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On a System of Coupled Langevin Equations in the Frame of Generalized Liouville–Caputo Fractional Derivatives

Hassan J. Al Salman [1], Muath Awadalla [1,*], Muthaiah Subramanian [2,*] and Kinda Abuasbeh [1]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Hofuf 319832, Al Ahsa, Saudi Arabia
[2] Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore 641407, India
* Correspondence: mawadalla@kfu.edu.sa (M.A.); subramanian.m@kpriet.ac.in (M.S.)

Abstract: We investigate the existence and uniqueness results for coupled Langevin differential equations of fractional order with Katugampola integral boundary conditions involving generalized Liouville–Caputo fractional derivatives. Furthermore, we discuss Ulam–Hyers stability in the context of the problem at hand. The results are shown with examples. Results are asymmetric when a generalised Liouville–Caputo fractional derivative (ρ) parameter is changed. With its novel results, this paper makes a significant contribution to the relevant literature.

Keywords: coupled system; Langevin equations; Katugampola integrals; generalized Liouville–Caputo derivatives; stability; existence; fixed point

MSC: 26A33; 34A08; 34B10

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1. Introduction

The Langevin equation (LE) is a powerful mathematical physics tool for describing processes such as anomalous diffusion in a systematic fashion. Price index fluctuations [1,2], harmonic oscillators, and other similar processes are examples of such processes. A general Langevin equation (GLE) for noise sources with correlations is also used a lot in the theory of critical dynamics [3]. To comprehend the nature of the quantum noise, a GLE [4] can be employed. When applied to systems of fractional order, such as reaction–diffusion systems of fractional order [5,6], the LE has a very elegant and rich function. The fractional analogue of the common LE is proposed for situations where the distinction between macroscopic and microscopic time scales is not apparent; this equation is sometimes referred to as the stochastic differential equation. See, for example, [1]. In [7], the author explored moments, variances, position, and velocity correlation for a generalized fractional LE of the Liouville–Caputo fractional derivative. The results were compared with those found for the same GLE. The aforementioned papers [8–13] and their relevant references contain recent findings on the LE with varied boundary conditions. In light of the numerous applications that it has in the fields of engineering, the social sciences, and the technical sciences, the study of fractional calculus has arisen as an important subject in which to do research. Due to their ability to reflect the history of current events and processes, integral and differential operators of fractional order based mathematical models are regarded as more realistic and applicable than their integer-order counterparts. This area of mathematics analysis has advanced significantly in recent years and currently includes a wide range of intriguing findings, such as [14–23]. Recently, the literature on the subject has included Hadamard, Caputo (Liouville–Caputo), Riemann–Liouville type derivatives, among others, as well as fractional order differential equations. See [24–33] and the list of references for some recent works on the subject. The authors [34] discussed the existence of solutions for fractional Langevin differential equations using Liouville–Caputo derivatives:

$$\begin{cases} {}^{\rho}_{c}\mathcal{D}^{\alpha}_{a^+}({}^{\rho}_{c}\mathcal{D}^{\beta}_{a^+}+\lambda)x(t)=f(t,x(t)), t\in\mathcal{J}\ :=[a,T], \\ \lambda\in\mathbb{R},\ 1<\alpha\leq 2,\ 0<\beta<1,\ \rho>0 \end{cases} \tag{1}$$

$$\Big\{ x(a)=0,\ x(\eta)=0,\ x(T)=\mu^{\rho}\mathcal{I}^{\gamma}_{a^+}x(\xi),\ \gamma>0,\rho>0, \tag{2}$$

where ${}^{\rho}_{c}\mathcal{D}^{\alpha}_{a^+}, {}^{\rho}_{c}\mathcal{D}^{\beta}_{a^+}$ denote the generalized Liouville–Caputo fractional derivatives (GLCFD), ${}^{\rho}\mathcal{I}^{\gamma}_{a^+}$ is the generalized fractional integral (Katugampola). The main results were proven using fixed point index theory. In [35], the authors studied existence results for coupled system of FDEs with Hilfer derivatives:

$$\begin{cases} ({}^{H}\mathcal{D}^{\alpha_1,\beta_1}_{0^+}x)(t)+\lambda_1({}^{H}\mathcal{D}^{\alpha_1-1,\beta_1}_{0^+}x)(t)=f(t,x(t),R^{(\delta_q,\dots,\delta_1)}x(t),y(t)),t\in[0,T], \\ ({}^{H}\mathcal{D}^{\alpha_2,\beta_2}_{0^+}y)(t)+\lambda_2({}^{H}\mathcal{D}^{\alpha_2-1,\beta_2}_{0^+}y)(t)=f(t,x(t),y(t),R^{(\zeta_q,\dots,\zeta_1)}y(t)),t\in[0,T], \end{cases} \tag{3}$$

$$\begin{cases} x(0)=0,\ y(0)=0, \\ x(T)=\sum_{i=1}^{m}\epsilon_i R^{(\mu_\rho,\dots,\mu_1)}y(\eta_i)\ \eta_i\in(0,T), \\ y(T)=\sum_{j=1}^{n}\theta_j R^{(\nu_\rho,\dots,\nu_1)}x(\xi_j)\ \xi_i\in(0,T), \end{cases} \tag{4}$$

where ${}^{H}\mathcal{D}^{\alpha_l,\beta_l}$ is the Hilfer derivative of fractional order α_l with parameters $\beta_l,\ l\in 1,2,\ 1<\alpha_l<2,\ 0\leq\beta_l\leq 1,\ \lambda_1,\lambda_2,\epsilon_i,\theta_j\in\mathcal{R}\backslash\{0\},\ i=1,2,\dots,m,\ j=1,2,\dots,n,$ $f,g:[0,T]\times\mathcal{R}\times\mathcal{R}\times\mathcal{R}\to\times\mathcal{R}$ are nonlinear continuous functions and $R^{(\phi_\tau,\dots,\phi_1)}$, $\phi_r\in\{\delta,\zeta,\mu,\nu\}$, $r\in\{q,p,\rho|q,p,\rho\in\mathcal{N}\}$, involves the iterated Riemann–Liouville and Hadamard fractional integral operators. The main results were proven using fixed point index theory. We demonstrate the existence, uniqueness of solutions and Ulam–Hyers stability for the following generalized Langevin fractional differential system, which was inspired by the previous research, by utilizing the fixed point theorems:

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi_1}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_1}_{0^+}+\phi_1)x(\iota)=f(\iota,x(\iota),y(\iota)),\iota\in\mathcal{E}:=[0,\mathcal{S}], \\ {}^{\rho}_{C}\mathcal{D}^{\xi_2}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_2}_{0^+}+\phi_2)y(\iota)=g(\iota,x(\iota),y(\iota)),\iota\in\mathcal{E}:=[0,\mathcal{S}], \end{cases} \tag{5}$$

$$\begin{cases} x(0)=0, \quad y(0)=0, \\ x(\mathcal{S})=\epsilon^{\rho}\mathcal{I}^{\varsigma}_{0+}y(\varpi),\ y(\mathcal{S})=\pi^{\rho}\mathcal{I}^{\varrho}_{0+}x(\sigma),\ 0<\sigma<\varpi<\mathcal{S}, \end{cases} \tag{6}$$

where ${}^{\rho}_{C}\mathcal{D}^{\xi_1}_{0^+},\ {}^{\rho}_{C}\mathcal{D}^{\xi_2}_{0^+},\ {}^{\rho}_{C}\mathcal{D}^{\zeta_1}_{0^+},\ {}^{\rho}_{C}\mathcal{D}^{\zeta_2}_{0^+}$ are the GLCFD of order $1<\xi_1,\xi_2\leq 2,\ 0<\zeta_1,\zeta_2<1$, ${}^{\rho}\mathcal{I}^{\varsigma}_{0^+},\ {}^{\rho}\mathcal{I}^{\varrho}_{0^+}$ are the Katugampola integral of fractional order $\varrho,\varsigma>0,\rho>0, f,g:\mathcal{E}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions, $\epsilon,\pi\in\mathbb{R}$. The requirement states that the unknown function's value at the right endpoint of the specified interval, $\iota=\mathcal{S}$, must be proportional to its values on strips of different lengths. The GLCFD is converted into the differentially effective Caputo sense when $\rho=1$. Similar to this, non-symmetric examples appear when $\rho=1$ and the Riemann–Liouville integrals are used instead of the Katugampola integrals. To the best of our knowledge, boundary value problems (BVPs) stability analysis is still in its development. The fundamental contribution of this research is to investigate the existence, uniqueness and stability analysis. In addition, we show the problem (5) and (6) used by the fixed-point theorems of Leray–Schauder and Banach to demonstrate the existence and uniqueness of solutions. The rest of the article is as follows: Section 2 presents the basic definitions, lemmas, and theorems that underpin our main conclusions. In Section 3, we show the existence and uniqueness of solutions to the given systems (5) and (6) using a variety of conditions and a few widely used fixed-point theorems. Section 4 looks at the Ulam–Hyers stability of the provided system (5) and (6) under particular circumstances. Examples are provided in Section 6 to further clarify the study's findings.

2. Preliminaries

We define space $\mathcal{P} = \{x(\iota) : x(\iota) \in \mathcal{C}(\mathcal{E}, \mathbb{R})\}$ equipped with the norm $||x|| = \sup\{|x(\iota)|, \iota \in \mathcal{E}\}$ is a Banach space. Also $\mathcal{Q} = \{y(\iota) : y(\iota) \in \mathcal{C}(\mathcal{E}, \mathbb{R})\}$ equipped with the norm is $||y|| = \sup\{|y(\iota)|, \iota \in \mathcal{E}\}$ is a Banach space. Then the product space $(\mathcal{P} \times \mathcal{Q}, ||(x,y)||)$ is also a Banach space with norm $||(x,y)|| = ||x|| + ||y||$.

