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# Bi-Condition of Existence for a Compatible Directed Order on an Arbitrary Field †

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† The paper has been finished in 2012, the results and the methods have been reported in Germany, Italy, Korea and China.

**Abstract:** One proves that a field carries a compatible directed order if and only if it has characteristic 0 and is real or has a transcendence degree of at least 1 over the field of rational numbers.

**Keywords:** Archimedean partial order; strongly Archimedean partial order; strong unit; partially ordered field; totally ordered field; totally ordered ring; real field; algebraic extension; Dedekind cut

**MSC:** 12J15; 06F25; 06F20

## 1. Introduction

Ordered fields are a diverse class of structures. The properties of ordered fields depend very much on the class of orders that are considered. Totally ordered fields and partially ordered fields with positive squares are well understood. Specifically, Yang [1] pointed out the equivalence between the division closed property (that is, $a > 0$ implies $\frac{1}{a} > 0$) and the positive square property (that is, $a^2 \geq 0$ for all $a$) for a lattice-ordered field and a skew field and thus generalized the original result of Bourbaki for positive square lattice-ordered fields to the skew case. Vaida [2] generalized the main theorem of [1] to more general situations.

Birkhoff and Pierce asked whether the field of complex numbers carries some lattice order such that it is a lattice-ordered field [3]. It is known that lattice-ordered algebraic number fields have to be real according to Schwartz [4,5]. However, despite considerable effort, the original question of Birkhoff and Pierce is still open. The same question can be asked about other classes of partial orders, for example, directed partial orders initiated by Yang in [6]. In this case, the answer proved to be more accessible: Given a field $F$ with a non-Archimedean total order and imaginary unit $i$ with $i^2 = -1$, Yang [6] (Corollary 2.3) first constructed a directed partial order on field $F(i)$. Then, Rump and Yang [7] characterized the construction of a directed order in [6] by means of the segment. Continuing in this direction, Schwartz and Yang [8] (Corollary 4.2) showed that almost all fields of characteristic 0, including the complex field, can be made into a directed partially ordered field. It is the only remained open question asked in [8]: whether non-real algebraic number fields can carry a directed partial order?

We show in this paper that the answer is negative (Corollary 2), which completely settles the question of which fields can be made into directed partially ordered fields (Theorem 4).

**Some notation and terminology**. A po-group (that is, a not-necessarily-abelian group $(G, +, 0)$ that is also a partially ordered set $(G, \leq)$ with the property that $a \leq b$ implies $c + a \leq c + b$ and $a + d \leq b + d$ for all $a, b, c, d \in G$; it is a lattice-ordered group if the partial order is a lattice, i.e., there are the least upper bound $a \vee b$ and the greatest lower bound $a \wedge b$ for all $a, b \in G$), a po-ring (that is, a ring $(R, +, \cdot)$ that is also a partially ordered set $(R, \leq)$ such that $(R, +, \leq, 0)$ is a partially ordered abelian group with the property that

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$a \leq b$ implies $ca \leq cb$ and $ad \leq bd$ for all $a, b \in R$ and $0 \leq c, d \in R$) or a po-field (that is, $(F, +, \cdot, \leq)$ is a partially ordered ring, and $F$ is a field; it is a directed partially ordered field if the order is a directed order, i.e., there are upper bounds and lower bounds for all $a, b \in F$) is a partially ordered group, ring or field.

Let $A$ be a po-group or po-ring. Then, $A^{\geq} = \{a \in A \mid a \geq 0\}$ is the positive cone, and $A^{>} = A^{\geq} \setminus \{0\}$.

If $A$ is a lattice-ordered group and $a \in A$, then symmetry notations are defined as $a^{+} = a \vee 0$ and $a^{-} = (-a) \vee 0$.

