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Higher-Order First Integrals of Autonomous Non-Riemannian Dynamical Systems

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Abstract: We consider autonomous holonomic dynamical systems defined by equations of the form $\ddot{q}^a = -\Gamma_{bc}^a(q)\dot{q}^b\dot{q}^c - Q^a(q)$, where $\Gamma_{bc}^a(q)$ are the coefficients of a symmetric (possibly non-metrical) connection and $-Q^a(q)$ are the generalized forces. We prove a theorem which for these systems determines autonomous and time-dependent first integrals (FIs) of any order in a systematic way, using the ‘symmetries’ of the geometry defined by the dynamical equations. We demonstrate the application of the theorem to compute linear, quadratic, and cubic FIs of various Riemannian and non-Riemannian dynamical systems.

Keywords: integrability; non-Riemannian autonomous system; higher order first integral; symmetric connection; generalized Killing tensor



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1. Introduction

A first integral (FI) of a second-order set of dynamical equations with generalized coordinates q^a and generalized velocities $\dot{q}^a \equiv \frac{dq^a}{dt}$ is a function $I(t, q^a, \dot{q}^a)$ satisfying the condition $\frac{dI}{dt} = 0$ along the dynamical equations. FIs are important because they can be used in order to reduce the order of the dynamical equations, and if they are ‘enough’ in number [1], to find the solution of the system by quadrature (Liouville integrability).

The standard method to compute the FIs of Lagrangian systems is Noether’s theorem. A different method is the direct method, which requires only the dynamical equations and was originally introduced by Whittaker [2–7]. In the latter method, one assumes a functional form for the FI I (e.g., a polynomial form in \dot{q}^a) and demands the condition $\frac{dI}{dt} = 0$. Using the dynamical equations to remove the terms \dot{q}^a whenever they appear, the FI condition leads to a system of partial differential equations (PDEs), whose solution provides the FIs.

In this work, we apply the direct method to autonomous holonomic dynamical systems in a space with a symmetric connection $\Gamma_{bc}^a(q)$ (not necessarily Riemannian) which is read from the dynamical equations. We computed the resulting system of PDEs and solved it in terms of the ‘symmetries’ of $\Gamma_{bc}^a(q)$. The result is stated as Theorem 1 and provides a systematic method to determine polynomial FIs in velocities of any order, time-dependent and autonomous, for dynamical systems of this type. In the special case that the symmetric connection $\Gamma_{bc}^a(q)$ is the Riemannian connection defined by the kinetic metric (kinetic energy) $\gamma_{ab}(q)$ of the system, the computed FIs are directly related by means of the inverse Noether theorem [8,9] to gauge generalized (i.e., velocity-dependent) weak Noether symmetries. Finally, we apply Theorem 1 in order to find new integrable and superintegrable systems which admit linear (LFIs), quadratic (QFIs), and cubic FIs (CFIs).

The structure of the paper is as follows. In Section 2, using the direct method, we derived the system of PDEs that must be satisfied by the coefficients of an m th-order FI of an autonomous (in general non-Riemannian) dynamical system. In Section 3, the ‘solution’ of the system of PDEs is stated as Theorem 1 (the proof of Theorem 1 is given in the Appendix A). In Section 4, we apply Theorem 1 for $m = 1$ in order to consider the LFIs. It is shown that there are three independent types of LFIs, and we determine their explicit formulae. In Section 5, we apply Theorem 1 for $m > 1$, and we found that there are three independent types of higher order FIs. In section 6, we discuss these independent FIs and provide a procedure by which symmetries of order smaller than the order of the FI can be removed. Using this procedure, we removed these lesser symmetries, and we found the complete forms of the FIs for an even order $m = 2\nu$ and an odd order $m = 2\nu + 1$. The results are collected in Propositions 5–7. In Section 7, we found a family of two-dimensional (2d) non-Riemannian autonomous dynamical systems, and by applying Theorem 1, we determined the LFIs. In Sections 8–10, we provide further applications of Theorem 1 for QFIs and CFIs, by extending existing results in the literature. Finally, in Section 11, we draw our conclusions.

2. The Conditions for m th-Order FIs

We consider *general* (i.e., Riemannian and non-Riemannian) autonomous dynamical systems of the form

$$\ddot{q}^a = -\Gamma_{bc}^a(q)\dot{q}^b\dot{q}^c - Q^a(q) \quad (1)$$

where q^a with $a = 1, 2, \dots, D$ are the generalized coordinates of the configuration space of the system, D is the dimension of the configuration space, a dot over a letter indicates derivation with respect to (wrt) the parameter t (time) along the trajectory $q^a(t)$, Einstein’s summation convention is applied, $\Gamma_{bc}^a(q)$ are the coefficients of a general connection, and $-Q^a(q)$ are the generalized forces. Since only the symmetric part $\Gamma_{(bc)}^a$ contributes to the dynamical equations—without loss of generality—the quantities $\Gamma_{bc}^a(q)$ are assumed to be symmetric.

We look for m th-order FIs of the general form:

$$I^{(m)} = \sum_{r=0}^m M_{i_1 i_2 \dots i_r}(t, q) \dot{q}^{i_1} \dot{q}^{i_2} \dots \dot{q}^{i_r} = M + M_{i_1} \dot{q}^{i_1} + M_{i_1 i_2} \dot{q}^{i_1} \dot{q}^{i_2} + \dots + M_{i_1 i_2 \dots i_m} \dot{q}^{i_1} \dot{q}^{i_2} \dots \dot{q}^{i_m} \quad (2)$$

where $M_{i_1 \dots i_r}(t, q)$ with $r = 0, 1, \dots, m$ are totally symmetric r -rank tensors, and the index $m \geq 1$ denotes the order of the FI. We note that when $r = 0$, the quantities $M_{i_1 \dots i_r}(t, q)$ reduce to the scalar $M(t, q)$. For $m = 1$, we have the LFIs; for $m = 2$, the QFIs; and for $m = 3$, the CFIs.

The FI condition

$$\frac{dI^{(m)}}{dt} = 0 \quad (3)$$

along the dynamical Equation (1) results in the following system of PDEs:

$$M_{i_1 i_2 \dots i_r, t} + M_{(i_1 i_2 \dots i_{r-1} | i_r)} - (r+1) M_{i_1 i_2 \dots i_r i_{r+1}} Q^{i_{r+1}} = 0, \quad r = 0, 1, 2, \dots, m, m+1 \quad (4)$$

which is expanded as follows:

$$M_{(i_1 i_2 \dots i_m | i_{m+1})} = 0 \quad (5)$$

$$M_{i_1 i_2 \dots i_m, t} + M_{(i_1 i_2 \dots i_{m-1} | i_m)} = 0 \quad (6)$$

$$M_{i_1 i_2 \dots i_r, t} + M_{(i_1 i_2 \dots i_{r-1} | i_r)} - (r+1) M_{i_1 i_2 \dots i_r i_{r+1}} Q^{i_{r+1}} = 0, \quad r = 1, 2, \dots, m-1, \quad m > 1 \quad (7)$$

$$M_{,t} - M_{i_1} Q^{i_1} = 0. \quad (8)$$

The symbol $|$ denotes the covariant derivative wrt the symmetric connection Γ_{bc}^a , a comma indicates partial derivative wrt q^a or t , round/square brackets indicate symmetrization/

antisymmetrization of the enclosed indices, and indices enclosed between wavy lines are overlooked by symmetrization or antisymmetrization symbols.

We note that Equation (5) is derived from (4) for $r = m + 1$; Equation (6) from (4) for $r = m$; and Equation (8) from (4) for $r = 0$.

Concerning the notation, we remark that:

$$M_{i_1 \dots i_{r-k}}(r=0) = \begin{cases} M, & k=0 \\ 0, & k \geq 1 \end{cases}, \quad M_{i_1 \dots i_r}(r > m) = 0.$$

Equations (5) and (6) are purely geometric equations, which are common to all systems of the form (1) that share the same symmetric connection. In particular, Equation (5) generalizes the concept of killing tensors (KTs) to a non-metrical geometry with a symmetric connection Γ_{bc}^a . In this context, $M_{i_1 i_2 \dots i_m}$ is a **generalized m th-order KT** for Γ_{bc}^a .

Equations (7) and (8) are of a dynamical character, because they relate the geometric elements with the generalized forces Q^a of the specific dynamical system.

Since the dynamical system (1) is autonomous, we should use the polynomial method described in [10] in order to solve the system of PDEs (5)–(8). According to this method, one assumes general polynomial expressions in the variable t for the tensor quantities $M_{i_1 \dots i_r}(t, q)$ with $r = 1, 2, \dots, m$ (see Equation (A1) in the Appendix A) and replaces these expressions in the system of PDEs (5)–(8). Then, the scalar $M(t, q)$ is determined exactly, and the remaining PDEs reduce to polynomial equations in t , whose coefficients are functions of q^a . The ‘solution’ of the latter system of PDEs is stated below as Theorem 1. A detailed proof is given in the Appendix A.

3. Theorem for m th-Order FIs of a General Autonomous Dynamical System

Theorem 1. *There are two types of m th-order FIs for the autonomous (in general non-Riemannian) dynamical system (1). These are the following:*

Integral 1.

$$I_n^{(m)} = \sum_{r=1}^m \left(\sum_{N=0}^n L_{(N)i_1 \dots i_r} t^N \right) \dot{q}^{i_1} \dots \dot{q}^{i_r} + s_0 \frac{t^{n+1}}{n+1} + \sum_{N=1}^{n \geq 0} L_{(N-1)c} Q^c \frac{t^N}{N} + G(q) \quad (9)$$

where $m \geq 1$, $n \geq 0$, $L_{(N)i_1 \dots i_m}(q)$ with $N = 0, 1, \dots, n$ are **m th-order generalized KT**s satisfying the condition

$$L_{(k)i_1 \dots i_m} = -\frac{1}{k} L_{(k-1)(i_1 \dots i_{m-1}|i_m)}, \quad k = 1, 2, \dots, n, \quad n > 0, \quad m > 1, \quad (10)$$

the totally symmetric tensor $L_{(n)i_1 \dots i_{m-1}}(q)$ with $m > 1$ is an **$(m-1)$ th-order generalized KT**; the constant s_0 is defined by the condition

$$L_{(n)a} Q^a = s_0 \quad (11)$$

and the function $G(q)$ and the totally symmetric tensors $L_{(N)i_1 \dots i_r}(q)$ satisfy the conditions:

$$G_{,i_1} = 2L_{(0)i_1 i_2}(m > 1)Q^{i_2} - L_{(1)i_1}(n > 0) \quad (12)$$

$$\begin{aligned} \left(L_{(k-1)c} Q^c \right)_{,i_1} &= 2kL_{(k)i_1 i_2}(m > 1)Q^{i_2} - k(k+1)L_{(k+1)i_1}(k < n), \\ &k = 1, 2, \dots, n, \quad n > 0 \end{aligned} \quad (13)$$

$$\begin{aligned} L_{(k)(i_1 \dots i_{r-1}|i_r)} &= (r+1)L_{(k)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (k+1)L_{(k+1)i_1 \dots i_r}(k < n), \\ &k = 0, 1, \dots, n, \quad r = 2, 3, \dots, m-1, \quad m > 2. \end{aligned} \quad (14)$$