Definition 1 ([36]). *The left- and right-sided GLCFD of $f \in \mathcal{AC}^n_\gamma[c,d]$ of order $\zeta \geq 0$*

$$ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{c+} f(z) = {}^{\rho}\mathcal{D}^{\zeta}_{c+}\left[f(\iota) - \sum_{k=0}^{n-1} \frac{\gamma^k f(c)}{k!}\left(\frac{\iota^\rho - c^\rho}{\rho}\right)^k\right](z), \gamma = z^{1-\rho}\frac{d}{dz}, \tag{7}$$

$$ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{d-} f(z) = {}^{\rho}\mathcal{D}^{\zeta}_{d-}\left[f(\iota) - \sum_{k=0}^{n-1} \frac{(-1)^k\gamma^k f(d)}{k!}\left(\frac{d^\rho - \iota^\rho}{\rho}\right)^k\right](z), \gamma = z^{1-\rho}\frac{d}{dz}, \tag{8}$$

when $n = [\zeta] + 1$.

Lemma 1 ([36]).

1. *if $\zeta \notin \mathbb{N}$*

$$ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{c+} f(\iota) = \frac{1}{\Gamma(n-\zeta)}\int_c^\iota \left(\frac{\iota^\rho - \theta^\rho}{\rho}\right)^{n-\zeta-1} \frac{(\gamma^n f)(\theta)d\theta}{\theta^{1-\rho}} = {}^{\rho}\mathcal{I}^{n-\zeta}_{c+}(\gamma^n f)(\iota), \tag{9}$$

$$ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{d-} f(\iota) = \frac{1}{\Gamma(n-\zeta)}\int_\iota^d \left(\frac{\theta^\rho - \iota^\rho}{\rho}\right)^{n-\zeta-1} \frac{(-1)^n(\gamma^n f)(\theta)d\theta}{\theta^{1-\rho}} = {}^{\rho}\mathcal{I}^{n-\zeta}_{d-}(\gamma^n f)(\iota). \tag{10}$$

2. *if $\zeta \in \mathbb{N}$*

$$ {}^{\rho}_{C}\mathcal{D}^{\zeta}_{c+} f = \gamma^n f, \qquad {}^{\rho}_{C}\mathcal{D}^{\zeta}_{d-} f = (-1)^n\gamma^n f. \tag{11}$$

Lemma 2 ([36]). *Let $f \in \mathcal{AC}^n_\gamma[c,d]$ or $\mathcal{C}^n_\gamma[c,d]$ and $\zeta \in \mathbb{R}$. Then,*

$$ {}^{\rho}\mathcal{I}^{\zeta}_{c+}\,{}^{\rho}_{C}\mathcal{D}^{\zeta}_{c+} f(z) = f(z) - \sum_{k=0}^{n-1} \frac{\gamma^k f(c)}{k!}\left(\frac{z^\rho - c^\rho}{\rho}\right)^k,$$

$$ {}^{\rho}\mathcal{I}^{\zeta}_{d-}\,{}^{\rho}_{C}\mathcal{D}^{\zeta}_{d-} f(z) = f(z) - \sum_{k=0}^{n-1} \frac{(-1)^k\gamma^k f(d)}{k!}\left(\frac{d^\rho - z^\rho}{\rho}\right)^k.$$

In particular, for $0 < \zeta \leq 1$, we have

$$ {}^{\rho}\mathcal{I}^{\zeta}_{c+}\,{}^{\rho}_{C}\mathcal{D}^{\zeta}_{c+} f(z) = f(z) - f(c), \qquad {}^{\rho}\mathcal{I}^{\zeta}_{d-}\,{}^{\rho}_{C}\mathcal{D}^{\zeta}_{d-} f(z) = f(z) - f(d).$$

In order to facilitate computation, we present the following notations:

$$ \widehat{\mathcal{E}}_1 = \frac{\mathcal{S}^{\rho\zeta_1}}{\rho^{\zeta_1}\Gamma(\zeta_1+1)}, \widehat{\mathcal{E}}_2 = \frac{\pi\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}, \tag{12}$$

$$ \mathcal{E}_1 = \frac{\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)}, \mathcal{E}_2 = \frac{\epsilon\omega^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}, \tag{13}$$

$$ \mathcal{G} = \widehat{\mathcal{E}}_1\mathcal{E}_2 - \mathcal{E}_1\widehat{\mathcal{E}}_2 \neq 0, \tag{14}$$

$$ \delta_1(\iota) = \left(\frac{\iota^{\rho\zeta_1}}{\rho^{\zeta_1}\Gamma(\zeta_1+1)\mathcal{G}}\right), \delta_2(\iota) = \left(\frac{\iota^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)\mathcal{G}}\right). \tag{15}$$

Lemma 3. *Let $\hat{f}, \hat{g} \in C(0,\mathcal{S}) \cup \mathcal{L}(0,\mathcal{S})$, $x, y \in \mathcal{AC}^2_\gamma(\mathcal{E})$ and $\Lambda \neq 0$. The solution of the system of coupled Langevin fractional BVP:*

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi_1}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_1}_{0^+}+\phi_1)x(\iota)=\hat{f}(\iota), \iota\in\mathcal{E}:=[0,\mathcal{S}],\\ {}^{\rho}_{C}\mathcal{D}^{\xi_2}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_2}_{0^+}+\phi_2)y(\iota)=\hat{g}(\iota), \iota\in\mathcal{E}:=[0,\mathcal{S}],\\ x(0)=0,\quad y(0)=0,\quad x(\mathcal{S})=\epsilon\,{}^{\rho}\mathcal{I}^{\varsigma}_{0+}y(\varpi),\quad y(\mathcal{S})=\pi\,{}^{\rho}\mathcal{I}^{\varrho}_{0+}x(\sigma)\quad 0<\sigma<\varpi<\mathcal{S}, \end{cases} \tag{16}$$

is given by

$$\begin{aligned} x(\iota) =& {}^{\rho}\mathcal{I}^{\xi_1+\zeta_1}_{0^+}\hat{f}(\iota)-\phi_1^{\rho}\mathcal{I}^{\zeta_1}_{0^+}x(\iota)\\ &+\delta_1(\iota)\Big[\mathcal{E}_2\Big(\epsilon^{\rho}\mathcal{I}^{\xi_2+\zeta_2+\varsigma}_{0^+}\hat{g}(\varpi)-\epsilon\phi_2^{\rho}\mathcal{I}^{\zeta_2+\varsigma}_{0^+}y(\varpi)-{}^{\rho}\mathcal{I}^{\xi_1+\zeta_1}_{0^+}\hat{f}(\mathcal{S})+\phi_1^{\rho}\mathcal{I}^{\zeta_1}_{0^+}x(\mathcal{S})\Big)\\ &+\mathcal{E}_1\Big(\pi^{\rho}\mathcal{I}^{\xi_1+\zeta_1+\varrho}_{0^+}\hat{f}(\sigma)-\pi\phi_1^{\rho}\mathcal{I}^{\zeta_1+\varrho}_{0^+}x(\sigma)-{}^{\rho}\mathcal{I}^{\xi_2+\zeta_2}_{0^+}\hat{g}(\mathcal{S})+\phi_2^{\rho}\mathcal{I}^{\zeta_2}_{0^+}y(\mathcal{S})\Big)\Big] \end{aligned} \tag{17}$$

and

$$\begin{aligned} y(\iota) =& {}^{\rho}\mathcal{I}^{\xi_2+\zeta_2}_{0^+}\hat{g}(\iota)-\phi_2^{\rho}\mathcal{I}^{\zeta_2}_{0^+}y(\iota)\\ &+\delta_2(\iota)\Big[\widehat{\mathcal{E}}_2\Big(\epsilon^{\rho}\mathcal{I}^{\xi_2+\zeta_2+\varsigma}_{0^+}\hat{g}(\varpi)-\epsilon\phi_2^{\rho}\mathcal{I}^{\zeta_2+\varsigma}_{0^+}y(\varpi)-{}^{\rho}\mathcal{I}^{\xi_1+\zeta_1}_{0^+}\hat{f}(\mathcal{S})+\phi_1^{\rho}\mathcal{I}^{\zeta_1}_{0^+}x(\mathcal{S})\Big)\\ &+\widehat{\mathcal{E}}_1\Big(\pi^{\rho}\mathcal{I}^{\xi_1+\zeta_1+\varrho}_{0^+}\hat{f}(\sigma)-\pi\phi_1^{\rho}\mathcal{I}^{\zeta_1+\varrho}_{0^+}x(\sigma)-{}^{\rho}\mathcal{I}^{\xi_2+\zeta_2}_{0^+}\hat{g}(\mathcal{S})+\phi_2^{\rho}\mathcal{I}^{\zeta_2}_{0^+}y(\mathcal{S})\Big)\Big]. \end{aligned} \tag{18}$$

Proof. Applying operators ${}^{\rho}\mathcal{I}^{\xi_1}_{0^+}, {}^{\rho}\mathcal{I}^{\xi_2}_{0^+}$ to (16) and using Lemma 2, we get

$$({}^{\rho}_{C}\mathcal{D}^{\zeta_1}_{0^+}+\phi_1)x(\iota) = {}^{\rho}\mathcal{I}^{\xi_1}_{0^+}\hat{f}(\iota)+a_1, \tag{19}$$

$$({}^{\rho}_{C}\mathcal{D}^{\zeta_2}_{0^+}+\phi_2)y(\iota) = {}^{\rho}\mathcal{I}^{\xi_2}_{0^+}\hat{g}(\iota)+b_1, \tag{20}$$

respectively, for some $a_1, b_1 \in \mathcal{R}$. When ${}^{\rho}\mathcal{I}^{\zeta_1}_{0^+}, {}^{\rho}\mathcal{I}^{\zeta_2}_{0^+}$ are applied to the FDEs in (19) and (20), the solution of the Langevin FDEs in (16) for $\iota \in \mathcal{E}$ is

$$x(\iota) = {}^{\rho}\mathcal{I}^{\xi_1+\zeta_1}_{0^+}\hat{f}(\iota)-\phi_1^{\rho}\mathcal{I}^{\zeta_1}_{0^+}x(\iota)+a_1\frac{\iota^{\rho\zeta_1}}{\rho^{\zeta_1}\Gamma(\zeta_1+1)}+a_2, \tag{21}$$

$$y(\iota) = {}^{\rho}\mathcal{I}^{\xi_2+\zeta_2}_{0^+}\hat{g}(\iota)-\phi_2^{\rho}\mathcal{I}^{\zeta_2}_{0^+}y(\iota)+b_1\frac{\iota^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)}+b_2, \tag{22}$$

