H\vartheta_1)}{2}\right\}.$$

If F is continuous, then the self-mapping $H : V \longrightarrow V$ must have a unique fixed point $\vartheta^* \in V$ and sequence $\{H^n\vartheta_0\}_{n\in\mathbb{N}}$ must converge to ϑ^*, for every $\vartheta_0 \in V$. Moreover, H is continuous at ϑ^* if and only if $\lim_{n\to+\infty} m(H^n\vartheta_0, \vartheta^*) = 0$.

Proof. Putting $\vartheta_1 = \vartheta_n, \vartheta_2 = \vartheta_{n-1}$, m_5 becomes

$$\begin{aligned} m_5(\vartheta_n, \vartheta_{n-1}) &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, H\vartheta_n), d(\vartheta_{n-1}, H\vartheta_{n-1}), \\ &\quad \frac{d(\vartheta_n, H\vartheta_{n-1}) + d(\vartheta_{n-1}, H\vartheta_n)}{2}\Big\} \\ &= \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_{n-1}, \vartheta_n), \\ &\quad \frac{d(\vartheta_n, \vartheta_n) + d(\vartheta_{n-1}, \vartheta_{n+1})}{2}\Big\} \\ &\leq \max\Big\{d(\vartheta_n, \vartheta_{n-1}), d(\vartheta_n, \vartheta_{n+1}), d(\vartheta_{n-1}, \vartheta_n), \\ &\quad \frac{d(\vartheta_{n-1}, \vartheta_n) + d(\vartheta_n, \vartheta_{n+1})}{2}\Big\} \\ &= d(\vartheta_{n-1}, \vartheta_n) + d(\vartheta_n, \vartheta_{n+1}). \end{aligned}$$

which shows that the proof is now similar to Theorem 9. □

3. Application

In order to show the usefulness of our findings, we explore that there is a solution which is unique to the second-order differential equation of a two-point boundary value problem for any continuous $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$.

$$\begin{cases} \frac{d^2\vartheta}{dt^2} = -f(t, \vartheta(t)), \quad t \in [0,1] \\ \vartheta(0) = 0 = \vartheta(1). \end{cases} \tag{17}$$

The differential equation's corresponding green function is

$$S(u, \vartheta) = \begin{cases} u(1-\vartheta), & 0 \le u \le \vartheta \le 1 \\ \vartheta(1-u), & 0 \le \vartheta \le u \le 1. \end{cases} \tag{18}$$

Now, we know that to find the solution of (17) is same as to find the solution $\vartheta(t) \in C[0,1]$ of the integral equation.

$$\vartheta(t) = \int_0^1 S(t,x) f(x, \vartheta(x)) dx, \text{ for all } t \in [0,1], \tag{19}$$

i.e., to find the solution of the operator $H : C[0,1] \longrightarrow C[0,1]$ defined by $H\vartheta(t) = \vartheta(t)$, where $C[0,1]$ is the complete metric space of all continuously real-valued maps on $[0,1]$ with its standard "sup" norm.

Theorem 10. *Equation (17) will have a unique solution if for some $\tau > 0$, the map $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that*

$$|f(t, \vartheta_1(t)) - f(t, \vartheta_2(t))| \le \frac{16}{3} e^{-\tau} \left\{ \frac{|\vartheta_1(t) - \vartheta_2(t) vert}{2} \right\}^{\frac{1}{1+m(\vartheta_1(t), \vartheta_2(t))}},$$

where

$$m\Big(\vartheta_1(t), \vartheta_2(t)\Big) = \max\Big\{ d\Big(\vartheta_1(t), \vartheta_2(t)\Big), d\Big(\vartheta_1(t), H\vartheta_1(t)\Big), d\Big(\vartheta_2(t), H\vartheta_2(t)\Big) \Big\}$$

