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On Stability of a General *n*-Linear Functional Equation

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Abstract: Let X be a linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, Y be a real or complex Banach space and $f\colon X^n \to Y$. With some fixed $a_{ji}, C_{i_1\ldots i_n} \in \mathbb{K}$ ($j \in \{1,\ldots,n\}$, $i, i_k \in \{1,2\}$, $k \in \{1,\ldots,n\}$), we study, using the direct and the fixed point methods, the stability and the general stability of the equation $f(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) = \sum_{1\le i_1,\ldots,i_n \le 2} C_{i_1\ldots i_n} f(x_{1i_1}, \ldots, x_{ni_n})$, for all $x_{ji_j} \in X$ ($j \in \{1,\ldots,n\}$, $i_j \in \{1,2\}$). Our paper generalizes several known results, among others concerning equations with symmetric coefficients, such as the multi-Cauchy equation or the multi-Jensen equation as well as the multi-Cauchy–Jensen equation. We also prove the hyperstability of the above equation in m-normed spaces with $m \ge 2$.

Keywords: Hyers–Ulam stability; generalized stability; functional equation; fixed point; nonlinear operator; linear operator

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1. Introduction

A general linear equation of the form

$$f(ax + by) = Af(x) + Bf(y), \tag{1}$$

with a function f acting between linear spaces over the same fields, has a long history. It has been studied for years by many mathematicians (see, e.g., [1–7]), e.g., it is known that, roughly speaking, f is a solution of (1) if and only if there exists an additive function φ and a constant δ such that $f(x) = \varphi(x) + \delta$, $\varphi(ax) = A\varphi(x)$, $\varphi(bx) = B\varphi(x)$, for all x, and $(A + B - 1)\delta = 0$. In [8], the authors were studying a counterpart of (1) for multivariable functions:

$$f(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) = \sum_{1\le i_1,\ldots,i_n \le 2} C_{i_1\ldots i_n} f(x_{1i_1}, \ldots, x_{ni_n}), \tag{2}$$

for all $x_{ji_j} \in X$, $j \in \{1,\ldots,n\}$, $i_j \in \{1,2\}$, where X, Y are linear spaces over a field $\mathbb{K}$, $f\colon X^n \to Y$, and some fixed $a_{ji}, C_{i_1\ldots i_n} \in \mathbb{K}$ for all $j \in \{1,\ldots,n\}$, $i, i_k \in \{1,2\}$, $k \in \{1,\ldots,n\}$.

Our purpose in the paper is to study the stability and the general stability of (2). Considering in general the stability problem we ask how much a slight disturbance of a state affects that state. Many physical processes are described by functional equations and while modeling such processes various deviations and errors occur. Therefore, it is natural to deal with stability problems in such situations. Speaking about stability of functional equations, one usually goes back to a problem posed in 1940 by Ulam (see [9]) which concerned the stability of homomorphisms. The first answer formulated by Hyers in 1941 (see [10]) started very rich and advanced stability investigations. For a comprehensive study of the subject we refer the reader, e.g., to the monograph [11].

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In our paper, we shall present two approaches to the stability problem – the so-called direct method and a fixed point method. The first method we shall apply in Section 2 while proving the Hyers–Ulam stability of (2), that is, when the equation is slightly perturbed and the considered difference is approximated by a constant. Even though the experienced reader could first have the impression that the computations here do not differ from those used for solving the equation in [8], the nature of the present problem needs in fact more sophisticated investigations.

The latter method we shall apply in Section 3 for proving the generalized stability of (2), when the mentioned difference is approximated by a function.

In Section 4 we present a short proof of hyperstability in m-normed spaces with $m \geq 2$. For the convenience of the reader we recall here the definition of m-normed spaces, which was introduced by A. Misiak (see [12]). For more details we refer the reader to [12,13]).

Let $m \in \mathbb{N} \setminus \{1\}$ and Y be an at least m-dimensional real linear space. If a mapping $\|\cdot, \dots, \cdot\| \colon Y^m \to \mathbb{R}$ fulfils the following four conditions:

(i) $\|x_1, \dots, x_m\| = 0$ if and only if $x_1, \dots, x_m$ are linearly dependent,
(ii) $\|x_1, \dots, x_m\|$ is invariant under permutation of $x_1, \dots, x_m$,
(iii) $\|\beta x_1, \dots, x_m\| = |\beta| \|x_1, \dots, x_m\|$,
(iv) $\|x_{11} + x_{12}, x_2, \dots, x_m\| \leq \|x_{11}, x_2, \dots, x_m\| + \|x_{12}, x_2, \dots, x_m\|$,

for every $\beta \in \mathbb{R}$ and $x_{11}, x_{12}, x_1, \dots, x_m \in Y$, then $\|\cdot, \dots, \cdot\|$ is called an m-norm on Y, and the pair $(Y, \|\cdot, \dots, \cdot\|)$ is said to be an m-normed space. We will use a well-known property which immediately follows from the above definition, namely,

if $x \in Y$ and $\|x, x_2, \dots, x_m\| = 0$, for all $x_2, \dots, x_m \in Y$, then $x = 0$.

The hyperstability phenomenon occurs when no deviation of a state affects that state (see, e.g., [14–16]). Proving our result we improve a result from [17], where the stability result was shown.

Our results generalize several known facts. Namely, as corollaries we obtain, for example, the stability results for the multi-Cauchy (3), multi-Jensen (4) and multi-Cauchy–Jensen (5) equations:

$$f(x_{11} + x_{12}, \dots, x_{n1} + x_{n2}) = \sum_{1 \leq i_1, \dots, i_n \leq 2} f(x_{1i_1}, \dots, x_{ni_n}), \tag{3}$$

$$f\Big(\frac{1}{2}x_{11} + \frac{1}{2}x_{12}, \dots, \frac{1}{2}x_{n1} + \frac{1}{2}x_{n2}\Big) = \sum_{1 \leq i_1, \dots, i_n \leq 2} \frac{1}{2^n} f(x_{1i_1}, \dots, x_{ni_n}), \tag{4}$$

$$\begin{aligned} f\Big(x_{11} + x_{12}, \dots, x_{k1} + x_{k2}, \frac{1}{2}x_{k+1,1} + \frac{1}{2}x_{k+1,2}, \dots, \frac{1}{2}x_{n1} + \frac{1}{2}x_{n2}\Big) \\ = \sum_{1 \leq i_1, \dots, i_n \leq 2} \frac{1}{2^{n-k}} f(x_{1i_1}, \dots, x_{ni_n}), \end{aligned} \tag{5}$$

for all $x_{ji_j} \in X$, $j \in \{1, \dots, n\}$, $i_j \in \{1, 2\}$ and fixed $k \in \{0, \dots, n\}$, where X, Y are linear spaces over a field $\mathbb{K}$ and $f \colon X^n \to Y$.

For the convenience of the reader, in what follows, we also cite a result from [8] describing the solutions of (1).