respectively, for some $a_2, b_2 \in \mathcal{R}$. By the conditions, $x(0) = y(0) = 0$ in (21) and (22) respectively, results $a_2 = b_2 = 0$. Then, using the generalized fractional integral operators, we obtain: ${}^{\rho}\mathcal{I}^{\varrho}_{0+}$, ${}^{\rho}\mathcal{I}^{\varsigma}_{0+}$ (21) and (22) respectively,

$${}^{\rho}\mathcal{I}^{\varrho}_{0+}x(\iota) = {}^{\rho}\mathcal{I}^{\xi_1+\zeta_1+\varrho}_{0^+}\hat{f}(\iota)-\phi_1^{\rho}\mathcal{I}^{\zeta_1+\varrho}_{0^+}x(\iota)+a_1\frac{\iota^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}, \tag{23}$$

$${}^{\rho}\mathcal{I}^{\varsigma}_{0+}y(\iota) = {}^{\rho}\mathcal{I}^{\xi_2+\zeta_2+\varsigma}_{0^+}\hat{g}(\iota)-\phi_2^{\rho}\mathcal{I}^{\zeta_2+\varsigma}_{0^+}y(\iota)+b_1\frac{\iota^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}, \tag{24}$$

which, when combined with the boundary conditions $x(\mathcal{S}) = \epsilon\,{}^{\rho}\mathcal{I}^{\varsigma}_{0+}y(\varpi)$, $y(\mathcal{S}) = \pi^{\rho}\mathcal{I}^{\varrho}_{0+}x(\sigma)$, gives the following results:

$$ {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}\hat{f}(\mathcal{S}) - \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\mathcal{S}) + a_1\frac{\mathcal{S}^{\rho\zeta_1}}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} = \epsilon^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}\hat{g}(\varpi) - \epsilon\phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}y(\varpi) + b_1\frac{\epsilon\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}, \tag{25} $$

$$ {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}\hat{g}(\mathcal{S}) - \phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2}y(\mathcal{S}) + b_1\frac{\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} = \pi^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}\hat{f}(\sigma) - \pi\phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}x(\sigma) + a_1\frac{\pi\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}. \tag{26} $$

Next, we obtain

$$ a_1\widehat{\mathcal{E}}_1 - b_1\mathcal{E}_1 = \epsilon^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}\hat{g}(\varpi) - \epsilon\phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}y(\varpi) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}\hat{f}(\mathcal{S}) + \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\mathcal{S}), \tag{27} $$

$$ b_1\mathcal{E}_2 - a_2\widehat{\mathcal{E}}_2 = \pi^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}\hat{f}(\sigma) - \pi\phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}x(\sigma) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}\hat{g}(\mathcal{S}) + \phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2}y(\mathcal{S}), \tag{28} $$

using the notations (13) in (25) and (26) respectively. When the system of equations is solved, we find that (27) and (28) for a_1 and b_1,

$$ \begin{aligned} a_1 =& \frac{1}{\mathcal{G}}\Bigg[\mathcal{E}_2\Big(\epsilon^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}\hat{g}(\varpi) - \epsilon\phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}y(\varpi) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}\hat{f}(\mathcal{S}) + \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\mathcal{S})\Big) \\ &+ \mathcal{E}_1\Big(\pi^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}\hat{f}(\sigma) - \pi\phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}x(\sigma) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}\hat{g}(\mathcal{S}) + \phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2}y(\mathcal{S})\Big)\Bigg], \end{aligned} \tag{29} $$

$$ \begin{aligned} b_1 =& \frac{1}{\mathcal{G}}\Bigg[\widehat{\mathcal{E}}_2\Big(\epsilon^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}\hat{g}(\varpi) - \epsilon\phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}y(\varpi) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}\hat{f}(\mathcal{S}) + \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\mathcal{S})\Big) \\ &+ \widehat{\mathcal{E}}_1\Big(\pi^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}\hat{f}(\sigma) - \pi\phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}x(\sigma) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}\hat{g}(\mathcal{S}) + \phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2}y(\mathcal{S})\Big)\Bigg]. \end{aligned} \tag{30} $$

Substituting the values of a_1, b_1 in (21) and (22) respectively, we obtain the BVP solution (16). □

3. Main Results

We propose a fixed point problem relevant to the problem by Lemma 3 as follows: $\Psi : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \times \mathcal{Q}$ by

$$ \Psi(x,y)(\iota) = (\Psi_1(x,y)(\iota), \Psi_2(x,y)(\iota)), \tag{31} $$

where

$$ \begin{aligned} \Psi_1(x,y)(\iota) =& {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}f(\iota,x(\iota),y(\iota)) - \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\iota) + \delta_1(\iota)\Bigg[\mathcal{E}_2\Big(\epsilon^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}g(\varpi,x(\varpi),y(\varpi)) \\ &- \epsilon\phi_2{}^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}y(\varpi) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}f(\mathcal{S},x(\mathcal{S}),y(\mathcal{S})) + \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\mathcal{S})\Big) \\ &+ \mathcal{E}_1\Big(\pi^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}f(\sigma,x(\sigma),y(\sigma)) - \pi\phi_1{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}x(\sigma) \\ &- {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}g(\mathcal{S},x(\mathcal{S}),y(\mathcal{S})) + \phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2}y(\mathcal{S})\Big)\Bigg], \end{aligned} \tag{32} $$

$$\Psi_2(x,y)(\iota) = {}^\rho\mathcal{I}_{0^+}^{\xi_2+\zeta_2} g(\iota, x(\iota), y(\iota)) - \phi_2^\rho \mathcal{I}_{0^+}^{\zeta_2} y(\iota) + \delta_2(\iota)\Bigg[\widehat{\mathcal{E}}_2\Big(\epsilon^\rho \mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma} g(\varpi, x(\varpi), y(\varpi))$$
$$- \epsilon\phi_2^\rho \mathcal{I}_{0^+}^{\zeta_2+\varsigma} y(\varpi) - {}^\rho\mathcal{I}_{0^+}^{\xi_1+\zeta_1} f(\mathcal{S}, x(\mathcal{S}), y(\mathcal{S})) + \phi_1^\rho \mathcal{I}_{0^+}^{\zeta_1} x(\mathcal{S})\Big)$$
$$+ \widehat{\mathcal{E}}_1\Big(\pi^\rho \mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho} f(\sigma, x(\sigma), y(\sigma)) - \pi\phi_1^\rho \mathcal{I}_{0^+}^{\zeta_1+\varrho} x(\sigma)$$
$$- {}^\rho\mathcal{I}_{0^+}^{\xi_2+\zeta_2} g(\mathcal{S}, x(\mathcal{S}), y(\mathcal{S})) + \phi_2^\rho \mathcal{I}_{0^+}^{\zeta_2} y(\mathcal{S})\Big)\Bigg]. \quad (33)$$

For brevity, use these notations:

$$\mathcal{U}_1 = \frac{\Big(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(1+|\delta_1||\mathcal{E}_2|)\Big)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}, \quad (34)$$

$$\mathcal{V}_1 = |\delta_1|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right), \quad (35)$$

$$\widehat{\mathcal{U}_1} = |\phi_1|\left(\frac{(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|))}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right), \quad (36)$$

$$\widehat{\mathcal{V}_1} = |\delta_1||\phi_2|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right), \quad (37)$$

$$\mathcal{U}_2 = |\delta_2|\left(\frac{\mathcal{S}^{\rho(\xi_1+\zeta_1)}|\widehat{\mathcal{E}_2}|}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}_1}|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right), \quad (38)$$

$$\mathcal{V}_2 = \frac{(1+|\delta_2||\widehat{\mathcal{E}_1}|)\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}_2}||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}, \quad (39)$$

$$\widehat{\mathcal{U}_2} = |\delta_2||\phi_1|\left(\frac{\mathcal{S}^{\rho\zeta_1}|\widehat{\mathcal{E}_2}|}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}_1}|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right), \quad (40)$$

$$\widehat{\mathcal{V}_2} = |\phi_2|\left(\frac{(1+|\delta_2||\widehat{\mathcal{E}_1}|)\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}_2}||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right), \quad (41)$$

$$\Phi = \min\{1 - [\psi_1(\mathcal{U}_1+\mathcal{U}_2) + \hat{\psi_1}(\mathcal{V}_1+\mathcal{V}_2) + \widehat{\mathcal{U}_1} + \widehat{\mathcal{U}_2}],$$
$$1 - [\psi_2(\mathcal{U}_1+\mathcal{U}_2) + \hat{\psi_2}(\mathcal{V}_1+\mathcal{V}_2) + \widehat{\mathcal{V}_1} + \widehat{\mathcal{V}_2}]\}. \quad (42)$$

Let $f, g : \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous functions.

