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Ordered Leonardo Quadruple Numbers

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Abstract: In this paper, we introduce a new quadruple number sequence by means of Leonardo numbers, which we call ordered Leonardo quadruple numbers. We determine the properties of ordered Leonardo quadruple numbers including relations with Leonardo, Fibonacci, and Lucas numbers. Symmetric and antisymmetric properties of Fibonacci numbers are used in the proofs. We attain some well-known identities, the Binet formula, and a generating function for these numbers. Finally, we provide illustrations of the identities.

Keywords: Fibonacci quaternions; Leonardo numbers; quadruple numbers

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1. Introduction

Two attractive subjects in integer sequences are Fibonacci and Lucas sequences. Mathematicians have studied them for a long time and continue to be attracted by their broad applications. Fibonacci numbers were discovered by Italian mathematician Leonardo Fibonacci in 1202. One of his books was Liber Abaci, which focused on elementary algebra and arithmetic. This book includes many ordinary problems, such as the famous rabbit problem (see [1]). A table was created for this problem, and the numbers 1, 1, 2, 3, 5,... in the bottom row of the table are called Fibonacci numbers. Moreover, the Fibonacci sequence is constituted by these numbers. Francois Edouard Anatole Lucas named this sequence in 1876, and although Fibonacci had several mathematical contributions, he is known for this sequence above all. The Fibonacci numbers arise throughout nature, for example, in flowers, trees, sunflowers, pinecones, artichokes, and pineapples. The number of petals in many flowers is often a Fibonacci number. The pentagonal shape with five pods is disclosed in the cross section of an apple. The Fibonacci number is also exhibited in a starfish, which has five limbs. Some spiral systems of leaves on the twigs of plants and trees hold Fibonacci numbers. Ripe sunflowers present Fibonacci numbers in a salient way, as the number of seeds from the center of the head to the exterior edge are wrapped in two different spirals in a clockwise and counterclockwise direction. With some exceptions, the number of spirals follows Fibonacci numbers. The spiral patterns on pineapples, pinecones, and artichokes are Fibonacci number examples. As can be seen, we encounter these numbers in numerous places. One other example is concerned with the Earth. The equatorial diameter of the Earth in miles and in kilometers is approximately the product of two alternate Fibonacci numbers and consecutive Fibonacci numbers, respectively (see [1]).

On the other hand, there is a relation between the golden ratio and Fibonacci numbers. The irrational number $\frac{1+\sqrt{5}}{2}$, which has the value 1.61803..., is defined as the golden ratio, and it is observed in many areas in mathematics and art. The relationship between the golden ratio and Fibonacci numbers is that as n increases in the ratio of successive Fibonacci numbers $\frac{F_n}{F_{n-1}}$, the result approaches the golden ratio.

The Fibonacci sequence comprises the numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

The n-th Fibonacci number F_n has the recursive definition as

$$F_n = F_{n-1} + F_{n-2}\,, \quad n \geq 3,$$

with the initial conditions $F_1 = F_2 = 1$.

Lucas numbers are intimately related to Fibonacci numbers and were discovered by Francois Edouard Anatole Lucas in the late nineteenth century. The numbers of the Lucas sequence are

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...,$$

with the initial conditions $L_0 = 2$ and $L_1 = 1$. The n-th Lucas number is indicated by L_n and satisfies the same recurrence relation of Fibonacci numbers. There are many relationships between Fibonacci and Lucas numbers, which can be seen in [1–3].

Falcon and Plaza generalized Fibonacci numbers and called them k-Fibonacci numbers in [4]. These numbers are connected with complex valued fuctions, and they give a generalization of the Pell sequence and Fibonacci sequence. After this study, mathematicians worked on this issue in [5,6]. Then, generalized Fibonacci numbers and k-Fibonacci numbers were studied with matrices in [7–11]. The descriptions and several properties of these matrices were given.