Theorem 1. *Let X, Y be linear spaces over a field $\mathbb{K}$. Let $a_{ji} \in \mathbb{K} \setminus \{0\}$, $C_{i_1 \dots i_n} \in \mathbb{K}$ for all $j \in \{1, \dots, n\}$, $i, i_k \in \{1, 2\}$, $k \in \{1, \dots, n\}$. A function $f \colon X^n \to Y$ satisfies (2) for all $x_{ji_j} \in X$, $j \in \{1, \dots, n\}$, $i_j \in \{1, 2\}$, if and only if there exist k-additive functions $g_{j_1 \dots j_k} \colon X^k \to Y$ $(1 \leq k \leq n)$ and $\delta \in Y$ such that for all $x_1, \dots, x_n \in X$, $k \in \{1, \dots, n\}$,*

$$f(x_1, \dots, x_n) = \delta + \sum_{k=1}^{n} \sum_{\substack{\{j_1, \dots, j_k\} \subseteq \{1, \dots, n\} \\ j_1 < \dots < j_k}} g_{j_1 \dots j_k}(x_{j_1}, \dots, x_{j_k}),$$

for each nonempty subset $\{j_1,\ldots,j_k\}$ *of* $\{1,\ldots,n\}$, $i_{j_1},\ldots,i_{j_k} \in \{1,2\}$ *and* $x_{j_1},\ldots,x_{j_k} \in X$,

$$g_{j_1\ldots j_k}(a_{j_1 i_{j_1}} x_{j_1},\ldots,a_{j_k i_{j_k}} x_{j_k}) = \sum_{\substack{1\le \nu_1,\ldots,\nu_n\le 2\\ \nu_{j_l}:=i_{j_l}, l=1,\ldots,k}} C_{\nu_1\ldots\nu_n} g_{j_1\ldots j_k}(x_{j_1},\ldots,x_{j_k}),$$

and

$$\delta\Big(1 - \sum_{1\le i_1,\ldots,i_n\le 2} C_{i_1\ldots i_n}\Big) = 0. \tag{6}$$

Unless stated differently, in the paper X will denote a linear space over the field of real or complex numbers, and $(Y,\|\cdot\|)$ will be a real or complex Banach space. By $\mathbb{R}_+$, $\mathbb{N}$, $\mathbb{N}_0$ we understand the sets of nonnegative real numbers, positive integers, nonnegative integers, respectively. To shorten the statements we use the notation $\mathbf{n} := \{1,\ldots,n\}$ for $n \in \mathbb{N}$.

2. Hyers–Ulam Stability of (2)

In what follows, let

$$(\Phi f)(x_{11},\ldots,x_{n1},x_{12},\ldots,x_{n2}) := f(a_{11}x_{11} + a_{12}x_{12},\ldots,a_{n1}x_{n1} + a_{n2}x_{n2}) - \sum_{i_1,\ldots,i_n\in\mathbf{2}} C_{i_1\ldots i_n} f(x_{1i_1},\ldots,x_{ni_n}),$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2} \in X$. Let us also denote $C := \sum_{i_1,\ldots,i_n\in\mathbf{2}} C_{i_1\ldots i_n}$.

We present a quite general result not depending on the coefficients a_{ji} (we assume only that they are non-zero) and $C_{i_1\ldots i_n}$. The price for this general approach is the size of the approximating constant. In our first result, we will use the direct method to prove the stability.

We start the section by recalling the Hyers–Ulam stability result for the multi-Cauchy equation (see, e.g., [18,19]).

Lemma 1. *Let* $(H,+)$ *be an abelian group. Given* $\varepsilon > 0$ *assume that* $g\colon H^k \to Y$ *satisfies*

$$\|g(x_{11}+x_{12},\ldots,x_{k1}+x_{k2}) - \sum_{i_1,\ldots,i_k\in\mathbf{2}} g(x_{1i_1},\ldots,x_{ki_k})\| \le \varepsilon,$$

for all $x_{j1},x_{j2} \in H$, $j \in \mathbf{k}$. *Then there exists a k-additive function* $G\colon H^k \to Y$ *such that*

$$\|g(x) - G(x)\| \le \frac{1}{2^k - 1}\varepsilon, \quad x \in H^k.$$

Now, we can present the main result of this section.

Theorem 2. *Given* $\varepsilon > 0$, *let* $f\colon X^n \to Y$ *be a mapping such that*

$$\|(\Phi f)(x_{11},\ldots,x_{n1},x_{12},\ldots,x_{n2})\| \le \varepsilon, \tag{7}$$

for $x_{11},x_{12},\ldots,x_{n1},x_{n2} \in X$. *Then there exists a solution* $F\colon X^n \to Y$ *of* (2) *such that*

$$\|f(x_1,\ldots,x_n) - f(0,\ldots,0) - F(x_1,\ldots,x_n)\| \le (3^n + 2^{n+1} - 3)\,\varepsilon, \tag{8}$$

for all $x_1,\ldots,x_n \in X$. *Moreover, if* $C \ne 1$ *then* F *is the unique solution of* (2) *such that* $f - F$ *is bounded.*

Proof. For each nonempty subset $\{j_1, \ldots, j_k\}$ of $\mathbf{n}$ with $j_1 < j_2 < \ldots < j_k$ we define

$$g_{j_1\ldots j_k}(x_{j_1}, \ldots, x_{j_k}) := \sum_{A \subseteq \{j_1,\ldots,j_k\}} (-1)^{k-|A|} f\big((x_{j_1}, \ldots, x_{j_k})^A\big), \tag{9}$$

where $|A| := \mathrm{card}(A)$, and

$$(x_{j_1}, \ldots, x_{j_k})^A := (z_1, \ldots, z_n), \quad z_l = \begin{cases} x_l, & \text{for } l \in A, \\ \\ 0, & \text{for } l \notin A, \end{cases} \qquad A \subseteq \{j_1, ..., j_k\}.$$

One can show (see [8] (Proof of Theorem 1 or Remark 1)) that

$$\sum_{k=1}^{n} \sum_{\substack{\{j_1,\ldots,j_k\}\subseteq \mathbf{n} \\ j_1<\ldots<j_k}} g_{j_1\ldots j_k}(x_{j_1}, \ldots, x_{j_k}) = f(x_1, \ldots, x_n) - f(0, \ldots, 0). \tag{10}$$

We will prove that for all $\emptyset \neq \{j_1, \ldots, j_k\} \subseteq \mathbf{n}$ and all $x_{ji_j} \in X$, $j \in \mathbf{n}$, $i_j \in \mathbf{2}$,

$$\|g_{j_1\ldots j_k}(x_{j_1 1} + x_{j_1 2}, \ldots, x_{j_k 1} + x_{j_k 2}) - \sum_{i_1,\ldots,i_k \in \mathbf{2}} g_{j_1\ldots j_k}(x_{j_1 i_1}, \ldots, x_{j_k i_k})\| \leq \varepsilon_k, \tag{11}$$

where $\varepsilon_k := (4^k + 2^k - 2)\varepsilon$.