## 2. Archimedean Po-Fields

Let $L$ be a directed po-field with positive cone $L^{\geq}$, i.e., $L^{\geq} \cap -L^{\geq} = \{0\}$, $L = L^{\geq} - L^{\geq}$, and the symmetry property with respect to addition and multiplication. i.e., $L^{\geq} + L^{\geq} \subseteq L^{\geq}$ and $L^{\geq} \cdot L^{\geq} \subseteq L^{\geq}$. We do not assume that $L^{\geq}$ is closed under division, contains all squares or contains 1. One can always extend the positive cone by forming the *quotient cone* $qL^{\geq} = \{\frac{a}{b} \mid a, b \in L^{\geq} \wedge b \neq 0\}$. It is easy to check that $qL^{\geq}$ is the positive cone of a division-closed partial order.

Frequently, we assume that $L$ is a po-algebra over an Archimedean totally ordered field $F$ (with positive cone $F^{\geq}$). Thus, $F$ is a subfield of $L$ that carries a total order, and $F^{\geq} \cdot L^{\geq} \subseteq L^{\geq}$. According to Hölder's theorem ([9] (p. 53), [10] (p. 35)), we may assume that $F$ is a subfield of $\mathbb{R}$. We emphasize that we do not assume $F^{\geq} = F \cap L^{\geq}$. However, $(L, qL^{\geq})$ is also a partially ordered $F$-algebra, and $F^{\geq} = F \cap qL^{\geq}$.

In general, a directed po-field needs not be a partially ordered $\mathbb{Q}$-algebra. For example, consider field $\mathbb{Q}$ with subset $P = \{x \in \mathbb{Q} \mid x > 1\}$. This is the positive cone of a directed partial order, but $(\mathbb{Q}, P)$ is not a partially ordered $\mathbb{Q}$-algebra. Given a directed po-field, set $\mathbb{Q}^{\geq} \cdot L^{\geq} = \{a \in L \mid \exists n \in \mathbb{N} : 1 \leq n \wedge n \cdot a \in L^{\geq}\}$ is a directed partial order. Clearly, this is the smallest partial order $P$ of $L$ such that $L^{\geq} \subseteq P$ and $(L, P)$ is a partially ordered $\mathbb{Q}$-algebra. Also note that $(L, qL^{\geq})$ is a partially ordered $\mathbb{Q}$-algebra.

We always assume that $L^{\geq} = \mathbb{Q}^{\geq} \cdot L^{\geq}$.

The following result can be used to reduce many questions about directed po-fields to countable fields.

**Proposition 1.** *Suppose that $L$ is a directed po-field.*

*(a) Suppose that $S \subseteq L$ is an at most countable subset. Then, there is a countable subfield $M \subseteq L$ such that partial order $M^{\geq} = M \cap L^{\geq}$ is directed and $S \subseteq M$.*

*(b) There is a directed* (for inclusion) *set $\mathcal{M}$ of countable subfields $M \subseteq L$ such that $M^{\geq} = M \cap L^{\geq}$ is a directed partial order and $L = \bigcup_{M \in \mathcal{M}} M$.*

**Proof.** (a). We produce field $M$ with recursive construction: We start with $M_0 = \mathbb{Q}(S)$. This is a countable field that contains $S$. The partial order $M_0^{\geq} = M_0 \cap L^{\geq}$ does not have to be directed. Define $A_0 = M_0^{\geq} - M_0^{\geq}$, which is a directed po-subring of $M_0$. For each of the many at most countable elements $a \in M_0 \setminus A_0$, there are $b_a, c_a \in L^{\geq}$ such that $a = b_a - c_a$. Define $S_1 = \{b_a \mid a \in M_0 \setminus A_0\}$, which is an at most countable set. Thus, field $M_1 = M_0(S_1)$ is countable. Iteration of the procedure leads to a sequence $M_0 \subseteq M_1 \subseteq \ldots$ of countable fields such that $M_n \subseteq M_{n+1}^{\geq} - M_{n+1}^{\geq}$ for all $n \in \mathbb{N}$. The union $M = \bigcup_{n \in \mathbb{N}} M_n$ is a countable field and contains $S$. The partial order $M^{\geq} = \bigcup_{n \in \mathbb{N}} M_n^{\geq} = M \cap L^{\geq}$ is directed. (b) is an immediate consequence of (a). □

