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# Decomposing Euler-Poincaré Flow on the Space of Hamiltonian Vector Fields 

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#### Abstract

The main result of this paper is a matched-pair decomposition of the space of symmetric contravariant tensors $\mathfrak{T Q}$. From this procedure two complementary Lie subalgebras of $\mathfrak{T} \mathcal{Q}$ under mutual interaction arise. Introducing a lift operator, the matched pair decomposition of the space of Hamiltonian vector fields is determined. According to this realization, the Euler-Poincaré flows on such spaces are decomposed into two subdynamics: one is the Euler-Poincaré formulation of isentropic fluid flows, and the other one corresponds with Euler-Poincaré equations on contravariant tensors of order $n \geqslant 2$.


Keywords: matched pair lie algebras; symmetric contravariant tensors; Hamiltonian vector fields; Euler-Poincaré equations

MSC: 17B66;37K30

Citation: Esen, O.; De Lucas, J.; Muñoz, C.S.; Zając, M. Decomposing Euler-Poincaré Flow on the Space of Hamiltonian Vector Fields. Symmetry 2023, 15, 23. https://doi.org/ 10.3390/sym15010023

Academic Editor: Jean-Pierre Magnot

Received: 30 October 2022
Revised: 5 December 2022
Accepted: 10 December 2022
Published: 22 December 2022


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## 1. Introduction

The modern geometric approach of classical mechanics changes the background of old Newtonian dynamics to a new framework where Lagrangian dynamics is defined over tangent bundles. Here, the positions and the velocities of the dynamical system are the elements of the tangent space $T \mathcal{Q}$ over the configuration manifold $\mathcal{Q}$ of the generalized coordinates [1-4]. If the system has a Lie group symmetry, then one is able to reduce the system to another with reduced dynamics (i.e., reduced number of degrees of freedom) on the quotient space [5]. This specific procedure is known as Lagrangian reduction [6]. In the case that the configuration space is a Lie group $G$ and the Lagrangian function on the tangent bundle $T G$ is invariant under the group action, then the reduction provides a Lagrangian on the Lie algebra $\mathfrak{g}$. This process is known as Euler-Poincaré reduction, whereas the reduced dynamics is given by the so-called Euler-Poincaré equations $[7,8]$. Many physical theories are formulated through this geometric procedure, e.g., the theory of the rigid body and fluid and plasma theories [9-12].

The problem addressed in this work is the decoupling of Euler-Poincaré dynamics available on the space of Hamiltonian vector fields. This can be regarded as the Lagrangian realization of Vlasov plasma particles. To observe this, we start with a set of plasma particles at rest on $\mathcal{Q} \subset \mathbb{R}^{3}$. Now, consider the cotangent bundle denoted by $T^{*} \mathcal{Q}$, which is by definition the dual of $T \mathcal{Q}$ and admits a canonical symplectic two form. The configuration group of Vlasov plasma is the infinite-dimensional group Diff can $\left(T^{*} \mathcal{Q}\right)$ of diffeomorphisms preserving the canonical form and its Lie algebra can be identified with the space of Hamiltonian vector fields $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ equipped with the negative Jacobi-Lie bracket. The minus sign is the manifestation of the symmetry due to the right action [7], which physically
corresponds to particle relabelling in plasma. In [13], a quadratic Lagrangian function is introduced on $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ by means of an operator so that a geodesic flow is obtained.

One can identify $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ with the space of (smooth) functions modulo constants. In this case, the dual space of the Lie algebra turns out to be the space of densities on $T^{*} \mathcal{Q}$ [14]. This identification enables us to recast the Vlasov motion into a Hamiltonian (Lie-Poisson) form [15]. In a recent study [16], it was shown that the Vlasov Lie-Poisson equation can be decoupled into two subdynamics under mutual interaction in a non-trivial way. The geometry introduced in this study is the theory of matched pair Lie algebras, which permits to (de)couple two Lie algebras under mutual interaction [17-19]. Such analysis is beyond the semidirect theory, since there is only one one-sided action in this theory $[6,7,20-22]$. The framework is the Hamiltonian matched pair theory given in [23], and referring to such decomposition, it is shown in [16] that the constitutive subdynamics of the Hamiltonian formulation of plasma's motion is obtained as the motion of an isentropic compressible fluid motion with kinetic moments of the plasma density function of order $n \geqslant 2$. Therefore, such algebraic/geometric decomposition of the Hamiltonian Vlasov theory is consistent with the physical intuition.

In this study, our aim is to present a similar decomposition for the geodesic Vlasov motion [13] in terms of the Euler-Poincaré formulation and obtain a matched pair decomposition of this system. The abstract framework of the Lagrangian matched pair theory is already available in the literature [24], and it provides matched Euler-Poincaré equations for classical Lie algebras.

Notation. We denote Lie algebras by $\mathfrak{g}, \mathfrak{h}$ and $\mathfrak{K}$. Throughout the work, we shall adopt the letters

$$
\tilde{\xi}, \tilde{\zeta} \in \mathfrak{g}, \quad \mu, \tilde{\mu} \in \mathfrak{g}^{*}, \quad \eta, \tilde{\eta}, \in \mathfrak{g}, \quad v, \tilde{v} \in \mathfrak{h}^{*}, \quad x, \tilde{x} \in \mathfrak{K}
$$

as elements of the given spaces. We shall use the notation $a d_{\xi} \tilde{\xi}=[\xi, \tilde{\xi}]$ for the infinitesimal left adjoint representation and we denote the infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^{*}$ by $a d_{\xi}^{*}$. The latter is defined to be the minus of the linear algebraic dual of $a d_{\xi}$, that is $\left\langle a d_{\tilde{\zeta}}^{*} \mu, \tilde{\zeta}\right\rangle=-\left\langle\mu, a d_{\xi} \tilde{\xi}\right\rangle$ for all $\xi, \tilde{\xi}$ in $\mathfrak{g}$ and $\mu \in \mathfrak{g}^{*}$.

## 2. Coupling of Euler-Poincaré Dynamics

In this subsection we recall the basics on matched pairs of Lie groups, Lie algebras, and Lie coalgebras. There is an extensive literature on these subjects, see e.g., [17-19,25-28] for further details. We also refer the reader to [23,24,29].

### 2.1. Matched Pair Lie Algebras

A matched-pair Lie algebra $\mathfrak{g} \bowtie \mathfrak{h}$ is a Lie algebra containing two non-intersecting Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ under mutual interaction. We represent the mutual actions by SEEMS TO FOLLOW THE SAME ORDER.

$$
\begin{array}{ll}
\triangleright: \mathfrak{h} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}, & \eta \otimes \xi \mapsto \eta \triangleright \xi,  \tag{1}\\
\triangleleft: \mathfrak{h} \otimes \mathfrak{g} \longrightarrow \mathfrak{h}, & \eta \otimes \xi \mapsto \eta \triangleleft \xi .
\end{array}
$$

In this case, the Lie algebra bracket defined on $\mathfrak{g} \bowtie \mathfrak{h}$ is given by

$$
\begin{equation*}
[\tilde{\zeta} \oplus \eta, \tilde{\zeta} \oplus \tilde{\eta}]=([\tilde{\xi}, \tilde{\zeta}]+\eta \triangleright \tilde{\xi}-\tilde{\eta} \triangleright \tilde{\xi}) \oplus([\eta, \tilde{\eta}]+\eta \triangleleft \tilde{\xi}-\tilde{\eta} \triangleleft \tilde{\xi}) \tag{2}
\end{equation*}
$$

The Jacobi identity for the matched pair Lie algebra bracket manifests the following compatibility conditions

$$
\begin{gather*}
\eta \triangleright[\tilde{\xi}, \tilde{\xi}]=[\eta \triangleright \xi, \tilde{\xi}]+[\xi, \eta \triangleright \tilde{\xi}]+(\eta \triangleleft \xi) \triangleright \tilde{\xi}-(\eta \triangleleft \tilde{\xi}) \triangleright \xi, \\
{[\eta, \tilde{\eta}] \triangleleft \xi=[\eta, \tilde{\eta} \triangleleft \xi]+[\eta \triangleleft \tilde{\xi}, \tilde{\eta}]+\eta \triangleleft(\tilde{\eta} \triangleright \tilde{\xi})-\tilde{\eta} \triangleleft(\eta \triangleright \tilde{\xi}) .} \tag{3}
\end{gather*}
$$

Now, we present here the following proposition, which will be useful in upcoming sections. We refer the reader to [19] ([Prop. 8.3.2]) for further details.

Proposition 1. Let $\mathfrak{s}$ be a Lie algebra with two Lie subalgebras $\mathfrak{g}$ and $\mathfrak{h}$ such that $\mathfrak{s}$ is isomorphic to the direct sum of $\mathfrak{g}$ and $\mathfrak{h}$ as vector spaces with the vector addition property in $\mathfrak{s}$. Then $\mathfrak{s}$ is isomorphic to the matched pair $\mathfrak{g} \bowtie \mathfrak{h}$ as Lie algebras, and the mutual actions are derived from

$$
\begin{equation*}
[\eta, \xi]=(\eta \triangleright \xi) \oplus(\eta \triangleleft \xi) . \tag{4}
\end{equation*}
$$

Here, the inclusions of the subalgebras are defined to be

$$
\begin{equation*}
\mathfrak{g} \longrightarrow \mathfrak{s}: \xi \mapsto(\xi \oplus 0), \quad \mathfrak{h} \longrightarrow \mathfrak{s}: \eta \mapsto(0 \oplus \eta) \tag{5}
\end{equation*}
$$

The decomposition on the Lie algebra determines a decomposition on the dual space as $\mathfrak{g}^{*} \oplus \mathfrak{h}^{*}$. Recall the Lie algebra actions in (1) and take the left action $\triangleright$ and fix an element $\eta$ in $\mathfrak{h}$ : this results in a linear mapping, denoted by $\eta \triangleright$, on $\mathfrak{g}$ and the linear algebraic dual of this mapping which reads

$$
\begin{equation*}
\stackrel{*}{\triangleleft} \eta: \mathfrak{g}^{*} \longrightarrow \mathfrak{g}^{*}, \quad\langle\mu \stackrel{*}{\triangleleft} \eta, \xi\rangle=\langle\mu, \eta \triangleright \xi\rangle . \tag{6}
\end{equation*}
$$

This is the right representation of $\mathfrak{h}$ on $\mathfrak{g}^{*}$. On the other hand, by fixing $\xi \in \mathfrak{g}$, we define a linear mapping $\mathfrak{b}_{\mathcal{\xi}}$ from $\mathfrak{h}$ to $\mathfrak{g}$ as

$$
\begin{equation*}
\mathfrak{b}_{\xi}: \mathfrak{h} \longrightarrow \mathfrak{g}, \quad \mathfrak{b}_{\xi}(\eta)=\eta \triangleright \xi . \tag{7}
\end{equation*}
$$

The dual of this mapping is

$$
\begin{equation*}
\mathfrak{b}_{\tilde{\xi}}^{*}: \mathfrak{g}^{*} \longrightarrow \mathfrak{h}^{*}, \quad\left\langle\mathfrak{b}_{\xi}^{*} \mu, \eta\right\rangle=\left\langle\mu, \mathfrak{b}_{\tilde{\zeta}} \eta\right\rangle=\langle\mu, \eta \triangleright \xi\rangle . \tag{8}
\end{equation*}
$$

Similarly, recall the right action in $\triangleleft$ and fix $\xi$ in $\mathfrak{g}$. This reads a linear mapping denoted by $\triangleleft \xi$ on the Lie algebra $\mathfrak{h}$. The dual of this mapping is

$$
\begin{equation*}
\xi \stackrel{*}{\triangleright}: \mathfrak{h}^{*} \longrightarrow \mathfrak{h}^{*}, \quad\left\langle\xi \stackrel{*}{\triangleright}^{\star} v, \eta\right\rangle=\langle v, \eta \triangleleft \xi\rangle . \tag{9}
\end{equation*}
$$

This is the left representation of $\mathfrak{h}$ on $\mathfrak{g}^{*}$. Now, we fix an element, say $\eta$ in $\mathfrak{h}$, in the right action $\triangleleft$. This enables us to define a linear mapping $\mathfrak{a}_{\eta}$ from $\mathfrak{g}$ to $\mathfrak{h}$ that is

$$
\begin{equation*}
\mathfrak{a}_{\eta}: \mathfrak{g} \longrightarrow \mathfrak{h}, \quad \mathfrak{a}_{\eta}(\xi)=\eta \triangleleft \xi \tag{10}
\end{equation*}
$$

along with the dual mapping

$$
\begin{equation*}
\mathfrak{a}_{\eta}^{*}: \mathfrak{h}^{*} \longrightarrow \mathfrak{g}^{*}, \quad\left\langle\mathfrak{a}_{\eta}^{*} v, \xi\right\rangle=\left\langle v, \mathfrak{a}_{\eta} \xi\right\rangle=\langle v, \eta \triangleleft \xi\rangle . \tag{11}
\end{equation*}
$$

Proposition 2. The infinitesimal coadjoint action ad* of an element $(\xi \oplus \eta)$ in $\mathfrak{g} \bowtie \mathfrak{h}$ onto an element $(\mu \oplus v)$ in the dual space $\mathfrak{g}^{*} \oplus \mathfrak{h}^{*}$ is computed to be

$$
\begin{equation*}
\operatorname{ad}_{(\xi \oplus \eta)}^{*}(\mu \oplus v)=\underbrace{\left(a d_{\xi}^{*} \mu-\mu \stackrel{*}{\triangleleft} \eta-\mathfrak{a}_{\eta}^{*} v\right)}_{\in \mathfrak{g}^{*}} \oplus \underbrace{\left(a d_{\eta}^{*} v+\xi \triangleright v+\mathfrak{b}_{\xi}^{*} \mu\right)}_{\in \mathfrak{h}^{*}} \tag{12}
\end{equation*}
$$

Here, (the italic) ad* represents the infinitesimal coadjoint actions of Lie subalgebras to their duals.