Integral 2.

$$I_e^{(m)} = \frac{e^{\lambda t}}{\lambda} \left(\lambda \sum_{r=1}^m L_{i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + L_c Q^c \right) \quad (15)$$

where $\lambda \neq 0$, $L_{i_1 \dots i_m}(q)$ is an ***m*th-order generalized KT** satisfying the condition

$$L_{i_1 \dots i_m} = -\frac{1}{\lambda} L_{(i_1 \dots i_{m-1} | i_m)}, \quad m > 1 \quad (16)$$

and the totally symmetric tensors $L_{i_1 \dots i_r}(q)$ satisfy the conditions:

$$(L_c Q^c)_{,i_1} = 2\lambda L_{i_1 i_2}(m > 1) Q^{i_2} - \lambda^2 L_{i_1} \quad (17)$$

$$L_{(i_1 \dots i_{r-1} | i_r)} = (r+1) L_{i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - \lambda L_{i_1 \dots i_r}, \quad r = 2, 3, \dots, m-1, \quad m > 2. \quad (18)$$

Concerning the notation, the symbol $I_n^{(m)}$ means the *m*th-order FI (upper index) with degree of time-dependence *n* (lower index), and $I_e^{(m)}$ indicates the *m*th-order FI with exponential time-dependence (lower index).

Using mathematical induction, one also proves the following recursion formulae.

Proposition 1. The independent *m*th-order FIs $I_n^{(m)}$ and $I_e^{(m)}$ satisfy the following recursion formulae:

- $I_n^{(k)} < I_n^{(k+1)}$; that is, each *k*th-order FI $I_n^{(k)}$ is a subcase of the next $(k+1)$ th-order FI $I_n^{(k+1)}$ with the same degree *n* of time-dependence for all integers $k \geq 1$.
- $I_\ell^{(m)} < I_{\ell+1}^{(m)}$; that is, the *m*th-order FI $I_\ell^{(m)}$ with time-dependence fixed by ℓ is a subcase of the *m*th-order FI $I_{\ell+1}^{(m)}$ with time-dependence $\ell+1$ for all integers $\ell \geq 0$.
- $I_e^{(k)} < I_e^{(k+1)}$; that is, each *k*th-order FI $I_e^{(k)}$ is a subcase of the next $(k+1)$ th-order FI $I_e^{(k+1)}$ for all integers $k \geq 1$.

We note that Theorem 1 is true for $m = 2$ (QFIs), and a Riemannian connection reduces to Theorem 3 of [9].

In the case of a Riemannian connection, by means of the inverse Noether theorem [8,9], the general *m*th-order FIs (2) are related to the generalized gauged weak Noether symmetry:

$$\left(\xi = 0, \quad \eta_{i_1} = -\frac{\partial I^{(m)}}{\partial \dot{q}^{i_1}}, \quad \phi_a, \quad f = I^{(m)} - \frac{\partial I^{(m)}}{\partial \dot{q}^{i_1}} \dot{q}^{i_1} \right) \quad \text{such that} \quad \phi_a \dot{q}^a + F^a \frac{\partial I^{(m)}}{\partial \dot{q}^a} = 0 \quad (19)$$

where $F^a(t, q, \dot{q})$ are the non-conservative generalized forces, $\phi^a(t, q, \dot{q})$ is an additional vector generator, $f(t, q, \dot{q})$ is the Noether function, $\mathbf{X} = \xi(t, q, \dot{q}) \partial_t + \eta^a(t, q, \dot{q}) \partial_{q^a}$ is the Lie generator, and the quantity

$$\begin{aligned} \frac{\partial I^{(m)}}{\partial \dot{q}^{i_1}} &= M_{i_1} + 2M_{i_1 i_2} \dot{q}^{i_2} + 3M_{i_1 i_2 i_3} \dot{q}^{i_2} \dot{q}^{i_3} + \dots + m M_{i_1 i_2 \dots i_m} \dot{q}^{i_2} \dots \dot{q}^{i_m} \\ &= \sum_{r=0}^{m-1} (r+1) M_{i_1 i_2 \dots i_{r+1}} \dot{q}^{i_2} \dots \dot{q}^{i_{r+1}}. \end{aligned}$$

4. FIs of Order $m = 1$: LFIs

Applying Theorem 1 for $m = 1$ (LFIs), we found the following proposition for the LFIs.

Proposition 2. There are two types of LFIs for the autonomous (in general non-Riemannian) dynamical system (1) which are the following:

LFI 1.

$$I_n^{(1)} = \sum_{N=0}^n L_{(N)a} t^N \dot{q}^a + s_0 \frac{t^{n+1}}{n+1} + \sum_{N=1}^{n \geq 0} L_{(N-1)a} Q^a \frac{t^N}{N} + G(q) \quad (20)$$

where $n \geq 0$, $G(q)$ is an arbitrary smooth function, and $L_{(N)a}(q)$ with $N = 0, 1, \dots, n$ are **generalized KVs** satisfying the conditions:

$$L_{(1)a}(n > 0) = -G_{,a}. \quad (21)$$

$$\left(L_{(k-1)b} Q^b \right)_{,a} = -k(k+1) L_{(k+1)a} (k < n), \quad k = 1, 2, \dots, n, \quad n > 0, \quad (22)$$

and the constant s_0 is defined by the condition

$$L_{(n)a} Q^a = s_0. \quad (23)$$

LFI 2.

$$I_e^{(1)} = \frac{e^{\lambda t}}{\lambda} (\lambda L_a \dot{q}^a + L_a Q^a) \quad (24)$$

where $\lambda \neq 0$ and $L_a(q)$ is a **generalized KV** satisfying the condition

$$\left(L_b Q^b \right)_{,a} = -\lambda^2 L_a. \quad (25)$$

The LFI (20) consists of two independent LFIs: (a) a LFI for the even vectors $L_{(2N)a}$, and (b) another one for the odd vectors $L_{(2N+1)a}$ and the scalar $G(q)$. This results directly from the fact that condition (21) involves only the odd vector $L_{(1)a}$ and the scalar $G(q)$, and conditions (22) are separated in two sets – one set involving only the even vectors $L_{(2N)a}$ (i.e., for the odd values $k = 1, 3, 5, \dots$), and another one involving only the odd vectors $L_{(2N+1)a}$ (i.e., for the even values $k = 2, 4, 6, \dots$). We note that the above holds for either an odd or an even time-dependence, n . In the following proposition, we give the explicit formulae of these two independent LFIs.

Proposition 3. The LFI $I_n^{(1)}$ given in (20) consists of the following two independent LFIs:

LFI 1.1.

$$I_\ell^{(1,1)} = \sum_{N=1}^{\ell} t^{2N-1} L_{(2N-1)a} \dot{q}^a + s_1 \frac{t^{2\ell}}{2\ell} + \sum_{N=1}^{\ell-1 \geq 1} L_{(2N-1)a} Q^a \frac{t^{2N}}{2N} + G(q) \quad (26)$$

where $\ell > 0$, $L_{(2N-1)a}(q)$ with $N = 1, 2, \dots, \ell$ are **generalized KVs** satisfying the condition

$$\left(L_{(2k-1)b} Q^b \right)_{,a} = -2k(2k+1) L_{(2k+1)a}, \quad k = 1, 2, \dots, \ell-1, \quad \ell > 1, \quad (27)$$

$G(q)$ is an arbitrary smooth function such that $L_{(1)a} = -G_{,a}$ is a **gradient generalized KV**, and the constant s_1 is defined by the condition

$$s_1 = L_{(2\ell-1)a} Q^a. \quad (28)$$

LFI 1.2.

$$I_\ell^{(1,2)} = \sum_{N=0}^{\ell} t^{2N} L_{(2N)a} \dot{q}^a + s_0 \frac{t^{2\ell+1}}{2\ell+1} + \sum_{N=0}^{\ell-1 \geq 0} L_{(2N)a} Q^a \frac{t^{2N+1}}{2N+1} \quad (29)$$

where $\ell \geq 0$, $L_{(2N)a}(q)$ with $N = 0, 1, \dots, \ell$ are **generalized KVs** satisfying the condition

$$\left(L_{(2k)b} Q^b \right)_{,a} = -2(2k+1)(k+1) L_{(2k+2)a}, \quad k = 0, 1, \dots, \ell-1, \quad \ell > 0, \quad (30)$$

and the constant s_0 is defined by the condition

$$L_{(2\ell)a}Q^a = s_0. \quad (31)$$

Therefore, the general autonomous dynamical system (1) admits three independent LFIs, which are given by the formulae (24), (26), and (29).

The notation $I_\ell^{(1,\alpha)}$, where $\alpha = 1, 2$, indicates one of the two types of independent LFIs of the type $I_n^{(1)}$ with time-dependence fixed by the lower index ℓ .

We note that condition (27) is derived from (22) for the even values $k = 2, 4, 6, \dots, 2\ell$ if we set $n = 2\ell$ and rename the index k as $2k$; condition (30) is derived from (22) for the odd values $k = 1, 3, 5, \dots, 2\ell - 1$ if we set $n = 2\ell$ and rename the index k as $2k + 1$.

5. FIs of Order $m > 1$

By applying Theorem 1 for $m > 1$, we found the following proposition.