We are mainly interested in Archimedean po-fields. In the literature, there exist several different notions of Archimedeanity. Therefore, we fix the terminology that we shall use. Let $A$ be an abelian and torsion-free po-group with positive cone $A^{\geq}$:

- $A$ is *Archimedean* if $n \cdot a < b$ for all $n \in \mathbb{Z}$ implies $a = 0$ [11] (p. 24).
- $A$ is *strongly Archimedean* if $n \cdot a < b$ for all $n \in \mathbb{N}$ implies $a \leq 0$ [12] (p. 918).
- $A$ has a *strong unit*, if there is an element $u \in A$ such $\forall a \in A \; \exists n \in \mathbb{N} : a < n \cdot u$ [11] (p. 122), [13] (p. 41).

Strongly Archimedean clearly implies Archimedean. However, the converse is false (see Example 1). If $A$ is a lattice-ordered group, then Archimedean also implies strongly Archimedean. Suppose that $n \cdot a < b$ for all $n \in \mathbb{N}$. Then, $n \cdot a^+ < b^+$ for all $n \in \mathbb{Z}$. According to Archimedeanity, it follows that $a^+ = 0$; hence, $a = -a^- \leq 0$. If $A$ is even totally ordered and Archimedean, then every positive element is a strong unit.

Given $a, b \in A^>$, we say that $a$ is *infinitesimal* compared with $b$ or that $b$ is *infinitely large* compared with $a$, with the following notation: $a \ll b$, if $n \cdot a < b$ for all $1 \leq n \in \mathbb{N}$. Suppose that $A$ is lattice-ordered. Then, $A$ is Archimedean if and only if there are no elements $a \ll b$. If $A$ is a ring, then the implication $a \ll b \Rightarrow c \cdot a \ll c \cdot b$ holds for every $c > 0$ that is not a zero divisor.

Suppose that $A$ is a po-group. Then, the following statements are equivalent:

- $A$ is Archimedean or strongly Archimedean.
- Every subgroup $U$ with partial order $U^\geq = U \cap A^\geq$ is Archimedean or strongly Archimedean.
- Every subgroup with two generators is Archimedean or strongly Archimedean.

Suppose that $A$ is an Archimedean po-group and that $P \subseteq A^\geq$ is a weaker partial order. Then, $(A, P)$ is also Archimedean. Note that a weakening of a strongly Archimedean partial order needs not be strongly Archimedean. Now, suppose that $A$ is a po-group with a strong unit $u$. Let $Q \supseteq A^\geq$ be a stronger partial order. Then, $u$ is also a strong unit for $Q$.

A strong unit in a po-group $A$ has a multiple that is positive—by applying the definition with $a = 0$, one finds some $n \in \mathbb{N}$ such that $0 < n \cdot u$. The existence of a strong unit implies that the po-group is directed. Given $a, b \in A$, there are $1 \leq n, p \in \mathbb{N}$ such that $a < n \cdot u$ and $b < p \cdot u$. It follows that $a, b < (n \cdot p) \cdot u$.

Real algebraists frequently call groups or rings Archimedean if they have a strong unit [14] (p. 421), [15] (p. 5). This terminology is in conflict with ours: The existence of a strong unit does not imply Archimedeanity as we define it. However, in the case of fields, we prove the below.

**Theorem 1.** *Suppose that $L$ is a po-field with a strong unit. Then, $L$ is Archimedean.*

**Proof.** Suppose that $u$ is a strong unit, which is positive, since it is assumed that $\mathbb{Q}^\geq \cdot L^\geq = L^\geq$. Note that there is some $1 \leq N \in \mathbb{N}$ such that $u^2 < N \cdot u$. Define

$$I = \{x \in L \mid \exists p \in \mathbb{N}\, \forall n \in \mathbb{Z} : n \cdot x < p \cdot u\}.$$

Trivially, $0 \in I$, $u \notin I$, and $x \in I$ implies $-x \in I$. We now show that $I$ is an ideal.