Proof. One way to prove this proposition is to use the direct definition of the coadjoint action. Then, by employing the matched Lie algebra bracket (2) in this definition, the result follows from a direct calculation [23]. Instead of presenting this calculation, we prefer to prove the proposition in a different, although a bit longer way, since we found this
alternative proof more appropriate for future aspects of the present paper. One advantage of this proof is to derive a distinction between two notations $\operatorname{ad}_{\xi} \mu$ (coadjoint action on the matched pair) and $a d_{\xi} \mu$ (coadjoint action on the Lie subalgebra level). In accordance with this, we start by rewriting the left hand side of (12) in the following form

$$
\begin{equation*}
\operatorname{ad}_{\xi+\eta}^{*}(\mu+v)=\operatorname{ad}_{\xi}^{*} \mu+\operatorname{ad}_{\xi}^{*} v+\operatorname{ad}_{\eta}^{*} \mu+\operatorname{ad}_{\eta}^{*} v, \tag{13}
\end{equation*}
$$

where, for example, $\xi$ denotes the element $\xi+0$ in $\mathfrak{g} \bowtie \mathfrak{h}$. We compute all these terms one by one in respective order to arrive at the right hand side of (12). Notice that all four terms on the right hand side are the coadjoint actions of $\mathfrak{g} \bowtie \mathfrak{h}$ on its dual space $(\mathfrak{g} \bowtie \mathfrak{h})^{*}$, so we should couple these terms with generic Lie algebra elements, say $\tilde{\xi}+\tilde{\eta}$ in $\mathfrak{g} \bowtie \mathfrak{h}$. For the first term in the right hand side of (13), we compute

$$
\begin{align*}
\left\langle\operatorname{ad}_{\tilde{\xi}}^{*} \mu, \tilde{\xi}+\tilde{\eta}\right\rangle & =\langle\mu,[\tilde{\xi}+\tilde{\eta}, \xi]\rangle=\langle\mu,[\tilde{\xi}, \xi]\rangle+\langle\mu,[\tilde{\eta}, \xi]\rangle \\
& =\left\langle a d_{\tilde{\xi}}^{*} \mu, \tilde{\xi}\right\rangle+\langle\mu, \tilde{\eta} \triangleright \tilde{\xi}\rangle+\langle\mu, \tilde{\eta} \triangleleft \tilde{\zeta}\rangle  \tag{14}\\
& =\left\langle a d_{\tilde{\xi}}^{*} \mu, \tilde{\xi}\right\rangle+0+\left\langle\mathfrak{b}_{\tilde{\xi}}^{*} \mu, \tilde{\eta}\right\rangle,
\end{align*}
$$

where we have employed (4) in the bracket $[\tilde{\eta}, \xi]$ in the second line. This computation shows that the projection of $\mathrm{ad}_{\xi}^{*} \mu$ on the dual space $\mathfrak{g}^{*}$ is $a d_{\xi}^{*} \mu$ whereas the projection of $\operatorname{ad}_{\xi}^{*} \mu$ on the dual space $\mathfrak{h}^{*}$ is $\mathfrak{b}_{\xi}^{*} \mu$. More formally, we write this as

$$
\begin{equation*}
\operatorname{ad}_{\xi}^{*} \mu=\left(a d_{\xi}^{*} \mu \oplus \mathfrak{b}_{\xi}^{*} \mu\right) \in \mathfrak{g}^{*} \oplus \mathfrak{h}^{*} . \tag{15}
\end{equation*}
$$

In other words, we conclude that $a d_{\xi}^{*} \mu$ is the restriction of $\operatorname{ad}_{\xi}^{*} \mu$ to $\mathfrak{g}^{*}$. Next, we study the second term on the right hand side of (13). Accordingly, for arbitrary $\tilde{\xi}+\tilde{\eta}$ in $\mathfrak{g} \bowtie \mathfrak{h}$ we have

$$
\begin{align*}
\left\langle\operatorname{ad}_{\xi}^{*} v, \tilde{\xi}+\tilde{\eta}\right\rangle & =\langle v,[\tilde{\xi}+\tilde{\eta}, \tilde{\xi}]\rangle=\langle v,[\tilde{\xi}, \xi]\rangle+\langle v,[\tilde{\eta}, \tilde{\xi}]\rangle \\
& =\langle v,[\tilde{\xi}, \tilde{\xi}]\rangle+\langle v, \tilde{\eta} \triangleright \tilde{\xi}\rangle+\langle v, \tilde{\eta} \triangleleft \tilde{\xi}\rangle  \tag{16}\\
& =0+0+\langle\xi \stackrel{*}{\triangleright} v, \tilde{\eta}\rangle .
\end{align*}
$$

The first and the second term in the second line are zero since all the possible pairings between $\mathfrak{h}^{*}$ and $\mathfrak{g}$ vanish. Therefore, we obtain that

$$
\begin{equation*}
\operatorname{ad}_{\zeta}^{*} v=(0 \oplus \xi \stackrel{*}{\triangleright} v) \in \mathfrak{g}^{*} \oplus \mathfrak{h}^{*} \tag{17}
\end{equation*}
$$

For the third term on the right hand side of (13) we compute

$$
\begin{align*}
\left\langle\operatorname{ad}_{\eta}^{*} \mu, \tilde{\xi}+\tilde{\eta}\right\rangle & =\langle\mu,[\tilde{\xi}+\tilde{\eta}, \eta]\rangle=\langle\mu,[\tilde{\xi}, \eta]\rangle+\langle\mu,[\tilde{\eta}, \eta]\rangle \\
& =-\langle\mu, \eta \triangleright \tilde{\xi}\rangle-\langle\mu, \eta \triangleleft \tilde{\xi}\rangle+\langle\mu,[\tilde{\eta}, \eta]\rangle  \tag{18}\\
& =-\langle\mu \stackrel{*}{\triangleleft} \eta\rangle-0+0 .
\end{align*}
$$

So that we record this as

$$
\begin{equation*}
\operatorname{ad}_{\eta}^{*} \mu=(-\mu \stackrel{*}{\triangleleft} \eta \oplus 0) \in \mathfrak{g}^{*} \oplus \mathfrak{h}^{*} \tag{19}
\end{equation*}
$$

Finally, we work on the fourth term on the right hand side of (13). We have

$$
\begin{align*}
\left\langle\operatorname{ad}_{\eta}^{*} v, \tilde{\xi}+\tilde{\eta}\right\rangle & =\langle v,[\tilde{\xi}+\tilde{\eta}, \eta]\rangle=\langle v,[\tilde{\xi}, \eta]\rangle+\langle v,[\tilde{\eta}, \eta]\rangle \\
& =-\langle v, \eta \triangleright \tilde{\xi}\rangle-\langle v, \eta \triangleleft \tilde{\xi}\rangle+\langle v,[\tilde{\eta}, \eta]\rangle  \tag{20}\\
& =-0-\left\langle\mathfrak{a}_{\eta}^{*} v, \tilde{\xi}\right\rangle+\left\langle a d_{\eta} v, \tilde{\eta}\right\rangle,
\end{align*}
$$

which means that

$$
\begin{equation*}
\operatorname{ad}_{\eta}^{*} v=\left(-\mathfrak{a}_{\eta}^{*} v \oplus a d_{\eta} v\right) \in \mathfrak{g}^{*} \oplus \mathfrak{h}^{*} \tag{21}
\end{equation*}
$$

Eventually, by adding all the results in (15), (17), (19) and (21) we arrive at the right hand side of (12).

### 2.2. Lie Algebra Homomorphisms

Lemma 1. Given two matched pair Lie algebras $\mathfrak{K}_{1}=\mathfrak{g}_{1} \bowtie \mathfrak{h}_{1}$ and $\mathfrak{K}_{2}=\mathfrak{g}_{2} \bowtie \mathfrak{h}_{2}$, a linear map $\varphi: \mathfrak{K}_{1} \mapsto \mathfrak{K}_{2}$ satisfying $\varphi\left(\mathfrak{g}_{1}\right) \subseteq \mathfrak{g}_{2}$ and $\varphi\left(\mathfrak{h}_{1}\right) \subseteq \mathfrak{h}_{2}$ is a Lie algebra homomorphism, if and only if

$$
\begin{equation*}
\varphi(\eta \triangleright \xi)=\varphi(\eta) \triangleright \varphi(\xi) \quad \varphi(\eta \triangleleft \xi)=\varphi(\eta) \triangleleft \varphi(\xi) \tag{22}
\end{equation*}
$$

for any $\xi \in \mathfrak{g}_{1}$ and any $\eta \in \mathfrak{h}_{1}$.
Assume that $\varphi$ is a homomorphism from a matched pair $\mathfrak{K}_{1}=\mathfrak{g}_{1} \bowtie \mathfrak{h}_{1}$ to another matched pair $\mathfrak{K}_{2}=\mathfrak{g}_{2} \bowtie \mathfrak{h}_{2}$. Let us also assume that this homomorphism respects the matched pair decompositions, that is, Lemma 1 is assumed. Then a straightforward calculation shows that

$$
\begin{equation*}
\varphi^{*}\left(\mathfrak{g}_{2}^{*}\right) \subset \mathfrak{g}_{1}^{*}, \quad \varphi^{*}\left(\mathfrak{h}_{2}^{*}\right) \subset \mathfrak{h}_{1}^{*}, \tag{23}
\end{equation*}
$$

where $\varphi^{*}$ is the dual operation. In the following two lemmas we exhibit commutation rules of the dual and the cross actions with the dual mapping.

Lemma 2. Assume a Lie algebra $\varphi$ as described in Lemma 1. Then the commutation rules

$$
\begin{equation*}
\varphi^{*} \circ \mathfrak{b}_{\varphi(\xi)}^{*}=\mathfrak{b}_{\xi}^{*} \circ \varphi^{*}, \quad \varphi^{*} \circ \mathfrak{a}_{\varphi(\eta)}^{*}=\mathfrak{a}_{\eta}^{*} \circ \varphi^{*} \tag{24}
\end{equation*}
$$

hold for the dual mapping $\varphi^{*}$ and the cross actions $\mathfrak{b}^{*}$ in (7) and $\mathfrak{a}^{*}$ in (11), respectively.
Proof. We start with the pairing of $\varphi(\eta \triangleright \xi)$ with an arbitrary element $\tilde{\mu}$ in $\mathfrak{g}_{2}^{*}$, i.e.,

$$
\begin{equation*}
\langle\varphi(\eta \triangleright \xi), \tilde{\mu}\rangle=\langle\varphi(\eta) \triangleright \varphi(\tilde{\xi}), \tilde{\mu}\rangle=\left\langle\varphi(\eta), \mathfrak{b}_{\varphi(\tilde{\xi})} \tilde{\mu}\right\rangle=\left\langle\eta, \varphi^{*} \circ \mathfrak{b}_{\varphi(\tilde{\xi})} \tilde{\mu}\right\rangle . \tag{25}
\end{equation*}
$$

Notice that we have employed the identity (22) in the first equality and then we have used the definition of $\mathfrak{b}^{*}$ from (8) in the second equality. On the other hand, we have that

$$
\begin{equation*}
\langle\varphi(\eta \triangleright \xi), \tilde{\mu}\rangle=\left\langle\eta \triangleright \xi, \varphi^{*}(\tilde{\mu})\right\rangle=\left\langle\eta, \mathfrak{b}_{\xi} \circ \varphi^{*}(\tilde{\mu})\right\rangle . \tag{26}
\end{equation*}
$$

Comparing the calculations in (25) and (26), we arrive at the first identity in (24) for an arbitrary $\tilde{\mu}$. For the second identity, start with pairing $\varphi(\eta \triangleleft \xi)$ with an arbitrary element $\tilde{v}$ in $\mathfrak{h}_{2}^{*}$. We have that

$$
\begin{equation*}
\langle\varphi(\eta \triangleleft \xi), \tilde{v}\rangle=\langle\varphi(\eta) \triangleleft \varphi(\tilde{\xi}), \tilde{v}\rangle=\left\langle\varphi(\xi), \mathfrak{a}_{\varphi(\eta)} \tilde{v}\right\rangle=\left\langle\tilde{\xi}, \varphi^{*} \circ \mathfrak{a}_{\varphi(\eta)} \tilde{v}\right\rangle, \tag{27}
\end{equation*}
$$

where identity (22) is used in the first equality, and definition (11) is employed in the second equality. On the other hand, one has

$$
\begin{equation*}
\langle\varphi(\eta \triangleleft \xi), \tilde{v}\rangle=\left\langle\eta \triangleleft \xi, \varphi^{*}(\tilde{v})\right\rangle=\left\langle\xi, \mathfrak{a}_{\eta} \circ \varphi^{*}(\tilde{v})\right\rangle . \tag{28}
\end{equation*}
$$

A comparison between (27) and (28) for an arbitrary $\tilde{v}$ results in the second identity in (24).

Lemma 3. Assume a Lie algebra $\varphi$ as described in Lemma 1. For the dual mapping $\varphi^{*}$ and the dual actions $\stackrel{*}{\triangleleft}$ in (6) and $\stackrel{*}{\triangleright}$ in (9), the following commutation rules hold

$$
\begin{equation*}
\left(\varphi^{*} \tilde{\mu}\right) \stackrel{*}{\triangleleft} \eta=\varphi^{*}(\tilde{\mu} \stackrel{*}{\triangleleft} \varphi(\eta)), \quad \tilde{\zeta}^{*} \varphi^{*}(\tilde{v})=\varphi^{*}(\varphi(\tilde{\xi}) \stackrel{*}{\triangleright} \tilde{v}) \tag{29}
\end{equation*}
$$

for arbitrary $\tilde{\mu}$ in $\mathfrak{g}_{2}^{*}$ and $\tilde{v}$ in $\mathfrak{h}_{2}^{*}$, respectively.
Proof. We couple $\varphi(\eta \triangleright \xi)$ with an arbitrary element $\tilde{\mu}$ in $\mathfrak{g}_{2}^{*}$ that is,

$$
\begin{align*}
\langle\varphi(\eta \triangleright \xi), \tilde{\mu}\rangle & =\langle\varphi(\eta) \triangleright \varphi(\xi), \tilde{\mu}\rangle=\langle\varphi(\xi), \tilde{\mu} \stackrel{*}{\triangleleft} \varphi(\eta)\rangle  \tag{30}\\
& =\left\langle\xi, \varphi^{*}(\tilde{\mu} \stackrel{*}{\triangleleft} \varphi(\eta))\right\rangle,
\end{align*}
$$

where we have employed the identity (22) in the first equality and we have used the definition of $\stackrel{*}{\triangleleft}$ in (6) in the second equality. On the other hand, we have that

$$
\begin{equation*}
\langle\varphi(\eta \triangleright \xi), \tilde{\mu}\rangle=\left\langle\eta \triangleright \xi, \varphi^{*}(\tilde{\mu})\right\rangle=\left\langle\xi,\left(\varphi^{*}(\tilde{\mu})\right) \stackrel{*}{\triangleleft} \eta\right\rangle . \tag{31}
\end{equation*}
$$

Comparing the calculations in (30) and (31), one arrives at the first identity in (29) for an arbitrary $\tilde{\mu}$. For the second identity, we start with the pairing of $\varphi(\eta \triangleleft \tilde{\xi})$ with an arbitrary element $\tilde{v}$ in $\mathfrak{h}_{2}^{*}$. We have that

$$
\begin{align*}
\langle\varphi(\eta \triangleleft \tilde{\xi}), \tilde{v}\rangle & =\langle\varphi(\eta) \triangleleft \varphi(\xi), \tilde{v}\rangle=\langle\varphi(\eta), \varphi(\xi) \stackrel{*}{\triangleright} \tilde{v}\rangle  \tag{32}\\
& =\left\langle\eta, \varphi^{*}(\varphi(\xi) \stackrel{*}{\triangleright} \tilde{v})\right\rangle,
\end{align*}
$$

where identity (22) is used in the first equality, and the definition of $\stackrel{*}{\triangleright}$ in (9) is employed in the second equality. Further,

$$
\begin{equation*}
\langle\varphi(\eta \triangleleft \xi), \tilde{v}\rangle=\left\langle\eta \triangleleft \xi, \varphi^{*}(\tilde{v})\right\rangle=\left\langle\eta, \xi^{*} \triangleright \varphi^{*}(\tilde{v})\right\rangle . \tag{33}
\end{equation*}
$$

(32) and (33) for an arbitrary $\tilde{v}$ give rise to the second identity in (29).