- $(\mathcal{A}_1)$ $\exists$ constants $\psi_m, \hat{\psi_m} \geq 0 (m = 1,2)$ and $\psi_0, \hat{\psi_0} > 0 \ni$

$$|f(\iota, o_1, o_2)| \leq \psi_0 + \psi_1|o_1| + \psi_2|o_2|,$$
$$|g(\iota, o_1, o_2)| \leq \hat{\psi_0} + \hat{\psi_1}|o_1| + \hat{\psi_2}|o_2|, \forall o_m \in \mathbb{R}, m = 1,2.$$

- $(\mathcal{A}_2)$ $\exists$ constants $\psi_m, \hat{\psi_m} \geq 0 (m = 1, 2) \ni$

$$\begin{aligned}|f(\iota, o_1, o_2) - f(\iota, \hat{o}_1, \hat{o}_2)| &\leq \psi_1 |o_1 - \hat{o}_1| + \psi_2 |o_2 - \hat{o}_2|,\\ |g(\iota, o_1, o_2) - g(\iota, \hat{o}_1, \hat{o}_2)| &\leq \hat{\psi}_1 |o_1 - \hat{o}_1| + \hat{\psi}_2 |o_2 - \hat{o}_2|, \forall o_m, \hat{o}_m \in \mathbb{R}, m = 1, 2.\end{aligned}$$

Theorem 1. *If the assumption $(\mathcal{A}_1)$ is satisfied, then the problem (5) and (6) has a at leat one solution on $\mathcal{E}$ if $\psi_1(\mathcal{U}_1 + \mathcal{U}_2) + \hat{\psi}_1(\mathcal{V}_1 + \mathcal{V}_2) + \widehat{\mathcal{U}_1} + \widehat{\mathcal{U}_2} < 1, \psi_2(\mathcal{U}_1 + \mathcal{U}_2) + \hat{\psi}_2(\mathcal{V}_1 + \mathcal{V}_2) + \widehat{\mathcal{V}_1} + \widehat{\mathcal{V}_2} < 1$. Where $\mathcal{U}_1, \mathcal{V}_1, \widehat{\mathcal{U}_1}, \widehat{\mathcal{V}_1}, \mathcal{U}_2, \mathcal{V}_2, \widehat{\mathcal{U}_2}, \widehat{\mathcal{V}_2}$ are given by (34)–(41) respectively.*

Proof. In the first phase, we define operator $\Psi : \mathcal{P} \times \mathcal{Q} \to \mathcal{P} \times \mathcal{Q}$ as being completely continuous. The operators Ψ_1 and Ψ_2 are continuous because the functions f and g are continuous. The operator Ψ is continuous as a result. For the purpose of illustrating how the uniformly bounded operator Ψ works, consider the bounded set $\Psi \subset \mathcal{P} \times \mathcal{Q}$. Then, $\hat{\mathcal{N}_1}$ and $\hat{\mathcal{N}_2}$ are positive constants such that $|f(\iota, x(\iota), y(\iota))| \leq \hat{\mathcal{N}_1}, |g(\iota, x(\iota), y(\iota))| \leq \hat{\mathcal{N}_2}$, $\forall (x, y) \in \Psi$. Then we have

$$\begin{aligned}|\Psi_1(x,y)(\iota)| \leq & {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|f(\iota,x(\iota),y(\iota))| + |\phi_1|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\iota)|\\ & + |\delta_1(\iota)|\Bigg[|\mathcal{E}_2|\Big(|\epsilon|{}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}|g(\varpi,x(\varpi),y(\varpi))|\\ & + |\epsilon||\phi_2|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}|y(\varpi)| + {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|f(\mathcal{S},x(\mathcal{S}),y(\mathcal{S}))| + |\phi_1|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\mathcal{S})|\Big)\\ & + |\mathcal{E}_1|\Big(|\pi|{}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}|f(\sigma,x(\sigma),y(\sigma))| + |\pi||\phi_1|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}|x(\sigma)|\\ & + {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}|g(\mathcal{S},x(\mathcal{S}),y(\mathcal{S}))| + |\phi_2|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_2}|y(\mathcal{S})|\Big)\Bigg]\\ \leq & \hat{\mathcal{N}_1}\left\{\frac{\left(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(1+|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right\}\\ & + \hat{\mathcal{N}_2}\left\{|\delta_1|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right)\right\}\\ & + \left\{|\phi_1|\left(\frac{(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|))}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\right\}\|x\|\\ & + \left\{|\delta_1||\phi_2|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\right\}\|y\|,\end{aligned}$$

when taking the norm and using (34)–(37), that yields for $(x, y) \in \Psi$,

$$||\Psi_1(x,y)|| \leq \mathcal{U}_1\hat{\mathcal{N}_1} + \widehat{\mathcal{U}_1}\|x\| + \mathcal{V}_1\hat{\mathcal{N}_2} + \widehat{\mathcal{V}_1}\|y\|. \tag{43}$$

Likewise, we obtain

$$
\begin{aligned}
||\Psi_2(x,y)|| \leq & \hat{\mathcal{N}_2}\left\{ \frac{(1+|\delta_2||\widehat{\mathcal{E}_1}|)\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}_2}||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)} \right\} \\
& + \hat{\mathcal{N}_1}\left\{ |\delta_2| \left(\frac{\mathcal{S}^{\rho(\xi_1+\zeta_1)}|\widehat{\mathcal{E}_2}|}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}_1}|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)} \right) \right\} \\
& + \left\{ |\phi_2| \left(\frac{(1+|\delta_2||\widehat{\mathcal{E}_1}|)\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}_2}||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)} \right) \right\} \|y\| \\
& + \left\{ |\delta_2||\phi_1| \left(\frac{\mathcal{S}^{\rho\zeta_1}|\widehat{\mathcal{E}_2}|}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}_1}|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)} \right) \right\} \|x\| \\
\leq & \mathcal{U}_2\hat{\mathcal{N}_1} + \widehat{\mathcal{U}_2}\|x\| + \mathcal{V}_2\hat{\mathcal{N}_2} + \widehat{\mathcal{V}_2}\|y\|,
\end{aligned}
\tag{44}
$$

using (38)–(41). We may infer that Ψ_1 and Ψ_2 are uniformly bounded based on the inequalities (43) and (44), which means that the operator Ψ is also uniformly bounded. Following that, we demonstrate that Ψ is equicontinuous. Let $\iota_1, \iota_2 \in \mathcal{E}$ with $\iota_1 < \iota_2$. Then we have

$$
\begin{aligned}
& |\Psi_1(x,y)(\iota_2) - \Psi_1(x,y)(\iota_1)| \\
& \leq |{}^\rho\mathcal{I}_{0+}^{\xi_1+\zeta_1} f(\iota_2, x(\iota_2), y(\iota_2)) - {}^\rho\mathcal{I}_{0+}^{\xi_1+\zeta_1} f(\iota_1, x(\iota_1), y(\iota_1))| + |\phi_1||{}^\rho\mathcal{I}_{0+}^{\zeta_1} x(\iota_2) - {}^\rho\mathcal{I}_{0+}^{\zeta_1} x(\iota_1)| \\
& + |\delta_1(\iota_2) - \delta_1(\iota_1)| \Bigg[|\mathcal{E}_2| \Big(|\epsilon| {}^\rho\mathcal{I}_{0+}^{\xi_2+\zeta_2+\varsigma} |g(\varpi, x(\varpi), y(\varpi))| \\
& \quad + |\epsilon||\phi_2| {}^\rho\mathcal{I}_{0+}^{\zeta_2+\varsigma} |y(\varpi)| + {}^\rho\mathcal{I}_{0+}^{\xi_1+\zeta_1} |f(\mathcal{S}, x(\mathcal{S}), y(\mathcal{S}))| + |\phi_1| {}^\rho\mathcal{I}_{0+}^{\zeta_1} |x(\mathcal{S})| \Big) \\
& \quad + |\mathcal{E}_1| \Big(|\pi| {}^\rho\mathcal{I}_{0+}^{\xi_1+\zeta_1+\varrho} |f(\sigma, x(\sigma), y(\sigma))| + |\pi||\phi_1| {}^\rho\mathcal{I}_{0+}^{\zeta_1+\varrho} |x(\sigma)| \\
& \quad + {}^\rho\mathcal{I}_{0+}^{\xi_2+\zeta_2} |g(\mathcal{S}, x(\mathcal{S}), y(\mathcal{S}))| + |\phi_2| {}^\rho\mathcal{I}_{0+}^{\zeta_2} |y(\mathcal{S})| \Big) \Bigg] \\
\leq & \frac{\rho^{1-(\xi_1+\zeta_1)}\hat{\mathcal{N}_1}}{\Gamma(\xi_1+\zeta_1)} \Bigg| \int_0^{\iota_1} \left[\frac{\theta^{\rho-1}}{(\iota_2^\rho - \theta^\rho)^{1-(\xi_1+\zeta_1)}} - \frac{\theta^{\rho-1}}{(\iota_1^\rho - \theta^\rho)^{1-(\xi_1+\zeta_1)}} \right] d\theta \\
& + \int_{\iota_1}^{\iota_2} \frac{\theta^{\rho-1}}{(\iota_2^\rho - \theta^\rho)^{1-(\zeta_1)}} d\theta \Bigg| \\
& \frac{\rho^{1-(\zeta_1)}\|x\|}{\Gamma(\zeta_1)} \Bigg| \int_0^{\iota_1} \left[\frac{\theta^{\rho-1}}{(\iota_2^\rho - \theta^\rho)^{1-(\zeta_1)}} - \frac{\theta^{\rho-1}}{(\iota_1^\rho - \theta^\rho)^{1-(\zeta_1)}} \right] d\theta \\
& + \int_{\iota_1}^{\iota_2} \frac{\theta^{\rho-1}}{(\iota_2^\rho - \theta^\rho)^{1-(\zeta_1)}} d\theta \Bigg| \\
& + |\delta_1(\iota_2) - \delta_1(\iota_1)| \Bigg[\hat{\mathcal{N}_1} \left\{ \frac{\left(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)} \right\} \\
& \quad + \hat{\mathcal{N}_2} \left\{ |\delta_1| \left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)} \right) \right\} \\
& \quad + \left\{ |\phi_1| \left(\frac{\left(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)} \right) \right\} \|x\| \\
& \quad + \left\{ |\delta_1||\phi_2| \left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)} \right) \right\} \|y\| \Bigg] \\
& \to 0 \text{ as } \iota_2 \to \iota_1.
\end{aligned}
\tag{45}
$$

independent of (x,y) with respect to $|f(\iota,x(\iota_1),y(\iota_1))| \leq \hat{\mathcal{N}_1}$ and $|g(\iota,x(\iota_1),y(\iota_1))| \leq \hat{\mathcal{N}_2}$. Similarly, we can express $|\Psi_2(x,y)(\iota_2) - \Psi_2(x,y)(\iota_1)| \to 0$ as $\iota_2 \to \iota_1$ independent of (x,y) in terms of the boundedness of f and g. The operator Ψ is equicontinuous due to the equicontinuity of Ψ_1 and Ψ_2. The operator is compact as a result of the Arzela–Ascoli theorem. Finally, we show that the set $\Pi(\Psi) = \{(x,y) \in \mathcal{P} \times \mathcal{Q} : \lambda\Psi(x,y); 0 < \lambda < 1\}$ is bounded. Let $(x,y) \in \Pi(\Psi)$.Then $(x,y) = \lambda\Psi(x,y)$. For any $\iota \in \mathcal{E}$, we have $x(\iota) = \lambda\Psi_1(x,y)(\iota), y(\iota) = \lambda\Psi_2(x,y)(\iota)$. By utilizing $(\mathcal{A}_1)$ in (32), we obtain