First we will show that each function g_j with $j \in \mathbf{n}$ is 4ε-additive, i.e., it satisfies

$$\|g_j(x_{j1} + x_{j2}) - g_j(x_{j1}) - g_j(x_{j2})\| \leq 4\varepsilon, \quad x_{j1}, x_{j2} \in X. \tag{12}$$

For $J := \{j\}$, $x_{j1}, x_{j2} \in X$, on account of (7) we have

$$\begin{aligned}
&\|g_j(a_{j1}x_{j1} + a_{j2}x_{j2}) - g_j(a_{j1}x_{j1}) - g_j(a_{j2}x_{j2})\| \\
&= \big\|f\big((a_{j1}x_{j1} + a_{j2}x_{j2})^J\big) - f\big((a_{j1}x_{j1})^J\big) - f\big((a_{j2}x_{j2})^J\big) + f(0, \ldots, 0)\big\| \\
&\leq \Big\|f\big((a_{j1}x_{j1} + a_{j2}x_{j2})^J\big) - \sum_{i_1,\ldots,i_n \in \mathbf{2}} C_{i_1\ldots i_n} f\big((x_{ji_j})^J\big)\Big\| \\
&\quad + \Big\| - f\big((a_{j1}x_{j1})^J\big) + \sum_{i_1,\ldots,i_n \in \mathbf{2},\, i_j:=1} C_{i_1\ldots i_n} f\big((x_{j1})^J\big) \\
&\qquad\qquad + \sum_{i_1,\ldots,i_n \in \mathbf{2},\, i_j:=2} C_{i_1\ldots i_n} f(0, \ldots, 0)\Big\| \\
&\quad + \Big\| - f\big((a_{j2}x_{j2})^J\big) + \sum_{i_1,\ldots,i_n \in \mathbf{2},\, i_j:=1} C_{i_1\ldots i_n} f(0, \ldots, 0) \\
&\qquad\qquad + \sum_{i_1,\ldots,i_n \in \mathbf{2},\, i_j:=2} C_{i_1\ldots i_n} f\big((x_{j2})^J\big)\Big\| \\
&\quad + \Big\|f(0, \ldots, 0) - \sum_{i_1,\ldots,i_n \in \mathbf{2},\, i_j:=1} C_{i_1\ldots i_n} f(0, \ldots, 0) \\
&\qquad\qquad - \sum_{i_1,\ldots,i_n \in \mathbf{2},\, i_j:=2} C_{i_1\ldots i_n} f(0, \ldots, 0)\Big\| \leq 4\varepsilon.
\end{aligned}$$

Since $a_{ji} \neq 0$, we have (12).

Now, for an arbitrary nonempty $J = \{j_1, \ldots, j_k\} \subseteq \mathbf{n}$ we obtain

$$
\begin{aligned}
&\Big\| g_{j_1\ldots j_k}(a_{j_1 1}x_{j_1 1} + a_{j_1 2}x_{j_1 2}, \ldots, a_{j_k 1}x_{j_k 1} + a_{j_k 2}x_{j_k 2}) \\
&\qquad - \sum_{i_{j_1},\ldots,i_{j_k}\in\mathbf{2}} g_{j_1\ldots j_k}(a_{j_1 i_{j_1}} x_{j_1 i_{j_1}}, \ldots, a_{j_k i_{j_k}} x_{j_k i_{j_k}}) \Big\| \\
=& \Big\| \sum_{A\subseteq J} (-1)^{k-|A|} f\big((a_{j_1 1}x_{j_1 1} + a_{j_1 2}x_{j_1 2}, \ldots, a_{j_k 1}x_{j_k 1} + a_{j_k 2}x_{j_k 2})^A\big) \\
&- \sum_{i_{j_1},\ldots,i_{j_k}\in\mathbf{2}} \sum_{A\subseteq J} (-1)^{k-|A|} f\big((a_{j_1 i_{j_1}} x_{j_1 i_{j_1}}, \ldots, a_{j_k i_{j_k}} x_{j_k i_{j_k}})^A\big) \Big\| \\
=& \Big\| \sum_{\emptyset\neq A\subseteq J} (-1)^{k-|A|} f\big((a_{j_1 1}x_{j_1 1} + a_{j_1 2}x_{j_1 2}, \ldots, a_{j_k 1}x_{j_k 1} + a_{j_k 2}x_{j_k 2})^A\big) \\
&- \sum_{i_{j_1},\ldots,i_{j_k}\in\mathbf{2}} \sum_{\emptyset\neq A\subseteq J} (-1)^{k-|A|} f\big((a_{j_1 i_{j_1}} x_{j_1 i_{j_1}}, \ldots, a_{j_k i_{j_k}} x_{j_k i_{j_k}})^A\big) \\
&- (-1)^k (2^k - 1) f(0, \ldots, 0) \Big\| \\
=& \Big\| \sum_{\emptyset\neq A\subseteq J} (-1)^{k-|A|} \Big[f\big((a_{j_1 1}x_{j_1 1} + a_{j_1 2}x_{j_1 2}, \ldots, a_{j_k 1}x_{j_k 1} + a_{j_k 2}x_{j_k 2})^A\big) \\
&- \sum_{\substack{i_{j_1},\ldots,i_{j_k}\in\mathbf{2} \\ \text{for } j_1,\ldots,j_k\in A}} \sum_{\substack{\nu_1,\ldots,\nu_n\in\mathbf{2} \\ \nu_{j_l}:=i_{j_l},\, j_l\in A}} C_{\nu_1\ldots\nu_n} f\big((x_{j_1 i_{j_1}}, \ldots, x_{j_k i_{j_k}})^A\big) \Big] \\
&- \sum_{i_{j_1},\ldots,i_{j_k}\in\mathbf{2}} \sum_{\emptyset\neq A\subseteq J} (-1)^{k-|A|} \Big[f\big((a_{j_1 i_{j_1}} x_{j_1 i_{j_1}}, \ldots, a_{j_k i_{j_k}} x_{j_k i_{j_k}})^A\big) \\
&- \sum_{A_1\subseteq A} \sum_{\substack{i_{j_1},\ldots,i_{j_k}\in\mathbf{2} \\ \text{for } j_1,\ldots,j_k\in A_1}} \sum_{\substack{\nu_1,\ldots,\nu_n\in\mathbf{2} \\ \nu_{j_l}:=i_{j_l},\, j_l\in A_1}} C_{\nu_1\ldots\nu_n} f\big((x_{j_1 i_{j_1}}, \ldots, x_{j_k i_{j_k}})^{A_1}\big) \Big] \\
&- (-1)^k (2^k - 1) \Big[f(0, \ldots, 0) - \sum_{\nu_1,\ldots,\nu_n\in\mathbf{2}} C_{\nu_1\ldots\nu_n} f(0, \ldots, 0) \Big] \Big\| \\
\leq& \sum_{\emptyset\neq A\subseteq J} \Big\| (-1)^{k-|A|} \Big[f\big((a_{j_1 1}x_{j_1 1} + a_{j_1 2}x_{j_1 2}, \ldots, a_{j_k 1}x_{j_k 1} + a_{j_k 2}x_{j_k 2})^A\big) \\
&- \sum_{\substack{i_{j_1},\ldots,i_{j_k}\in\mathbf{2} \\ \text{for } j_1,\ldots,j_k\in A}} \sum_{\substack{\nu_1,\ldots,\nu_n\in\mathbf{2} \\ \nu_{j_l}:=i_{j_l},\, j_l\in A}} C_{\nu_1\ldots\nu_n} f\big((x_{j_1 i_{j_1}}, \ldots, x_{j_k i_{j_k}})^A\big) \Big] \\
&- \sum_{i_{j_1},\ldots,i_{j_k}\in\mathbf{2}} (-1)^{k-|A|} \Big[f\big((a_{j_1 i_{j_1}} x_{j_1 i_{j_1}}, \ldots, a_{j_k i_{j_k}} x_{j_k i_{j_k}})^A\big) \\
&- \sum_{A_1\subseteq A} \sum_{\substack{i_{j_1},\ldots,i_{j_k}\in\mathbf{2} \\ \text{for } j_1,\ldots,j_k\in A_1}} \sum_{\substack{\nu_1,\ldots,\nu_n\in\mathbf{2} \\ \nu_{j_l}:=i_{j_l},\, j_l\in A_1}} C_{\nu_1\ldots\nu_n} f\big((x_{j_1 i_{j_1}}, \ldots, x_{j_k i_{j_k}})^{A_1}\big) \Big] \Big\| \\
&+ (2^k - 1) \Big\| f(0, \ldots, 0) - \sum_{\nu_1,\ldots,\nu_n\in\mathbf{2}} C_{\nu_1\ldots\nu_n} f(0, \ldots, 0) \Big\| \\
\leq& \sum_{\emptyset\neq A\subseteq J} \Big\| f\big((a_{j_1 1}x_{j_1 1} + a_{j_1 2}x_{j_1 2}, \ldots, a_{j_k 1}x_{j_k 1} + a_{j_k 2}x_{j_k 2})^A\big) \\
&- \sum_{\substack{i_{j_1},\ldots,i_{j_k}\in\mathbf{2} \\ \text{for } j_1,\ldots,j_k\in A}} \sum_{\substack{\nu_1,\ldots,\nu_n\in\mathbf{2} \\ \nu_{j_l}:=i_{j_l},\, j_l\in A}} C_{\nu_1\ldots\nu_n} f\big((x_{j_1 i_{j_1}}, \ldots, x_{j_k i_{j_k}})^A\big) \Big\| \\
&+ \sum_{\emptyset\neq A\subseteq J} \sum_{i_{j_1},\ldots,i_{j_k}\in\mathbf{2}} \Big\| f\big((a_{j_1 i_{j_1}} x_{j_1 i_{j_1}}, \ldots, a_{j_k i_{j_k}} x_{j_k i_{j_k}})^A\big) \\
&- \sum_{A_1\subseteq A} \sum_{\substack{i_{j_1},\ldots,i_{j_k}\in\mathbf{2} \\ \text{for } j_1,\ldots,j_k\in A_1}} \sum_{\substack{\nu_1,\ldots,\nu_n\in\mathbf{2} \\ \nu_{j_l}:=i_{j_l},\, j_l\in A_1}} C_{\nu_1\ldots\nu_n} f\big((x_{j_1 i_{j_1}}, \ldots, x_{j_k i_{j_k}})^{A_1}\big) \Big\| \\
&+ (2^k - 1)\varepsilon \\
\leq& (2^k - 1 + (2^k - 1)2^k + 2^k - 1)\varepsilon = (4^k + 2^k - 2)\varepsilon.
\end{aligned}
$$