Pick elements $x, y \in I$, e.g., $n \cdot x < p \cdot u$, $n \cdot y < q \cdot u$, for all $n \in \mathbb{Z}$. Then, $n \cdot (x + y) < (p + q) \cdot u$ for all $n \in \mathbb{Z}$; hence, $x + y \in I$.

Pick $x \in I$, e.g., $n \cdot x < p \cdot u$, for all $n \in \mathbb{Z}$, and $y \in L$. We claim that $x \cdot y \in I$. Since $u$ is a strong unit, there is some $m \in \mathbb{N}$ such that $0 < m \cdot u + y, m \cdot u - y$. For each $k \in \mathbb{N}$, the following inequalities are obtained, which finish the proof that $I$ is an ideal:

$$\begin{aligned}
0 &< (m \cdot u - y) \cdot (p \cdot u + k \cdot x) \\
&= (m \cdot p) \cdot u^2 + (m \cdot u) \cdot (k \cdot x) - (p \cdot u) \cdot y - k \cdot x \cdot y \\
&< (m \cdot p) \cdot u^2 + (m \cdot u) \cdot (p \cdot u - k \cdot x) + (m \cdot u) \cdot (k \cdot x) \\
&\quad + (p \cdot u) \cdot (m \cdot u + y) - (p \cdot u) \cdot y - k \cdot x \cdot y \\
&= 3 \cdot m \cdot p \cdot u^2 - k \cdot x \cdot y \\
&< 3 \cdot N \cdot m \cdot p \cdot u - k \cdot x \cdot y
\end{aligned}$$

$$\begin{aligned} 0 &< (m \cdot u + y) \cdot (p \cdot u + k \cdot x) \\ &= (m \cdot p) \cdot u^2 + (m \cdot u) \cdot (k \cdot x) + (p \cdot u) \cdot y + k \cdot x \cdot y \\ &< (m \cdot p) \cdot u^2 + (m \cdot u) \cdot (p \cdot u - k \cdot x) + (m \cdot u) \cdot (k \cdot x) \\ &\quad + (p \cdot u) \cdot (m \cdot u - y) + (p \cdot u) \cdot y + k \cdot x \cdot y \\ &= 3 \cdot m \cdot p \cdot u^2 + k \cdot x \cdot y \\ &< 3 \cdot N \cdot m \cdot p \cdot u + k \cdot x \cdot y. \end{aligned}$$

Ideal $I$ is proper in field $L$; hence, $I = \{0\}$.

To finish the proof, assume that $n \cdot a < b$ for all $n \in \mathbb{Z}$. Pick $q \in \mathbb{N}$ such that $b < q \cdot u$. Then, $a \in I$; hence, $a = 0$. □

The next result is a topological criterion for strong units. We use the following terminology: Suppose that $F$ is an Archimedean totally ordered field and that $V$ is an $F$-vector space. For every $0 < \varepsilon \in F$ and every $F$-basis $B$ of $V$, the following subset of $V$ is called a $(\varepsilon, B)$-*neighborhood of* 0:

$$N_{\varepsilon,B} = \{x \in V \mid x = \sum_{b \in B} x_b \cdot b \Rightarrow \forall b \in B : |x_b| < \varepsilon\}.$$

An element $x \in V$ is an interior of a subset $S \subseteq V$ if there is some $N_{\varepsilon,B}$ such that $x + N_{\varepsilon,B} \subseteq S$.

**Theorem 2.** *Suppose that $F$ is an Archimedean totally ordered field and that $L$ is a directed partially ordered extension:*

*(a) If $u$ is an interior point of $L^{\geq}$, then $u$ is a strong unit, and $L$ is Archimedean.*
*(b) Suppose that $\dim_F L \leq \aleph_0$ and that $u$ is a strong unit. Then, $u$ is an interior point of $L^{\geq}$.*

**Proof.** (a). Suppose that $u + N_{\varepsilon,B} \subseteq L^{\geq}$. Pick any element $a \in L$, say $a = \sum_{b \in B} a_b \cdot b$. There is some $1 \leq n \in \mathbb{N}$ such that $|\frac{1}{n} \cdot a_b| < \varepsilon$ for all $b \in B$. Then, $u - \frac{1}{n} \cdot a = u - \sum_{b \in B} \frac{1}{n} \cdot a_b \cdot b \in L^{\geq}$, and it follows that $a < (n+1) \cdot u$. This proves that $u$ is a strong unit; Archimedeanity follows from Theorem 1.