In more recent times, Catarino and Borges defined a new type of number related to Fibonacci numbers, called Leonardo numbers, in [12]. They proved some properties comprising this new sequence. Some sum and product statements, including elements of this sequence, were presented. New identities of Leonardo numbers were obtained by Alp and Koçer in [13]. The same authors identified hybrid Leonardo numbers and found well-known properties for them in [14]. Additionally, Leonardo numbers were generalized by Shannon in [15], and using the Leonardo Pisano numbers and hybrid numbers, Kürüz et al. investigated Leonardo Pisano polynomials and hybrinomials [16].

Quaternions are a four dimensional number system, invented as an extension of complex numbers by Irish mathematician W. R. Hamilton in 1843 [17]. Quaternions are noncommutative with multiplication operations; in addition, they are associative and constitute a group recognized as a quaternion group. Quaternions correspond to rotation in three dimensional space, are used in vectorial studies, and help to interpret some physical equations and spherical geometry. Especially in recent years, the utilization of quaternions has broadened, including geometry, physics, mechanics, kinematics, vectorial analysis, animation, computer graphics, and the technology of robots.

Many types of quaternions, just as dual quaternions, split quaternions, bi-quaternions, hyperbolic quaternions, segre quaternions, ellipsoidal quaternions, hyperboloidal quaternions, degenerate quaternions, degenerate pseudo quaternions, null quaternions, and generalized quaternions have been studied by researchers. Split quaternions were introduced firstly by Cockle [18] in 1849. Bi-quaternions were given by Clifford in [19]. Quaternions of Fibonacci type and generalized Fibonacci quaternions were investigated in [20–22]. Kula and Yaylı studied split quaternions in semi-Euclidean space E_2^4 in [23]. Moreover, split Fibonacci quaternions and their properties were given by Akyiğit et al. in [24]. Dual quaternions with real number coefficients were defined by Majernik in [25]. Additionally, Majernik expressed the special Galilean transformation in the algebraic ring of the dual four-component numbers in [26]. Nurkan and Güven identified dual Fibonacci and Lucas quaternions with dual number coefficients in [27]. After that, from a different viewpoint, Yüce and Aydın gave dual Fibonacci and Lucas quaternions with real number coefficients in [28]. In addition, quaternion-like structures have been a popular topic of study in recent years. Özdemir defined a new noncommutative number system called hybrid numbers in [29]. Kızılateş and Kone dealt with developing a new class of quaternions, octonions, and sedenions called higher order Fibonacci 2^m -ions (or-higher order Fibonacci hyper complex numbers) whose components were higher order Fibonacci numbers in [30]. Gu discussed algebraic equations and obtained some interesting relations by means of hypercomplex numbers in [31].

To summarize, number systems play a special role in defining different types of quaternions. Combining the fundamental properties of numbers and quaternions enables the determination of new features. Considering the numbers mentioned above, the quaternions with different number components have been studied by several authors from many points of view.

Motivated by these ideas, in this work, we combined Leonardo numbers and dual quaternions. Since dual quaternions are a four dimensional number system, we were inspired by dual quaternions, and we called these new numbers "ordered Leonardo quadruple numbers", which are ordered quadruple numbers with Leonardo number components. We imputed the properties of ordered Leonardo quadruple numbers and established some relations between quadruple numbers and Leonardo numbers. Additionally, special identities, which are Cassini identities, and Binet's formula were obtained. Moreover, a generating function for these quadruple numbers was given. Finally, we illustrated the identities with examples.

2. Preliminaries

In this part, some basic terms are recollected in relation to dual quaternions and Leonardo numbers.

In [12], the authors stated that Leonardo numbers are terms of the Leonardo sequence, which is expressed as $\{Le_n\}_{n=0}^{\infty}$. The n-th Leonardo number is denoted by Le_n. The following recurrence relation

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 2 \tag{1}$$

defines this sequence, and $Le_0 = Le_1 = 1$ are the initial conditions. The Leonardo numbers of the Leonardo sequence are

$$1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, \ldots$$

In deference to the n-th Fibonacci number F_n, the relation between Fibonacci numbers and Leonardo numbers is given by

$$Le_n = 2F_{n+1} - 1, \tag{2}$$

for $n \geq 0$.