Proposition 4. *There are two types of m th-order FIs with $m > 1$ for the autonomous (in general, non-Riemannian) dynamical system (1), which are the following:*

Integral 1.

$$I_n^{(m>1)} = \sum_{r=1}^m \left(\sum_{N=0}^n L_{(N)i_1 \dots i_r} t^N \right) \dot{q}^{i_1} \dots \dot{q}^{i_r} + s_0 \frac{t^{n+1}}{n+1} + \sum_{N=1}^{n>0} L_{(N-1)c} Q^c \frac{t^N}{N} + G(q) \quad (32)$$

where $m > 1$, $n \geq 0$, $L_{(N)i_1 \dots i_m}(q)$ with $N = 0, 1, \dots, n$ are m th-order generalized KTs satisfying the condition

$$L_{(k)i_1 \dots i_m} = -\frac{1}{k} L_{(k-1)(i_1 \dots i_{m-1}|i_m)}, \quad k = 1, 2, \dots, n, \quad n > 0, \quad (33)$$

the totally symmetric tensor $L_{(n)i_1 \dots i_{m-1}}(q)$ is an $(m-1)$ th-order generalized KT; the constant s_0 is defined by the condition

$$L_{(n)a}Q^a = s_0 \quad (34)$$

and the function $G(q)$ and the remaining totally symmetric tensors $L_{(N)i_1 \dots i_r}(q)$ satisfy the conditions:

$$G_{,i_1} = 2L_{(0)i_1 i_2} Q^{i_2} - L_{(1)i_1} \quad (n > 0) \quad (35)$$

$$\left(L_{(k-1)c} Q^c \right)_{,i_1} = 2kL_{(k)i_1 i_2} Q^{i_2} - k(k+1)L_{(k+1)i_1} \quad (k < n), \quad k = 1, 2, \dots, n, \quad n > 0 \quad (36)$$

$$L_{(k)(i_1 \dots i_{r-1}|i_r)} = (r+1)L_{(k)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (k+1)L_{(k+1)i_1 \dots i_r} \quad (k < n), \\ k = 0, 1, \dots, n, \quad r = 2, 3, \dots, m-1, \quad m > 2. \quad (37)$$

Integral 2.

$$I_e^{(m>1)} = \frac{e^{\lambda t}}{\lambda} \left(\lambda \sum_{r=1}^m L_{i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + L_c Q^c \right) \quad (38)$$

where $\lambda \neq 0$ and $L_{i_1 \dots i_m}(q)$ is an m th-order generalized KT satisfying the condition

$$L_{i_1 \dots i_m} = -\frac{1}{\lambda} L_{(i_1 \dots i_{m-1}|i_m)} \quad (39)$$

and the remaining totally symmetric tensors $L_{i_1 \dots i_r}(q)$ with $r = 1, 2, \dots, m-1$ satisfy the conditions:

$$(L_c Q^c)_{,i_1} = 2\lambda L_{i_1 i_2} Q^{i_2} - \lambda^2 L_{i_1} \quad (40)$$

$$L_{(i_1 \dots i_{r-1} | i_r)} = (r+1) L_{i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - \lambda L_{i_1 \dots i_r}, \quad r = 2, 3, \dots, m-1, \quad m > 2. \quad (41)$$

From Proposition 4, we observe that the conditions (33)–(37) of the m th-order FI (32) are divided in two classes, according to if the tensor quantities $L_{(N)i_1 \dots i_r}(q)$ are of even or odd order, and the associated index N is even or odd. Therefore, the FI (32) consists of two independent FIs:

(a) One FI, say $I_\ell^{(m,1)}$, which contains tensor quantities $L_{(2N-1)i_1 \dots i_r}(q)$ of odd order and $L_{(2N)i_1 \dots i_r}(q)$ of even order.

(b) Another FI, say $I_\ell^{(m,2)}$, which contains tensor quantities $L_{(2N-1)i_1 \dots i_r}(q)$ of even order and $L_{(2N)i_1 \dots i_r}(q)$ of odd order.

We note that the order r of the involved totally symmetric tensors $L_{(N)i_1 \dots i_r}(q)$ takes values from 1 to $m > 1$, where m is the order of the considered FI (32).

The independent FIs $I_\ell^{(m,1)}$ and $I_\ell^{(m,2)}$ are given by the following explicit formulae:

a.

$$\begin{aligned} I_\ell^{(m>1,1)} = & \sum_{r=1, \text{odd}}^m \sum_{N=1}^{\ell>0} t^{2N-1} L_{(2N-1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \sum_{r=1, \text{even}}^m \sum_{N=0}^{\ell} t^{2N} L_{(2N)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \\ & + \sum_{N=1}^{\ell>0} L_{(2N-1)c} Q^c \frac{t^{2N}}{2N} + G(q) \end{aligned} \quad (42)$$

where the time-dependence $\ell \geq 0$ and the involved quantities satisfy the conditions:

$$G_{,i_1} = 2L_{(0)i_1 i_2} Q^{i_2} - L_{(1)i_1} \quad (\ell > 0) \quad (43)$$

$$\left(L_{(2N-1)c} Q^c \right)_{,i_1} = 4NL_{(2N)i_1 i_2} Q^{i_2} - 2N(2N+1)L_{(2N+1)i_1} \quad (N < \ell), \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (44)$$

$$\begin{aligned} L_{(2N-1)(i_1 \dots i_{r-1} | i_r)} = & (r+1) L_{(2N-1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - 2NL_{(2N)i_1 \dots i_r}, \\ & N = 1, 2, \dots, \ell, \quad r = 2, 4, 6, \dots, \quad \ell > 0 \end{aligned} \quad (45)$$

$$\begin{aligned} L_{(2N)(i_1 \dots i_{r-1} | i_r)} = & (r+1) L_{(2N)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (2N+1) L_{(2N+1)i_1 \dots i_r} \quad (N < \ell), \\ & N = 0, 1, \dots, \ell, \quad r = 3, 5, 7, \dots \end{aligned} \quad (46)$$

In conditions (45) and (46), it holds that $r \leq m-1$, where $m > 2$.

b.

$$\begin{aligned} I_\ell^{(m>1,2)} = & \sum_{r=1, \text{odd}}^m \sum_{N=0}^{\ell} t^{2N} L_{(2N)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \sum_{r=1, \text{even}}^m \sum_{N=1}^{\ell>0} t^{2N-1} L_{(2N-1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \\ & + s_0 \frac{t^{2\ell+1}}{2\ell+1} + \sum_{N=0}^{\ell-1 \geq 0} L_{(2N)c} Q^c \frac{t^{2N+1}}{2N+1} \end{aligned} \quad (47)$$

where the time-dependence $\ell \geq 0$ and the involved quantities satisfy the conditions:

$$L_{(2\ell)a} Q^a = s_0 \quad (48)$$

$$\left(L_{(2N-2)c} Q^c \right)_{,i_1} = 2(2N-1) L_{(2N-1)i_1 i_2} Q^{i_2} - 2N(2N-1) L_{(2N)i_1}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (49)$$

$$\begin{aligned} L_{(2N)(i_1 \dots i_{r-1} | i_r)} = & (r+1) L_{(2N)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (2N+1) L_{(2N+1)i_1 \dots i_r} \quad (N < \ell), \\ & N = 0, 1, \dots, \ell, \quad r = 2, 4, 6, \dots, \end{aligned} \quad (50)$$

$$\begin{aligned} L_{(2N-1)(i_1 \dots i_{r-1} | i_r)} = & (r+1) L_{(2N-1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - 2NL_{(2N)i_1 \dots i_r}, \\ & N = 1, 2, \dots, \ell, \quad r = 3, 5, 7, \dots, \quad \ell > 0. \end{aligned} \quad (51)$$

In conditions (50) and (51), it holds that $r \leq m - 1$ where $m > 2$.

For the FIs (42) and (47), we have also that $L_{(2\ell)i_1 \dots i_{m-1}}(q)$ is an **$(m-1)$ th-order generalized KT**, and $L_{(N)i_1 \dots i_m}(q)$ with $N = 0, 1, \dots, 2\ell$ are **m th-order generalized KTs** such that:

$$L_{(2N)i_1 \dots i_m} = -\frac{1}{2N} L_{(2N-1)(i_1 \dots i_{m-1}|i_m)}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (52)$$

$$L_{(2N-1)i_1 \dots i_m} = -\frac{1}{2N-1} L_{(2N-2)(i_1 \dots i_{m-1}|i_m)}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0. \quad (53)$$

We note that the sum $\sum_{r=1, \text{odd}}^m$ is over the odd values of r , and the sum $\sum_{r=1, \text{even}}^m$ is over the even values of r .

6. How to Remove the $(m-1)$ th-Order Geometric Symmetries from the Two Independent FIs $I_\ell^{(m>1,1)}$ and $I_\ell^{(m>1,2)}$: The Complete Forms

As we have seen, the two independent FIs (42) and (47) include geometric symmetries of order smaller from the order $m > 1$ of the FIs. These symmetries are described by the $(m-1)$ th-order generalized KT $L_{(2\ell)i_1 \dots i_{m-1}}(q)$. We observe that if $m = 2\nu$ (even) where $\nu > 0$, the considered $(m-1)$ th-order generalized KT appears in the FI (47); and in the case that $m = 2\nu + 1$ (odd), appears in the FI (42). Moreover, for an even order m , the condition (52) accompanies the FI (42) and the condition (53) accompanies the FI (47); and for an odd order m , the condition (52) accompanies the FI (47) and the condition (53) accompanies the FI (42).

The $(m-1)$ th-order generalized KT $L_{(2\ell)i_1 \dots i_{m-1}}(q)$ can be removed from expressions (42) and (47) by applying the following procedure:

(1) For an even order $m = 2\nu$, where $\nu > 0$.

In this case, the $(2\nu-1)$ th-order generalized KT $L_{(2\ell)i_1 \dots i_{2\nu-1}}(q)$ appears in the independent FI (47). We can remove this $(2\nu-1)$ th-order symmetry by introducing a sequence of an even rank totally symmetric tensors of a higher time-dependence; that is, we add in the FI (47) terms of the form

$$t^{2\ell+1} L_{(2\ell+1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r}, \quad r = 2, 4, \dots, 2\nu, \quad \nu > 0 \quad (54)$$

such that $L_{(2\ell+1)i_1 \dots i_{2\nu}}$ is a (2ν) th-order generalized KT given by the relation

$$L_{(2\ell+1)i_1 \dots i_{2\nu}} = -\frac{1}{2\ell+1} L_{(2\ell)(i_1 \dots i_{2\nu-1}|i_{2\nu})}. \quad (55)$$

Then, the condition (48) must be generalized as

$$\left(L_{(2\ell)c} Q^c \right)_{,i_1} = 2(2\ell+1) L_{(2\ell+1)i_1 i_2} Q^{i_2}; \quad (56)$$

from the condition (50), the restriction $N < \ell$ must be removed because now quantities of the form $L_{(2N+1)i_1 \dots i_r}$ do exist for $N = \ell$; and we must add the condition

$$L_{(2\ell+1)(i_1 \dots i_{r-1}|i_r)} = (r+1) L_{(2\ell+1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}}, \quad r = 3, 5, 7, \dots, 2\nu-1, \quad \nu > 1. \quad (57)$$

Observe that the condition (55) removes the original $(2\nu-1)$ th-order symmetry by allowing the $(2\nu-1)$ th-order generalized KT $L_{(2\ell)i_1 \dots i_{2\nu-1}}(q)$ to be a totally symmetric tensor of the same rank, which is not necessarily a KT. Then, the case of the generalized KT is derived as a subcase. This shows that symmetries of an order lesser than the order of the FI can always be absorbed in a higher-order term.