$$\begin{aligned}
|x(\iota)| \leq & {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}(\psi_0, \psi_1|x(\iota)|, \psi_2|y(\iota)|) + |\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\iota)| \\
& + |\delta_1(\iota)|\Bigg[|\mathcal{E}_2|\Big(|\epsilon|^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}(\hat{\psi}_0 + \hat{\psi}_1|x(\varpi)| + \hat{\psi}_2|y(\varpi)|) \\
& + |\epsilon||\phi_2|^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}|y(\varpi)| + {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}(\psi_0, \psi_1|x(\mathcal{S})|, \psi_2|y(\mathcal{S})|) + |\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\mathcal{S})|\Big) \\
& + |\mathcal{E}_1|\Big(|\pi|^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}(\psi_0, \psi_1|x(\sigma)|, \psi_2|y(\sigma)|) + |\pi||\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}|x(\sigma)| \\
& + {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}(\hat{\psi}_0 + \hat{\psi}_1|x(\mathcal{S})| + \hat{\psi}_2|y(\mathcal{S})|) + |\phi_2|^{\rho}\mathcal{I}_{0^+}^{\zeta_2}|y(\mathcal{S})|\Big)\Bigg],
\end{aligned}$$

which is obtained when the norm for $\iota \in \mathcal{E}$ is taken,

$$||x|| \leq (\psi_0 + \psi_1||x|| + \psi_2||y||)\mathcal{U}_1 + (\hat{\psi}_0 + \hat{\psi}_1||x|| + \hat{\psi}_2||y||)\mathcal{V}_1 + \|x\|\widehat{\mathcal{U}_1} + \|y\|\widehat{\mathcal{V}_1}. \quad (46)$$

Likewise, we have the ability to get that

$$||y|| \leq (\hat{\psi}_0 + \hat{\psi}_1||x|| + \hat{\psi}_2||y||)\mathcal{V}_2 + (\psi_0 + \psi_1||x|| + \psi_2||y||)\mathcal{U}_2 + \|x\|\widehat{\mathcal{U}_2} + \|y\|\widehat{\mathcal{V}_2}. \quad (47)$$

From (46) and (47), we get

$$\begin{aligned}
||x|| + ||y|| = & \psi_0(\mathcal{U}_1 + \mathcal{U}_2) + \hat{\psi}_0(\mathcal{V}_1 + \mathcal{V}_2) + ||x||\Big[\psi_1(\mathcal{U}_1 + \mathcal{U}_2) + \hat{\psi}_1(\mathcal{V}_1 + \mathcal{V}_2) + \widehat{\mathcal{U}_1} + \widehat{\mathcal{U}_2}\Big] \\
& + ||y||\Big[\psi_1(\mathcal{U}_1 + \mathcal{U}_2) + \hat{\psi}_1(\mathcal{V}_1 + \mathcal{V}_2) + \widehat{\mathcal{V}_1} + \widehat{\mathcal{V}_2}\Big],
\end{aligned}$$

which results, with $||(x,y)|| = ||x|| + ||y||$,

$$||(x,y)|| \leq \frac{\psi_0(\mathcal{U}_1 + \mathcal{U}_2) + \hat{\psi}_0(\mathcal{V}_1 + \mathcal{V}_2)}{\Phi}.$$

Thus, $\Pi(\Psi)$ is bounded. Hence the operator Ψ has a fixed point by Leray–Schauder nonlinear alternative [37], which corresponds to at least one solution of the problem (5) and (6) on $\mathcal{E}$. □

Theorem 2. *If the assumption $(\mathcal{A}_2)$ is satisfied, then the problem (5) and (6) has a unique solution on $\mathcal{E}$. And there exist $\mathcal{S}_1, \mathcal{S}_2 > 0$ such that $|f(\iota,0,0)| \leq \mathcal{S}_1, |g(\iota,0,0)| \leq \mathcal{S}_2$, Then, given that*

$$(\mathcal{U}_1 + \mathcal{U}_2)(\psi_1 + \psi_2) + (\mathcal{V}_1 + \mathcal{V}_2)(\hat{\psi}_1 + \hat{\psi}_2) + (\widehat{\mathcal{U}_1} + \widehat{\mathcal{U}_2}) + (\widehat{\mathcal{V}_1} + \widehat{\mathcal{V}_2}) < 1, \quad (48)$$

where $\mathcal{U}_1, \mathcal{V}_1, \widehat{\mathcal{U}_1}, \widehat{\mathcal{V}_1}, \mathcal{U}_2, \mathcal{V}_2, \widehat{\mathcal{U}_2}, \widehat{\mathcal{V}_2}$ are given by (34)–(41) respectively.

Proof. Let us fix $\varphi \leq \frac{(\mathcal{U}_1+\mathcal{U}_2)\mathcal{S}_1+(\mathcal{V}_1+\mathcal{V}_2)\mathcal{S}_2}{1-((\mathcal{U}_1+\mathcal{U}_2)(\psi_1+\psi_2)+(\mathcal{V}_1+\mathcal{V}_2)(\hat{\psi}_1+\hat{\psi}_2)+\widehat{\mathcal{U}_2})+(\widehat{\mathcal{V}_1}+\widehat{\mathcal{V}_2}))}$ and demonstrate that $\Psi\mathcal{B}_\varphi \subset \mathcal{B}_\varphi$ when operator Ψ is given by (31) and $\mathcal{B}_\varphi = \{(x,y) \in \mathcal{P} \times \mathcal{Q} : ||(x,y)|| \leq \varphi\}$. For $(x,y) \in \mathcal{B}_\varphi, \iota \in \mathcal{E}$

$$\begin{aligned}
|f(\iota, x(\iota), y(\iota))| & \leq \psi_1|x(\iota)| + \psi_2|y(\iota)| + \mathcal{S}_1 \\
& \leq \psi_1||x|| + \psi_2||y|| + \mathcal{S}_1,
\end{aligned}$$

and

$$|g(\iota,x(\iota),y(\iota))| \leq \hat{\psi}_1|x(\iota)| + \hat{\psi}_2|y(\iota)| + \mathcal{S}_2 \\ \leq \hat{\psi}_1||x|| + \hat{\psi}_2||y|| + \mathcal{S}_2. \tag{49}$$

This guides to

$$\begin{aligned}
&|\Psi_1(x,y)(\iota)| \\
\leq &{}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|f(\iota,x(\iota),y(\iota)) - f(\iota,0,0)| + |f(\iota,0,0)| + |\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\iota)| \\
&+ |\delta_1(\iota)|\Bigg[|\mathcal{E}_2|\Big(|\epsilon|^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}|g((\varpi,x(\varpi),y(\varpi)) - g(\varpi,0,0)| + |g(\varpi,0,0)|) \\
&+ |\epsilon||\phi_2|^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}|y(\varpi)| + {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|f(\mathcal{S},x(\mathcal{S}),y(\mathcal{S})) - f(\mathcal{S},0,0)| + |f(\mathcal{S},0,0)| \\
&\quad + |\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\mathcal{S})|\Big) \\
&+ |\mathcal{E}_1|\Big(|\pi|^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}|f(\sigma,x(\sigma),y(\sigma)) - f(\sigma,0,0)| + |f(\sigma,0,0)| \\
&\quad + |\pi||\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}|x(\sigma)| \\
&+ {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}|g((\mathcal{S},x(\mathcal{S}),y(\mathcal{S})) - g(\mathcal{S},0,0)| + |g(\mathcal{S},0,0)|) + |\phi_2|^{\rho}\mathcal{I}_{0^+}^{\zeta_2}|y(\mathcal{S})|\Big)\Bigg]
\end{aligned}$$

$$\begin{aligned}
\leq &(\psi_1||x|| + \psi_2||y|| + \mathcal{S}_1)\left\{\frac{\left(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(1+|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right\} \\
&+ (\hat{\psi}_1||x|| + \hat{\psi}_2||y|| + \mathcal{S}_2)\left\{|\delta_1|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right)\right\} \\
&+ \left\{|\phi_1|\left(\frac{(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|))}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\right\}\|x\| \\
&+ \left\{|\delta_1||\phi_2|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\right\}\|y\|
\end{aligned}$$

$$||\Psi_1(x,y)|| \leq (\psi_1||x|| + \psi_2||y|| + \mathcal{S}_1)\mathcal{U}_1 + (\hat{\psi}_1||x|| + \hat{\psi}_2||y|| + \mathcal{S}_2)\mathcal{V}_1 + \|x\|\widehat{\mathcal{U}_1} + \|y\|\widehat{\mathcal{V}_1}. \tag{50}$$

Similarly, we obtain

$$\begin{aligned}
|\Psi_2(x,y)(\iota)| \leq &(\hat{\psi}_1||x|| + \hat{\psi}_2||y|| + \mathcal{S}_2)\left\{\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right\} \\
&+ (\psi_1||x|| + \psi_2||y|| + \mathcal{S}_1)\left\{|\delta_2|\left(\frac{\mathcal{S}^{\rho(\xi_1+\zeta_1)}|\widehat{\mathcal{E}}_2|}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right)\right\} \\
&+ \left\{|\phi_2|\left(\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\right\}\|y\| \\
&+ \left\{|\delta_2||\phi_1|\left(\frac{\mathcal{S}^{\rho\zeta_1}|\widehat{\mathcal{E}}_2|}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\right\}\|x\|
\end{aligned}$$

$$||\Psi_2(x,y)|| \leq (\hat{\psi}_1||x|| + \hat{\psi}_2||y|| + \mathcal{S}_2)\mathcal{K}_2 + (\psi_1||x|| + \psi_2||y|| + \mathcal{S}_1)\mathcal{J}_2 + \|x\|\widehat{\mathcal{U}_2} + \|y\|\widehat{\mathcal{V}_2}. \tag{51}$$