By Lemma 1, for every nonempty $\{j_1,\dots,j_k\} \subseteq \mathbf{n}$ and function $g_{j_1\dots j_k}$ given by (9) there exists a k-additive function $G_{j_1\dots j_k}\colon X^k \to Y$ such that

$$\|g_{j_1\dots j_k}(x) - G_{j_1\dots j_k}(x)\| \leq \frac{1}{2^k-1}\varepsilon_k, \quad x \in X^k.$$

Putting

$$F(x_1,\dots,x_n) := \sum_{k=1}^{n} \sum_{\substack{\{j_1,\dots,j_k\}\subseteq \mathbf{n} \\ \mathbf{j_1<\dots<j_k}}} G_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}),$$

and using (10) and (11) we obtain

$$\begin{aligned}
&\|f(x_1,\dots,x_n) - f(0,\dots,0) - F(x_1,\dots,x_n)\| \\
&\leq \sum_{k=1}^{n} \sum_{\substack{\{j_1,\dots,j_k\}\subseteq \mathbf{n} \\ \mathbf{j_1<\dots<j_k}}} \|g_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}) - G_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k})\| \\
&\leq \sum_{k=1}^{n} \binom{n}{k} \frac{\varepsilon_k}{2^k-1} = \sum_{k=1}^{n} \binom{n}{k} \frac{(4^k+2^k-2)\varepsilon}{2^k-1} = \sum_{k=1}^{n} \binom{n}{k} (2^k+2)\varepsilon \\
&= \big(3^n - 1 + 2\cdot(2^n-1)\big)\varepsilon = (3^n + 2^{n+1} - 3)\varepsilon.
\end{aligned}$$

For the proof of the uniqueness, assume that $C \neq 1$ and suppose that F' is another function satisfying (2) and such that

$$\|f(x_1,\dots,x_n) - f(0,\dots,0) - F'(x_1,\dots,x_n)\| \leq M,$$

for some positive constant $M \in \mathbb{R}$. Therefore, F' is of the form (cf., Theorem 1)

$$F'(x_1,\dots,x_n) := \sum_{k=1}^{n} \sum_{\substack{\{j_1,\dots,j_k\}\subseteq \mathbf{n} \\ j_1<\dots<j_k}} G'_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}),$$

for all $x_1,\dots,x_n \in X$, with k-additive functions $G'_{j_1\dots j_k}$ and with $\delta' = 0$ in the case $C \neq 1$, on account of (6). We have for all $x_1,\dots,x_n \in X$, $l \in \mathbb{N}$,

$$\|F(lx_1,\dots,lx_n) - F'(lx_1,\dots,lx_n)\| \leq (3^n + 2^{n+1} - 3)\varepsilon + M,$$

$$\Big\|\sum_{k=1}^{n} \sum_{\substack{\{j_1,\dots,j_k\}\subseteq \mathbf{n} \\ \mathbf{j_1<\dots<j_k}}} \big(G_{j_1\dots j_k}(lx_{j_1},\dots,lx_{j_k}) - G'_{j_1\dots j_k}(lx_{j_1},\dots,lx_{j_k})\big)\Big\| \leq (3^n + 2^{n+1} - 3)\varepsilon + M,$$

$$\begin{aligned}
&\Big\| l^n\big(G_{1\dots n}(x_1,\dots,x_n) - G'_{1\dots n}(x_1,\dots,x_n)\big) \\
&+ \sum_{k=1}^{n-1} \sum_{\substack{\{j_1,\dots,j_k\}\subseteq \mathbf{n} \\ \mathbf{j_1<\dots<j_k}}} l^k\big(G_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}) - G'_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k})\big)\Big\| \\
&\qquad\leq (3^n + 2^{n+1} - 3)\varepsilon + M.
\end{aligned}$$