(2) For an odd order $m = 2\nu + 1$, where $\nu > 0$.

In this case, the (2ν) th-order generalized KT $L_{(2\ell)i_1 \dots i_{2\nu}}(q)$ appears in the independent FI (42). We can remove this (2ν) th-order symmetry by introducing a sequence of an odd

rank totally symmetric tensors of a higher time-dependence; that is, we add in the FI (42) the sum

$$\sum_{r=1, \text{odd}}^{2\nu+1} t^{2\ell+1} L_{(2\ell+1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + s_1 \frac{t^{2\ell+2}}{2\ell+2} \quad (58)$$

such that $L_{(2\ell+1)i_1 \dots i_{2\nu+1}}$ is a $(2\nu+1)$ th-order generalized KT given by the relation

$$L_{(2\ell+1)i_1 \dots i_{2\nu+1}} = -\frac{1}{2\ell+1} L_{(2\ell)(i_1 \dots i_{2\nu} | i_{2\nu+1})} \quad (59)$$

and the constant s_1 is defined by the relation

$$L_{(2\ell+1)c} Q^c = s_1. \quad (60)$$

Then, from conditions (44) and (46), the restriction $N < \ell$ must be removed because now quantities of the form $L_{(2N+1)i_1 \dots i_r}$ do exist for $N = \ell$; and we must add the condition

$$L_{(2\ell+1)(i_1 \dots i_{r-1} | i_r)} = (r+1) L_{(2\ell+1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}}, \quad r = 2, 4, 6, \dots, 2\nu. \quad (61)$$

Observe that the condition (59) removes the original (2ν) th-order symmetry defined by allowing the (2ν) th-order generalized KT $L_{(2\ell)i_1 \dots i_{2\nu}}(q)$ to be a totally symmetric tensor of the same rank, which is not necessarily a KT. Then, the case of the generalized KT is derived as a subcase. This shows that symmetries of an order lesser than the order of the FI can always be absorbed into a higher-order term.

We collected the above results in the following propositions.

Proposition 5. For an even order $m = 2\nu$, where $\nu > 0$, the **complete forms** (i.e., expressions without geometric symmetries of order less than m) of the independent FIs (42) and (47) are the following:

Integral 1.1.

$$\begin{aligned} J_\ell^{(2\nu,1)} &= \sum_{r=1, \text{odd}}^{2\nu} \sum_{N=1}^{\ell > 0} t^{2N-1} L_{(2N-1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \sum_{r=1, \text{even}}^{2\nu} \sum_{N=0}^{\ell} t^{2N} L_{(2N)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \\ &+ \sum_{N=1}^{\ell > 0} L_{(2N-1)c} Q^c \frac{t^{2N}}{2N} + G(q) \end{aligned} \quad (62)$$

where $\ell \geq 0$, $L_{(2N)i_1 \dots i_{2\nu}}(q)$ with $N = 0, 1, \dots, \ell$ are (2ν) th-order generalized KTs given by the relation

$$L_{(2N)i_1 \dots i_{2\nu}} = -\frac{1}{2N} L_{(2N-1)(i_1 \dots i_{2\nu-1} | i_{2\nu})}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (63)$$

and the involved quantities satisfy the conditions:

$$G_{,i_1} = 2L_{(0)i_1 i_2} Q^{i_2} - L_{(1)i_1} (\ell > 0) \quad (64)$$

$$\left(L_{(2N-1)c} Q^c \right)_{,i_1} = 4N L_{(2N)i_1 i_2} Q^{i_2} - 2N(2N+1) L_{(2N+1)i_1} (N < \ell), \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (65)$$

$$\begin{aligned} L_{(2N-1)(i_1 \dots i_{r-1} | i_r)} &= (r+1) L_{(2N-1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - 2N L_{(2N)i_1 \dots i_r}, \\ N &= 1, 2, \dots, \ell, \quad r = 2, 4, 6, \dots, 2\nu-2, \quad \ell > 0, \quad \nu > 1 \end{aligned} \quad (66)$$

$$\begin{aligned} L_{(2N)(i_1 \dots i_{r-1} | i_r)} &= (r+1) L_{(2N)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (2N+1) L_{(2N+1)i_1 \dots i_r} (N < \ell), \\ N &= 0, 1, \dots, \ell, \quad r = 3, 5, 7, \dots, 2\nu-1, \quad \nu > 1. \end{aligned} \quad (67)$$

Integral 1.2.

$$\begin{aligned}
J_{\ell}^{(2\nu,2)} &= \sum_{r=1, \text{odd}}^{2\nu} \sum_{N=0}^{\ell} t^{2N} L_{(2N)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \sum_{r=1, \text{even}}^{2\nu} \sum_{N=1}^{\ell+1} t^{2N-1} L_{(2N-1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \\
&+ s_0 \frac{t^{2\ell+1}}{2\ell+1} + \sum_{N=0}^{\ell-1 \geq 0} L_{(2N)c} Q^c \frac{t^{2N+1}}{2N+1}
\end{aligned} \quad (68)$$

where $\ell \geq 0$, $L_{(2N-1)i_1 \dots i_{2\nu}}(q)$ with $N = 1, 2, \dots, \ell + 1$ are (2ν) th-order generalized KTs given by the relation

$$L_{(2N-1)i_1 \dots i_{2\nu}} = -\frac{1}{2N-1} L_{(2N-2)(i_1 \dots i_{2\nu-1}|i_{2\nu})}, \quad N = 1, 2, \dots, \ell + 1 \quad (69)$$

and the involved quantities satisfy the conditions:

$$\begin{aligned}
\left(L_{(2N-2)c} Q^c \right)_{,i_1} &= 2(2N-1) L_{(2N-1)i_1 i_2} Q^{i_2} - 2N(2N-1) L_{(2N)i_1} (N < \ell + 1), \\
&N = 1, 2, \dots, \ell + 1
\end{aligned} \quad (70)$$

$$\begin{aligned}
L_{(2N)(i_1 \dots i_{r-1}|i_r)} &= (r+1) L_{(2N)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (2N+1) L_{(2N+1)i_1 \dots i_r}, \\
&N = 0, 1, \dots, \ell, \quad r = 2, 4, 6, \dots, 2\nu - 2, \quad \nu > 1
\end{aligned} \quad (71)$$

$$\begin{aligned}
L_{(2N-1)(i_1 \dots i_{r-1}|i_r)} &= (r+1) L_{(2N-1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - 2N L_{(2N)i_1 \dots i_r} (N < \ell + 1), \\
&N = 1, 2, \dots, \ell + 1, \quad r = 3, 5, 7, \dots, 2\nu - 1, \quad \nu > 1.
\end{aligned} \quad (72)$$

Proposition 6. For an odd order $m = 2\nu + 1$, where $\nu > 0$, the complete forms (i.e., expressions without geometric symmetries of order less than m) of the independent FIs (42) and (47) are the following:

Integral 1.1.

$$\begin{aligned}
J_{\ell}^{(2\nu+1,1)} &= \sum_{r=1, \text{odd}}^{2\nu+1} \sum_{N=1}^{\ell+1} t^{2N-1} L_{(2N-1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \sum_{r=1, \text{even}}^{2\nu+1} \sum_{N=0}^{\ell} t^{2N} L_{(2N)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \\
&+ s_1 \frac{t^{2\ell+2}}{2\ell+2} + \sum_{N=1}^{\ell \geq 0} L_{(2N-1)c} Q^c \frac{t^{2N}}{2N} + G(q)
\end{aligned} \quad (73)$$

where $\ell \geq 0$, $L_{(2N-1)i_1 \dots i_{2\nu+1}}(q)$ with $N = 1, \dots, \ell + 1$ are $(2\nu + 1)$ th-order generalized KTs given by the relation

$$L_{(2N-1)i_1 \dots i_{2\nu+1}} = -\frac{1}{2N-1} L_{(2N-2)(i_1 \dots i_{2\nu}|i_{2\nu+1})}, \quad N = 1, 2, \dots, \ell + 1 \quad (74)$$

and s_1 is a constant defined by the relation

$$L_{(2\ell+1)a} Q^a = s_1, \quad (75)$$

and the involved quantities satisfy the conditions:

$$G_{,i_1} = 2L_{(0)i_1 i_2} Q^{i_2} - L_{(1)i_1} (\ell > 0) \quad (76)$$

$$\left(L_{(2N-1)c} Q^c \right)_{,i_1} = 4N L_{(2N)i_1 i_2} Q^{i_2} - 2N(2N+1) L_{(2N+1)i_1}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (77)$$

$$\begin{aligned}
L_{(2N-1)(i_1 \dots i_{r-1}|i_r)} &= (r+1) L_{(2N-1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - 2N L_{(2N)i_1 \dots i_r} (N < \ell + 1), \\
&N = 1, 2, \dots, \ell + 1, \quad r = 2, 4, 6, \dots, 2\nu
\end{aligned} \quad (78)$$

$$\begin{aligned}
L_{(2N)(i_1 \dots i_{r-1}|i_r)} &= (r+1) L_{(2N)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (2N+1) L_{(2N+1)i_1 \dots i_r}, \\
&N = 0, 1, \dots, \ell, \quad r = 3, 5, 7, \dots, 2\nu - 1, \quad \nu > 1.
\end{aligned} \quad (79)$$

Integral 1.2.

$$\begin{aligned}
J_{\ell}^{(2\nu+1,2)} = & \sum_{r=1, \text{odd}}^{2\nu+1} \sum_{N=0}^{\ell} t^{2N} L_{(2N)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \sum_{r=1, \text{even}}^{2\nu+1} \sum_{N=1}^{\ell > 0} t^{2N-1} L_{(2N-1)i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + \\
& + s_0 \frac{t^{2\ell+1}}{2\ell+1} + \sum_{N=0}^{\ell-1 \geq 0} L_{(2N)c} Q^c \frac{t^{2N+1}}{2N+1}
\end{aligned} \quad (80)$$

where $\ell \geq 0$, $L_{(2N)i_1 \dots i_{2\nu+1}}(q)$ with $N = 0, 1, \dots, \ell$ are $(2\nu + 1)$ th-order generalized KTs given by the relation

$$L_{(2N)i_1 \dots i_{2\nu+1}} = -\frac{1}{2N} L_{(2N-1)(i_1 \dots i_{2\nu} | i_{2\nu+1})}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (81)$$

and the involved quantities satisfy the conditions:

$$L_{(2\ell)a} Q^a = s_0 \quad (82)$$

$$\left(L_{(2N-2)c} Q^c \right)_{,i_1} = 2(2N-1) L_{(2N-1)i_1 i_2} Q^{i_2} - 2N(2N-1) L_{(2N)i_1}, \quad N = 1, 2, \dots, \ell, \quad \ell > 0 \quad (83)$$

$$\begin{aligned}
L_{(2N)(i_1 \dots i_{r-1} | i_r)} &= (r+1) L_{(2N)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (2N+1) L_{(2N+1)i_1 \dots i_r} \quad (N < \ell), \\
&N = 0, 1, \dots, \ell, \quad r = 2, 4, 6, \dots, 2\nu
\end{aligned} \quad (84)$$

$$\begin{aligned}
L_{(2N-1)(i_1 \dots i_{r-1} | i_r)} &= (r+1) L_{(2N-1)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - 2N L_{(2N)i_1 \dots i_r}, \\
&N = 1, 2, \dots, \ell, \quad r = 3, 5, 7, \dots, 2\nu-1, \quad \ell > 0, \quad \nu > 1.
\end{aligned} \quad (85)$$

Therefore, concerning the higher-order FIs of a general autonomous dynamical system, we have the following general result.