As a result, (50) and (51) follow $||\Psi(x,y)|| \leq \varphi$, and thus $\Psi\mathcal{B}_\varphi \subset \mathcal{B}_\varphi$. Now, for $(x_1,y_1),(x_2,y_2) \in \mathcal{P} \times \mathcal{Q}$ and any $\iota \in \mathcal{E}$, we get

$$
\begin{aligned}
&|\Psi_1(x_1,y_1)(\iota)-\Psi_1(x_2,y_2)(\iota)| \\
\leq&{}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|f(\iota,x_1(\iota),y_1(\iota))-f(\iota,x_2(\iota),y_2(\iota))|+|\phi_1|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x_1(\iota)-x_2(\iota)| \\
&+|\delta_1(\iota)|\bigg[|\mathcal{E}_2|\Big(|\epsilon|{}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}|g(\varpi,x_1(\varpi),y_1(\varpi))-g(\varpi,x_2(\varpi),y_2(\varpi))| \\
&+|\epsilon||\phi_2|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}|y_1(\varpi)-y_2(\varpi)| \\
&+{}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|f(\mathcal{S},x_1(\mathcal{S}),y_1(\mathcal{S}))-f(\mathcal{S},x_2(\mathcal{S}),y_2(\mathcal{S}))|+|\phi_1|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x_1(\mathcal{S})-x_2(\mathcal{S})|\Big) \\
&+|\mathcal{E}_1|\Big(|\pi|{}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}|f(\sigma,x_1(\sigma),y_1(\sigma))-f(\sigma,x_2(\sigma),y_2(\sigma))| \\
&+|\pi||\phi_1|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}|x_1(\sigma)-x_2(\sigma)| \\
&+{}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}|g(\mathcal{S},x_1(\mathcal{S}),y_1(\mathcal{S}))-g(\mathcal{S},x_2(\mathcal{S}),y_2(\mathcal{S}))|+|\phi_2|{}^{\rho}\mathcal{I}_{0^+}^{\zeta_2}|y_1(\mathcal{S})-y_2(\mathcal{S})|\Big)\bigg] \\
\leq&(\psi_1||x_1-x_2||+\psi_2||y_1-y_2||)\left\{\frac{\left(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(1+|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)}+\frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right\} \\
&+(\hat{\psi}_1||x_1-x_2||+\hat{\psi}_2||y_1-y_2||)\left\{|\delta_1|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)}+\frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right)\right\} \\
&+\left\{|\phi_1|\left(\frac{\left(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\zeta_1}\Gamma(\zeta_1+1)}+\frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\right\}||x_1-x_2|| \\
&+\left\{|\delta_1||\phi_2|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)}+\frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\right\}||y_1-y_2|| \\
\leq&\ (\mathcal{U}_1(\psi_1+\psi_2)+\mathcal{V}_1(\hat{\psi}_1+\hat{\psi}_2)+\widehat{\mathcal{U}_1}+\widehat{\mathcal{V}_1})(||x_1-x_2||+||y_1-y_2||).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&|\Psi_2(x_1,y_1)(\iota)-\Psi_2(x_2,y_2)(\iota)| \\
\leq&(\hat{\psi}_1||x_1-x_2||+\hat{\psi}_2||y_1-y_2||)\left\{\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)}+\frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right\} \\
&+(\psi_1||x_1-x_2||+\psi_2||y_1-y_2||)\left\{|\delta_2|\left(\frac{\mathcal{S}^{\rho(\xi_1+\zeta_1)}|\widehat{\mathcal{E}}_2|}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)}+\frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right)\right\} \\
&+\left\{|\phi_2|\left(\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)}+\frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\right\}||y_1-y_2|| \\
&+\left\{|\delta_2||\phi_1|\left(\frac{\mathcal{S}^{\rho\zeta_1}|\widehat{\mathcal{E}}_2|}{\rho^{\zeta_1}\Gamma(\zeta_1+1)}+\frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\right\}||x_1-x_2|| \\
\leq&\ (\mathcal{U}_2(\psi_1+\psi_2)+\mathcal{V}_2(\hat{\psi}_1+\hat{\psi}_2)+\widehat{\mathcal{U}_2}+\widehat{\mathcal{V}_2})(||x_1-x_2||+||y_1-y_2||).
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
||\Psi_1(x_1,y_1)(\iota)-\Psi_1(x_2,y_2)(\iota)||\leq(\mathcal{U}_1(\psi_1+\psi_2)+\mathcal{V}_1(\hat{\psi}_1+\hat{\psi}_2)+\widehat{\mathcal{U}_1}+\widehat{\mathcal{V}_1}) \\
(||x_1-x_2||+||y_1-y_2||).
\end{aligned} \tag{52}
$$

In a similar manner,

$$||\Psi_2(x_1,y_1)(\iota)-\Psi_2(x_2,y_2)(\iota)|| \leq (\mathcal{U}_2(\psi_1+\psi_2)+\mathcal{V}_2(\hat{\psi}_1+\hat{\psi}_2)+\widehat{\mathcal{U}_2}+\widehat{\mathcal{V}_2}) \\ (||x_1-x_2||+||y_1-y_2||). \tag{53}$$

Hence , using (52) and (53) we can get

$$||\Psi(x_1,y_1)(\iota)-\Psi(x_2,y_2)(\iota)|| \leq ((\mathcal{U}_1+\mathcal{U}_2)(\psi_1+\psi_2)+(\mathcal{V}_1+\mathcal{V}_2)(\hat{\psi}_1+\hat{\psi}_2) \\ +(\widehat{\mathcal{U}_1}+\widehat{\mathcal{U}_2})+(\widehat{\mathcal{V}_1}+\widehat{\mathcal{V}_2}))(||x_1-x_2||+||y_1-y_2||).$$

As a consequence of condition $((\mathcal{U}_1+\mathcal{U}_2)(\psi_1+\psi_2)+(\mathcal{V}_1+\mathcal{V}_2)(\hat{\psi}_1+\hat{\psi}_2)+(\widehat{\mathcal{U}_1}+\widehat{\mathcal{U}_2})+(\widehat{\mathcal{V}_1}+\widehat{\mathcal{V}_2}))<1$, Ψ is an operator for contraction. Hence the operator has a unique fixed point by Banach's contraction principle [37], which corresponds to a unique solution of the problem (5), and (6). □

4. Example

Consider the following problem:

$$\begin{cases} {}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{5}{4}}_{0+}({}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{1}{2}}_{0+}+\frac{1}{100})x(\iota)=f(\iota,x(\iota),y(\iota)), \iota\in\mathcal{E}:=[0,1], \\ {}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{7}{4}}_{0+}({}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{4}{5}}_{0+}+\frac{1}{150})y(\iota)=g(\iota,x(\iota),y(\iota)), \iota\in\mathcal{E}:=[0,1], \end{cases} \tag{54}$$

$$\Big\{x(0)=0,\ y(0)=0,\ x(1)=\tfrac{9}{50}{}^{\frac{19}{25}}\mathcal{I}^{\frac{33}{50}}y(\tfrac{18}{25}),\ y(1)=\tfrac{4}{25}{}^{\frac{19}{25}}\mathcal{I}^{\frac{43}{50}}x(\tfrac{11}{20}), \tag{55}$$

where $\xi_1=\frac{5}{4},\xi_2=\frac{7}{4},\zeta_1=\frac{1}{2},\zeta_2=\frac{4}{5},\rho=\frac{19}{25},\mathcal{S}=1,\epsilon=\frac{9}{50},\varpi=\frac{18}{25},\pi=\frac{4}{25},\sigma=\frac{11}{20},\varsigma=\frac{33}{50},\varrho=\frac{43}{50}$ and

$$f(\iota,x(\iota),y(\iota))=\frac{(\iota+1)}{200}\left(\frac{|y(\iota)|}{1+|y(\iota)|}+\frac{1}{4}\cos(x(\iota))+5\iota\right), \tag{56}$$

$$g(\iota,x(\iota),y(\iota))=\frac{e^{-\iota}}{100}\left(\frac{1+\sqrt{\iota}}{4}+\frac{1}{7}\cos(y(\iota))+\frac{|x(\iota)|}{3(1+|x(\iota)|)}\right). \tag{57}$$

With $\psi_0=\frac{1}{40},\psi_1=\frac{1}{800},\psi_2=\frac{1}{200},\hat{\psi}_0=\frac{1}{400},\hat{\psi}_1=\frac{1}{300}$, and $\hat{\psi}_2=\frac{1}{700}$, using $(\mathcal{A}_1)$ condition. Next, we find that $\mathcal{U}_1=3.8212184102805633,\mathcal{V}_1=13.84209017173257,\mathcal{U}_2=2.2659612415346384$, $\mathcal{V}_2=14.222771283122185$, $\widehat{\mathcal{U}_1}=0.07786603271240089$, $\widehat{\mathcal{V}_1}=0.27980001669983334,\widehat{\mathcal{U}_2}=0.053808908037849844,\widehat{\mathcal{V}_2}=0.2869951632752146$, $\mathcal{U}_i,\mathcal{V}_i,\widehat{\mathcal{U}_i},\widehat{\mathcal{V}_i}(i=1,2)$ are respectively given by (34)–(41). Thus $\psi_1(\mathcal{U}_1+\mathcal{U}_2)+\hat{\psi}_1(\mathcal{V}_1+\mathcal{V}_2)+\widehat{\mathcal{U}_1}+\widehat{\mathcal{U}_2}\approxeq 0.23283345349786894<1,\psi_2(\mathcal{U}_1+\mathcal{U}_2)+\hat{\psi}_2(\mathcal{V}_1+\mathcal{V}_2)+\widehat{\mathcal{V}_1}+\widehat{\mathcal{V}_2}\approxeq 0.637323737455345<1$, hence, Theorem 1 can be applied to the problem (54) and (55).