Consider a Lie algebra homomorphism $\varphi$ from a Lie algebra $\mathfrak{K}_{1}$ to $\mathfrak{K}_{2}$. The commutation rule between the dual mapping $\varphi^{*}$ and the coadjoint action ad* is computed to be

$$
\begin{equation*}
\operatorname{ad}_{x}^{*} \circ \varphi^{*}=\varphi^{*} \circ \operatorname{ad}_{\varphi(x)}^{*} \tag{34}
\end{equation*}
$$

for all $x$ in $\mathfrak{K}_{1}$. Notice that the coadjoint action on the left hand side is the one on $\mathfrak{K}_{2}^{*}$, whereas the coadjoint action on the right hand side is the one on $\mathfrak{K}_{1}^{*}$. This reads that $\varphi^{*}$ is a Poisson mapping preserving the Lie-Poisson brackets and the coadjoint flows. Now we are ready to study matched Lie-Poisson dynamics under Lie algebra homomorphisms preserving the matched pair decompositions.

Proposition 3. Assume that $\varphi$ is a Lie algebra homomorphism from a matched pair Lie algebra $\mathfrak{K}_{1}=\mathfrak{g}_{1} \bowtie \mathfrak{h}_{1}$ to a matched pair Lie algebra $\mathfrak{K}_{2}=\mathfrak{g}_{2} \bowtie \mathfrak{h}_{2}$ respecting decompositions. Then we have the following commutation law for the coadjoint actions and the pull-back $\varphi^{*}$

$$
\begin{equation*}
\operatorname{ad}_{\xi \oplus \eta}^{*}\left(\varphi^{*} \tilde{\mu} \oplus \varphi^{*} \tilde{v}\right)=\varphi^{*} \circ \operatorname{ad}_{\varphi(\tilde{\xi}) \oplus \varphi(\eta)}^{*}(\tilde{\mu} \oplus \tilde{v}) \tag{35}
\end{equation*}
$$

for any $\mathfrak{\xi} \oplus \eta$ in $\mathfrak{g}_{1} \bowtie \mathfrak{h}_{1}$ and for any $\tilde{\mu} \oplus \tilde{v}$ in $\mathfrak{g}_{2} \bowtie \mathfrak{h}_{2}$.
Proof. One way to prove the proposition above is to apply directly the inclusions in (23) to the identity (34). However, we prefer once more a bit longer proof in order to identify how the terms in the coadjoint actions behave under the Poisson mapping. For this end, we start
with the left hand side of (35) for the explicit expression of the coadjoint representation in (12). Accordingly, we compute

$$
\begin{align*}
& \operatorname{ad}_{\tilde{\zeta} \oplus \eta}^{*}\left(\varphi^{*} \tilde{\mu} \oplus \varphi^{*} \tilde{v}\right) \\
& \begin{array}{l}
=\left(a d_{\xi}^{*} \circ \varphi^{*} \tilde{\mu}-\left(\varphi^{*} \tilde{\mu}\right) \stackrel{*}{\triangleleft} \eta-\mathfrak{a}_{\eta}^{*} \circ \varphi^{*} \tilde{v}\right) \\
\quad \oplus\left(a d_{\eta}^{*} \circ \varphi^{*} \tilde{v}+\xi^{*} \triangleright^{*}\left(\varphi^{*} \tilde{v}\right)+\mathfrak{b}_{\xi}^{*} \circ \varphi^{*} \tilde{\mu}\right) \\
=\left(\varphi^{*} \circ a d_{\varphi(\xi)}^{*} \tilde{\mu}-\varphi^{*}(\tilde{\mu} \stackrel{*}{\triangleleft} \varphi(\eta))-\varphi^{*} \circ \mathfrak{a}_{\varphi(\eta)}^{*} \tilde{v}\right) \\
\quad \oplus\left(\varphi^{*} \circ a d_{\varphi(\eta)^{*}}^{\sim}+\varphi^{*}(\varphi(\tilde{\xi}) \stackrel{*}{\triangleright} \tilde{v})+\varphi^{*} \circ \mathfrak{b}_{\varphi(\tilde{\xi})}^{*} \tilde{\mu}\right) \\
=\varphi^{*}\left(a d_{\varphi(\tilde{)})}^{*} \tilde{\mu}-(\tilde{\mu} \stackrel{*}{\triangleleft} \varphi(\eta))-\mathfrak{a}_{\varphi(\eta)}^{*} \tilde{v}\right) \\
\quad \oplus\left(a d_{\varphi(\eta)}^{*} \tilde{v}+(\varphi(\tilde{\zeta}) \stackrel{*}{\triangleright} \tilde{v})+\mathfrak{b}_{\varphi(\tilde{\xi})}^{*} \tilde{\mu}\right) \\
=\varphi^{*} \circ \operatorname{ad}_{\varphi(\xi) \oplus \varphi(\eta)}^{*}(\tilde{\mu} \oplus \tilde{v}),
\end{array}
\end{align*}
$$

where we have employed the identities presented in Lemmas 2 and 3 in the second line of the calculation, and we have used the linearity of the dual mapping in the third line.

### 2.3. Euler-Poincaré Equations

Consider a Lagrangian function $\mathfrak{L}$ defined on a Lie algebra $\mathfrak{g}$. To arrive at the equations of motion governed by $\mathfrak{L}$, one takes variations of the action functional

$$
\begin{equation*}
\delta \int_{a}^{b} \mathfrak{L}(\xi) d t=\int_{a}^{b}\left\langle\frac{\delta \mathfrak{L}}{\delta \xi}, \delta \xi\right\rangle_{e} d t \tag{37}
\end{equation*}
$$

Applying the reduced variational principle $\delta \xi=\dot{\eta}+[\xi, \eta]$ to the Lie algebra element, one arrives at the Euler-Poincaré equations $[8,30]$

$$
\begin{equation*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \xi}=-\operatorname{ad}_{\xi}^{*} \frac{\delta \mathfrak{L}}{\delta \tilde{\xi}} \tag{38}
\end{equation*}
$$

Our aim in this section is to couple two different Euler-Poincaré dynamics described by (38). This can be accomplished in several ways: the naive approach is to put the two equations together, a procedure that we call direct coupling. In such case, one has two Lie algebras, say $\mathfrak{g}$ and $\mathfrak{h}$, and assumes only trivial representations of each other. Physically, this corresponds to the situation of each system having its own individual motion. Instead of two trivial actions, one can assume only a one-sided action, i.e., a non-trivial right action of $\mathfrak{g}$ on $\mathfrak{h}$. This corresponds algebraically with the Lie algebra structure

$$
\begin{equation*}
[(\xi \oplus \eta),(\tilde{\xi} \oplus \tilde{\eta})]=[\xi, \tilde{\zeta}] \oplus([\eta, \tilde{\eta}]+\eta \triangleleft \tilde{\xi}-\tilde{\eta} \triangleleft \tilde{\xi}) \tag{39}
\end{equation*}
$$

on the semi-direct product Lie algebra $\mathfrak{g} \triangleleft \mathfrak{h}$. It is immediate to see that the Lie algebra bracket (39) is a particular instance of the matched pair Lie algebra in (2), where the left action in assumed to be trivial. In this realization, referring to Proposition 2, on the dual space $\mathfrak{g}^{*} \oplus \mathfrak{h}^{*}$, the coadjoint action is computed to be

$$
\begin{equation*}
\operatorname{ad}_{(\xi \oplus \eta)}^{*}(\mu \oplus v)=\underbrace{\left(a d_{\xi}^{*} \mu-\mathfrak{a}_{\eta}^{*} v\right)}_{\in \mathfrak{g}^{*}} \oplus \underbrace{\left(a d_{\eta}^{*} v+\xi^{*} \triangleright v\right)}_{\in \mathfrak{h}^{*}} \tag{40}
\end{equation*}
$$

Then, assuming a Lagrangian function $\mathfrak{L}=\mathfrak{L}(\xi \oplus \eta)$ on the Lie algebra $\mathfrak{g} \triangleleft \mathfrak{h}$ and identifying the variations

$$
\begin{equation*}
\frac{\delta \mathfrak{L}}{\delta(\mathfrak{\xi} \oplus \eta)}=\frac{\delta \mathfrak{L}}{\delta \mathfrak{\xi}} \oplus \frac{\delta \mathfrak{L}}{\delta \eta} \in \mathfrak{g}^{*} \oplus \mathfrak{h}^{*} \tag{41}
\end{equation*}
$$

one can write the Euler-Poincaré equations on the semi-direct product Lie algebra. Referring to (38), the dynamical equations are computed to be

$$
\begin{equation*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \xi}=-\operatorname{ad}_{\xi}^{*} \frac{\delta \mathfrak{L}}{\delta \tilde{\xi}}+\underbrace{\mathfrak{a}_{\eta}^{*} \frac{\delta \mathfrak{L}}{\delta \eta}}_{\text {action of } \mathfrak{g}}, \quad \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \eta}=-\operatorname{ad}_{\eta}^{*} \frac{\delta \mathfrak{L}}{\delta \eta}-\underbrace{\xi \triangleright \frac{\delta \mathfrak{L}}{\delta \eta}}_{\text {action of } \mathfrak{g}} . \tag{42}
\end{equation*}
$$

The first term on the right hand side of the Equation (46) are the individual EulerPoincaré motions on the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. To see this, compare those terms with the Euler-Poincaré equation in (38). We have labelled the rest of the terms on the right hand side of the equations to exhibit the manifestations of the right action. We refer to a surely incomplete list [6,7,20-22] for more details on Lagrangian dynamics on semidirect products.

Instead of a nontrivial right action, one can consider a nontrivial left action of $\mathfrak{h}$ on $\mathfrak{g}$ and arrive alternatively at the semidirect product Lie algebra $\mathfrak{g} \triangleright \mathfrak{h}$ with a Lie bracket

$$
\begin{equation*}
[(\xi \oplus \eta),(\tilde{\xi} \oplus \tilde{\eta})]=([\tilde{\xi}, \tilde{\zeta}]+\eta \triangleright \tilde{\xi}-\tilde{\eta} \triangleright \tilde{\xi}) \oplus[\eta, \tilde{\eta}] \tag{43}
\end{equation*}
$$

Once more, we have derived this bracket from the matched pair Lie algebra (2) by employing a trivial right action. In this case, the coadjoint action given in Proposition 2 reduces to

$$
\begin{equation*}
\operatorname{ad}_{(\xi \oplus \eta)}^{*}(\mu \oplus v)=\underbrace{\left(a d_{\xi}^{*} \mu-\mu \stackrel{*}{\triangleleft} \eta\right)}_{\in \mathfrak{g}^{*}} \oplus \underbrace{\left(a d_{\eta}^{*} v+\mathfrak{b}_{\xi}^{*} \mu\right)}_{\in \mathfrak{h}^{*}} . \tag{44}
\end{equation*}
$$

Now, we are ready to recast the Euler-Poincaré equations on the semidirect product Lie algebras $\mathfrak{g} \triangleright \mathfrak{h}$ as

$$
\begin{equation*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \tilde{\zeta}}=-\operatorname{ad}_{\xi}^{*} \frac{\delta \mathfrak{L}}{\delta \xi}+\underbrace{\frac{\delta \mathfrak{L}}{\delta \xi} \triangleleft \eta}_{\text {action of } \mathfrak{h}}, \quad \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \eta}=-\operatorname{ad}_{\eta}^{*} \frac{\delta \mathfrak{L}}{\delta \eta}-\underbrace{\mathfrak{b}_{\zeta}^{*} \frac{\delta \mathfrak{L}}{\delta \xi}}_{\text {action of } \mathfrak{h}} \tag{45}
\end{equation*}
$$

As a last commentary on the semidirect theory, we recall the matched pair compatibility conditions in (3). It is evident that for the semidirect product Lie algebras these conditions reduce to the simple relations determining the left or right character of the actions.

It is easy to observe now that mutual actions are beyond the realm of the semidirect product theory. We think that matched pair Lie algebras are proper for studying mutually acting Euler-Poincaré flows. Let us depict this realization and discuss how matched pair Euler-Poincaré equations contain the semidirect product theories as a particular instance. For this general theory, we consider two Lie algebras, say $\mathfrak{h}$ and $\mathfrak{g}$, under mutual interaction as given in (1) assuming the conditions in (3). Then, it is immediate to observe that one can define a matched pair Lie algebra bracket (2) on the product space $\mathfrak{g} \bowtie \mathfrak{h}$. For a Lagrangian function(al) $\mathfrak{L}=\mathfrak{L}(\xi, \eta)$ depending on $\xi$ in $\mathfrak{g}$, and $\eta$ in $\mathfrak{h}$, the matched Euler-Poincaré equations are computed to be

$$
\begin{align*}
& \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \xi}=-\operatorname{ad}_{\xi}^{*} \frac{\delta \mathfrak{L}}{\delta \tilde{\xi}}+\underbrace{\frac{\delta \mathfrak{L}}{\delta \xi} \triangleleft \eta}_{\text {action of } \mathfrak{h}}+\underbrace{\mathfrak{a}_{\eta}^{*} \frac{\delta \mathfrak{L}}{\delta \eta}}_{\text {action of } \mathfrak{g}}, \\
& \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \eta}=-\operatorname{ad}_{\eta}^{*} \frac{\delta \mathfrak{L}}{\delta \eta}-\underbrace{\xi^{*} \frac{\delta \mathfrak{L}}{\delta \eta}}_{\text {action of } \mathfrak{g}}-\underbrace{\mathfrak{b}_{\xi}^{*} \frac{\delta \mathfrak{L}}{\delta \xi}}_{\text {action of } \mathfrak{h}} \tag{46}
\end{align*}
$$

where we have used the matched pair coadjoint action in Proposition 2. In (46), the second term on the right hand side of the first equation and the third term on the right hand side of the second equation are obtained by dualizing the left action of the Lie algebra $\mathfrak{h}$ on $\mathfrak{g}$. So that, if this action is trivial, that is, we have a semi-direct product Lie algebra
$\mathfrak{g} \triangleright \mathfrak{h}$, then the Euler-Poincaré dynamics is the one in (42) without these terms. On the other hand, the third term on the right hand side of the first line, and the second equation on the right hand side of the second line are manifestation of the action of $\mathfrak{g}$ on $\mathfrak{h}$. If the action is trivial in this case, that is, those terms are identically zero, then we arrive at the Euler-Poincaré equations on the semi-direct product Lie algebra $\mathfrak{g} \triangleright \mathfrak{h}$ and hence, the semidirect product Euler-Poincaré equations in (45). In this regard, the matched pair Euler-Poincaré Equation (46) involve both of these semi-direct theories so that it permits mutual interactions. The first approach to study the Euler-Poincaré equations from the point of view of the matched pair theory is presented in [23] and we refer to [31] for the matching of higher order Euler-Poincaré equations.