(b). Since $L$ is directed, there is a countable basis $(b_n)_{n \in \mathbb{N}}$ in $L^{\geq}$. For each $n$, there is some $p_n \in \mathbb{N}$ such that $b_n < p_n \cdot u$. Replace $b_n$ with $\frac{1}{p_n} \cdot b_n$ and assume now that $b_n < u$ for each $n \in \mathbb{N}$. Next, replace $b_n$ with $\frac{1}{2^n} \cdot b_n$. Pick a finite subset $M \subseteq \mathbb{N}$ and a family $(x_n)_{n \in M}$ in $F$ such that $|x_n| < \frac{1}{2}$ for each $n \in M$. The inequality

$$\sum_{n \in M} x_n \cdot b_n \leq \frac{1}{2} \cdot \sum_{n \in M} b_n < \frac{1}{2} \cdot \sum_{n \in M} \frac{1}{2^n} \cdot u < u$$

proves the claim. □

Recall that Archimedean lattice-ordered fields need not have a strong unit (cf. [16] (Example 3.3)). The following result generalizes [5] (Lemma 7).

**Corollary 1.** *Suppose that $F$ is an Archimedean totally ordered field and that $L$ is a directed partially ordered algebraic extension. Then, $L$ is Archimedean. If $L$ is finite over $F$, then $L$ has a strong unit.*

**Proof.** First, suppose that $[L : F] = n \in \mathbb{N}$. There is an element $a \in L^{\geq}$ such that $L = F[a]$. The elements $a, a^2, \ldots, a^n$ are a positive basis for $L$ over $F$. The element $u = \sum_{i=1}^{n} a^i$ is an interior point of $L^{\geq}$ and thus is a strong unit according to Theorem 2.

Now, drop the assumption of finiteness. Suppose that $n \cdot a < b$ for all $n \in \mathbb{Z}$. Field $F[a, b]$ is finite over $F$. The partial order $F[a, b]^{\geq} = F[a, b] \cap L^{\geq}$ is directed, since $F[a, b]$ is generated by positive elements $b - a$ and $b$. Thus, $F[a, b]$ has a strong unit; hence, it is Archimedean. We conclude that $a = 0$. □

If the directed po-field $L$ has a strong unit, then $L$ is Archimedean (cf. Theorem 1). One may ask whether $L$ is even strongly Archimedean. We show with an example that this is not true in general.

**Example 1.** *Consider field $\mathbb{Q}[\sqrt{2}]$, partially ordered as*

$$\mathbb{Q}[\sqrt{2}]^{\geq} = \{x + y \cdot \sqrt{2} \mid x = y = 0 \vee (x > 0 \wedge y \geq 0)\}.$$

*The partial order is directed, and $\mathbb{Q}[\sqrt{2}]$ is a partially ordered $\mathbb{Q}$-algebra. According to Corollary 1, there is a strong unit, and $\mathbb{Q}[\sqrt{2}]$ is Archimedean. However, $\mathbb{Q}[\sqrt{2}]$ is not strongly Archimedean, for $\sqrt{2} \notin \mathbb{Q}[\sqrt{2}]^{\geq}$, but $n \cdot (-\sqrt{2}) < 1$ for all $n \in \mathbb{N}$.*

## 3. Extending Partial Orders to Total Orders

It is a long-standing (and still open) question whether every lattice-ordered field is formally real. One can ask the same question about directed po-fields [6]. We showed in [8] (Corollary 4.2) that almost all fields of characteristic 0 carry a directed partial order. It remains an open question whether non-real number fields have directed partial orders. It is a consequence of the following result that the answer to this question is negative.