Proof. We have

$$\begin{aligned} \int_0^1 S(t,x)dx &= \int_0^t S(t,x)dx + \int_t^1 S(t,x)dx \\ &= \int_0^t x(1-t)dx + \int_t^1 t(1-x)dx \\ &= \left[\frac{(1-t)x^2}{2} \right]_{x=0}^t + \left[t\Big(x - \frac{x^2}{2}\Big) \right]_{x=t}^1 \\ &= \left[\frac{t^2 - t^3}{2} \right] + \left[\frac{t - 2t^2 + t^3}{2} \right] \\ &= \frac{t}{2} - \frac{t^2}{2}, \text{ for all } t \in [0,1]. \end{aligned}$$

and

$$\begin{aligned}|H\vartheta_1(t)-H\vartheta_2(t)| &= \Big|\int_0^1 S(t,x)(f(x,\vartheta_1(x))-f(x,\vartheta_2(x)))dx\Big| \\ &\leq \int_0^1 S(t,x)|f(x,\vartheta_1(x))-f(x,\vartheta_2(x)))|dx \\ &\leq \frac{16}{3}e^{-\tau}\Big\{\frac{|\vartheta_1(x)-\vartheta_2(x)|}{2}\Big\}^{\frac{1}{1+m(\vartheta_1(x),\vartheta_2(x))}}\int_0^1 S(t,x)dx \\ &\leq \frac{2}{3}e^{-\tau}\Big\{\frac{|\vartheta_1(x)-\vartheta_2(x)|}{2}\Big\}^{\frac{1}{1+m(\vartheta_1(x),\vartheta_2(x))}}, \\ &\quad \text{since } \max\int_0^1 S(t,x)dx=\frac{1}{8}. \\ &\leq \frac{2}{3}e^{-\tau}\Big\{\frac{m(\vartheta_1(x),\vartheta_2(x))}{2}\Big\}^{\frac{1}{1+m(\vartheta_1(x),\vartheta_2(x))}}\end{aligned}$$

which implies

$$\frac{3}{2}e^{\tau}d(H\vartheta_1(x),H\vartheta_2(x)) \leq \Big\{\frac{m(\vartheta_1(x),\vartheta_2(x))}{2}\Big\}^{\frac{1}{1+m(\vartheta_1(x),\vartheta_2(x))}}$$

Now, if $F(r)=\ln(\frac{r}{2}+\sin r)$, $\beta(r)=\frac{1}{1+r}$ and $\alpha: C[0,1]\times[0,1]\longrightarrow\mathbb{R}$ defined by

$$\alpha(\vartheta_1(x),\vartheta_2(x))=\begin{cases}1 & \text{if} \quad \vartheta_1(x)\in C[0,1] \\ 0 & \text{otherwise}\end{cases},$$

then

$$\begin{aligned}&e^{\tau}\Big(\frac{d(H\vartheta_1(x),H\vartheta_2(x))}{2}+\sin d(H\vartheta_1(x),H\vartheta_2(x))\Big) \\ &\quad\leq \frac{3}{2}e^{\tau}d(H\vartheta_1(x),H\vartheta_2(x)) \\ &\quad\leq \Big\{\frac{m(\vartheta_1(x),\vartheta_2(x))}{2}\Big\}^{\frac{1}{1+m(\vartheta_1(x),\vartheta_2(x))}} \\ &\quad\leq \Big\{\frac{m(\vartheta_1(x),\vartheta_2(x))}{2}+\sin m(\vartheta_1(x),\vartheta_2(x))\Big\}^{\frac{1}{1+m(\vartheta_1(x),\vartheta_2(x))}}\end{aligned}$$

Taking ln on both sides, we have

$$\Rightarrow \tau+F(d(H\vartheta_1(x),H\vartheta_2(x)))\leq\beta(m(\vartheta_1(x),\vartheta_2(x)))F(m(\vartheta_1(x),\vartheta_2(x))).$$

Thus, Theorem 8 proves the theorem. □

4. Conclusions

In the framework of metric space, we defined two types of generalized contractions, namely (α,β,F^*)-weak Geraghty contraction and (α,β,F^{**})-weak Geraghty contraction, which extends all the composition types of α, β, F contractions. We proved our main results with examples that the mappings satisfying such weak contraction conditions must have a unique fixed point. In addition, we stated some corollaries that can be easily concluded from the main results. In the end, to show the usefulness of our result, we presented one application in the literature. So, our results consist of original ideas that are meaningful and can be used for further extensions; also, in the future, we can use the extended results in different related fields such as differential equations, nonlinear analysis, fractional calculus models, etc.

Author Contributions: Conceptualization, K.H.A., N.S. and Y.R. Formal analysis, N.S. and Y.R. Investigation, N.S. and K.H.A. Writing—original draft preparation, N.S. and K.H.A. Writing—review and editing, N.S., Y.R. and K.H.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data used to support the findings of this study are available from the corresponding author upon request.