Let us examine now the behaviour of the Euler-Poincaré formalism under differentiable transformations. Assume a Lie algebra homomorphism $\varphi: \mathfrak{K}_{1} \rightarrow \mathfrak{K}_{2}$. Then, according to the identity (34), the pull-back operation for a Lagrangian function $\mathfrak{L}=\mathfrak{L}(\varphi(x))$ defined on $\mathfrak{K}_{2}$ reads

$$
\begin{align*}
\varphi^{*} \frac{d}{d t}\left(\frac{\delta \mathfrak{L}}{\delta \varphi(x)}\right) & =-\varphi^{*} \circ \operatorname{ad}_{\varphi(x)}^{*}\left(\frac{\delta \mathfrak{L}}{\delta \varphi(x)}\right)=-\operatorname{ad}_{x}^{*} \circ \varphi^{*}\left(\frac{\delta \mathfrak{L}}{\delta \varphi(x)}\right)  \tag{47}\\
& =\frac{d}{d t} \varphi^{*}\left(\frac{\delta \mathfrak{L}}{\delta \varphi(x)}\right)
\end{align*}
$$

We conclude that a Lie algebra homomorphism respects the Euler-Poincaré flows. Furthermore, Proposition 3 verifies that, if both of the domain and the image space admit matched pair decompositions in the realm of the Equation (35), then the coadjoint actions are properly conserved.

## 3. Symmetric Tensor Spaces

In this section, we summarize the notation and main definitions in the spaces of symmetric contravariant and covariant tensors.

### 3.1. Symmetric Contravariant Tensors

We denote the space of $k$-th order symmetric contravariant tensor fields on a manifold $\mathcal{Q}$ by $\mathfrak{T}^{\mathbf{k}} \mathcal{Q}$. The space of zeroth order tensors $\mathfrak{T}^{0} \mathcal{Q}$ is the space $\mathcal{F}(\mathcal{Q})$, whereas the first order tensors $\mathfrak{T}^{1} \mathcal{Q}$ are precisely smooth vector fields $\mathfrak{X}(\mathcal{Q})$. We take the sum of all orders to define the space of symmetric contravariant tensor fields

$$
\begin{equation*}
\mathfrak{T} \mathcal{Q}:=\sum_{\mathbf{k}=0}^{\infty} \mathfrak{T}^{\mathbf{k}} \mathcal{Q} \tag{48}
\end{equation*}
$$

A bold super script $\mathbf{k}$ in the notation $\mathfrak{T} \mathcal{Q}$ stands to denote the $k$-th order symmetric contravariant tensor fields. We reserve the bold notation to distinguish the space with indices. Accordingly, on a local chart $\left(q^{i}\right)$ over $\mathcal{Q}$, an element of $\mathfrak{T} \mathcal{Q}$ is written as

$$
\begin{equation*}
\mathbb{X}=\sum_{\mathbf{k}=0}^{\infty} \mathbb{X}^{\mathbf{k}}=\sum_{k=0}^{\infty} \mathbb{X}^{i_{1} i_{2} \ldots i_{k}}(q) \partial q^{i_{1}} \otimes \ldots \otimes \partial q^{i_{k}} \tag{49}
\end{equation*}
$$

where $\mathbb{X}^{i_{1} i_{2} \ldots i_{k}}$ are functions on $Q$.
For a $k$-th order symmetric contravariant tensor field $\mathbb{X}^{\mathbf{k}}$ and an $m$-th order field $\mathbb{Y}^{\mathbf{m}}$, where $k+m \geqslant 1$, the symmetric Schouten concomitant is defined to be [32-34]

$$
\begin{gather*}
{\left[\mathbb{X}^{\mathbf{k}}, \mathbb{Y} \mathbf{m}\right]:=\left(k \mathbb{X}^{i_{m+1} \ldots i_{m+k-1} \ell} \mathbb{Y}_{, \ell}^{i_{1} \ldots i_{m}}-m \mathbb{Y}^{i_{k+1} \ldots i_{k+m-1} \ell} \mathbb{X}_{, \ell}^{i_{1} i_{2} \ldots i_{k}}\right.}  \tag{50}\\
\partial q^{i_{1}} \otimes \ldots \otimes \partial q^{i_{k+m-1}} .
\end{gather*}
$$

See that, the result is a symmetric contravariant tensor field of order $m+k-1$. Assuming $\left[\mathbb{X}^{\mathbf{0}}, \mathbb{Y}^{\mathbf{0}}\right]=0$, we define a Lie algebra structure on $\mathfrak{T} \mathcal{Q}$ as follows. For $\mathbb{X}=\sum_{\mathbf{k} \geqslant \mathbf{0}} \mathbb{X}^{\mathbf{k}}$ and $\mathbb{Y}=\sum_{\mathbf{m} \geqslant 0} \mathbb{Y}^{\mathbf{m}}$, the symmetric Schouten concomitant is

$$
\begin{equation*}
[\mathbb{X}, \mathbb{Y}]_{S}=\sum_{\mathbf{k}, \mathbf{m}=0}^{\infty}\left[\mathbb{X}^{\mathbf{k}}, \mathbb{Y}^{\mathbf{m}}\right]_{S} \tag{51}
\end{equation*}
$$

If $\mathbb{X}^{\mathbf{k}}$ is a first order tensor field $X$ in (50), then the bracket reduces to the Lie derivative

$$
\begin{equation*}
\left[X, \mathbb{Y}^{\mathbf{m}}\right]=\mathcal{L}_{X} \mathbb{Y}^{\mathbf{m}}=\left(X^{\ell} \mathbb{Y}_{, \ell}^{i_{1} \ldots i_{m}}-m \mathbb{Y}^{i_{2} \ldots i_{m} \ell} X_{, \ell}^{i_{1}}\right) \partial q^{i_{1}} \otimes \ldots \otimes \partial q^{i_{m}} \tag{52}
\end{equation*}
$$

of the tensor field $\mathbb{Y}^{m}$ in the direction of $X$. We introduce the divergence

$$
\begin{align*}
\operatorname{div}: & \mathfrak{T}^{\mathbf{k}} \mathcal{Q} \longrightarrow \mathfrak{T}^{\mathbf{k}-\mathbf{1}} \mathcal{Q}, \\
& \mathbb{X}^{i_{1} i_{2} \ldots i_{k}} \partial q^{i_{1}} \otimes \cdots \otimes \partial q^{i_{k}} \mapsto k \mathbb{X}_{, \ell}^{\ell_{2} \ldots i_{k}} \partial q^{i_{2}} \otimes \cdots \otimes \partial q^{i_{k}}, \tag{53}
\end{align*}
$$

for $k>0$. Notice that, if $k=1$, then the operation in (53) turns out to be the classical divergence of a vector field. We define $\operatorname{div} \mathbb{X}^{0}$ as 0 .

Let us point out two Lie subalgebras of $\mathfrak{T Q}$. The first one is

$$
\begin{equation*}
\mathfrak{s}:=\sum_{\mathbf{k}=0}^{1} \mathfrak{T}^{\mathbf{k}} \mathcal{Q}=\mathcal{F}(\mathcal{Q}) \triangleright \mathfrak{X}(\mathcal{Q}) \tag{54}
\end{equation*}
$$

In this case, the symmetric Schouten concomitant (51) reduces to the semi-direct product Lie algebra structure

$$
\begin{equation*}
[(\rho, Z),(\sigma, Y)]=\left(Z(\sigma)-Y(\rho),[Z, Y]_{J L}\right) \tag{55}
\end{equation*}
$$

for $(\rho, Z)$ and $(\sigma, Y)$ in $\mathfrak{s}$. Notice that the complementary subspace of $\mathfrak{s}$ is determined by

$$
\begin{equation*}
\mathfrak{n}:=\sum_{\mathbf{k}=2}^{\infty} \mathfrak{T}^{\mathbf{k}} \mathcal{Q} \tag{56}
\end{equation*}
$$

and it is a subalgebra of $\mathfrak{T} \mathcal{Q}$ as well. Therefore, we have two complementary Lie subalgebras of $\mathfrak{T} \mathcal{Q}$. Due to the universal character of the matched pair decomposition, and in the light of Proposition $1, \mathfrak{T} \mathcal{Q}$ is a matched pair Lie algebra. We record this fact in the following statement and we refer the reader to [16] for the proof.

Proposition 4. The pair of Lie subalgebras $\mathfrak{s}$ and $\mathfrak{n}$ exhibited in (54) and (56) of the space $\mathfrak{T Q}$ of symmetric contravariant tensor fields is a matched pair of Lie algebras, and

$$
\begin{equation*}
\mathfrak{T Q}=\mathfrak{s} \bowtie \mathfrak{n}, \quad \mathbb{X}=(\sigma, Y) \bowtie \mathbf{X} \tag{57}
\end{equation*}
$$

where $(\sigma, Y)$ is an element of $\mathfrak{s}$ whereas $\mathbf{X}=\sum_{k=2}^{\infty} \mathbb{X}^{k}$ is in $\mathfrak{n}$. Mutual actions are computed to be

$$
\begin{array}{ll}
\triangleright: \mathfrak{n} \otimes \mathfrak{s} \mapsto \mathfrak{s}, & \mathbf{X} \triangleright(\sigma, Y)=\left(0,\left[\mathbb{X}^{2}, \sigma\right]\right), \\
\triangleleft: \mathfrak{n} \otimes \mathfrak{s} \mapsto \mathfrak{n}, & \mathbf{X} \triangleleft(\sigma, Y)=\sum_{k=2}^{\infty}\left(\left[\mathbb{X}^{k+1}, \sigma\right]-\mathcal{L}_{Y} \mathbb{X}^{k}\right) . \tag{58}
\end{array}
$$

### 3.2. Symmetric Covariant Tensors

We shall now consider the space $\mathfrak{T}_{\mathbf{k}}^{*} \mathcal{Q}$ of symmetric covariant (compactly supported) tensor fields of order $k$ as the dual of the space of symmetric contravariant tensor fields $\mathfrak{T}_{\mathbf{k}} \mathcal{Q}$ [35]. We denote the sum of all dual spaces by

$$
\mathfrak{T}^{*} \mathcal{Q}:=\sum_{\mathbf{k}=0}^{\infty} \mathfrak{T}_{\mathbf{k}}^{*} \mathcal{Q}
$$

In local coordinates $\left(q^{i}\right)$ on $\mathcal{Q}$, an element of $\mathfrak{T}^{*} \mathcal{Q}$ can be written as

$$
\mathbb{A}=\bigoplus_{\mathbf{m}=0}^{\infty} \mathbb{A}_{\mathbf{m}}=\bigoplus_{m=0}^{\infty} \mathbb{A}_{i_{1} \ldots i_{m}}(q) d q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}}
$$

Given the local tensor fields

$$
\mathbb{A}_{\mathbf{m}}=\mathbb{A}_{i_{1} \ldots i_{m}} d q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q}, \quad \mathbb{X}^{\mathbf{k}}=\mathbb{X}^{i_{1} \ldots i_{k}} \partial q^{i_{1}} \otimes \cdots \otimes \partial q^{i_{k}} \in \mathfrak{T}^{\mathbf{k}} \mathcal{Q}
$$

we denote the tensor contraction operation by $\lrcorner$, and

$$
\left.\mathbb{X}^{\mathbf{k}}\right\lrcorner \mathbb{A}_{\mathbf{m}}:= \begin{cases}\mathbb{X}^{i_{1} i_{2} \ldots i_{k}} \mathbb{A}_{i_{1} i_{2} \ldots i_{m}} d q^{i_{k+1}} \otimes \cdots \otimes d q^{i_{m}} \in \mathfrak{T}_{\mathbf{m}-\mathbf{k}}^{*} \mathcal{Q} & \text { if } m>k  \tag{59}\\ \mathbb{X}^{i_{1} i_{2} \ldots i_{k}} \mathbb{A}_{i_{1} i_{2} \ldots i_{m}} \partial q^{i_{m+1}} \otimes \cdots \otimes \partial q^{i_{k}} \in \mathfrak{T}^{\mathbf{k}-\mathbf{m}} \mathcal{Q} & \text { if } k>m\end{cases}
$$

As an application and for future reference, using the operation (53), we compute the following contraction

$$
\begin{equation*}
\left.\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \mathbb{A}_{\mathbf{k}+\mathbf{m}-\mathbf{1}}=k \mathbb{X}_{, \ell}^{\ell i_{m+1} \ldots i_{k+m-1}} \mathbb{A}_{i_{1} \ldots i_{k+m-1}} d q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}} \tag{60}
\end{equation*}
$$

After fixing a volume form $d \mathbf{q}$ on $Q$, the duality between $\mathfrak{T}_{\mathbf{k}}^{*} \mathcal{Q}$ and $\mathfrak{T}^{\mathbf{k}} \mathcal{Q}$ is expressed in a multiply-and-integrate form

$$
\begin{equation*}
\langle\mathbb{A}, \mathbb{X}\rangle=\sum_{k \geqslant 0} \int_{\mathcal{Q}} \mathbb{A}_{i_{1} \ldots i_{k}}(q) \mathbb{X}^{i_{1} \ldots i_{k}}(q) d \mathbf{q} . \tag{61}
\end{equation*}
$$

We assume the topological conditions which make the pairing (61) convergent and we make use of the abbreviations given in [16]

$$
\begin{align*}
& \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}} \star \mathbb{X}^{\mathbf{k}}=m \mathbb{A}_{i_{1} \ldots i_{m-1} i_{m+1} \ldots i_{m+k}} \mathbb{X}_{i_{m}}^{i_{m+1} \ldots i_{m+k}} d q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q}  \tag{62}\\
& \mathbb{X}^{\mathbf{k}} * \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}=k \mathbb{X}^{i_{m+1} \ldots i_{m+k-1} \ell} \mathbb{A}_{i_{1} \ldots i_{m+k-1}, \ell} \ell q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q}
\end{align*}
$$

for $k \geqslant 0$ and $m+k-1 \geqslant 0$. Then by simply adding these two operations we introduce

$$
\begin{equation*}
\mathrm{L}_{\mathbb{X} \mathbf{k}} \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}=\mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}} \star \mathbb{X}^{\mathbf{k}}+\mathbb{X}^{\mathbf{k}} * \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q} \tag{63}
\end{equation*}
$$