Moreover, we will employ

$$f(\iota,x(\iota),y(\iota))=\frac{\iota}{20}+\frac{1}{500}\frac{|y(\iota)|}{1+|y(\iota)|}+\frac{3}{800}\cos(x(\iota)), \tag{58}$$

$$g(\iota,x(\iota),y(\iota))=\frac{(e^{-\iota}+1)}{50}+\frac{1}{400}\cos(y(\iota))+\frac{1}{700}\frac{|x(\iota)|}{1+|x(\iota)|}, \tag{59}$$

to illustrate Theorem 2. Using the assumption $(\mathcal{A}_2)$ with $\psi_1=\frac{3}{800},\psi_2=\frac{1}{500},\hat{\psi}_1=\frac{1}{700}$ and $\hat{\psi}_2=\frac{1}{400}$. The assumptions of Theorem 2 are also satisfied with $(\mathcal{U}_1+\mathcal{U}_2)(\psi_1+\psi_2)+(\mathcal{V}_1+\mathcal{V}_2)(\hat{\psi}_1+\hat{\psi}_2)+(\widehat{\mathcal{U}_1}+\widehat{\mathcal{U}_2})+(\widehat{\mathcal{V}_1}+\widehat{\mathcal{V}_2})\approxeq 0.8437262165815941<1$. Hence, Theorem 2 can be applied to the problem (54) and (55).

For brevity, use these notations:

$$\mathcal{U}_1 = \frac{\left(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(1+|\delta_1||\mathcal{E}_2|)\right)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)} + |\phi_1|\left(\frac{(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|))}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right), \tag{60}$$

$$\mathcal{V}_1 = |\delta_1|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right) + |\delta_1||\phi_2|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right), \tag{61}$$

$$\mathcal{U}_2 = |\delta_2|\left(\frac{\mathcal{S}^{\rho(\xi_1+\zeta_1)}|\widehat{\mathcal{E}}_2|}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right) + |\delta_2||\phi_1|\left(\frac{\mathcal{S}^{\rho\zeta_1}|\widehat{\mathcal{E}}_2|}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right), \tag{62}$$

$$\mathcal{V}_2 = \frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)} + |\phi_2|\left(\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right), \tag{63}$$

5. Ulam–Hyers Stability Results

With the help of an integral formulation of the solution provided by

$$x(\iota) = \Psi_1(x,y)(\iota),\ y(\iota) = \Psi_2(x,y)(\iota), \tag{64}$$

where Ψ_1 and Ψ_2 are given by (32) and (33). We analyse the Ulam–Hyers stability for problem (5) and (6) in this section. Consider the following definitions of nonlinear operators

$$\mathcal{H}_1, \mathcal{H}_2 \in \mathcal{C}(\mathcal{E},\mathbb{R}) \times \mathcal{C}(\mathcal{E},\mathbb{R}) \to \mathcal{C}(\mathcal{E},\mathbb{R}),$$

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi_1}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_1}_{0^+} + \phi_1)x(\iota) - f(\iota, x(\iota), y(\iota)) = \mathcal{H}_1(x,y)(\iota), \iota \in \mathcal{E}, \\ {}^{\rho}_{C}\mathcal{D}^{\xi_2}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_2}_{0^+} + \phi_2)y(\iota) - g(\iota, x(\iota), y(\iota)) = \mathcal{H}_1(x,y)(\iota), \iota \in \mathcal{E}. \end{cases}$$

For some $\hat{\lambda}_1, \hat{\lambda}_2 > 0$, taking into consideration the following inequality::

$$||\mathcal{H}_1(x,y)|| \le \hat{\lambda}_1, ||\mathcal{H}_2(x,y)|| \le \hat{\lambda}_2. \tag{65}$$

Definition 2. *The system (5) and (6) is U–H stable if* $\exists\, \mathcal{V}_1, \mathcal{V}_2 > 0 \ni$, *for every solution* $(x^*, y^*) \in \mathcal{C}(\mathcal{E},\mathbb{R}) \times \mathcal{C}(\mathcal{E},\mathbb{R})$ *of inequality* (65), $\exists$ *a unique solution on* $(x,y) \in \mathcal{C}(\mathcal{E},\mathbb{R}) \times \mathcal{C}(\mathcal{E},\mathbb{R})$, *of problem (5) and (6) with*

$$||(x,y) - (x^*, y^*)|| \le \mathcal{V}_1\hat{\lambda}_1 + \mathcal{V}_2\hat{\lambda}_2.$$

Theorem 3. *If the assumption* $(\mathcal{A}_2)$ *is satisfied, then the BVP (5) and (6) is U–H stable.*

Proof. Let $(x,y) \in \mathcal{C}(\mathcal{E},\mathbb{R}) \times \mathcal{C}(\mathcal{E},\mathbb{R})$ be the solution of the BVP (5) and (6) satisfying (32) and (33). Let (x,y) be any solution satisfying (65):

$$\begin{cases} {}^{\rho}_{C}\mathcal{D}^{\xi_1}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_1}_{0^+} + \phi_1)x(\iota) = f(\iota, x(\iota), y(\iota)) + \mathcal{H}_1(x,y)(\iota), \iota \in \mathcal{E}, \\ {}^{\rho}_{C}\mathcal{D}^{\xi_2}_{0^+}({}^{\rho}_{C}\mathcal{D}^{\zeta_2}_{0^+} + \phi_2)y(\iota) = g(\iota, x(\iota), y(\iota)) + \mathcal{H}_1(x,y)(\iota), \iota \in \mathcal{E}, \end{cases}$$

so,

$$
\begin{aligned}
x^*(\iota) = {} & \Psi_1(x^*,y^*)(\iota) +^{\rho} \mathcal{I}_{0^+}^{\xi_1+\zeta_1}\mathcal{H}_1(x,y)(\iota) - \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\iota) \\
& + \delta_1(\iota)\Bigg[\mathcal{E}_2\Big(\epsilon^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}\mathcal{H}_2(x,y)(\varpi) \\
& - \epsilon\phi_2{}^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}y(\varpi) - {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}\mathcal{H}_1(x,y)(\mathcal{S}) + \phi_1^{\rho}\mathcal{I}_{0^+}^{\zeta_1}x(\mathcal{S})\Big) \\
& + \mathcal{E}_1\Big(\pi^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}\mathcal{H}_1(x,y)(\sigma) - \pi\phi_1{}^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}x(\sigma) \\
& - {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}\mathcal{H}_2(x,y)(\mathcal{S}) + \phi_2^{\rho}\mathcal{I}_{0^+}^{\zeta_2}y(\mathcal{S})\Big)\Bigg].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& |\Psi_1(x^*,y^*)(\iota) - x^*(\iota)| \\
\le & {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|\mathcal{H}_1(x,y)(\iota)| + |\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\iota)| \\
& + |\delta_1(\iota)|\Bigg[|\mathcal{E}_2|\Big(|\epsilon|^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2+\varsigma}|\mathcal{H}_2(x,y)(\varpi)| \\
& + |\epsilon||\phi_2|^{\rho}\mathcal{I}_{0^+}^{\zeta_2+\varsigma}|y(\varpi)| + {}^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1}|\mathcal{H}_1(x,y)(\mathcal{S})| + |\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1}|x(\mathcal{S})|\Big) \\
& + |\mathcal{E}_1|\Big(|\pi|^{\rho}\mathcal{I}_{0^+}^{\xi_1+\zeta_1+\varrho}\mathcal{H}_1(x,y)(\sigma) + |\pi||\phi_1|^{\rho}\mathcal{I}_{0^+}^{\zeta_1+\varrho}|x(\sigma)| \\
& + {}^{\rho}\mathcal{I}_{0^+}^{\xi_2+\zeta_2}|\mathcal{H}_2(x,y)(\mathcal{S})| + |\phi_2|^{\rho}\mathcal{I}_{0^+}^{\zeta_2}|y(\mathcal{S})|\Big)\Bigg] \\
\le & \ \hat{\lambda}_1\Bigg\{\frac{\Big(\mathcal{S}^{\rho(\xi_1+\zeta_1)}(1+|\delta_1||\mathcal{E}_2|)\Big)}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)} \\
& + |\phi_1|\left(\frac{(\mathcal{S}^{\rho\zeta_1}(1+|\delta_1||\mathcal{E}_2|))}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\delta_1||\pi||\mathcal{E}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\Bigg\} \\
& + \hat{\lambda}_2\Bigg\{|\delta_1|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)}\right) \\
& + |\delta_1||\phi_2|\left(\frac{|\mathcal{E}_1|\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\mathcal{E}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\Bigg\} \\
\le & \mathcal{U}_1\hat{\lambda}_1 + \mathcal{V}_1\hat{\lambda}_2.
\end{aligned}
$$

Similarly , we obtain

$$
\begin{aligned}
& |\Psi_2(x^*,y^*)(\iota) - y^*(\iota)| \\
\le & \hat{\lambda}_2\Bigg\{\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho(\xi_2+\zeta_2)}}{\rho^{\xi_2+\zeta_2}\Gamma(\xi_2+\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\xi_2+\zeta_2+\varsigma)}}{\rho^{\xi_2+\zeta_2+\varsigma}\Gamma(\xi_2+\zeta_2+\varsigma+1)} \\
& + |\phi_2|\left(\frac{(1+|\delta_2||\widehat{\mathcal{E}}_1|)\mathcal{S}^{\rho\zeta_2}}{\rho^{\zeta_2}\Gamma(\zeta_2+1)} + \frac{|\delta_2||\widehat{\mathcal{E}}_2||\epsilon|\varpi^{\rho(\zeta_2+\varsigma)}}{\rho^{\zeta_2+\varsigma}\Gamma(\zeta_2+\varsigma+1)}\right)\Bigg\} \\
& + \hat{\lambda}_1\Bigg\{|\delta_2|\left(\frac{\mathcal{S}^{\rho(\xi_1+\zeta_1)}|\widehat{\mathcal{E}}_2|}{\rho^{\xi_1+\zeta_1}\Gamma(\xi_1+\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\xi_1+\zeta_1+\varrho)}}{\rho^{\xi_1+\zeta_1+\varrho}\Gamma(\xi_1+\zeta_1+\varrho+1)}\right) \\
& + |\delta_2||\phi_1|\left(\frac{\mathcal{S}^{\rho\zeta_1}|\widehat{\mathcal{E}}_2|}{\rho^{\zeta_1}\Gamma(\zeta_1+1)} + \frac{|\pi||\widehat{\mathcal{E}}_1|\sigma^{\rho(\zeta_1+\varrho)}}{\rho^{\zeta_1+\varrho}\Gamma(\zeta_1+\varrho+1)}\right)\Bigg\} \\
\le & \mathcal{U}_2\hat{\lambda}_1 + \mathcal{V}_2\hat{\lambda}_2,
\end{aligned}
$$