**Theorem 3.** *Suppose that $F$ is an Archimedean totally ordered field and that $L$ is a directed partially ordered algebraic extension. Then, there is a total order on $L$ that extends the partial order.*

**Proof.** We form the quotient order $qL^{\geq}$, which contains $L^{\geq}$ and is also directed. It suffices to prove that the quotient order can be extended to a total order. Thus, we may assume that $L^{\geq}$ is quotient-closed. In particular, $1 \in L^{\geq}$ and $-1 \notin L^{\geq}$; hence, $L^{\geq}$ is an infinite preprime of $L$ (cf. [17], [14] (p. 421)).

We claim that the preprime is Archimedean in the sense that 1 is a strong unit (cf. [14] (p. 421)). Pick an element $a \in L$. Since the partial order is directed, we can write $a = b - c$ with $b, c \in L^{\geq}$. Field $F[b, c]$ is finite over $F$, and $\{b^r \cdot c^s \mid r, s \in \mathbb{N}\} \subseteq L^{\geq}$ is a generating set for $F$-vector space $F[b, c]$. Therefore, partial order $F[b, c]^{\geq} = F[b, c] \cap L^{\geq}$ is directed. According to Corollary 1, there is a strong unit $u \in F[b, c]$ for this partial order. Pick $p \in \mathbb{N}$ such that $a \cdot u < p \cdot u$. Note that $F[b, c]^{\geq}$ is quotient-closed; hence, $a < p$.

Apply [14] with the Archimedean infinite preprime $L^{\geq}$. Let $S$ be any subset of $L$ that is maximal with the following properties:

- $L^{\geq} \subseteq S$, $-1 \notin S$.
- $L^{\geq} \cdot S \subseteq S$.
- $S + S \subseteq S$.

Maximal subsets with these properties exist according to Zorn's lemma. There exists a ring homomorphism $\varphi : L \to \mathbb{R}$ such that $\varphi(S) \subseteq \mathbb{R}^{\geq 0}$, [14] (Satz 5). However, $L^{\geq} \subseteq S \subseteq \varphi^{-1}(\mathbb{R}^{\geq 0})$. Since $L$ is a field, we conclude that $\varphi^{-1}(\mathbb{R}^{\geq 0})$ is a total order. □

**Corollary 2.** *Suppose that $L$ is a directed partially ordered number field. Then, $L$ is a real field.*

**Proof.** The quotient order $qL^{\geq}$ is a directed partial order on $L$; hence, $(L, qL^{\geq})$ satisfies the hypotheses of Theorem 3. Therefore, the partial order can be extended to a total order, which implies that $L$ is a real field. □

**Example 2.** *All algebraic number fields with $i$ cannot be directed partially ordered. Both fields $\mathbb{Q}(\pi)[i]$ and $\mathbb{C} = \mathbb{R}[i]$ can be directed partially ordered.*

## 4. Conclusions

Note that Corollary 2 not only strengthens the results in [4,5] that say that lattice-ordered number fields are real. Furthermore, Corollary 2 and the main theorem of [8] tell us that a field carries a compatible directed order if and only if it has characteristic 0 and is

real or has a transcendence degree of at least 1 over the field of rational numbers (that is, a subfield of the complex number field contains $i$ and at least one transcendental number), answering the question in [8].

**Theorem 4.** *A field can be directed partially ordered if and only if it has characteristic 0 and is real or has a transcendence degree of at least 1 over the field of rational numbers.*

This means that we have finished the question introduced in [6] on determining which fields can be directed partially ordered. We list some directions for further work:

1. Can one construct directed partial orders on the complex number field $\mathbb{C}$, which is stronger than the ones constructed in [6–8], for example, Riesz directed partial orders?
2. Generalize the construction idea to the non-commutative case.
3. Develop complex analysis and geometry based on one of the directed partial orders defined in [6–8].
4. Considering that a great deal is known about totally ordered fields and very little is known about partially ordered fields, it is reasonable to try to relate partially ordered fields to totally ordered fields.

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