If $m=0$ then $\mathbb{A}_{\mathbf{k}-\mathbf{1}} \star \mathbb{X}^{\mathbf{k}}$ identically vanishes for all $k$, so that

$$
\begin{equation*}
\mathrm{L}_{\mathbb{X}^{\mathbf{k}}} \mathbb{A}_{\mathbf{k}-\mathbf{1}}=\mathbb{A}_{\mathbf{k}-\mathbf{1}} \star \mathbb{X}^{\mathbf{k}}+\mathbb{X}^{\mathbf{k}} * \mathbb{A}_{\mathbf{k}-\mathbf{1}}=\mathbb{X}^{\mathbf{k}} * \mathbb{A}_{\mathbf{k}-\mathbf{1}} \in \mathfrak{T}_{0}^{*} \mathcal{Q} \tag{64}
\end{equation*}
$$

In this case, $k$ must be greater than 0 . If, on the other hand, $k=0$ (that is $\mathbb{X}^{0}=\mathfrak{s}$ is a smooth function) in (63), then $\sigma * \mathbb{A}_{\mathbf{m}-\mathbf{1}}$ identically vanishes for all $m$, so that

$$
\begin{align*}
\mathrm{L}_{\sigma} \mathbb{A}_{\mathbf{m}-\mathbf{1}} & =\mathbb{A}_{\mathbf{m}-\mathbf{1}} \star \sigma+\sigma * \mathbb{A}_{\mathbf{m}-\mathbf{1}}=\mathbb{A}_{\mathbf{m}-\mathbf{1}} \star \sigma \\
& =m \mathbb{A}_{i_{1} \ldots i_{m-1}} \sigma_{, i_{m}} d q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q} \tag{65}
\end{align*}
$$

provided that $m>0$. If, further $m=1$, then

$$
\begin{equation*}
\mathrm{L}_{\mathbb{X}^{0}} \mathbb{A}_{0}=\mathrm{L}_{\sigma} \rho=\sigma * \rho+\rho \star \sigma=0+\rho \star \sigma=\rho d \sigma . \tag{66}
\end{equation*}
$$

Finally, if $k=1$ then the notation reduces to the classical definition of Lie derivative

$$
\begin{align*}
\mathrm{L}_{\mathbb{X}^{1}} \mathbb{A}_{\mathbf{m}} & =\mathcal{L}_{Y} \mathbb{A}_{\mathbf{m}} \\
& =\left(m \mathbb{A}_{i_{1} \ldots i_{m-1} \ell} Y_{i_{m}}^{\ell}+Y^{\ell} \mathbb{A}_{i_{1} \ldots i_{m}, \ell}\right) d q^{i_{1}} \otimes \cdots \otimes d q^{i_{m}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q} \tag{67}
\end{align*}
$$

The dual spaces of the Lie subalgebras $\mathfrak{s}$ in (54) and $\mathfrak{n}$ in (56) are

$$
\begin{align*}
\mathfrak{s}^{*} & :=\bigoplus_{\mathbf{m}=0}^{1} \mathbb{A}_{\mathbf{m}}=\mathbb{A}_{0} \oplus \mathbb{A}_{1}  \tag{68}\\
\mathfrak{n}^{*} & :=\bigoplus_{\mathbf{k} \geqslant 2} \mathbb{A}_{\mathbf{k}}=\mathbb{A}_{2} \oplus \mathbb{A}_{3} \oplus \ldots \tag{69}
\end{align*}
$$

respectively. Therefore, we arrive at the decomposition

$$
\begin{equation*}
\mathfrak{T}^{*} \mathcal{Q}=\mathfrak{s}^{*} \oplus \mathfrak{n}^{*}, \quad \mathbb{A}=(\rho, M) \oplus \mathbf{A} \tag{70}
\end{equation*}
$$

where $(\rho, M)$ in $\mathfrak{s}^{*}$ and $\mathbf{A}=\sum_{k=2}^{\infty} \mathbb{A}_{\mathbf{k}}$ in $\mathfrak{n}^{*}$.

## 4. Diffeomorphim Groups and Hamiltonian Vector Fields

### 4.1. Diffeomorphims and Vector Fields

Let $\mathcal{M}$ be a smooth volume manifold. The group of diffeomorphisms, denoted by $\operatorname{Diff}(\mathcal{M})$, on $\mathcal{M}$ is an infinite dimensional Lie group with multiplication

$$
\begin{equation*}
\operatorname{Diff}(\mathcal{M}) \times \operatorname{Diff}(\mathcal{M}) \longrightarrow \operatorname{Diff}(\mathcal{M}), \quad(\varphi, \psi) \mapsto \varphi \circ \psi \tag{71}
\end{equation*}
$$

and inverse $\varphi \rightarrow \varphi^{-1}$. The unit element of the group is the identity automorphism id. As a manifold, $\operatorname{Diff}(\mathcal{M})$ is locally diffeomorphic to an infinite dimensional vector space, which can be a Banach, Hilbert or Fréchet space, and consequently, the group is called respectively a Banach Lie group, Hilbert Lie group or Fréchet Lie Group [36]. We will not discuss the details of the functional analysis and refer to $[37,38]$.

The elements of the tangent space $T_{\varphi} \operatorname{Diff}(\mathcal{M})$ at $\varphi$ are the material velocity fields

$$
\begin{equation*}
V_{\varphi}: \mathcal{M} \longrightarrow T \mathcal{M} \tag{72}
\end{equation*}
$$

satisfying $\tau_{\operatorname{Diff}(\mathcal{M})} \circ V_{\varphi}=\varphi$. In particular, the tangent space at the identity $T_{\mathrm{id}} \operatorname{Diff}(\mathcal{M})$ is the space of smooth vector fields on $\mathcal{M}$, that is,

$$
\begin{equation*}
T_{\mathrm{id}} \operatorname{Diff}(\mathcal{M})=\mathfrak{X}(\mathcal{M}) \tag{73}
\end{equation*}
$$

A vector field on $\operatorname{Diff}(\mathcal{M})$ is a map $V$ on $\operatorname{Diff}(\mathcal{M})$ taking values on the tangent bundle $T \operatorname{Diff}(\mathcal{M})$. A particular value of a vector field at $\varphi$ in $\operatorname{Diff}(\mathcal{M})$ is the material velocity field $V_{\varphi}$ in $T_{\varphi} \operatorname{Diff}(\mathcal{M}) . V_{\varphi}$ can be represented as a composition of a diffeomorphism $\varphi$ and a vector field $X$, that is

$$
\begin{equation*}
V_{\varphi}=X \circ \varphi \tag{74}
\end{equation*}
$$

This is the manifestation of the parallelizable character of $\operatorname{Tiff}(\mathcal{M})$.
We assume that a continuum rests at $\mathcal{M}$ and $\operatorname{Diff}(\mathcal{M})$ operates by left action on the space $\mathcal{M}$ that is

$$
\begin{equation*}
\operatorname{Diff}(\mathcal{M}) \times \mathcal{M} \longrightarrow \mathcal{M}, \quad(\varphi, \mathbf{x}) \mapsto \varphi(\mathbf{x}) \tag{75}
\end{equation*}
$$

to reproduce the motion of particles. The right action of $\operatorname{Diff}(\mathcal{M})$ commutes with the particle motion and constitutes an infinite dimensional kinematical symmetry group. This is known as particle relabelling symmetry [39].

The inner automorphism on the $\operatorname{group} \operatorname{Diff}(\mathcal{M})$ is

$$
\begin{equation*}
I_{\psi}\left(\varphi^{t}\right)=\psi \circ \varphi^{t} \circ \psi^{-1} \tag{76}
\end{equation*}
$$

and its differentiation at $t=0$ along the direction $X$ provides an adjoint operator, that is

$$
\begin{align*}
A d_{\psi}(X) & =T_{e} I_{\psi}(X)=T_{e} I_{\psi}\left(\left.\frac{d}{d t} \varphi^{t}\right|_{t=0}\right)=\left.\frac{d}{d t} I_{\psi} \varphi^{t}\right|_{t=0} \\
& =\left.\frac{d}{d t} \psi \circ \varphi^{t} \circ \psi^{-1}\right|_{t=0}=T \psi \circ X \circ \psi^{-1}=\psi_{*} X \tag{77}
\end{align*}
$$

Thus, the adjoint action of $\operatorname{Diff}(\mathcal{M})$ on its Lie algebra $\mathfrak{X}(\mathcal{M})$ is the push-forward operation

$$
\begin{equation*}
A d_{\psi}(X)=\psi_{*} X \tag{78}
\end{equation*}
$$

The tangent space of $\operatorname{Diff}(\mathcal{M})$ at the identity id consists of vector fields on $\mathcal{M}$. The Lie algebra bracket on $T_{\mathrm{id}} \operatorname{Diff}(\mathcal{M})$ can be calculated as the differential of the adjoint representation at the identity. We differentiate $A d_{\psi^{t}}(X)$ with respect to $t$ at $t=0$ and in the direction of $Y$ to obtain

$$
\begin{equation*}
[Y, X]_{\operatorname{Diff}(\mathcal{M})}=a d_{Y} X=\left.\frac{d}{d t} \psi_{*}^{t} X\right|_{t=0}=-[Y, X]_{J L}=-\mathcal{L}_{Y} X, \tag{79}
\end{equation*}
$$

where $[,]_{J L}$ is the standard Jacobi-Lie bracket of vector fields and $\mathcal{L}_{Y} X$ is the Lie derivative of $X$ with respect to $Y$. Thus, the Lie algebra structure is minus the Jacobi-Lie bracket.

The dual space of the Lie algebra $\mathfrak{X}(\mathcal{M})$ is the space of one-forms densities on $\mathcal{M}$, that is,

$$
\begin{equation*}
\mathfrak{X}^{*}(\mathcal{M}) \simeq \Lambda^{1}(\mathcal{M}) \otimes \operatorname{Den}(\mathcal{M}) \tag{80}
\end{equation*}
$$

The pairing is

$$
\begin{equation*}
\langle\alpha \otimes \omega, X\rangle=\int_{\mathcal{M}}\langle\alpha, X\rangle \omega \tag{81}
\end{equation*}
$$

where $X \in \mathfrak{X}(\mathcal{M}), \alpha \in \Lambda^{1}(\mathcal{M})$ and $\omega$ is a volume form on $\mathcal{M}$. The pairing inside the integral is the natural pairing of finite dimensional spaces $T_{x} \mathcal{M}$ and $T_{x}^{*} \mathcal{M}$. The dual $a d^{*}$ of the adjoint action ad is defined by

$$
\begin{equation*}
\left\langle a d_{X}^{*}(\alpha \otimes \mu), Y\right\rangle=-\left\langle(\alpha \otimes \mu), a d_{X} Y\right\rangle=\int_{\mathcal{M}}\left\langle\alpha,[X, Y]_{J L}\right\rangle \omega, \tag{82}
\end{equation*}
$$

and after integrating by parts, we find the explicit expression

$$
\begin{equation*}
a d_{X}^{*}(\alpha \otimes \mu)=-\left(\mathcal{L}_{X} \alpha+\left(\operatorname{div}_{\omega} X\right) \alpha\right) \otimes \omega \tag{83}
\end{equation*}
$$

of the coadjoint action $a d^{*}$, where $\operatorname{div}_{\mathscr{\omega}} X$ is the divergence of the vector field $X$ with respect to the volume form $\omega$. In the case of divergence-free vector fields, (83) reduces to

$$
\begin{equation*}
a d_{X}^{*} \alpha=-\mathcal{L}_{X} \alpha \tag{84}
\end{equation*}
$$

### 4.2. Canonical Diffeomorphisms

The group of canonical diffeomorphisms $\operatorname{Diff}_{\text {can }}\left(T^{*} \mathcal{Q}\right)$ on the canonical symplectic manifold $T^{*} \mathcal{Q}$ consists of diffeomorphisms $\varphi$ preserving the symplectic form $\Omega_{\mathcal{Q}}$, that is, $\varphi^{*} \Omega_{\mathcal{Q}}=\Omega_{\mathcal{Q}}$. This reads the conservation $\mathfrak{L}_{X} \Omega_{\mathcal{Q}}=0$ and the Cartan's formula $\mathfrak{L}_{X}=d \iota_{X}+\iota_{X} d$ leads to $d \iota_{X} \Omega_{\mathcal{Q}}=0$.

Therefore, the Lie algebra of $\operatorname{Diff}_{\text {can }}\left(T^{*} \mathcal{Q}\right)$ can be identified with the space of (locally) Hamiltonian vector fields $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$, see [40-42]. The following equalities

$$
\begin{equation*}
\left[X_{h}, X_{f}\right]_{\mathfrak{X}}=-\left[X_{h}, X_{f}\right]_{J L}=X_{\{h, f\}} \tag{85}
\end{equation*}
$$

set a correspondence between the space $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ and the space of smooth functions $\mathcal{F}\left(T^{*} \mathcal{Q}\right)$. We write here the following Lie algebra homomorphism for future reference

$$
\begin{equation*}
\varphi: \mathcal{F}\left(T^{*} \mathcal{Q}\right) \longrightarrow \mathfrak{X}_{\mathrm{ham}}\left(T^{*} \mathcal{Q}\right), \quad h \mapsto X_{h} \tag{86}
\end{equation*}
$$

It is easy to see that the kernel of this mapping is the space of constant functions.
The nonzero elements of the dual space of the Lie algebra $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ are given by

$$
\begin{equation*}
\mathfrak{X}_{h a m}^{*}\left(T^{*} \mathcal{Q}\right)=\left\{\Pi \in \Lambda^{1}\left(T^{*} \mathcal{Q}\right): \operatorname{div}_{\Omega_{T^{*} Q}} \Pi^{\sharp} \neq 0\right\} \tag{87}
\end{equation*}
$$

To find the precise definition of the dual space $\mathfrak{X}_{\text {ham }}^{*}\left(T^{*} \mathcal{Q}\right)$, we require that the $L_{2}$ pairing $\left\langle X_{h}, \Pi\right\rangle$ is nondegenerate. We take the volume $d \mathbf{q} d \mathbf{p}$ and compute

$$
\begin{aligned}
\int_{T^{*} \mathcal{Q}}\left\langle X_{h}, \Pi\right\rangle d \mathbf{q} d \mathbf{p} & =-\int_{T^{*} \mathcal{Q}}\left\langle d h, \Pi^{\sharp}\right\rangle d \mathbf{q} d \mathbf{p}=-\int_{T^{*} \mathcal{Q}} \iota_{\Pi^{\sharp}}(d h) d \mathbf{q} d \mathbf{p} \\
& =-\int_{T^{*} \mathcal{Q}} d h \wedge \iota_{\Pi^{\sharp}}(d \mathbf{q} d \mathbf{p})=\int_{T^{*} \mathcal{Q}} h d \iota_{\Pi^{\sharp}}(d \mathbf{q} d \mathbf{p}) \\
& =\int_{T^{*} \mathcal{Q}} h \operatorname{div}_{\Omega_{T^{*} Q}} \Pi^{\sharp}(d \mathbf{q} d \mathbf{p}),
\end{aligned}
$$

where we have used the musical isomorphism $\Omega_{\mathcal{Q}}^{\sharp}$ induced from the symplectic two-form $\Omega_{\mathcal{Q}}$ in the first step, and we have applied integration by parts in the last step. Thus, in Darboux' coordinates, we arrive at the following map

$$
\begin{equation*}
\Pi \longrightarrow f(q, p)=\operatorname{div}_{\Omega_{T^{*} Q}} \Pi^{\sharp}(\mathbf{z}), \quad \Pi_{i} d q^{i}+\Pi^{i} d p_{i} \mapsto \frac{\partial \Pi^{i}}{\partial q^{i}}-\frac{\partial \Pi_{i}}{\partial p_{i}} \tag{88}
\end{equation*}
$$

which is defined to be the density function. Note that, if

$$
\begin{equation*}
\Pi=\delta_{i j} \frac{\partial \psi}{\partial p_{i}} d q^{j}-\delta^{i j} \frac{\partial \psi}{\partial q^{i}} d p_{j} \tag{89}
\end{equation*}
$$

for some function $\psi$, then the identification in (88) reduces to the following Laplace equation $f=\Delta \psi$.