Dividing the above inequality by l^n side by side and letting l tend to infinity we derive that $G_{1\dots n} = G'_{1\dots n}$, and consequently,

$$\Big\|\sum_{k=1}^{n-1} \sum_{\substack{\{j_1,\dots,j_k\}\subset \mathbf{n} \\ j_1<\dots<j_k}} l^k\big(G_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}) - G'_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k})\big)\Big\| \leq (3^n + 2^{n+1} - 3)\varepsilon + M.$$

Dividing the above inequality by l^{n-1} side by side and letting l tend to infinity we obtain

$$\sum_{\substack{\{j_1,\dots,j_{n-1}\}\subset\mathbf{n}\\ j_1<\dots<j_{n-1}}} G_{j_1\dots j_{n-1}}(x_{j_1},\dots,x_{j_{n-1}}) = \sum_{\substack{\{j_1,\dots,j_{n-1}\}\subset\mathbf{n}\\ j_1<\dots<j_{n-1}}} G'_{j_1\dots j_{n-1}}(x_{j_1},\dots,x_{j_{n-1}}), \tag{13}$$

for all $x_1,\dots,x_n \in X$. Fix an arbitrary $\{j_1,\dots,j_{n-1}\} \subset \mathbf{n}$ and let $\mathbf{n} \setminus \{j_1,\dots,j_{n-1}\} =: \{j\}$. Substituting $x_j := 0$ in (13) we obtain $G_{j_1\dots j_{n-1}} = G'_{j_1\dots j_{n-1}}$, and consequently,

$$\Big\| \sum_{k=1}^{n-2} \sum_{\substack{\{j_1,\dots,j_k\}\subset\mathbf{n}\\ j_1<\dots<j_k}} l^k \big(G_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}) - G'_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k})\big) \Big\| \leq (3^n + 2^{n+1} - 3)\varepsilon + M.$$

Proceeding analogously, we derive that all corresponding k-additive functions coincide, which results in $F = F'$ and completes the proof. □

Remark 1. *A thorough inspection of the proof of Theorem 2 shows that in the case $C = 1$ we have a better approximation. Namely, if $f\colon X^n \to Y$ is a mapping satisfying (7) for $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$, then there exists a solution $F\colon X^n \to Y$ of (2) such that*

$$\|f(x_1,\dots,x_n) - f(0,\dots,0) - F(x_1,\dots,x_n)\| \leq (3^n + 2^n - 2)\,\varepsilon,$$

for all $x_1,\dots,x_n \in X$.

Remark 2. *It is also easy to observe that in the case $C = 1$ the function F in (8) is not uniquely determined. Indeed, each function $\overline{F}\colon X^n \to Y$,*

$$\overline{F}(x_1,\dots,x_n) := \sum_{k=1}^{n} \sum_{\substack{\{j_1,\dots,j_k\}\subseteq\mathbf{n}\\ j_1<\dots<j_k}} G_{j_1\dots j_k}(x_{j_1},\dots,x_{j_k}) + \delta'$$

with $G_{j_1\dots j_k}$ defined as in the proof of Theorem 2 with $\delta' \in Y$ such that

$$\|\delta'\| \leq (2^n - 1)\varepsilon$$

satisfies, on account of Remark 1, conditions (2) and (8).

Remark 3. *By Theorem 2 and Remark 1 we obtain, for example, the Hyers–Ulam stability result for multi-Jensen Equation (4). However, due to the general nature of our considerations, the obtained estimation $(3^n + 2^n - 2)\,\varepsilon$ is not optimal (cf., [20]).*

3. Generalized Stability of (2) in Banach Spaces

This section provides some results concerning generalized stability with given approximation functions. Let us denote $a_j := a_{j1} + a_{j2}$ for $j \in \mathbf{n}$, $\alpha z := (\alpha z_1,\dots,\alpha z_n)$, for $\alpha \in \mathbb{K}$, $z = (z_1,\dots,z_n) \in X^n$, $n \in \mathbb{N}$. We also keep the notation for Φf and C from the previous section.

Theorem 3. *Suppose that $C \neq 0$, $a_j \neq 0$, $j \in \mathbf{n}$. Let $f\colon X^n \to Y$ and $\theta\colon X^{2n} \to \mathbb{R}_+$ be mappings satisfying the inequality*

$$\|(\Phi f)(x_{11},\dots,x_{n1},x_{12},\dots,x_{n2})\| \leq \theta(x_{11},\dots,x_{n1},x_{12},\dots,x_{n2}), \tag{14}$$

for $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. *Assume, further, that for some* $s \in \{-1, 1\}$ *(depending on* a_j, C*) we have*

$$\varepsilon^*(x_1, \ldots, x_n) := \sum_{m=0}^{\infty} \frac{\theta\left(a_1^{sm+\frac{s-1}{2}} x_1, \ldots, a_n^{sm+\frac{s-1}{2}} x_n, a_1^{sm+\frac{s-1}{2}} x_1, \ldots, a_n^{sm+\frac{s-1}{2}} x_n\right)}{|C|^{sm+\frac{s+1}{2}}} < \infty, \tag{15}$$

for all $x_1, \ldots, x_n \in X$ *and*

$$\lim_{m\to\infty} \frac{\theta\left(a_1^{sm} x_{11}, \ldots, a_n^{sm} x_{n1}, a_1^{sm} x_{12}, \ldots, a_n^{sm} x_{n2}\right)}{|C|^{sm}} = 0, \tag{16}$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. *Then there exists a unique solution* $F \colon X^n \to Y$ *of* (2) *such that*

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n)\| \le \varepsilon^*(x_1, \ldots, x_n), \tag{17}$$

for all $x_1, \ldots, x_n \in X$.

Proof. Putting $x_{j1} = x_{j2} = x_j$ for $j \in \mathbf{n}$ in (14) we obtain

$$\|f(a_1 x_1, \ldots, a_n x_n) - Cf(x_1, \ldots, x_n)\| \le \theta(x_1, \ldots, x_n, x_1, \ldots, x_n),$$

for all $x_1, \ldots, x_n \in X$, hence,

$$\left\|\frac{f(a_1 x_1, \ldots, a_n x_n)}{C} - f(x_1, \ldots, x_n)\right\| \le \frac{1}{|C|}\theta(x_1, \ldots, x_n, x_1, \ldots, x_n), \tag{18}$$

for all $x_1, \ldots, x_n \in X$. Similarly, putting $x_{j1} = x_{j2} = \frac{x_j}{a_j}$ for $j \in \mathbf{n}$ in (14) we obtain

$$\left\|f(x_1, \ldots, x_n) - Cf\left(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}\right)\right\| \le \theta\left(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}, \frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}\right), \tag{19}$$

for all $x_1, \ldots, x_n \in X$.