Proposition 7. The autonomous (non-Riemannian in general) dynamical system (1) admits three independent m th-order FIs—autonomous or time-dependent—with $m > 0$, which are given by the following formulae:

- i. For $m = 1$, we have the LFIs (24), (26) and (29).
- ii. For $m = 2\nu$ with $\nu > 0$, we have the FIs (38), (62) and (68).
- iii. For $m = 2\nu + 1$ with $\nu > 0$, we have the FIs (38), (73) and (80).

In the following sections, we give applications of the above general results.

7. Application 1: A Family of 2d Non-Riemannian Autonomous Dynamical Systems

We consider two-dimensional (2d) autonomous dynamical systems of the form:

$$\ddot{x} = -Q^1(x, y) - \Gamma_{11}^1(x, y) \dot{x}^2 \quad (86)$$

$$\ddot{y} = -Q^2(x, y) - \Gamma_{22}^2(x, y) \dot{y}^2 \quad (87)$$

where $q^a = (x, y)$ are the generalized coordinates, Q^1, Q^2 are the components of the generalized forces, and $\Gamma_{11}^1, \Gamma_{22}^2$ are the non-zero symmetric connection coefficients of the system (all the other connection coefficients vanish).

We will find conditions for which the dynamical system (86)–(87) is non-Riemannian. In that case, a kinetic metric (i.e., a regular Lagrangian or a kinetic energy) cannot be defined, and standard methods (e.g., Noether's theorem) for the determination of FIs cannot be applied. Therefore, Theorem 1 is the only systematic method we have in order to determine the FIs of the system (86)–(87).

7.1. Conditions for a Non-Riemannian Connection

It is well-known that the symmetric connection coefficients Γ_{bc}^a define a Riemannian connection iff there exists a (kinetic) metric γ_{ab} with zero metricity; that is, $\gamma_{ab|c} = 0$. Then,

$$\Gamma_{bc}^a = \frac{1}{2} \gamma^{ad} (\gamma_{bd,c} + \gamma_{cd,b} - \gamma_{bc,d}).$$

Question: In which cases are the dynamical system (86)–(87) defined over a Riemannian configuration space, or equivalently, when is the associated symmetric connection Riemannian?

In order to answer this question, we assume that there exists a kinetic metric $\gamma_{ab}(x, y)$ such that

$$\gamma_{ab|c} = 0 \implies \gamma_{ab,c} - \gamma_{db} \Gamma_{ac}^d - \gamma_{ad} \Gamma_{bc}^d = 0. \quad (88)$$

Setting $\Gamma_{11}^1 \equiv f_1(x, y)$ and $\Gamma_{22}^2 \equiv f_2(x, y)$, condition (88) gives the following system of PDEs:

$$\begin{cases} \gamma_{11,x} = 2\gamma_{11}f_1, & \gamma_{12,x} = \gamma_{12}f_1, & \gamma_{22,x} = 0 \implies \gamma_{22} = \gamma_{22}(y) \\ \gamma_{22,y} = 2\gamma_{22}f_2, & \gamma_{12,y} = \gamma_{12}f_2, & \gamma_{11,y} = 0 \implies \gamma_{11} = \gamma_{11}(x) \end{cases} \implies$$

$$\frac{d\gamma_{11}}{dx} = 2\gamma_{11}(x)f_1 \quad (89)$$

$$\frac{d\gamma_{22}}{dy} = 2\gamma_{22}(y)f_2 \quad (90)$$

$$\gamma_{12,x} = \gamma_{12}(x, y)f_1 \quad (91)$$

$$\gamma_{12,y} = \gamma_{12}(x, y)f_2. \quad (92)$$

In order to have a well-defined metric γ_{ab} (i.e., the inverse γ^{ab} exists), it must also hold that

$$\det[\gamma_{ab}] \neq 0 \implies \gamma_{11}\gamma_{22} - \gamma_{12}^2 \neq 0. \quad (93)$$

We consider the following cases:

(1) Case $\gamma_{12} \neq 0$.

1.1. Subcase $\gamma_{11} = \gamma_{22} = 0$.

Conditions (89), (90), and (93) are satisfied identically.

The remaining conditions (91) and (92) give the Riemannian connection coefficients:

$$f_1 = \frac{F_{,x}}{F}, \quad f_2 = \frac{F_{,y}}{F}$$

where $\gamma_{12} \equiv F(x, y)$.

The associated kinetic metric is $\gamma_{ab} = F(x, y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The constrained geodesics of this metric has been discussed for various cases of the function $F(x, y)$ in [11], and more recently, in [12].

1.2. Subcase $\gamma_{11} \equiv f(x) \neq 0$ and $\gamma_{22} = 0$.

Conditions (90) and (93) are satisfied identically.

Condition (89) implies that $f_1 = \frac{f_{,x}}{2f}$. By replacing f_1 in (91), we find that $\gamma_{12} = h(y)\sqrt{f(x)}$, which when substituted into the remaining condition (92) gives $f_2 = \frac{h_{,y}}{h}$.

The associated kinetic metric is $\gamma_{ab} = \begin{pmatrix} f(x) & h(y)\sqrt{f} \\ h(y)\sqrt{f} & 0 \end{pmatrix}$.

1.3. Subcase $\gamma_{22} \equiv h(y) \neq 0$ and $\gamma_{11} = 0$.

Conditions (89) and (93) are satisfied identically.

Condition (90) implies that $f_2 = \frac{h_{,y}}{2h}$. By replacing f_2 in (92), we find that $\gamma_{12} = f(x)\sqrt{h(y)}$, which when replaced with the remaining condition (91) gives $f_1 = \frac{f_{,x}}{f}$.

The associated kinetic metric is $\gamma_{ab} = \begin{pmatrix} 0 & f(x)\sqrt{h} \\ f(x)\sqrt{h} & h(y) \end{pmatrix}$.

1.4. Subcase $\gamma_{11} \equiv f(x)$, $\gamma_{22} \equiv h(y)$, and $f(x)h(y) \neq 0$.

Conditions (89) and (90) imply that $f_1 = \frac{f_x}{2f}$ and $f_2 = \frac{h_y}{2h}$, respectively. By replacing f_1, f_2 in the remaining conditions (91) and (92), we found that $\gamma_{12} = c_0 \sqrt{f(x)h(y)}$, where c_0 is an arbitrary non-zero constant.

The associated kinetic metric is $\gamma_{ab} = \begin{pmatrix} f(x) & c_0 \sqrt{fh} \\ c_0 \sqrt{fh} & h(y) \end{pmatrix}$.

From the condition (93), we found that $c_0 \neq \pm 1$.

(2) Case $\gamma_{12} = 0$.

Condition (93) implies that $\gamma_{11}\gamma_{22} \neq 0$.

We set $\gamma_{11} \equiv f(x)$ and $\gamma_{22} \equiv h(y)$.

Conditions (91) and (92) are satisfied identically.

The remaining conditions (89) and (90) give $f_1 = \frac{f_x}{2f}$ and $f_2 = \frac{h_y}{2h}$, respectively.

The associated kinetic metric is $\gamma_{ab} = \begin{pmatrix} f(x) & 0 \\ 0 & h(y) \end{pmatrix}$.

We collected the above results in the following Proposition.

Proposition 8. Autonomous dynamical systems of the form (86)–(87) admit a Riemannian geometry (i.e., the quantities Γ_{bc}^a are Riemannian connection coefficients defined by a kinetic metric γ_{ab}) in the following cases:

(1) For $\Gamma_{11}^1 = \frac{F_x}{F}$ and $\Gamma_{22}^2 = \frac{F_y}{F}$, where $F(x, y)$ is a non-zero arbitrary smooth function. The associated kinetic metric is $\gamma_{ab} = F(x, y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(2) For $\Gamma_{11}^1 = \frac{f_x}{2f}$ and $\Gamma_{22}^2 = \frac{h_y}{h}$, where $f(x)$ and $h(y)$ are non-zero arbitrary smooth functions. Then, $\gamma_{ab} = \begin{pmatrix} f(x) & h(y)\sqrt{f} \\ h(y)\sqrt{f} & 0 \end{pmatrix}$.

(3) For $\Gamma_{11}^1 = \frac{f_x}{f}$ and $\Gamma_{22}^2 = \frac{h_y}{2h}$. Then, $\gamma_{ab} = \begin{pmatrix} 0 & f(x)\sqrt{h} \\ f(x)\sqrt{h} & h(y) \end{pmatrix}$.

(4) For $\Gamma_{11}^1 = \frac{f_x}{2f}$ and $\Gamma_{22}^2 = \frac{h_y}{2h}$. Then, $\gamma_{ab} = \begin{pmatrix} f(x) & c_0 \sqrt{fh} \\ c_0 \sqrt{fh} & h(y) \end{pmatrix}$ where the (possibly zero) constant $c_0 \neq \pm 1$.

From Proposition 8, we found the following proposition.