It is important to remark that the action of $\operatorname{Diff}_{\mathrm{can}}\left(T^{*} \mathcal{Q}\right)$ on $T^{*} \mathcal{Q}$ is a canonical action with momentum map

$$
\begin{equation*}
\mathbf{J}: T^{*} \mathcal{Q} \longrightarrow \mathfrak{X}_{\text {ham }}^{*}\left(T^{*} \mathcal{Q}\right), \quad\left\langle\mathbf{J}(\mathbf{z}), X_{h}\right\rangle=h(\mathbf{z}) \tag{90}
\end{equation*}
$$

where $X_{h}$ is the Hamiltonian vector field for the Hamiltonian function $h$.

### 4.3. Generalized Complete Cotangent Lift

The lifting of a symmetric $k$-covariant tensor field $\mathbb{X}^{\mathbf{k}}$ on $\mathcal{Q}$ to a function on the cotangent bundle $T^{*} \mathcal{Q}$ is defined to be

$$
\begin{equation*}
\kappa: \mathfrak{T}^{\mathbf{k}} \mathcal{Q} \longrightarrow \mathcal{F}\left(T^{*} \mathcal{Q}\right), \quad \mathbb{X}^{\mathbf{k}} \mapsto \hat{\mathbb{X}}^{\mathbf{k}}:=\theta_{\mathcal{Q}}^{k}\left(\mathbb{X}^{\mathbf{k}}\right) \tag{91}
\end{equation*}
$$

where $\theta_{\mathcal{Q}}^{k}=\theta_{\mathcal{Q}} \otimes \ldots \otimes \theta_{\mathcal{Q}}$ is the $k$-th tensor power of the canonical-one form $\theta_{\mathcal{Q}}$ on $T^{*} \mathcal{Q}$. In the local picture of $\mathbb{X}^{\mathbf{k}}$ in (49), this operation reads a $p$-polynomial

$$
\hat{\mathbb{X}}^{\mathbf{k}}=\mathbb{X}^{i_{1} \ldots i_{k}}(q) p_{i_{1}} \ldots p_{i_{k}}
$$

It is evident that the mapping (91) is far from being surjective if one focuses on the smooth category. In this case, the space of flat functions with respect to the momentum variables cannot be obtained in the image space, whereas in the analytical category, the image space is equal to the functions on $T^{*} \mathcal{Q}$. Since flat functions cannot be physically interpreted [43], we determine $\mathcal{F}\left(T^{*} \mathcal{Q}\right)$ as the image space of the mapping $\kappa$. For a more general discussion involving the flat functions we refer once more to [16]. Assuming the canonical Poisson bracket on $\mathcal{F}\left(T^{*} \mathcal{Q}\right)$, we have that $\kappa$ is a Lie algebra anti-homomorphism, that is

$$
\begin{equation*}
\widehat{[\mathbb{X}, \mathbb{Y}]_{S}}=-\{\hat{\mathbb{X}}, \hat{\mathbb{Y}}\} \tag{92}
\end{equation*}
$$

where, referring to (91), we have that

$$
\begin{equation*}
\mathbb{X}=\sum_{\mathbf{k}=0}^{\infty} \mathbb{X}^{\mathbf{k}} \mapsto \hat{\mathbb{X}}:=\sum_{\mathbf{k}=0}^{\infty} \hat{\mathbb{X}}^{\mathbf{k}} \tag{93}
\end{equation*}
$$

Composing $\kappa$ with the mapping (86) and multiplying by the negative sign, we define the generalized complete cotangent lift, abbreviated as GCCL,

$$
\begin{equation*}
\text { GCCL }: \varphi \circ \kappa=\mathfrak{T} \mathcal{Q} \longrightarrow \mathfrak{X}_{\mathrm{ham}}\left(T^{*} \mathcal{Q}\right), \quad \mathbb{X}=\sum_{\mathbf{k}=0}^{\infty} \mathbb{X}^{\mathbf{k}} \mapsto-X_{\hat{\mathbb{X}}}:=-\sum_{k=0}^{\infty} X_{\hat{\mathbb{X}}^{\mathbf{k}}} \tag{94}
\end{equation*}
$$

See that (94) takes a contravariant tensor field $\mathbb{X}$ on $\mathcal{Q}$ to minus the Hamiltonian vector field $X_{\hat{\mathbb{X}}}$ generated by the Hamiltonian function $\hat{\mathbb{X}}$ in the polynomial form (91). In the Darboux' coordinates, GCCL is given by

$$
\begin{equation*}
\operatorname{GCCL}\left(\mathbb{X}^{\mathbf{k}}\right)=-X_{\widehat{\mathbb{X}}^{\mathbf{k}}}=-k p_{i_{1}} p_{i_{2}} \ldots p_{i_{k-1}} \mathbb{X}^{i_{1} \ldots i_{k-1} \ell} \partial_{q^{\ell}}+p_{i_{1}} p_{i_{2}} \ldots p_{i_{k}} \frac{\partial \mathbb{X}^{i_{1} i_{2} \ldots i_{k}}}{\partial q^{\ell}} \partial_{p_{\ell}} \tag{95}
\end{equation*}
$$

Lemma 4. The generalized complete cotangent lift (94) is a Lie algebra homomorphism, that is,

$$
\begin{equation*}
\operatorname{GCCL}[\mathbb{X}, \mathbb{Y}]_{S}=[\operatorname{GCCL}(\mathbb{X}), \operatorname{GCCL}(\mathbb{Y})]_{\mathfrak{X}}, \tag{96}
\end{equation*}
$$

where $[\bullet, \bullet]_{S}$ is the Schouten commutator (51) of tensor fields, and $[\bullet, \bullet]_{\mathfrak{X}}$ is the negative Jacobi-Lie bracket of vector fields.

Let us provide the proof of this assertion. Accordingly, we compute

$$
\begin{align*}
{[\operatorname{GCCL}(\mathbb{X}), \operatorname{GCCL}(\mathbb{Y})]_{\mathfrak{X}} } & =-[\operatorname{GCCL}(\mathbb{X}), \operatorname{GCCL}(\mathbb{Y})]_{J L}=-\left[-X_{\widehat{X}},-X_{\hat{Y}}\right]_{J L} \\
& =-\left[X_{\hat{\mathbb{X}}}, X_{\hat{\mathbb{Y}}}\right]_{J L}=X_{\{\hat{\mathbb{X}}, \hat{\mathbb{Y}}\}}=-X_{\widehat{\mathbb{X}, \mathbb{Y}}]_{S}}  \tag{97}\\
& =\operatorname{GCCL}[\mathbb{X}, \mathbb{Y}]_{S}
\end{align*}
$$

where we have employed the definition of GCCL in the first line, identity (85) in the second equality displayed in the second line, and identity (92) in the second equality displayed in the third equality.

We now exhibit the image of the constitutive Lie subalgebras $\mathfrak{s}$ and $\mathfrak{n}$ in the matched pair decomposition of $\mathfrak{T} \mathcal{Q}$ in Proposition 4 under GCCL. A direct calculation shows that the restriction of GCCL to $\mathfrak{s}$ reads

$$
\begin{equation*}
\mathfrak{s} \longrightarrow \mathfrak{s}^{c}:=\operatorname{GCCL}(\mathfrak{s}), \quad(\sigma, Y) \mapsto-X_{\hat{\sigma}}-X_{\widehat{Y}}=\sigma_{, i} \partial_{p_{i}}-X^{i} \partial_{q^{i}}+p_{j} X_{, i}^{j} \partial_{p_{i}} \tag{98}
\end{equation*}
$$

whereas the restriction of GCCL to the Lie subalgebra $\mathfrak{n}$ is

$$
\begin{equation*}
\mathfrak{n} \longrightarrow \mathfrak{n}^{c}:=\operatorname{GCCL}(\mathfrak{n}), \quad \mathbf{X}=\sum_{\mathbf{k}=2}^{\infty} \mathbb{X}^{\mathbf{k}} \mapsto-X_{\hat{\mathbf{X}}}:=-\sum_{\mathbf{k}=2}^{\infty} X_{\widehat{\mathbb{X}} \mathbf{k}} \tag{99}
\end{equation*}
$$

We refer to [16] for proof of the following statement.
Proposition 5. Let the pair of Lie algebras $\left(\mathfrak{s}^{c}, \mathfrak{n}^{c}\right)$ be the ones given by (98). Then, (99) determines a matched pair decomposition of the space of Hamiltonian vector fields generated by non-flat smooth functions on $T^{*} \mathcal{Q}$ as

$$
\begin{equation*}
\mathfrak{X}_{\mathrm{ham}}\left(T^{*} \mathcal{Q}\right) \cong \mathfrak{s}^{\mathcal{c}} \bowtie \mathfrak{n}^{c}, \tag{100}
\end{equation*}
$$

where $\mathfrak{s}^{c}$ is the image of $\mathcal{F}\left(T^{*} \mathcal{Q}\right) / \mathbb{R}$ under the mapping $\varphi$ in (86).
Let us present the result of Proposition 5 by computing the mutual actions for future reference. Notice that decomposition (100) asserts that a Hamiltonian vector field $-X_{h}$ can be written as the pair

$$
\begin{equation*}
-X_{h}=X_{h}^{\mathfrak{s}}+X_{h}^{\mathfrak{n}} \tag{101}
\end{equation*}
$$

where the constitutive vector fields are defined to be

$$
\begin{equation*}
X_{h}^{\mathfrak{s}}=\operatorname{GCCL}(\sigma, Y)=-X_{\widehat{(\sigma, Y)^{\prime}}}, \quad X_{h}^{\mathfrak{n}}=\operatorname{GCCL}(\mathbf{X})=-X_{\hat{\mathbf{x}}} \tag{102}
\end{equation*}
$$

for some $(\sigma, Y)$ in $\mathfrak{s}$ and $\mathbf{X}$ in $\mathfrak{n}$. In order to compute the mutual actions, we may refer to the universal property in Proposition 1 once more, or instead, (22) given in Lemma (1), so that we have

$$
\begin{align*}
X_{h}^{\mathfrak{n}} \triangleright X_{h}^{\mathfrak{s}} & =\left(-X_{\hat{\mathbf{x}}}\right) \triangleright\left(-X_{\widehat{(\sigma, Y)}}\right)=-X_{\mathbf{X} \triangleright(\sigma, Y)}  \tag{103}\\
& =-X_{\widehat{\left[\mathbb{X}^{2}, \sigma\right]^{\prime}}}
\end{align*}
$$

where we have used the left action in (58). In the light of the identity in (22), the right action is computed to be

$$
\begin{align*}
X_{h}^{\mathfrak{n}} \triangleleft X_{h}^{\mathfrak{s}} & =\left(-X_{\hat{\mathbf{X}}}\right) \triangleleft\left(-X_{(\sigma, Y)}\right)=-X_{\mathbf{X} \triangleleft(\sigma, Y)} \\
& =\sum_{\mathbf{k}=2}^{\infty}\left(X_{\widehat{\mathcal{L}_{Y} \mathbb{X}^{\mathbf{k}}}}-X_{\left[\widehat{\left.X^{\mathbf{k}+1}, \sigma\right]}\right.}\right) . \tag{104}
\end{align*}
$$

## 5. Euler-Poincaré Flows

In this section we introduce the decomposition of Euler-Poincaré dynamics on two spaces: the space of contravariant tensor fields and on Hamiltonian vector fields.

### 5.1. EP Dynamics on the Space Contravariant Tensor Fields

Being a Lie algebra, the space $\mathfrak{T \mathcal { Q }}$ of contravariant fields determines a coadjoint action on the space $\mathfrak{T \mathcal { Q }}$ of covariant fields $\mathfrak{T} \mathcal{Q}$ as follows

$$
\begin{equation*}
\left\langle\operatorname{ad}_{\mathbb{X}}^{*} \mathbb{A}, \mathbb{Y}\right\rangle=\left\langle\mathbb{A},[\mathbb{Y}, \mathbb{X}]_{S}\right\rangle \tag{105}
\end{equation*}
$$

for all $\mathbb{X}$ and $\mathbb{Y}$ in $\mathfrak{T} \mathcal{Q}$ and $\mathbb{A}$ in $\mathfrak{T}^{*} \mathcal{Q}$. Here, the pairing is the one in (81), whereas the bracket is the Schouten commutator (51). To have the explicit expression of the coadjoint action, we consider $\mathbb{X}^{\mathbf{k}}, \mathbb{Y}^{m}$, and $\mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}$, after fixing the volume form $d \mathbf{q}$ on $\mathcal{Q}$, we perform the following calculation

$$
\begin{align*}
& \left\langle a d_{\mathbb{X} \mathbf{k}}^{*} \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}, \mathbb{Y}^{\mathbf{m}}\right\rangle=\left\langle\mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}},\left[\mathbb{Y}^{\mathbf{m}}, \mathbb{X}^{\mathbf{k}}\right]\right\rangle \\
& =\int_{\mathcal{Q}} \mathbb{A}_{i_{1} \ldots i_{k+m-1}}\left(m \mathbb{Y}^{i_{k+1} \ldots i_{k+m-1} \ell} \mathbb{X}_{, \ell}^{i_{1} \ldots i_{k}}-k \mathbb{X}^{i_{m+1} \ldots i_{k+m-1} \ell} \mathbb{Y}_{, \ell}^{i_{1} \ldots i_{m}}\right) d \mathbf{q} \\
& =  \tag{106}\\
& m \int_{\mathcal{Q}} \mathbb{A}_{i_{1} \ldots i_{k+m-1}} \mathbb{Y}^{i_{k+1} \ldots i_{k+m-1} \ell} \mathbb{X}_{, \ell}^{i_{1} \ldots i_{k}} d \mathbf{q} \\
& \quad+k \int_{\mathcal{Q}} \mathbb{A}_{i_{1} \ldots i_{k+m-1}, \ell} \mathbb{X}^{i_{m+1} \ldots i_{k+m-1} \ell} \mathbb{Y}^{i_{1} \ldots i_{m}} d \mathbf{q} \\
& \quad+k \int_{\mathcal{Q}} \mathbb{A}_{i_{1} \ldots i_{k+m-1}} \mathbb{X}_{, \ell}^{i_{m+1} \ldots i_{k+m-1} \ell} \mathbb{Y}_{1}^{i_{1} \ldots i_{m}} d \mathbf{q} \\
& \left.=\left\langle\mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}} \star \mathbb{X}^{\mathbf{k}}, \mathbb{Y}^{\mathbf{m}}\right\rangle+\left\langle\mathbb{X}^{\mathbf{k}} * \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}, \mathbb{Y}^{\mathbf{m}}\right\rangle+\left\langle\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \mathbb{A}_{\mathbf{k}+\mathbf{m}-\mathbf{1}}, \mathbb{Y}^{\mathbf{m}}\right\rangle,
\end{align*}
$$

where, in the second line, we have employed the explicit expression of the Schouten commutator in (50) and in the last line, we have referred to the definitions of $\star$ and $*$ in (62), and the contraction of the divergence in (60). Recalling the notation in (63), we collect the first two terms in the last line of the calculation, providing the following realization of the coadjoint action

$$
\begin{equation*}
\left.\mathrm{ad}_{\mathbb{X}^{\mathbf{k}}}^{*} \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}=\mathrm{L}_{\mathbb{X}_{\mathbf{k}}} \mathbb{A}_{\mathbf{m}+\mathbf{k}-\mathbf{1}}+\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \mathbb{A}_{\mathbf{k}+\mathbf{m}-\mathbf{1}} \in \mathfrak{T}_{\mathbf{m}}^{*} \mathcal{Q} . \tag{107}
\end{equation*}
$$