Define

$$(\mathcal{T}\xi)(x_1, \ldots, x_n) := \frac{1}{C^s}\xi(a_1^s x_1, \ldots, a_n^s x_n),$$

for all $\xi \in Y^{X^n}$, $x_1, \ldots, x_n \in X$, and

$$\varepsilon(x_1, \ldots, x_n) := \begin{cases} \frac{1}{|C|}\theta(x_1, \ldots, x_n, x_1, \ldots, x_n), & \text{for } s = 1, \\ \theta\left(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}, \frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}\right), & \text{for } s = -1, \end{cases}$$

for all $x_1, \ldots, x_n \in X$. Then, for any $\xi, \mu \colon X^n \to Y$, $x_1, \ldots, x_n \in X$ we have

$$\begin{aligned} &\|(\mathcal{T}\xi)(x_1, \ldots, x_n) - (\mathcal{T}\mu)(x_1, \ldots, x_n)\| \\ &\qquad = \frac{1}{|C|^s}\left\|\xi\left(a_1^s x_1, \ldots, a_n^s x_n\right) - \mu\left(a_1^s x_1, \ldots, a_n^s x_n\right)\right\|, \end{aligned}$$

and by (18) and (19),

$$\|(\mathcal{T}f)(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)\| \le \varepsilon(x_1, \ldots, x_n),$$

for all $x_1, \ldots, x_n \in X$.

Next, put

$$(\Lambda\eta)(x_1, \ldots, x_n) := \frac{1}{|C|^s}\eta\left(a_1^s x_1, \ldots, a_n^s x_n\right),$$

for all $\eta \in \mathbb{R}_+^{X^n}$, $x_1, \ldots, x_n \in X$. As one can check,

$$(\Lambda^m \varepsilon)(x_1, \ldots, x_n) = \frac{\varepsilon\big(a_1^{sm}x_1, \ldots, a_n^{sm}x_n\big)}{|C|^{sm}}$$
$$= \begin{cases} \dfrac{\theta(a_1^m x_1, \ldots, a_n^m x_n, a_1^m x_1, \ldots, a_n^m x_n)}{|C|^{m+1}}, & \text{for } s = 1, \\ |C|^m \theta\Big(\dfrac{x_1}{a_1^{m+1}}, \ldots, \dfrac{x_n}{a_n^{m+1}}, \dfrac{x_1}{a_1^{m+1}}, \dfrac{x_n}{a_n^{m+1}}\Big), & \text{for } s = -1, \end{cases}$$

for all $x_1, \ldots, x_n \in X$, $m \in \mathbb{N}_0$.

The operators $\mathcal{T}\colon Y^{X^n} \to Y^{X^n}$ and $\Lambda\colon \mathbb{R}_+^{X^n} \to \mathbb{R}_+^{X^n}$ satisfy the assumptions of [21] (Theorem 1), therefore, there exists a unique fixed point $F\colon X^n \to Y$ of $\mathcal{T}$ such that (17) holds. Moreover,

$$F(x_1, \ldots, x_n) = \lim_{m\to\infty} (\mathcal{T}^m f)(x_1, \ldots, x_n),$$

for all $x_1, \ldots, x_n \in X$.

Now, we prove that for any $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $m \in \mathbb{N}_0$ we have

$$\begin{aligned} &\big\|\big(\Phi(\mathcal{T}^m f)\big)(x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2})\big\| \\ &\quad \leq \frac{\theta\big(a_1^{sm}x_{11}, \ldots, a_n^{sm}x_{n1}, a_1^{sm}x_{12}, \ldots, a_n^{sm}x_{n2}\big)}{|C|^{sm}}. \end{aligned} \tag{20}$$

Since the case $m = 0$ is just (14), fix an $m \in \mathbb{N}_0$ and assume that (20) holds for any $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$. Then for any $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ we obtain

$$\begin{aligned} &\big\|\big(\Phi(\mathcal{T}^{m+1} f)\big)(x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2})\big\| \\ &= \Big\|\big(\mathcal{T}(\mathcal{T}^m f)\big)(a_{11}x_{11} + a_{12}x_{12}, \ldots, a_{n1}x_{n1} + a_{n2}x_{n2}) \\ &\quad - \sum_{i_1,\ldots,i_n \in \mathbf{2}} C_{i_1\ldots i_n}\big(\mathcal{T}(\mathcal{T}^m f)\big)(x_{1i_1}, \ldots, x_{ni_n})\Big\| \\ &= \Big\|\frac{1}{C^s}(\mathcal{T}^m f)\big(a_1^s(a_{11}x_{11} + a_{12}x_{12}), \ldots, a_n^s(a_{n1}x_{n1} + a_{n2}x_{n2})\big) \\ &\quad - \sum_{i_1,\ldots,i_n \in \mathbf{2}} C_{i_1\ldots i_n}\frac{1}{C^s}(\mathcal{T}^m f)(a_1^s x_{1i_1}, \ldots, a_n^s x_{ni_n})\Big\| \\ &= \frac{1}{|C|^s}\big\|\big(\Phi(\mathcal{T}^m f)\big)(a_1^s x_{11}, \ldots, a_n^s x_{n1}, a_1^s x_{12}, \ldots, a_n^s x_{n2})\big\| \\ &\leq \frac{\theta\big(a_1^{s(m+1)}x_{11}, \ldots, a_n^{s(m+1)}x_{n1}, a_1^{s(m+1)}x_{12}, \ldots, a_n^{s(m+1)}x_{n2}\big)}{|C|^{s(m+1)}}, \end{aligned}$$

and thus, (20) holds for any $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$ and $m \in \mathbb{N}_0$.

Letting $m \to \infty$ in (20) and using (16) we finally obtain

$$(\Phi F)(x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}) = 0,$$

for all $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X$, which means that F satisfies (2).

For the proof of the uniqueness, suppose that F' is another function satisfying (2) and (17). We have for all $x_1, \dots, x_n \in X$, $l \in \mathbb{N}_0$,

$$\begin{aligned}
&\|F(x_1, \dots, x_n) - F'(x_1, \dots, x_n)\| \\
&= \left\| \frac{1}{C^{sl}} F(a_1^{sl} x_1, \dots, a_n^{sl} x_n) - \frac{1}{C^{sl}} F'(a_1^{sl} x_1, \dots, a_n^{sl} x_n) \right\| \\
&\le \frac{1}{|C|^{sl}} \big(\|F(a_1^{sl} x_1, \dots, a_n^{sl} x_n) - f(a_1^{sl} x_1, \dots, a_n^{sl} x_n)\| \\
&\quad + \|F'(a_1^{sl} x_1, \dots, a_n^{sl} x_n) - f(a_1^{sl} x_1, \dots, a_n^{sl} x_n)\| \big) \\
&\le 2 \sum_{m=0}^{\infty} \frac{\theta\big(a_1^{s(m+l)+\frac{s-1}{2}} x_1, \dots, a_n^{s(m+l)+\frac{s-1}{2}} x_n, a_1^{s(m+l)+\frac{s-1}{2}} x_1, \dots, a_n^{s(m+l)+\frac{s-1}{2}} x_n\big)}{|C|^{s(m+l)+\frac{s+1}{2}}} \\
&= 2 \sum_{m=l}^{\infty} \frac{\theta\big(a_1^{sm+\frac{s-1}{2}} x_1, \dots, a_n^{sm+\frac{s-1}{2}} x_n, a_1^{sm+\frac{s-1}{2}} x_1, \dots, a_n^{sm+\frac{s-1}{2}} x_n\big)}{|C|^{sm+\frac{s+1}{2}}},
\end{aligned}$$

hence letting $l \to \infty$ and using (15) we obtain $F = F'$, which finishes the proof. $\square$

From Theorem 3 we can derive several consequences.