Proposition 9. The dynamical system (86)–(87) is non-Riemannian if the non-zero symmetric connection coefficients $\Gamma_{11}^1(x, y)$ and $\Gamma_{22}^2(x, y)$ satisfy the condition

$$\Gamma_{11,y}^1 \neq \Gamma_{22,x}^2. \quad (94)$$

Proof. - Case (1) of Proposition 8 implies that $\Gamma_{11}^1 = (\ln F)_{,x}$ and $\Gamma_{22}^2 = (\ln F)_{,y}$. Taking the integrability condition $(\ln F)_{,xy} = (\ln F)_{,yx}$, we found $\Gamma_{11,y}^1 = \Gamma_{22,x}^2$. Therefore, for a non-Riemannian connection, the condition (94) is required.

- For the remaining cases (2)–(4) of Proposition 8, we observe that $\Gamma_{11}^1(x)$ and $\Gamma_{22}^2(y)$. This implies that $\Gamma_{11,y}^1 = \Gamma_{22,x}^2 = 0$, which is a subcase of the condition $\Gamma_{11,y}^1 = \Gamma_{22,x}^2$. Therefore, for a non-Riemannian connection, we find again the condition (94). \square

7.2. LFI for the Non-Riemannian Dynamical System (86)–(87)

We assume that the connection of the autonomous dynamical system (86)–(87) is non-Riemannian; therefore, the condition (94) applies.

By applying Theorem 1 for $m = 1$ (LFIs) and $n = 0$ (zero degree of time-dependence), we found for the considered dynamical system the autonomous LFI

$$I = L_1(x, y)\dot{x} + L_2(x, y)\dot{y} + s_0 t \quad (95)$$

where $L_a(x, y)$ is a generalized KV of the connection $\Gamma_{bc}^a(x, y)$ defined by the dynamical Equations (86) and (87) and s_0 is a constant defined by the condition

$$L_1 Q^1 + L_2 Q^2 = s_0. \quad (96)$$

The generalized KV condition $L_{(a|b)} = 0$ implies the following system of PDEs:

$$L_{1,x} - L_1 \Gamma_{11}^1 = 0 \quad (97)$$

$$L_{2,y} - L_2 \Gamma_{22}^2 = 0 \quad (98)$$

$$L_{1,y} + L_{2,x} = 0. \quad (99)$$

We have an overdetermined system of four PDEs (96)–(99), which has six unknown functions, $L_a(x, y)$, $Q^a(x, y)$, $\Gamma_{11}^1(x, y)$, $\Gamma_{22}^2(x, y)$, and one free parameter s_0 . Therefore, in order to solve it, we should fix either the dynamics of the system (i.e., the generalized forces Q^a) or the non-Riemannian geometry of the system (i.e., the non-Riemannian connection coefficients; it holds that $\Gamma_{11,y}^1 \neq \Gamma_{22,x}^2$).

Two Linearly Coupled Harmonic Oscillators with a Non-Riemannian Quadratic Damping Term

As a first application, we fixed the generalized forces Q^a . We assume that

$$Q^a = \begin{pmatrix} kx - py \\ ky + px \end{pmatrix} \quad (100)$$

where k, p are arbitrary non-zero constants.

For the choice (100), the dynamical system of (86) and (87) becomes:

$$\ddot{x} = -kx + py - \Gamma_{11}^1(x, y)\dot{x}^2 \quad (101)$$

$$\ddot{y} = -ky - px - \Gamma_{22}^2(x, y)\dot{y}^2 \quad (102)$$

which describes two linearly coupled harmonic oscillators with a non-Riemannian (i.e., $\Gamma_{11,y}^1 \neq \Gamma_{22,x}^2$) quadratic damping term.

Replacing Q^a from (100), Equation (96) gives

$$L_1 = -\frac{ky + px}{kx - py} L_2 + \frac{s_0}{kx - py} = \frac{1}{py - kx} [(ky + px)L_2 - s_0] \quad (103)$$

which when substituted into (99) implies that

$$L_2 = (py - kx)F_1(p(y^2 - x^2) - 2kxy) - \frac{s_0 x}{p(y^2 - x^2) - 2kxy} \quad (104)$$

where $F_1(p(y^2 - x^2) - 2kxy)$ is an arbitrary smooth function of its argument.

By replacing (104) in (103), we find

$$L_1 = (ky + px)F_1(p(y^2 - x^2) - 2kxy) + \frac{s_0 x(ky + px)}{(kx - py)[p(y^2 - x^2) - 2kxy]} + \frac{s_0}{kx - py}. \quad (105)$$

Using (104) and (105), the remaining PDEs (97) and (98) determine the connection coefficients as follows:

$$\Gamma_{11}^1 = \frac{L_{1,x}}{L_1}, \quad \Gamma_{22}^2 = \frac{L_{2,y}}{L_2}. \quad (106)$$

In order to have a non-Riemannian connection, the condition (94) must be satisfied. Therefore,

$$\Gamma_{11,y}^1 \neq \Gamma_{22,x}^2 \implies (\ln |L_1|)_{,xy} \neq (\ln |L_2|)_{,yx} \implies \left(\ln \left| \frac{L_1}{L_2} \right| \right)_{,xy} \neq 0 \quad (107)$$

which leads to restrictions among the parameters k, p, s_0 and the function F_1 .

We note that the vector L_a whose components are given by the relations (104) and (105) is a generalized KV of the connection (106).

Finally, by replacing (106) in the dynamical Equations (101) and (102), we find the family of dynamical systems:

$$\ddot{x} = -kx + py - \frac{L_{1,x}}{L_1} \dot{x}^2 \quad (108)$$

$$\ddot{y} = -ky - px - \frac{L_{2,y}}{L_2} \dot{y}^2 \quad (109)$$

parameterized by the constant s_0 and the function F_1 , which admits the time-dependent LFI (95)

In order to get a specific example, we fix the parameters s_0 and F_1 as follows:

- Case $s_0 = 0$ and $F_1 = 1$.

Then, Equations (104) and (105) give the generalized KV

$$L_a = \begin{pmatrix} ky + px \\ py - kx \end{pmatrix} \quad (110)$$

which when substituted into the relations (106) determines the connection coefficients:

$$\Gamma_{11}^1 = \frac{p}{ky + px}, \quad \Gamma_{22}^2 = \frac{p}{py - kx}. \quad (111)$$

In order to have a non-Riemannian connection, the quantities (111) must satisfy the condition (107). We compute:

$$\Gamma_{11,y}^1 = -\frac{kp}{(ky + px)^2} \neq \Gamma_{22,x}^2 = \frac{kp}{(py - kx)^2} \implies$$

$$(py - kx)^2 + (ky + px)^2 \neq 0 \implies (k^2 + p^2)(x^2 + y^2) \neq 0 \implies k \neq \pm ip.$$

Therefore, the connection (111) is non-Riemannian only when $k \neq \pm ip$.

Using (110), the dynamical system of (108) and (109) becomes:

$$\ddot{x} = -kx + py - \frac{p}{ky + px} \dot{x}^2 \quad (112)$$

$$\ddot{y} = -ky - px - \frac{p}{py - kx} \dot{y}^2 \quad (113)$$

and the associated autonomous LFI (95) is

$$I_1 = (ky + px)\dot{x} + (py - kx)\dot{y}. \quad (114)$$

Remark: In the case that $k = \pm ip$, the connection coefficients (111) reduce to the Riemannian connection:

$$\Gamma_{11\pm}^1 = \frac{1}{x \pm iy}, \quad \Gamma_{22\pm}^2 = \frac{1}{y \mp ix}$$

which, according to Proposition 8, is associated with a metric of the form (1). Indeed, for an arbitrary function $F(x, y)$, we have:

$$\begin{cases} (\ln F_{\pm})_{,x} = \frac{1}{x \pm iy} \\ (\ln F_{\pm})_{,y} = \frac{1}{y \mp ix} \end{cases} \implies F_{+} = \frac{x^2 + y^2}{y + ix}, \quad F_{-} = y + ix$$

and the corresponding metric is $\gamma_{ab} = F_{\pm} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

8. Application 2: The QFIs of a Non-Riemannian Dynamical System

Consider the dynamical system:

$$\ddot{u} = -\frac{8\beta}{u^3} (u\dot{u}\dot{w} - w\dot{u}^2) - \frac{1}{u^2} \quad (115)$$

$$\ddot{w} = -\frac{4\beta}{u^3} (u\dot{w}^2 - 4w\dot{u}\dot{w}) + \frac{2w}{u^3} \quad (116)$$

where β is an arbitrary real constant. This system is autonomous holonomic of the form (1) with variables

$$q^a = \begin{pmatrix} u \\ w \end{pmatrix}, \quad Q^a = \frac{1}{u^2} \begin{pmatrix} 1 \\ -\frac{2w}{u} \end{pmatrix}.$$

The symmetric connection coefficients are read from the dynamical equations and are:

$$\Gamma_{22}^1 = \Gamma_{11}^2 = 0, \quad \Gamma_{11}^1 = \Gamma_{12}^2 = -8\beta \frac{w}{u^3}, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = \frac{4\beta}{u^2}. \quad (117)$$

The non-zero components of the curvature tensor $R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{sc}\Gamma^s_{bd} - \Gamma^a_{sd}\Gamma^s_{bc}$ are:

$$R^1_{112} = R^2_{221} = -R^2_{212} = -R^1_{121} = -\frac{32b^2w}{u^5}, \quad R^2_{112} = -R^2_{121} = \frac{24bw}{u^4}.$$

Solving the generalized KT condition $C_{(ab|c)} = 0$, we found that the connection (117) admits only the second order generalized KT

$$C_{ab} = ke^{\frac{12\beta w}{u^2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (118)$$

where k is an arbitrary constant.

Solving the generalized killing vector (KV) condition $L_{(a|b)} = 0$, we found $L_a = 0$; therefore, generalized KVs do not exist.

Moreover, it can be shown that non-zero vectors B_a , which generate reducible generalized KTs of the form $B_{(a|b)}$, do not exist as well.

By applying Theorem 1 for $m = 2$, we find that the system admits only one QFI which is the

$$I = e^{\frac{12\beta w}{u^2}} \left(\dot{u}\dot{w} + \frac{1}{12\beta} \right). \quad (119)$$

To prove that the given system is integrable, one needs one more independent autonomous FI of higher order in involution.

9. Application 3: A New Superintegrable Potential Which Admits Time-Dependent QFIs

In [13], using the separability of the corresponding Hamilton–Jacobi equation in more than two coordinate systems, all minimally and maximally superintegrable potentials in the Euclidean space E^3 that admit autonomous QFIs are determined. We extend this result to the case where time-dependent FIs are considered.