This calculation excludes the case $m=0$. To have that, we use (64).
We collect all of the discussions into the following proposition [16].
Proposition 6. The coadjoint action of $\mathfrak{T} \mathcal{Q}$ on $\mathfrak{T}^{*} \mathcal{Q}$ is given by

$$
\begin{equation*}
\operatorname{ad}_{\mathbb{X}}^{*} \mathbb{A}=\bigoplus_{m=0}^{\infty} \widetilde{\mathbb{A}}_{m}, \tag{108}
\end{equation*}
$$

where

$$
\begin{align*}
& \left.\widetilde{\mathbb{A}}_{0}=\sum_{k=1}^{\infty} \mathbb{X}^{k} * \mathbb{A}_{k-1}+\operatorname{div} \mathbb{X}^{k}\right\lrcorner \mathbb{A}_{k-1}  \tag{109}\\
& \left.\widetilde{\mathbb{A}}_{m}=\sum_{k=0}^{\infty} \mathrm{L}_{\mathbb{X}^{k}} \mathbb{A}_{m+k-1}+\operatorname{div} \mathbb{X}^{k}\right\lrcorner \mathbb{A}_{k+m-1}, \quad m \geqslant 1 .
\end{align*}
$$

Now, we assume a Lagrangian function $\mathfrak{L}=\mathfrak{L}(\mathbb{X})$ on $\mathfrak{T} \mathcal{Q}$. The generic formulation in (38) determines the Euler-Poincaré equations on the spce of contravariant tensor fields as

$$
\begin{equation*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}}=-\operatorname{ad}_{\mathbb{X}}^{*} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}} \tag{110}
\end{equation*}
$$

We write an element of $\mathfrak{T \mathcal { Q }}$ as $\left(\sigma, Y, \sum_{\mathbf{k}=2} \mathbb{X}^{\mathbf{k}}\right)$. We assume that the variation of the Lagrangian is

$$
\begin{equation*}
\frac{\delta \mathfrak{L}}{\delta \mathbb{X}}=\frac{\delta \mathfrak{L}}{\delta\left(\sum_{\mathbf{m} \geqslant \mathbf{0}} \mathbb{X}^{\mathbf{m}}\right)}=\bigoplus_{\mathbf{m}=0} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}=\left(\frac{\delta \mathfrak{L}}{\delta \sigma} \oplus \frac{\delta \mathfrak{L}}{\delta Y}\right) \bigoplus_{\mathbf{m}=2} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}} \tag{111}
\end{equation*}
$$

Then, in the light of Proposition (6), we write the Euler-Poincaré equation explicitly as

$$
\begin{align*}
& \left.\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \sigma}=-\left(\mathcal{L}_{Y}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}\right)+\operatorname{div}(Y) \frac{\delta \mathfrak{L}}{\delta \sigma}\right)-\left(\mathbb{X}^{2} * \frac{\delta \mathfrak{L}}{\delta Y}+\operatorname{div} \mathbb{X}^{2}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta Y}\right) \\
& \left.-\sum_{\mathbf{k}=2}^{\infty}\left(\mathbb{X}^{\mathbf{k}+\mathbf{1}} * \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}+\mathbf{1}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}\right) \\
& \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta Y}=-\left(\frac{\delta \mathfrak{L}}{\delta \sigma} d \sigma+\mathcal{L}_{Y}\left(\frac{\delta \mathfrak{L}}{\delta Y}\right)+\operatorname{div}(Y) \frac{\delta \mathfrak{L}}{\delta Y}\right) \\
& \left.-\sum_{\mathbf{k}=2}^{\infty}\left(\mathrm{L}_{\mathbb{X}^{\mathbf{k}}} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}\right),  \tag{112}\\
& \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{2}}}=-\left(\frac{\delta \mathfrak{L}}{\delta Y} \star \sigma\right)-\left(\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{2}}+\operatorname{div} Y \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{2}}\right) \\
& \left.-\sum_{\mathbf{k}=3}^{\infty}\left(\mathrm{L}_{\mathbb{X}^{\mathbf{k}-1}} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}-\mathbf{1}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}\right) \\
& \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}=-\left(\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}+\operatorname{div} Y \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}+\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}-\mathbf{1}}} \star \sigma\right) \\
& \left.-\sum_{\mathbf{k}=2}^{\infty}\left(\mathrm{L}_{\mathbb{X}^{\mathbf{k}}} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}+\mathbf{k}-\mathbf{1}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}+\mathbf{k}-\mathbf{1}}}\right),
\end{align*}
$$

where the last equality is valid if $\mathbf{m} \geqslant 2$. Recall (62) for coordinate realizations of $\star$ and $*$ and (60) for the contractions involving the divergence operators.

### 5.2. Decomposition of EP Dynamics on the Space Contravariant Tensor Fields

We are interested in two subdynamics of the Euler-Poincare formulation on the space of contravariant tensor fields (112). One is on the Lie subalgebra $\mathfrak{s}$ in (54) and the other is on the complementary Lie subalgebra $\mathfrak{s}$ in (56). The former one corresponds to the Euler-Poincaré formulation of an isentropic compressible fluid flow. Let us examine this case. We recall the Lie algebra bracket (55) on the subalgebra $\mathfrak{s}=\mathcal{F}(\mathcal{Q}) \triangleright \mathfrak{X}(\mathcal{Q})$. A direct observation reads that, by choosing $\mathfrak{g}=\mathcal{F}(\mathcal{Q})$ and $\mathfrak{h}=\mathfrak{X}(\mathcal{Q})$, the bracket (55) precisely fits the abstract framework in (43). Here, the left action of $\mathfrak{X}(\mathcal{Q})$ on $\mathcal{F}(\mathcal{Q})$ is the directional derivative. So, it determines a left semidirect product algebra where the algebra on $\mathcal{F}(\mathcal{Q})$ is trivial. In this case, the coadjoint action is trivial on $\mathfrak{g}$. Accordingly, we refer to (45) for the Euler-Poincaré equations generated by a Lagrangian function $\mathfrak{L}=\mathfrak{L}(\sigma, Y)$ on $\mathfrak{s}$, as follows

$$
\begin{equation*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \sigma}=\frac{\delta \mathfrak{L}}{\delta \sigma} \stackrel{*}{\triangleleft} Y, \quad \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta Y}=-\operatorname{ad}_{Y}^{*} \frac{\delta \mathfrak{L}}{\delta Y}-\mathfrak{b}_{\sigma}^{*} \frac{\delta \mathfrak{L}}{\delta \sigma} . \tag{113}
\end{equation*}
$$

To have a more explicit calculation, we need to determine the right hand side of the equations in (113). A direct computation shows that

$$
\begin{align*}
\frac{\delta \mathfrak{L}}{\delta \sigma} \stackrel{*}{\triangleleft} Y & =-\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta \sigma}-\frac{\delta \mathfrak{L}}{\delta \sigma} \operatorname{div} Y \\
\operatorname{ad}_{Y}^{*} \frac{\delta \mathfrak{L}}{\delta Y} & =\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta Y}+\operatorname{div} Y \frac{\delta \mathfrak{L}}{\delta Y}  \tag{114}\\
\mathfrak{b}_{\sigma}^{*} \frac{\delta \mathfrak{L}}{\delta \sigma} & =\frac{\delta \mathfrak{L}}{\delta \sigma} d \sigma .
\end{align*}
$$

As a result, the Euler-Poincaré equations are

$$
\begin{equation*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \sigma}=-\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta \sigma}-\frac{\delta \mathfrak{L}}{\delta \sigma} \operatorname{div} Y, \quad \frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta Y}=-\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta Y}-\operatorname{div} Y \frac{\delta \mathfrak{L}}{\delta Y}-\frac{\delta \mathfrak{L}}{\delta \sigma} d \sigma . \tag{115}
\end{equation*}
$$

Now, we will examine the Euler-Poincaré flow on higher order ( $\mathbf{m} \geqslant 2$ ) covariant tensor spaces. As mentioned before, the space of symmetric covariant tensors of order $\mathbf{m} \geqslant 2$ determines a Lie subalgebra $\mathfrak{n}$ which is depicted in (56). This permits us to determine
an Euler-Poincaré flow over $\mathfrak{n}$. Assume a Lagrangian function $\mathfrak{L}=\mathfrak{L}\left(\sum_{\mathbf{k}=2} \mathbb{X}^{\mathbf{k}}\right)$ depending on higher order tensors. Then, we write

$$
\begin{equation*}
\left.\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}=-\sum_{\mathbf{k}=2}^{\infty}\left(\mathrm{L}_{\mathbb{X}^{\mathbf{k}}} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}+\mathbf{k}-\mathbf{1}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}+\mathbf{k}-\mathbf{1}}}\right) \tag{116}
\end{equation*}
$$

where we employed the notations in (63) and (60).
A direct observation suggests that merely putting together the Euler-Poincare equations (115) governing the fluid motion and the Euler-Poincaré equations (116) governing the covariant tensors of order $\mathbf{m} \geqslant 2$ will not be equal to the collective motion of the all covariant tensors fields on $\mathfrak{T} \mathcal{Q}$ exhibited in (112). The extra terms arising in this coupling are a manifestation of the mutual actions of the constitutive subalgebras $\mathfrak{s}$ and $\mathfrak{n}$ on each other. These mutual actions are computed in an abstract framework in (58). Considering the matched pair Euler-Poincaré equations (46), we first compute the dual actions [16]

$$
\begin{align*}
\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right) \stackrel{*}{\triangleleft}\left(\sum_{\mathbf{k}=2} \mathbb{X}^{\mathbf{k}}\right)= & \left.\left(-\mathbb{X}^{2} * \frac{\delta \mathfrak{L}}{\delta Y}-\operatorname{div} \mathbb{X}^{2}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta Y^{\prime}}, 0\right)  \tag{117}\\
(\sigma, Y) \stackrel{*}{\triangleright}\left(\bigoplus_{\mathbf{m}=2} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right)= & \left(\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{2}}+\operatorname{div} Y \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{2}}},\right.  \tag{118}\\
& \left.\bigoplus_{\mathbf{m}=3}^{\infty}\left(\mathcal{L}_{Y} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}+\operatorname{div} Y \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}+\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}-\mathbf{1}}} \star \sigma\right)\right)
\end{align*}
$$

where the first one takes values in $\mathfrak{s}^{*}$ and the second takes values in $\mathfrak{n}^{*}$. Further, we compute the cross actions

$$
\begin{align*}
\mathfrak{a}_{\left(\sum_{\mathbf{k}=2}^{*} \mathbb{X}^{\mathbf{k}}\right)}^{\bigoplus_{\mathbf{m}=2}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right)=} & \left(-\sum_{\mathbf{k}=2}^{\infty}\left(\mathbb{X}^{\mathbf{k}+\mathbf{1}} * \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}+\mathbf{1}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}\right)  \tag{119}\\
& \left.\left.-\sum_{\mathbf{k}=2}^{\infty}\left(\mathrm{L}_{\mathbb{X}^{\mathbf{k}}} \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}+\operatorname{div} \mathbb{X}^{\mathbf{k}}\right\lrcorner \frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{k}}}\right)\right) \\
\mathfrak{b}_{(\sigma, Y)}^{*}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right)= & \frac{\delta \mathfrak{L}}{\delta Y} \star \sigma \in \mathfrak{T}_{2}^{*} \mathcal{Q} . \tag{120}
\end{align*}
$$

Now, we collect all these terms together and decompose the Euler-Poincaré equations (110) in the form of matched Euler-Poincaré (46) as

$$
\begin{align*}
& \frac{d}{d t}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right)=-\operatorname{ad}_{(\sigma, Y)}^{*}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right)+\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right) \stackrel{*}{\triangleleft}\left(\sum_{\mathbf{k}=2} \mathbb{X}^{\mathbf{k}}\right) \\
& +\mathfrak{a}_{\left(\sum_{\mathbf{k}=2}^{*} \mathbb{X}^{\mathbf{k}}\right)}^{\bigoplus_{\mathbf{m}=2}}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right),  \tag{121}\\
& \bigoplus_{\mathbf{m}=2} \frac{d}{d t}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right)=-\operatorname{ad}_{\left(\Sigma_{\mathbf{k}=2} \mathbb{X} \mathbf{k}\right)}^{\bigoplus_{\mathbf{m}=2}}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right)-(\sigma, Y) \stackrel{*}{\triangleright} \bigoplus_{\mathbf{m}=2}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right) \\
& -\mathfrak{b}_{(\sigma, Y)}^{*}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right) .
\end{align*}
$$

Here, the first terms on the right hand side are those available in individual EP equations for isentropic fluid flow and the contravariant tensor fields on $\mathbf{m} \geqslant 2$. The second terms on the right hand side of both lines are the dual operations in (117) and (118), respectively. The third terms are cross actions in (119) and (120), respectively.