where $\mathcal{U}_1, \mathcal{V}_1, \mathcal{U}_2$, and $\mathcal{V}_2$ are defined in (60)–(63), respectively. Consequently, based on the fixed point property of the operator Ψ, provided in (32) and (33), we derive that

$$\begin{aligned} |x(\iota) - x^*(\iota)| =& |x(\iota) - \Psi_1(x^*, y^*)(\iota) + \Psi_1(x^*, y^*)(\iota) - x^*(\iota)| \\ \leq& |\Psi_1(x, y)(\iota) - \Psi_1(x^*, y^*)(\iota)| + |\Psi_1(x^*, y^*)(\iota) - x^*(\iota)| \\ \leq& ((\mathcal{U}_1\psi_1 + \mathcal{V}_1\hat{\psi}_1) + (\mathcal{U}_1\psi_2 + \mathcal{V}_1\hat{\psi}_2))||(x, y) - (x^*, y^*)|| \\ &+ \mathcal{U}_1\hat{\lambda}_1 + \mathcal{V}_1\hat{\lambda}_2. \end{aligned} \tag{66}$$

$$\begin{aligned} |y(\iota) - y^*(\iota)| =& |y(\iota) - \Psi_2(x^*, y^*)(\iota) + \Psi_2(x^*, y^*)(\iota) - y^*(\iota)| \\ \leq& |\Psi_2(x, y)(\iota) - \Psi_2(x^*, y^*)(\iota)| + |\Psi_2(x^*, y^*)(\iota) - y^*(\iota)| \\ \leq& ((\mathcal{U}_2\psi_1 + \mathcal{V}_2\hat{\psi}_1) + (\mathcal{U}_2\psi_2 + \mathcal{V}_2\hat{\psi}_2))||(x, y) - (x^*, y^*)|| \\ &+ \mathcal{U}_2\hat{\lambda}_1 + \mathcal{V}_2\hat{\lambda}_2. \end{aligned} \tag{67}$$

From the above equations (66) and (67) it follows that

$$\begin{aligned} ||(x, y) - (x^*, y^*)|| \leq& (\mathcal{U}_1 + \mathcal{U}_2)\hat{\lambda}_1 + (\mathcal{V}_1 + \mathcal{V}_2)\hat{\lambda}_2 \\ &+ ((\mathcal{U}_1 + \mathcal{U}_2)(\psi_1 + \psi_2) + (\mathcal{V}_1 + \mathcal{V}_2)(\hat{\psi}_1 + \hat{\psi}_2))||(x, y) - (x^*, y^*)||. \end{aligned}$$

$$\begin{aligned} ||(x, y) - (x^*, y^*)|| \leq& \frac{(\mathcal{U}_1 + \mathcal{U}_2)\hat{\lambda}_1 + (\mathcal{V}_1 + \mathcal{V}_2)\hat{\lambda}_2}{1 - ((\mathcal{U}_1 + \mathcal{U}_2)(\psi_1 + \psi_2) + (\mathcal{V}_1 + \mathcal{V}_2)(\hat{\psi}_1 + \hat{\psi}_2))} \\ \leq& \mathcal{V}_1\hat{\lambda}_1 + \mathcal{V}_2\hat{\lambda}_2, \end{aligned}$$

with

$$\begin{aligned} \mathcal{V}_1 &= \frac{\mathcal{U}_1 + \mathcal{U}_2}{1 - ((\mathcal{U}_1 + \mathcal{U}_2)(\psi_1 + \psi_2) + (\mathcal{V}_1 + \mathcal{V}_2)(\hat{\psi}_1 + \hat{\psi}_2))}, \\ \mathcal{V}_2 &= \frac{\mathcal{V}_1 + \mathcal{V}_2}{1 - ((\mathcal{U}_1 + \mathcal{U}_2)(\psi_1 + \psi_2) + (\mathcal{V}_1 + \mathcal{V}_2)(\hat{\psi}_1 + \hat{\psi}_2))}. \end{aligned}$$

Thus, the BVP (5) and (6) is U–H stable. □

6. Example

Consider the following problem:

$$\begin{cases} {}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{5}{4}}_{0+}({}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{1}{2}}_{0+} + \frac{1}{100})x(\iota) = \frac{\sqrt{\iota}}{30} + \frac{1}{40(16+\iota)}\frac{|y(\iota)|}{1+|y(\iota)|} + \frac{4}{900}\cos(x(\iota)), \iota \in [0,1], \\ {}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{7}{4}}_{0+}({}^{\frac{19}{25}}_{C}\mathcal{D}^{\frac{4}{5}}_{0+} + \frac{1}{150})y(\iota) = \frac{\iota}{70} + \frac{1}{400}\cos(y(\iota)) + \frac{1}{800}\frac{|x(\iota)|}{1+|x(\iota)|}, \iota \in [0,1], \end{cases} \tag{68}$$

$$\begin{cases} x(0) = 0,\ y(0) = 0,\ x(1) = \frac{9}{50}\,{}^{\frac{19}{25}}\mathcal{I}^{\frac{33}{50}}y(\frac{18}{25}),\ y(1) = \frac{4}{25}\,{}^{\frac{19}{25}}\mathcal{I}^{\frac{43}{50}}x(\frac{11}{20}), \end{cases} \tag{69}$$

where $\xi_1 = \frac{5}{4}, \xi_2 = \frac{7}{4}, \zeta_1 = \frac{1}{2}, \zeta_2 = \frac{4}{5}, \rho = \frac{19}{25}, \mathcal{S} = 1, \epsilon = \frac{9}{50}, \varpi = \frac{18}{25}, \pi = \frac{4}{25}, \sigma = \frac{11}{20}, \varsigma = \frac{33}{50}, \varrho = \frac{43}{50}$ and

$$|f(\iota, x_1(\iota), y_1(\iota)) - f(\iota, x_2(\iota), y_2(\iota))| = \frac{4}{900}|x_1(\iota) - x_2(\iota)| + \frac{1}{640}|y_1(\iota) - y_2(\iota)|, \tag{70}$$

$$|g(\iota, x_1(\iota), y_1(\iota)) - g(\iota, x_2(\iota), y_2(\iota))| = \frac{1}{800}|x_1(\iota) - x_2(\iota)| + \frac{1}{400}|y_1(\iota) - y_2(\iota)|. \tag{71}$$

With $\psi_1 = \frac{4}{900}, \psi_2 = \frac{1}{640}, \hat{\psi}_1 = \frac{1}{800}$, and $\hat{\psi}_2 = \frac{1}{400}$, the $(\mathcal{A}_2)$ condition. Next, we find that $\mathcal{U}_1 = 3.8990844429929643, \mathcal{V}_1 = 14.121890188432403, \mathcal{U}_2 = 2.3197701495724883, \mathcal{V}_2 = 14.5097664463974, \mathcal{U}_i, \mathcal{V}_i$ are respectively given by (60)–(63). Thus $((\mathcal{U}_1 + \mathcal{U}_2)(\psi_1 + \psi_2) + (\mathcal{V}_1 + \mathcal{V}_2)(\hat{\psi}_1 + \hat{\psi}_2)) \approxeq 0.8402787011728106 < 1$, hence, Theorem 3 can be applied to the problem (68) and (69).

7. Asymmetric Results

Remark 1. *The generalized Langevin FDEs in the problem (5) simplify to the Caputo Langevin FDEs if* $\rho = 1$.

$$\begin{cases} {}_C\mathcal{D}^{\xi_1}_{0^+}({}_C\mathcal{D}^{\zeta_1}_{0^+} + \phi_1)x(\iota) = f(\iota, x(\iota), y(\iota)), \iota \in \mathcal{E} := [0, \mathcal{S}], \\ {}_C\mathcal{D}^{\xi_2}_{0^+}({}_C\mathcal{D}^{\zeta_2}_{0^+} + \phi_2)y(\iota) = g(\iota, x(\iota), y(\iota)), \iota \in \mathcal{E} := [0, \mathcal{S}]. \end{cases} \tag{72}$$

Remark 2. *If the value of* $\rho = 1$ *in the boundary condition (6) Katugampola integral reduces to the fractional integral of Riemann–Liouville.*

$$\begin{cases} x(0) = 0, \quad y(0) = 0, \\ x(\mathcal{S}) = \epsilon \mathcal{I}^{\varsigma}_{0+} y(\varpi) = \frac{\epsilon}{\Gamma(\varsigma)} \int_0^{\varpi} (\varpi - \theta)^{\varsigma - 1} y(\theta) d\theta, \\ y(\mathcal{S}) = \pi \mathcal{I}^{\varrho}_{0+} x(\sigma) = \frac{\pi}{\Gamma(\varrho)} \int_0^{\sigma} (\sigma - \theta)^{\varrho - 1} x(\theta) d\theta, \\ 0 < \sigma < \varpi < \mathcal{S}. \end{cases} \tag{73}$$

Remark 3. *If the value of* $\rho = 1$ *and* $\varsigma = \varrho = 1$ *in the boundary condition (6) Katugampola integral reduces to the integral of classical.*

$$\left\{ x(0) = 0, \; y(0) = 0, \; x(\mathcal{S}) = \epsilon \int_0^{\varpi} y(\theta) d\theta, \; y(\mathcal{S}) = \pi \int_0^{\sigma} x(\theta) d\theta \; 0 < \sigma < \varpi < \mathcal{S}. \right. \tag{74}$$

8. Conclusions

In our contribution, we presented the existence of solutions for a generalized Liouville–Caputo type fractional Langevin coupled system with certain boundary conditions utilizing the generalized fractional integrals. To get at our result, we used the Leray–Schauder and Banach fixed-point theorems, and we included examples to help explain our study results. By use of conventional functional analysis, we demonstrated Ulam–Hyers stability. Our findings in this context are original and contribute to the body of knowledge on generalized fractional integral operators that are used to resolve coupled generalized Langevin fractional differential equations with nonlocal boundary conditions. We have highlighted the topic's asymmetries in the remarks. The form of the solution in these kinds of statements can be used to conduct additional research on the positive solution and its asymmetry.

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