Applying Theorem 1 for $m = 2$ (QFIs), $q^a = (x, y, z)$, $\Gamma^a_{bc} = 0$, and $Q^a = V^a$, where $V(x, y, z)$ denotes the potential, we found the new maximally superintegrable potential in E^3

$$V(x, y, z) = -\frac{\lambda^2}{2}R^2 + \frac{kx}{y^2R} + \frac{c_1}{y^2} - \frac{\lambda^2}{8}z^2 + \frac{c_2}{z^2} \quad (120)$$

where $\lambda \neq 0, k, c_1, c_2$ are arbitrary constants, and $R = \sqrt{x^2 + y^2}$.

The potential (120) admits the following independent (autonomous and time-dependent) QFIs:

$$I_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\lambda^2}{2}R^2 + \frac{kx}{y^2R} + \frac{c_1}{y^2} - \frac{\lambda^2}{8}z^2 + \frac{c_2}{z^2} \quad (121)$$

$$I_2 = \frac{1}{2}M_3^2 + \frac{(kR + c_1x)x}{y^2} \quad (122)$$

$$I_3 = \frac{1}{2}\dot{z}^2 - \frac{\lambda^2}{8}z^2 + \frac{c_2}{z^2} \quad (123)$$

$$I_4 = e^{\lambda t} \left[M_3(\dot{y} - \lambda y) + \frac{2c_1x}{y^2} + \frac{k(y^2 + 2x^2)}{y^2R} \right] \quad (124)$$

$$I_5 = e^{\lambda t} \left[\left(\dot{z} - \frac{\lambda}{2}z \right)^2 + \frac{2c_2}{z^2} \right]. \quad (125)$$

The QFI I_1 is the Hamiltonian of the system and the vector $M_i = (y\dot{z} - z\dot{y}, z\dot{x} - x\dot{z}, x\dot{y} - y\dot{x})$ with $i = 1, 2, 3$ is the angular momentum.

10. Application 4: A New Superintegrable Separable Potential Which Admits an Autonomous CFI

It is well-known that separable Newtonian potentials of the form $V(x, y) = F_1(x) + F_2(y)$ admit the QFIs $J_1 = \frac{1}{2}\dot{x}^2 + F_1(x)$ and $J_2 = \frac{1}{2}\dot{y}^2 + F_2(y)$, where F_1 and F_2 are arbitrary smooth functions of their arguments. The question is if there are functions F_1 and F_2 for which the corresponding potential $V(x, y)$ is superintegrable.

A partial answer to this problem has been given in [14] by considering autonomous CFIs as the third FI. One of the third order superintegrable potentials found in [14] is the

$$V(x, y) = c_1y^2 + F(x) \quad (126)$$

where c_1 is an arbitrary non-zero constant and $F(x)$ is an arbitrary smooth function satisfying the condition

$$k_2x^2 + 4k_1^2 + (9F - c_1x^2)(F - c_1x^2)^3 - 4k_1(F - c_1x^2)(3F + c_1x^2) = 0 \quad (127)$$

where k_1 and k_2 are arbitrary constants.

Using Theorem 1, we generalize the above result and determine a class of superintegrable potentials, which contains the potential (126) as a special case.

We applied Theorem 1 for $m = 3$ (CFIs), $q^a = (x, y)$, $\Gamma_{bc}^a = 0$ and $Q^a = V^{,a}$, where $V(x, y)$ denotes potentials of the form $V = F_1(x) + F_2(y)$. We found that the potential (126) is superintegrable due to the three independent autonomous FIs:

$$I_1 = \frac{1}{2}\dot{x}^2 + F(x) \quad (128)$$

$$I_2 = \frac{1}{2}\dot{y}^2 + c_1y^2 \quad (129)$$

$$I_3 = L\dot{x}^2 - (3yF - c_1x^2y + k_3y)\dot{x} + \frac{F'}{2c_1}(3F - c_1x^2 + k_3)y \quad (130)$$

where $L \equiv x\dot{y} - y\dot{x}$ is the angular momentum, $F' \equiv \frac{dF}{dx}$, and the function $F(x)$ satisfies the condition

$$0 = k_2x^2 + 4k_1^2 + (9F - c_1x^2)(F - c_1x^2)^3 - 4k_1(F - c_1x^2)(3F + c_1x^2) + 4k_3(3F - c_1x^2)(F - c_1x^2)^2 + 4k_3^2(F - c_1x^2)^2 - \frac{8k_1k_3}{3}(3F - c_1x^2) \quad (131)$$

where k_1, k_2, k_3 are arbitrary constants. We note that for $k_3 = 0$ condition (131) reduces to condition (127). Therefore, the superintegrable potential (C.6) of [14] is a subcase of (126).

11. Conclusions

We draw the following conclusions:

- (a) We have developed a direct systematic method to compute the m th-order FIs of the autonomous (in general non-Riemannian) dynamical systems (1) in terms of the 'symmetries' of the geometric objects (symmetric connection or kinetic metric, depending on the case) defined by the dynamical equations.
- (b) This method applies to non-Riemannian geometries with a symmetric connection. It has been shown that the m th-order FIs require the generalized KTs and KVs defined by the symmetric connection $\Gamma_{bc}^a(q)$. The case of a Riemannian connection is a special case, where the m th-order FIs can be related to a gauged weak generalized Noether symmetry by means of the inverse Noether theorem.
- (c) The system of PDEs (5)–(8) resulting from the condition $\frac{dI^{(m)}}{dt} = 0$ along the dynamical equations consists of two parts: a geometric part (Equations (5) and (6)) common to all systems which share the same connection, and a dynamical part (Equations (7) and (8)) which includes the generalized forces Q^a of the specific system.
- (d) We determined the condition which the connection coefficients must satisfy in order the 2d dynamical systems (86) and (87) to be non-Riemannian.

Obviously, Theorem 1 provides a new systematic way to determine the higher order FIs, autonomous and time-dependent, of autonomous (in general non-Riemannian) dynamical systems of the form (1).

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Appendix A

We assume that the totally symmetric tensor quantities

$$M_{i_1 \dots i_r}(t, q) = \sum_{N_r=0}^{n_r} L_{(N_r)i_1 \dots i_r}(q) t^{N_r}, \quad r = 1, 2, \dots, m, \quad m \geq 1 \quad (\text{A1})$$

where $L_{(N_r)i_1 \dots i_r}(q)$ and $N_r = 0, 1, \dots, n_r$, are arbitrary r -rank totally symmetric tensors and $n_r \geq 0$ is the degree of the polynomial associated with the r -rank tensor $M_{i_1 \dots i_r}(t, q)$. We note that the degrees n_r of the above polynomial expressions of t may be infinite.

We consider the following cases.

I. Case with n finite.

By substituting (A1) in the system of PDEs (5)–(8), we obtain the following system of polynomial equations in t :

$$0 = \sum_{N=0}^n L_{(N)(i_1 \dots i_m | i_{m+1})} t^N \quad (\text{A2})$$

$$0 = M_{,t} - \sum_{N=0}^n L_{(N)i_1} Q^{i_1} t^N \quad (\text{A3})$$

$$0 = M_{,i_1} + \sum_{N=1}^{n \geq 0} \left[N L_{(N)i_1} - 2 L_{(N-1)i_1 i_2} Q^{i_2} \right] t^{N-1} - 2 L_{(n)i_1 i_2} Q^{i_2} t^n, \quad m > 1 \quad (\text{A4})$$

$$0 = \sum_{N=0}^{n-1 \geq 0} \left[(N+1) L_{(N+1)i_1 \dots i_r} + L_{(N)(i_1 \dots i_{r-1} | i_r)} - (r+1) L_{(N)i_1 \dots i_{r+1}} Q^{i_{r+1}} \right] t^N + \\ + \left[L_{(n)(i_1 \dots i_{r-1} | i_r)} - (r+1) L_{(n)i_1 \dots i_{r+1}} Q^{i_{r+1}} \right] t^n, \quad r = 2, 3, \dots, m-1, \quad m > 2 \quad (\text{A5})$$

$$0 = M_{,i_1} + \sum_{N=1}^{n \geq 0} N L_{(N)i_1} t^{N-1}, \quad m = 1 \quad (\text{A6})$$

$$0 = \sum_{N=0}^{n-1 \geq 0} \left[(N+1) L_{(N+1)i_1 \dots i_m} + L_{(N)(i_1 \dots i_{m-1} | i_m)} \right] t^N + L_{(n)(i_1 \dots i_{m-1} | i_m)} t^n, \quad m > 1 \quad (\text{A7})$$

where—without loss of generality—the polynomial expressions (A1) of t are assumed to be of the same degree; that is, $n = n_r$ for all values of r . All the results with $n \neq n_r$ are derived as subcases from the case $n = n_r$. We note also that: Equation (A2) is derived from (5), (A3) from (8), (A4) from (7) for $r = 1$, (A5) from (7) for $r = 2, 3, \dots, m-1$, (A6) from (6) for $m = 1$, and (A7) from (6) for $m > 1$.

Equation (A2) implies that the quantities $L_{(N)i_1 \dots i_m}$ with $N = 0, 1, \dots, n$ are m th-order generalized KTs. For $m = 1$, $L_{(N)i_1}$ are generalized KVs.

By integrating Equation (A3), we find that

$$M = \sum_{N=0}^n L_{(N)i_1} Q^{i_1} \frac{t^{N+1}}{N+1} + G(q) \quad (\text{A8})$$

where $G(q)$ is an arbitrary smooth function. We note that the integrability conditions of the scalar $M(t, q)$ have been replaced by the integrability conditions $G_{,[ab]} = 0$ of the function $G(q)$.

By replacing M from (A8), Equation (A4) gives:

$$0 = G_{,i_1} + L_{(1)i_1} (n > 0) - 2 L_{(0)i_1 i_2} Q^{i_2} + \\ + \sum_{N=1}^{n-1 \geq 1} \frac{1}{N} \left[\left(L_{(N-1)c} Q^c \right)_{,i_1} + N(N+1) L_{(N+1)i_1} - 2 N L_{(N)i_1 i_2} Q^{i_2} \right] t^N + \\ + \frac{1}{n} \underbrace{\left[\left(L_{(n-1)c} Q^c \right)_{,i_1} - 2 n L_{(n)i_1 i_2} Q^{i_2} \right]}_{n > 0} t^n + \left(L_{(n)c} Q^c \right)_{,i_1} \frac{t^{n+1}}{n+1}, \quad m > 1 \implies$$

$$G_{,i_1} = 2 L_{(0)i_1 i_2} Q^{i_2} - L_{(1)i_1} (n > 0), \quad m > 1 \quad (\text{A9})$$

$$\left(L_{(k-1)c} Q^c \right)_{,i_1} = 2 k L_{(k)i_1 i_2} Q^{i_2} - k(k+1) L_{(k+1)i_1}, \quad k = 1, 2, \dots, n-1, \quad n > 1, \quad m > 1 \quad (\text{A10})$$

$$\left(L_{(n-1)c} Q^c \right)_{,i_1} = 2 n L_{(n)i_1 i_2} Q^{i_2}, \quad n > 0, \quad m > 1 \quad (\text{A11})$$

$$L_{(n)i_1} Q^{i_1} = s_0, \quad m > 1 \quad (\text{A12})$$

where s_0 is an arbitrary constant. The notation $L_{(1)i_1} (n > 0)$ indicates that the vector $L_{(1)i_1}$ exists only when the degree of the polynomial $n > 0$ —that is, when $n = 0$, the vector $L_{(1)i_1}$ vanishes.