Let us comment on the Lagrangian dynamics in (121) in more physical terms. Consider the dynamics

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right)=-\operatorname{ad}_{(\sigma, Y)}^{*}\left(\frac{\delta \mathfrak{L}}{\delta \sigma}, \frac{\delta \mathfrak{L}}{\delta Y}\right) \tag{122}
\end{equation*}
$$

This is exactly an adjoint flow (in Euler-Poincaré form) and it is precisely the fluid dynamics presented in (115). We cite [7] for some further analysis on the fluid motion in this semi-direct geometry. So that we argue that one of the pair in the analysis is the fluid flow. Consider now the dynamics in the second line of (121) removing the actions. This reads

$$
\begin{equation*}
\bigoplus_{\mathbf{m}=2} \frac{d}{d t}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right)=-\mathrm{ad}_{\left(\sum_{\mathbf{k}=2}^{*} \mathbb{X}^{\mathbf{k}}\right)}^{\bigoplus_{\mathbf{m}=2}\left(\frac{\delta \mathfrak{L}}{\delta \mathbb{X}^{\mathbf{m}}}\right) . . . . .} \tag{123}
\end{equation*}
$$

This dynamic is also an adjoint flow (in the Euler-Poincaré form) and represents the Lagrangian dynamic of the kinetic momenta of order $\geqslant 2$ given in (116). The momenta are investigated in detail in $[13,35,44]$. So, we can argue that the motion determined in (121) is the collective motion of fluid dynamics (see (122)) and the higher order ( $\geqslant 2$ ) momentum dynamic (see (123)). However, as one can see, the matched pair dynamic (121) contains extra terms from individual motions. These are the second and the third terms on the right hand sides of (121). Matched pair geometry exhibits these mutual interaction terms in terms of algebraic relations between the underlying Lie (sub)algebras. As a result, we can argue that matched pair analysis of the whole momentum dynamics provides a decomposition of it into two subdynamics, both in Euler-Poincaré form. The dynamical decomposition is far from trivial due to mutual interactions. So, it is performed by referring to the underlying algebraic/geometric frameworks. Accordingly, to learn more about qualitative or quantitative details of the motion, one can concentrate on the constitutive subdynamics, which is fairly simpler, and referring to the algebraic relations, one can transfer the results of these observations to the whole system.

Further, the decomposition presented in this subsection determines a relationship between two characters of the continuum. One is its behaviour as a compressible fluid, and the other is its behaviour as projected momenta from the cotangent bundle level.

### 5.3. Decomposition of EP Dynamics on Hamiltonian Vector Fields

In (94), we have represented the GCCL as a map from the space $\mathfrak{T} \mathcal{Q}$ of contravariant tensors to the space $\mathfrak{X}_{\text {ham }}^{*}\left(T^{*} \mathcal{Q}\right)$ of Hamiltonian vector fields. In Proposition 5, we have used it to arrive at a matched pair decomposition of $\mathfrak{X}_{\text {ham }}^{*}\left(T^{*} \mathcal{Q}\right)$. This observation and Proposition 3 motivate us to determine the dual mapping of GCCL for the matched pair analysis of the Euler-Poincaré flow on $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$. Let us begin by deriving the explicit realization of the dual mapping of GCCL .

Being the dual of a Lie algebra homomorphism, the dual of GCCL is a momentum and Poisson mapping. Recall the dual space $\mathfrak{X}_{\text {ham }}^{*}\left(T^{*} \mathcal{Q}\right)$ given in (87). The dual mapping of GCCL is computed to be

$$
\begin{equation*}
\mathrm{GCCL}^{*}: \mathfrak{X}_{\mathrm{ham}}^{*}\left(T^{*} \mathcal{Q}\right) \longrightarrow \mathfrak{T}^{*} \mathcal{Q}, \quad \Pi \mapsto \bigoplus_{k=0}^{\infty} \int_{T_{p}^{*} \mathcal{Q}}\left(\theta_{\mathcal{Q}}^{k-1} \otimes \vartheta\right) d \mathbf{p} \tag{124}
\end{equation*}
$$

where $\theta_{\mathcal{Q}}^{k-1}$ is the $(k-1)$-th tensor power of the canonical one form $\theta_{\mathcal{Q}}$, and $\theta_{\mathcal{Q}}^{-1}$ is assumed to be 1 . Here, $\vartheta$ is a one-form on $T^{*} \mathcal{Q}$ given locally by

$$
\begin{equation*}
\vartheta=-\left(k \Pi_{i}+\frac{\partial \Pi^{j}}{\partial q^{j}} p_{i}\right) d q^{i} . \tag{125}
\end{equation*}
$$

According to the matched pair decomposition of the Lie algebra $\mathfrak{X}_{\text {ham }}^{*}\left(T^{*} \mathcal{Q}\right)$ as in Proposition (5), we can decompose the dual space as

$$
\begin{equation*}
\mathfrak{X}_{\mathrm{ham}}^{*}\left(T^{*} \mathcal{Q}\right)=\mathfrak{s}^{\mathfrak{c} *} \oplus \mathfrak{n}^{\mathfrak{c} *} \quad \Pi \mapsto\left(\Pi^{\mathfrak{s} *}, \Pi^{\mathfrak{n} *}\right), \tag{126}
\end{equation*}
$$

where $\mathfrak{s}^{\mathfrak{c} *}$ and $\mathfrak{n}^{\mathcal{c} *}$ are dual spaces of $\mathfrak{s}^{\mathcal{c}}$ and $\mathfrak{n}^{c}$ given in (98). The identities in (23) read that

$$
\begin{equation*}
\text { GCCL }^{*}: \mathfrak{s}^{\mathfrak{c} *} \longrightarrow \mathfrak{s}^{*}, \quad \Pi^{\mathfrak{s} *} \mapsto(\rho, M) \tag{127}
\end{equation*}
$$

where we have

$$
\begin{equation*}
\rho(q)=-\int_{T_{q}^{*} \mathcal{Q}} \frac{\partial \Pi^{i}}{\partial q^{i}} d \mathbf{p}, \quad M_{i}(q)=-\int_{T_{q}^{*} \mathcal{Q}}\left(\Pi_{i}+p_{i} \frac{\partial \Pi^{j}}{\partial q^{j}}\right) d \mathbf{p} \tag{128}
\end{equation*}
$$

where $\rho$ is a real valued function on $\mathcal{Q}$, and $M=M_{i} d q^{i}$ is a differential one-form on $\mathcal{Q}$. In the density formulation, the first two momenta (128) are known as the plasma-to-fluid map in the momentum formulation [8]. For the complementary dual space

$$
\begin{equation*}
\mathrm{GCCL}^{*}: \mathfrak{n}^{\mathfrak{c}} \longrightarrow \mathfrak{n}^{*}, \quad \Pi^{\mathfrak{n} *} \mapsto \mathbf{A}=\bigoplus_{k=2}^{\infty} \int_{T_{p}^{*} \mathcal{Q}}\left(\theta_{\mathcal{Q}}^{k-1} \otimes \vartheta\right) d \mathbf{p} \tag{129}
\end{equation*}
$$

Now, we rewrite the matched pair Euler-Poincaré equation for the present case for a Lagrangian $\mathfrak{L}=\mathfrak{L}\left(X_{h}^{\mathfrak{s}}, X_{h}^{\mathfrak{n}}\right)$ as

$$
\begin{align*}
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}} & =-\operatorname{ad}_{X_{h}^{\mathfrak{s}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}}+\frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}} \stackrel{*}{\triangleleft} X_{h}^{\mathfrak{n}}+\mathfrak{a}_{X_{h}^{\mathfrak{n}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}}  \tag{130}\\
\frac{d}{d t} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}} & =-\operatorname{ad}_{X_{h}^{\mathfrak{n}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}}-X_{h}^{\mathfrak{s}} \stackrel{*}{\triangleright} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}}-\mathfrak{b}_{X_{h}^{\mathfrak{s}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}} .
\end{align*}
$$

We wish to determine each of the components in (130). First, we recall the identity (34) for the individual coadjoint flows

$$
\begin{align*}
\mathrm{GCCL}^{*} \circ \operatorname{ad}_{X_{h}^{\mathfrak{s}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}} & =-\operatorname{ad}_{X_{(\sigma, Y)}}^{*} \circ \mathrm{GCCL}^{*}\left(\frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}}\right), \\
\mathrm{GCCL}^{*} \circ \operatorname{ad}_{X_{h}^{\mathfrak{n}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}} & =-\operatorname{ad}_{X_{\widehat{\mathbf{x}}}^{*}}^{*} \circ \mathrm{GCCL}^{*}\left(\frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}}\right), \tag{131}
\end{align*}
$$

Further, referring to Lemma (3), we have the second terms on the right hand side of (130) as

$$
\begin{align*}
\left(\operatorname{GCCL}^{*}\left(\frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}}\right)\right) \stackrel{*}{\triangleleft} \mathbf{X} & =\operatorname{GCCL}^{*}\left(\frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}} \stackrel{*}{\triangleleft} X_{h}^{\mathfrak{n}}\right), \\
(\sigma, Y) \stackrel{*}{\triangleright} \operatorname{GCCL}^{*}\left(\frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}}\right) & =\operatorname{GCCL}^{*}\left(X_{h}^{\mathfrak{s}} \stackrel{*}{\triangleright} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}}\right) . \tag{132}
\end{align*}
$$

In addition, we refer to Lemma (2) for the third term on the right hand side of (130) as

$$
\begin{align*}
\mathrm{GCCL}^{*} \circ \mathfrak{b}_{X_{h}^{\mathfrak{s}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}} & =-\mathfrak{b}_{(\sigma, Y)}^{*} \circ \mathrm{GCCL}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{s}}}  \tag{133}\\
\mathrm{GCCL}^{*} \circ \mathfrak{a}_{X_{h}^{\mathfrak{n}}}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}} & =-\mathfrak{a}_{\mathrm{X}}^{*} \circ \mathrm{GCCL}^{*} \frac{\delta \mathfrak{L}}{\delta X_{h}^{\mathfrak{n}}} .
\end{align*}
$$

As an example to this geometry, and following [13], we define a pure quadratic Lagrangian functional on the space $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ of Hamiltonian vector fields as follows

$$
\begin{equation*}
\mathfrak{L}\left(X_{h}\right)=\frac{1}{2} \int_{T^{*} \mathcal{Q}}\left\langle\hat{Q} X_{h}, X_{h}\right\rangle \mu, \tag{134}
\end{equation*}
$$

where $\hat{Q}$ is a positive-definite symmetric operator on $\mathfrak{X}_{\text {ham }}\left(T^{*} \mathcal{Q}\right)$ and $\mu$ is the symplectic volume. It is easy to see that the variation of the quadratic Lagrangian $\mathfrak{L}$ with respect to $X_{h}$ is $\hat{Q} X_{h}$. We assume that there exists a one-form $\Pi$ so that $\Pi=\hat{Q} X_{h}$ and the Hamiltonian vector field $X_{h}$ is written as a sum of two Hamiltonian vector fields $X_{h}^{\mathfrak{s}}$ and $X_{h}^{\mathfrak{n}}$ according to the decomposition in (5). In this decomposition $X_{h}^{\mathfrak{s}}$ is in the Lie subalgebra $\mathfrak{s}$, whereas
$X_{h}^{\mathfrak{n}}$ is in the Lie subalgebra $\mathfrak{n}$. Further, the positive-definite symmetric operator $\hat{Q}$ is decomposible as

$$
\hat{Q}=\left(\begin{array}{cc}
\hat{Q}^{\mathfrak{s}} & 0  \tag{135}\\
0 & \hat{Q}^{\mathfrak{n}}
\end{array}\right)
$$

so that the Lagrangian $\mathfrak{L}$ in (134) turns out to be a sum

$$
\begin{align*}
\mathfrak{L}\left(X_{h}^{\mathfrak{s}}, X_{h}^{\mathfrak{n}}\right) & =\mathfrak{L}^{\mathfrak{s}}\left(X_{h}^{\mathfrak{s}}\right)+\mathfrak{L}^{\mathfrak{n}}\left(X_{h}^{\mathfrak{n}}\right) \\
& =\frac{1}{2} \int_{T^{*} \mathcal{Q}}\left\langle\hat{Q}^{\mathfrak{s}} X_{h}^{\mathfrak{s}}, X_{h}^{\mathfrak{s}}\right\rangle d \mathbf{q} d \mathbf{p}+\frac{1}{2} \int_{T^{*} \mathcal{Q}}\left\langle\hat{Q}^{\mathfrak{n}} X_{h}^{\mathfrak{n}}, X_{h}^{\mathfrak{n}}\right\rangle d \mathbf{q} d \mathbf{p}, \tag{136}
\end{align*}
$$

see that

$$
\begin{equation*}
\frac{\delta \mathfrak{L}}{\delta X_{h}}=\left(\frac{\delta \mathfrak{L}^{\mathfrak{s}}}{\delta X_{h}^{\mathfrak{s}}}, \frac{\delta \mathfrak{L}^{\mathfrak{n}}}{\delta X_{h}^{\mathfrak{n}}}\right)=\left(\hat{Q}^{\mathfrak{s}} X_{h}^{\mathfrak{s}}, \hat{Q}^{\mathfrak{n}} X_{h}^{\mathfrak{n}}\right)=\left(\Pi^{\mathfrak{s}}, \Pi^{\mathfrak{n}}\right) \tag{137}
\end{equation*}
$$

See that the algebra of momenta and the algebra of Hamiltonian vector fields are in one-to-one correspondence if one omits the flat functions. This is achieved by the GCCL mapping. So, the matched pair decomposition of the Euler-Poincaré flow on the momentum hierarchy given in (121) can be transferred to the Lagrangian dynamics on the space of Hamiltonian vector fields as well. The result is that the matched pair decomposition of the Euler-Poincaré flow on Hamiltonian vector fields is (130). As in the case of momentum decomposition, the whole Euler-Poincaré flow is now a matched pair of two Euler-Poincaré subdynamics, whilst the extra terms are presented in algebraically. To learn more about the collective motion, one can start with the subdynamics and transfer it to the whole picture by means of the geometry in this subsection.

Author Contributions: Investigation, O.E.; Writing-original draft, C.S.M.; Supervision, J.D.L. and M.Z. All authors have read and agreed to the published version of the manuscript.

Funding: TÜBİTAK (the Scientific and Technological Research Council of Turkey) under the title "Matched pairs of Lagrangian and Hamiltonian Systems" with project number 117F426.

Data Availability Statement: No data supporting the reported results can be found.
Acknowledgments: OE gratefully acknowledges the support of TÜBITAK (the Scientific and Technological Research Council of Turkey) under the title "Matched pairs of Lagrangian and Hamiltonian Systems" with project number 117F426. OE is grateful to Hasan Gümral and Serkan Sütlü for many discussions done around the problem addressed in the present work.

Conflicts of Interest: The authors declare no conflict of interest.

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