Remark 4. *Putting $n = 2$ in Theorem 3 we obtain [5] (Theorem 3).*

Remark 5. *Applying Theorem 3 with $a_{j1} = a_{j2} = 1$ $(a_j = 2)$, for $j \in \mathbf{n}$ and $C_{i_1 \dots i_n} = 1$, $i_1, \dots, i_n \in \mathbf{2}$, $(C = 2^n)$ we obtain immediately the well known result on generalized stability of the multi-Cauchy Equation (3) characterizing multiadditive mappings (see [18,19]).*

Remark 6. *The conditions imposed on θ in Theorem 3 exclude its application for the multi-Jensen Equation (4). Indeed, with $C = 1$ and $a_j = 1$, $j \in \mathbf{n}$, the series $\sum_{m=0}^{\infty} \theta(x_1, \dots, x_n, x_1, \dots, x_n)$, for all $x_1, \dots, x_n \in X$, is not convergent for any non-zero θ, therefore condition (15) is not satisfied. However, this situation changes completely if at least for one $j \in \mathbf{n}$ there is $a_{j1} = a_{j2} = 1$, that is, we have the multi-Cauchy–Jensen Equation (5) with $k \in \mathbf{n}$. Namely, we have the following.*

Corollary 1. *Let $f\colon X^n \to Y$ and $\theta\colon X^{2n} \to \mathbb{R}_+$ be mappings satisfying for a fixed $k \in \mathbf{n}$ the inequality*

$$\begin{aligned}
&\Big\| f\Big(x_{11} + x_{12}, \dots, x_{k1} + x_{k2}, \frac{1}{2} x_{k+1,1} + \frac{1}{2} x_{k+1,2}, \dots, \frac{1}{2} x_{n1} + \frac{1}{2} x_{n2}\Big) \\
&\quad - \sum_{i_1, \dots, i_n \in \mathbf{2}} \frac{1}{2^{n-k}} f(x_{1i_1}, \dots, x_{ni_n}) \Big\| \le \theta(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}),
\end{aligned}$$

for $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$. Assume, further, that for some $s \in \{-1, 1\}$ we have

$$\begin{aligned}
&\varepsilon^*(x_1, \dots, x_n) := \\
&\quad \sum_{m=0}^{\infty} \frac{\theta\big(2^{\alpha_m} x_1, \dots, 2^{\alpha_m} x_k, x_{k+1}, \dots, x_n, 2^{\alpha_m} x_1, \dots, 2^{\alpha_m} x_k, x_{k+1}, \dots, x_n\big)}{2^{k(\alpha_m+1)}} < \infty,
\end{aligned}$$

for $x_1, \dots, x_n \in X$, where $\alpha_m := sm + \frac{s-1}{2}$, and

$$\lim_{m \to \infty} \frac{\theta\big(2^{sm} x_{11}, \dots, 2^{sm} x_{k1}, x_{k+1,1}, \dots, x_{n1}, 2^{sm} x_{12}, \dots, 2^{sm} x_{k2}, x_{k+1,2}, \dots, x_{n2}\big)}{2^{k(sm)}} = 0,$$

for $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$. Then there exists a unique solution $F\colon X^n \to Y$ of (5) such that

$$\|f(x_1, \dots, x_n) - F(x_1, \dots, x_n)\| \le \varepsilon^*(x_1, \dots, x_n), \qquad x_1, \dots, x_n \in X.$$

Proof. It is enough to take $a_{j1} = a_{j2} = 1$ for $j \in \mathbf{n}$ ($a_1 = \ldots = a_k = 2$, $a_{k+1} = \ldots = a_n = 1$), $C_{i_1 \ldots i_n} = \frac{1}{2^{n-k}}$ for all $i_1, \ldots, i_n \in \mathbf{2}$, ($C = 2^k$), and apply Theorem 3. □

Remark 7. *Analyzing the proof of Theorem 3 we derive that for the results concerning the multi-Cauchy and multi-Cauchy–Jensen equations (cf., Remark 5 and Corollary 1) it is enough to assume about X that it is a commutative semigroup uniquely divisible by 2 with the identity element 0 (see also [18] (Theorem 3.3), [22] (Theorem 6)). In fact, if $s = 1$ in case (3) we may assume even less, namely, we do not need to assume divisibility in X (see [18] (Theorem 3.2)).*

Theorem 3 with $\theta(x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}) =: \varepsilon > 0$ and with additional assumption that $|C| \neq 1$ gives immediately the classical Hyers–Ulam stability result for (2). Namely, we have the following corollary.

Corollary 2. *Let $\varepsilon > 0$, $C \neq 0$, $|C| \neq 1$, $a_j \neq 0$ for $j \in \mathbf{n}$. If $f\colon X^n \to Y$ satisfies the inequality*

$$\|(\Phi f)(x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2})\| \leq \varepsilon, \quad x_{11}, x_{12}, \ldots, x_{n1}, x_{n2} \in X,$$

then there exists a unique solution $F\colon X^n \to Y$ of (2) such that

$$\|f(x_1, \ldots, x_n) - F(x_1, \ldots, x_n)\| \leq \frac{\varepsilon}{|1 - |C||}, \quad x_1, \ldots, x_n \in X.$$

Proof. From (15) we have

$$\begin{aligned}
\varepsilon^*(x_1, \ldots, x_n) &= \begin{cases} \sum\limits_{m=0}^{\infty} \dfrac{\varepsilon}{|C|^{m+1}}, \text{ for } |C| > 1 \\ \sum\limits_{m=0}^{\infty} |C|^m \varepsilon, \text{ for } 0 < |C| < 1 \end{cases} \\
&= \begin{cases} \dfrac{\varepsilon}{|C| - 1}, \text{ for } |C| > 1 \\ \dfrac{\varepsilon}{1 - |C|}, \text{ for } 0 < |C| < 1 \end{cases} \\
&= \frac{\varepsilon}{|1 - |C||}, \quad \text{for } C \in \mathbb{R} \setminus \{-1, 0, 1\}. \quad \square
\end{aligned}$$

Remark 8. *If $|C| > 1$ then $\varepsilon^*(x_1, \ldots, x_n) = \frac{\varepsilon}{|C|-1}$ and Corollary 2 coincides with the result of Ciepliński from [17] (Theorem 2).*

From Corollary 2 we obtain Hyers–Ulam stability for multi-additive functions ($C = 2^n$) and multi-Cauchy–Jensen mappings ($C = 2^k$).