By replacing (A8) in (A6), we find the following conditions:

$$G_{,i_1} = -L_{(1)i_1} (n > 0), \quad m = 1 \quad (\text{A13})$$

$$\left(L_{(k-1)c} Q^c \right)_{,i_1} = -k(k+1) L_{(k+1)i_1}, \quad k = 1, 2, \dots, n-1, \quad n > 1, \quad m = 1 \quad (\text{A14})$$

$$L_{(n-1)i_1} Q^{i_1} = s_1, \quad n > 0, \quad m = 1 \quad (\text{A15})$$

$$L_{(n)i_1} Q^{i_1} = s_0, \quad m = 1 \quad (\text{A16})$$

where s_1 is an arbitrary constant.

We observe that conditions (A13)–(A16) are emerged from conditions (A9)–(A12) if we rewrite the latter in the following compact form:

$$G_{,i_1} = 2L_{(0)i_1 i_2} (m > 1) Q^{i_2} - L_{(1)i_1} (n > 0) \quad (\text{A17})$$

$$\left(L_{(k-1)c} Q^c \right)_{,i_1} = 2kL_{(k)i_1 i_2} (m > 1) Q^{i_2} - k(k+1) L_{(k+1)i_1}, \quad k = 1, 2, \dots, n-1, \quad n > 1 \quad (\text{A18})$$

$$\left(L_{(n-1)c} Q^c \right)_{,i_1} = 2nL_{(n)i_1 i_2} (m > 1) Q^{i_2}, \quad n > 0 \quad (\text{A19})$$

$$L_{(n)i_1} Q^{i_1} = s_0. \quad (\text{A20})$$

The notation $L_{(0)i_1 i_2} (m > 1)$ indicates that the quantities $L_{(0)i_1 i_2}$ exist only when the degree of the FI $m > 1$; that is, when $m = 1$, the quantities $L_{(0)i_1 i_2}$ vanish.

Conditions (A18) and (A19) are written compactly as follows:

$$\begin{aligned} \left(L_{(k-1)c} Q^c \right)_{,i_1} &= 2kL_{(k)i_1 i_2} (m > 1) Q^{i_2} - k(k+1) L_{(k+1)i_1} (k < n), \\ &k = 1, 2, \dots, n, \quad n > 0. \end{aligned} \quad (\text{A21})$$

The notation $L_{(k+1)i_1} (k < n)$ indicates that the vector $L_{(k+1)i_1}$ exists only when $k < n$; that is, if $k \geq n$, the vector $L_{(k+1)i_1}$ vanishes.

Equation (A5) implies that:

$$\begin{aligned} L_{(k)(i_1 \dots i_{r-1} | i_r)} &= (r+1) L_{(k)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (k+1) L_{(k+1)i_1 \dots i_r}, \\ &k = 0, 1, \dots, n-1, \quad r = 2, 3, \dots, m-1, \quad n > 0, \quad m > 2 \end{aligned} \quad (\text{A22})$$

$$L_{(n)(i_1 \dots i_{r-1} | i_r)} = (r+1) L_{(n)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}}, \quad r = 2, 3, \dots, m-1, \quad m > 2. \quad (\text{A23})$$

Conditions (A22) and (A23) are written compactly as follows:

$$\begin{aligned} L_{(k)(i_1 \dots i_{r-1} | i_r)} &= (r+1) L_{(k)i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - (k+1) L_{(k+1)i_1 \dots i_r} (k < n), \\ &k = 0, 1, \dots, n, \quad r = 2, 3, \dots, m-1, \quad m > 2. \end{aligned} \quad (\text{A24})$$

The notation $L_{(k+1)i_1 \dots i_r} (k < n)$ indicates that the quantities $L_{(k+1)i_1 \dots i_r}$ exist only when $k < n$; that is, if $k \geq n$, the quantities $L_{(k+1)i_1 \dots i_r}$ vanish.

Equation (A7) implies that the quantities $L_{(n)i_1 \dots i_{m-1}}$ with $m > 1$ are the components of an $(m-1)$ th-order generalized KT and the m th-order generalized KTs:

$$L_{(k)i_1 \dots i_m} = -\frac{1}{k} L_{(k-1)(i_1 \dots i_{m-1} | i_m)}, \quad k = 1, 2, \dots, n, \quad n > 0, \quad m > 1. \quad (\text{A25})$$

The m th-order FI (2) is

$$I_n^{(m)} = \sum_{r=1}^m \left(\sum_{N=0}^n L_{(N)i_1 \dots i_r} t^N \right) \dot{q}^{i_1} \dots \dot{q}^{i_r} + s_0 \frac{t^{n+1}}{n+1} + \sum_{N=1}^{n>0} L_{(N-1)c} Q^c \frac{t^N}{N} + G(q) \quad (\text{A26})$$

where $m \geq 1$, $n \geq 0$, $L_{(N)i_1 \dots i_m}(q)$ with $N = 0, 1, \dots, n$ are m th-order generalized KT satisfying the condition (A25); $L_{(n)i_1 \dots i_{m-1}}(q)$ with $m > 1$ is an $(m-1)$ th-order generalized KT; and the constant s_0 is given by (A20). The function $G(q)$ and the totally symmetric tensors $L_{(N)i_1 \dots i_r}(q)$ satisfy the conditions (A17), (A21), and (A24).

The notation $I_n^{(m)}$ means the m th-order FI (upper index) with time-dependence n (lower index). For example, the $I_n^{(2)}$ is a QFI whose coefficients are expressed as polynomials of t of degree fixed by n .

II. Case with n infinite.

The polynomial expressions (A1) as $n_r = n \rightarrow \infty$ turns into the infinite sum (series):

$$M_{i_1 \dots i_r}(t, q) = \sum_{N=0}^{\infty} L_{(N)i_1 \dots i_r}(q) t^N, \quad r = 1, 2, \dots, m, \quad m \geq 1. \quad (\text{A27})$$

It is found that new results—different from those found with n finite—are derived in the case that

$$L_{(N)i_1 \dots i_r}(q) = \frac{\lambda_r^N}{N!} L_{i_1 \dots i_r}(q) \quad (\text{A28})$$

where λ_r are arbitrary non-zero constants and $L_{i_1 \dots i_r}(q)$ are r -rank totally symmetric tensors.

Replacing (A28) in (A27), we found

$$M_{i_1 \dots i_r}(t, q) = L_{i_1 \dots i_r}(q) \sum_{N=0}^{\infty} \frac{(\lambda_r t)^N}{N!} = e^{\lambda_r t} L_{i_1 \dots i_r}(q). \quad (\text{A29})$$

By substituting (A29) into the system of PDEs (5)–(8), we obtain the following system of equations:

$$0 = L_{(i_1 \dots i_m | i_{m+1})} \quad (\text{A30})$$

$$0 = M_{,t} - e^{\lambda t} L_c Q^c \quad (\text{A31})$$

$$0 = M_{,i_1} + e^{\lambda t} (\lambda L_{i_1} - 2L_{i_1 i_2} Q^{i_2}), \quad m > 1 \quad (\text{A32})$$

$$0 = L_{(i_1 \dots i_{r-1} | i_r)} + \lambda L_{i_1 \dots i_r} - (r+1) L_{i_1 \dots i_r i_{r+1}} Q^{i_{r+1}}, \quad r = 2, 3, \dots, m-1, \quad m > 2 \quad (\text{A33})$$

$$0 = M_{,i_1} + \lambda e^{\lambda t} L_{i_1}, \quad m = 1 \quad (\text{A34})$$

$$0 = \lambda L_{i_1 \dots i_m} + L_{(i_1 \dots i_{m-1} | i_m)}, \quad m > 1 \quad (\text{A35})$$

where—without loss of generality—all the non-zero constants λ_r are fixed to the same non-zero constant λ . The m th-order FI produced from this assumption contains as subcases all the FIs associated with constants λ_r which are not all the same.

Equation (A30) implies that $L_{i_1 \dots i_m}$ is an m th-order generalized KT.

By integrating Equation (A31), we find

$$M = \frac{e^{\lambda t}}{\lambda} L_c Q^c + G(q) \quad (\text{A36})$$

where $G(q)$ is an arbitrary smooth function.

By replacing (A36) in Equations (A32) and (A34), we find that $G(q) = \text{const} \equiv 0$ and the condition:

$$(L_c Q^c)_{,i_1} = 2\lambda L_{i_1 i_2} (m > 1) Q^{i_2} - \lambda^2 L_{i_1}. \quad (\text{A37})$$

Equation (A33) gives the condition

$$L_{(i_1 \dots i_{r-1} | i_r)} = (r+1) L_{i_1 \dots i_r i_{r+1}} Q^{i_{r+1}} - \lambda L_{i_1 \dots i_r}, \quad r = 2, 3, \dots, m-1, \quad m > 2 \quad (\text{A38})$$

and the remaining condition (A35) implies that

$$L_{i_1 \dots i_m} = -\frac{1}{\lambda} L_{(i_1 \dots i_{m-1} | i_m)}, \quad m > 1. \quad (\text{A39})$$

The associated m th-order FI (2) is

$$I_e^{(m)} = \frac{e^{\lambda t}}{\lambda} \left(\lambda \sum_{r=1}^m L_{i_1 \dots i_r} \dot{q}^{i_1} \dots \dot{q}^{i_r} + L_c Q^c \right) \quad (\text{A40})$$

where $\lambda \neq 0$, $L_{i_1 \dots i_m}(q)$ is an m th-order generalized KT satisfying the condition (A39), and the remaining totally symmetric tensors $L_{i_1 \dots i_r}(q)$ with $r = 1, 2, \dots, m-1$ and $m > 1$ satisfy the conditions (A37) and (A38).

The notation $I_e^{(m)}$ indicates the m th-order FI (upper index) with exponential time-dependence (lower index).

The above completes the proof of Theorem 1.

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