Acknowledgments: The authors are thankful to the editor and anonymous referees for their valuable comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math* **1922**, *3*, 133–181. [CrossRef]
2. Geraghty, M.A. On contractive mappings. *Proc. Am. Math. Soc.* **1973**, *40*, 604–608. [CrossRef]
3. Samet, B.; Vetro, C.; Vetro, P. Fixed point theorems for α-ψ-contractive type mappings. *Nonlinear Anal. Theory Methods Appl.* **2012**, *75*, 2154–2165. [CrossRef]
4. Karapinar, E.; Kumam, P.; Salimi, P. On α-ψ-Meir-Keeler contractive mappings. *Fixed Point Theory Appl.* **2013**, *2013*, 94. [CrossRef]
5. Cho, S.H.; Bae, J.S.; Karapinar, E. Fixed point theorems for α-Geraghty contraction type maps in metric spaces. *Fixed Point Theory Appl.* **2013**, *2013*, 329. [CrossRef]
6. Chandok, S. Some fixed point theorems for (α,β)-admissible Geraghty type contractive mappings and related results. *Math. Sci.* **2015**, *9*, 127–135. [CrossRef]
7. Wardowski, D. Fixed points of a new type of contractive mappings in complete metric spaces. *Fixed Point Theory Appl.* **2012**, *2013*, 94. [CrossRef]
8. Wardowski, D.; Van Dung, N. Fixed points of F-weak contractions on complete metric spaces. *Demonstr. Math.* **2014**, *47*, 146–155. [CrossRef]
9. Priyobarta, N.; Rohen, Y.; Thounaojam, S.; Radenović, S. Some remarks on α-admissibility in S-metric spaces. *J. Inequalities Appl.* **2022**, *2022*, 34. [CrossRef]
10. Devi, M.B.; N,P.; Rohen, Y. Some common best proximity points theorems for generalised rational α-φ-Geraghaty proximal contractions. *J. Math. Comput. Sci.* **2020**, *10*, 713–727.
11. Zhou, M.; Saleem, N.; Bashir, S. Solution of fractional integral equations via fixed point results. *J. Inequalities Appl.* **2022**, *2022*, 148. [CrossRef]
12. Agwu, I.K.; Ishtiaq, U.; Saleem, N.; Igbokwe, D.I.; Jarad, F. Equivalence of novel IH-implicit fixed point algorithms for a general class of contractive maps. *AIMS Math.* **2023**, *8*, 841–872. [CrossRef]
13. Wang, M.; Saleem, N.; Bashir, S.; Zhou, M. Fixed Point of Modified F-Contraction with an Application. *Axioms* **2022**, *11*, 413. [CrossRef]
14. Piri, H.; Kumam, P. Some fixed point theorems concerning F-contraction in complete metric spaces. *Fixed Point Theory Appl.* **2014**, *2014*, 210. [CrossRef]
15. Alfaqih, W.M.; Imdad, M.; Gubran, R. An observation on F-weak contractions and discontinuity at the fixed point with an application. *J. Fixed Point Theory Appl.* **2020**, *22*, 66. [CrossRef]
16. Bedre, S. Remarks on F-weak contractions and discontinuity at the fixed point. *Adv. Theory Nonlinear Anal. Appl.* **2020**, *4*, 260–265. [CrossRef]
17. Neugebauer, J.T. The role of symmetry and concavity in the existence of solutions of a difference equation with Dirichlet boundary conditions. *Int. J. Differ. Equ.* **2020**, *15*, 483–491.
18. Saleem, N.; Agwu, I.K.; Ishtiaq, U.; Radenovic, S. Strong Convergence Theorems for a Finite Family of Enriched Strictly Pseudocontractive Mappings and Φ_T-Enriched Lipschitizian Mappings Using a New Modified Mixed-Type Ishikawa Iteration Scheme with Error. *Symmetry* **2022**, *14*, 1032. [CrossRef]
19. Sain, D. Birkhoff-James orthogonality of linear operators on finite dimensional Banach spaces. *J. Math. Anal. Appl.* **2017**, *447*, 860–866. [CrossRef]

20. Edelstein, M. On fixed and periodic points under contractive mappings. *J. Lond. Math. Soc.* **1962**, *1*, 74–79. [CrossRef]
21. Berzig, M.; Karapinar, E.; Roldán-López-de-Hierro, A.F. Discussion on generalized-(α, ψ, β)-contractive mappings via generalized altering distance function and related fixed point theorems. *Abstr. Appl. Anal.* **2014**, *2014*, 259768. [CrossRef]