Remark 9. *Comparing the results in Corollary 2 and Theorem 2 we observe that the approximating constant in the theorem is much bigger. This is, however, the price for assuming less about the coefficients.*

Remark 10. *Studying the proof of Theorem 3 one can make several further observations:*

- *We do not demand that the coefficients a_{ji} are non-zero (only $a_j \neq 0$).*
- *If $C = 0$ then for the series in (15) to be convergent we take $s = -1$. If also $a_l \neq 0$ for $l \in \mathbf{n}$, then in Theorem 3, f satisfies the condition*

$$\|f(x_1, \ldots, x_n)\| \leq \theta\Big(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}, \frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}\Big),$$

for all $x_1, \dots, x_n \in X$, and in Corollary 2, f is bounded by ε. Both, in the theorem and in the corollary, we have then

$$\begin{aligned} F(x_1, \dots, x_n) &= \lim_{m\to\infty} (\mathcal{T}^m f)(x_1, \dots, x_n) \\ &= \lim_{m\to\infty} C^m f\Big(\frac{x_1}{a_1^m}, \dots, \frac{x_n}{a_n^m}\Big) = 0, \end{aligned}$$

for all $x_1, \dots, x_n \in X$.

- *If $a_1 = \dots = a_n = 0$ (and $|C| > 1$, for (15) to be satisfied), we take $s = 1$, and we have*

$$\Big\| f(x_1, \dots, x_n) - \frac{f(0, \dots, 0)}{C} \Big\| \le \frac{1}{|C|} \theta(x_1, \dots, x_n, x_1, \dots, x_n), \tag{21}$$

for all $x_1, \dots, x_n \in X$, in Theorem 3, and with $\theta(x_1, \dots, x_n, x_1, \dots, x_n) = \varepsilon$, in Corollary 2. Then

$$F(x_1, \dots, x_n) = \lim_{m\to\infty} (\mathcal{T}^m f)(x_1, \dots, x_n) = \lim_{m\to\infty} \frac{1}{C^m} f(0, \dots, 0) = 0.$$

From (21), it follows that in Theorem 3, f is majorized by the function

$$X^n \ni (x_1, \dots, x_n) \mapsto \frac{1}{|C|} \theta(x_1, \dots, x_n, x_1, \dots, x_n) + \frac{\theta(0, \dots, 0)}{|C-1||C|},$$

and in Corollary 2, it is simply bounded.

- *If $a_1 = \dots = a_k = 0$ ($1 \le k < n$) and $a_l \neq 0$ for $l \in \{k+1, \dots, n\}$ (and $|C| > 1$) then $s = 1$ and the approximating function F depends only on $n-k$ last variables*

$$\begin{aligned} F(x_1, \dots, x_n) &= \lim_{m\to\infty} (\mathcal{T}^m f)(x_1, \dots, x_n) \\ &= \lim_{m\to\infty} \frac{1}{C^m} f(0, \dots, 0, a_{k+1}^m x_{k+1}, \dots, a_n^m x_n), \end{aligned}$$

for all $x_1, \dots, x_n \in X$. We have an analogous approach for $a_{j_1}, \dots, a_{j_k} = 0$ and $a_{j_l} \neq 0$ for $l \in \{k+1, \dots, n\}$, where $\{j_1, \dots, j_k\}$ is a nonempty subset of $\mathbf{n}$ and $\{j_{k+1}, \dots, j_n\} := \mathbf{n} \setminus \{j_1, \dots, j_k\}$.

At the end of this section we should point out that without any additional assumptions imposed on θ we are not able to obtain any stability result. Our Theorem 3 describes some sufficient conditions for the generalized stability of the general n-linear Equation (2). The set of conditions affects considerably the method of the proof. And the fact that we were not able to apply Theorem 3 for proving stability of (4) was, therefore, caused by the assumptions imposed on θ, and consequently, by the method of the proof, and not by θ itself. In Section 2, by use of the direct method we proved for example the Hyers–Ulam stability (with $\theta(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}) \equiv \varepsilon$) of (4) (cf., Remark 3).

4. Hyperstability of (2) in m-Normed Spaces with $m \ge 2$

In [17] (compare also [23]), the author has proved the Hyers–Ulam stability of (2) in m-Banach spaces with $m \ge 2$ under the additional assumption that $|\sum_{i_1,\dots,i_n \in \mathbf{2}} C_{i_1,\cdots,i_n}| > 1$. In fact, we are able to obtain more, namely, the hyperstability of (2), that is, we do not only obtain an approximation of f by a function satisfying the equation, but f itself has to satisfy already the equation.

In order to simplify the notation we write

$$\|x, z\| := \|x, z_1, \dots, z_{m-1}\|, \qquad x \in Y,\ z = (z_1, \cdots, z_{m-1}) \in Y^{m-1}.$$

Now, we are in the position to present the main result of this section

Theorem 4. *Let $\varepsilon > 0$, $m \in \mathbb{N} \setminus \{1\}$ and $(Y, \|\cdot, \dots, \cdot\|)$ be an m-normed space. Assume also that $f\colon X^n \to Y$ is a mapping satisfying*

$$\|f(a_{11}x_{11} + a_{12}x_{12}, \dots, a_{n1}x_{n1} + a_{n2}x_{n2}) - \sum_{i_1,\dots,i_n \in \mathbf{2}} C_{i_1,\dots,i_n} f(x_{1i_1}, \dots, x_{ni_n}), z\| \le \varepsilon,$$

for all $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$ and all $z \in Y^{m-1}$. Then f satisfies (2).

Proof. Fix $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$ and $z \in Y^{m-1}$. Then, for all $k \in \mathbb{N}$,

$$\|(\Phi f)(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}), kz\| \le \varepsilon,$$

therefore, on account of properties (ii) and (iii) of the m-norm,

$$k^{m-1}\|(\Phi f)(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}), z\| \le \varepsilon,$$

and consequently, for all $k \in \mathbb{N}$,

$$\|(\Phi f)(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}), z\| \le \frac{\varepsilon}{k^{m-1}}.$$

Hence, letting $k \to \infty$, for every $x_{11}, x_{12}, \dots, x_{n1}, x_{n2} \in X$ and $z \in Y^{m-1}$, we obtain $\|(\Phi f)(x_{11}, \dots, x_{n1}, x_{12}, \dots, x_{n2}), z\| = 0$, which means that f satisfies (2). □

5. Concluding Remarks

In our paper, we were dealing with the stability problem for (2) obtaining various approximations. Often, this kind of investigations originate a discussion on the optimality of the estimates. In our results, e.g., the optimality of the constants occurring in Theorem 2 or Corollary 2 is an open problem.

One can observe the connections between general n-linear functional equations and the behaviors of approximate homomorphisms and derivations on Banach algebras (see, e.g., [24–26]), therefore it is recommended to proceed with some research in this direction.

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