Quaternions are a four dimensional number system discovered by W. R. Hamilton in 1843 and are given as the elements of the set [17]

$$H = \{q = q_0 + iq_1 + jq_2 + kq_3 \ \mid \ q_0, q_1, q_2, q_3 \in R\},$$

where i, j, and k are the standard orthonormal basis in R^3 and

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

A dual quaternion, as an extension of a dual number in four dimensions, is defined by the same form with different multiplication conditions for quaternionic units as in [25]

$$i^2 = j^2 = k^2 = ijk = 0, \quad ij = -ji = jk = -kj = ki = -ik = 0.$$

The set of dual quaternions

$$H_{\mathbb{D}} = \{q = a + ib + jc + kd \ \mid \ a, b, c, d \in R\},$$

which is isomorphic to Galilean 4-space, forms a commutative division algebra under addition and multiplication. Furthermore, using the dual quaternions, one can express the Galilean transformation in one quaternionic equation [25].

3. Ordered Leonardo Quadruple Numbers

In this part, by inspiring dual quaternions and Leonardo numbers, ordered Leonardo quadruple numbers are given. We define an *ordered Leonardo quadruple number* set as

$$\mathbb{Q}_{\mathbb{L}} = \{Q_n \mid Q_n = Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3} \ \textit{and} \ Le_n, \ n\text{-th} \ \textit{Leonardo number}\}, \tag{3}$$

where

$$i^2 = j^2 = k^2 = ijk = 0, \quad ij = -ji = jk = -kj = ki = -ik = 0. \tag{4}$$

Note that throughout this paper, we denote the n-th ordered Leonardo quadruple number with Q_n.

Lemma 1. *Let the set* $A = \{(Le_n, Le_{n+1}, Le_{n+2}, Le_{n+3}) \mid Le_n \in \{Le_n\}_{n=0}^{\infty}\}$ *be a set containing ordered quadruple numbers. The function* $F : A \longrightarrow \mathbb{Q}_{\mathbb{L}}$ *given as*

$$F(Le_n, Le_{n+1}, Le_{n+2}, Le_{n+3}) = Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3}$$

is an isomorphism.

Proof. Formally, an isomorphism is a bijective morphism. In other words, the term isomorphism is mainly used for algebraic structures. In this case, mappings are called homomorphisms, and a homomorphism is an isomorphism if and only if it is bijective. It can be easily seen that the function F is bijective. So F is an isomorphism, and the ordered quadruple number set A and ordered Leonardo quadruple number set $\mathbb{Q}_{\mathbb{L}}$ are isomorphic. □

Remark 1. *The n-th ordered Leonardo quadruple number* $Q_n = Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3}$ *can be represented as the quadruple number* $Q_n = (Le_n, Le_{n+1}, Le_{n+2}, Le_{n+3})$.

Let us take two ordered Leonardo quadruple numbers from $\mathbb{Q}_{\mathbb{L}}$ as follows:

$$Q_n = Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3}$$

and

$$Q_m = Le_m + iLe_{m+1} + jLe_{m+2} + kLe_{m+3}.$$

The addition and subtraction operations of the ordered Leonardo quadruple numbers are defined by

$$\begin{aligned} Q_n \pm Q_m &= (Le_n \pm Le_m) + i(Le_{n+1} \pm Le_{m+1}) + j(Le_{n+2} \pm Le_{m+2}) \\ &\quad + k(Le_{n+3} \pm Le_{m+3}). \end{aligned} \tag{5}$$

Multiplication of two ordered Leonardo quadruple numbers is defined by

$$\begin{aligned} Q_n Q_m &= (Le_n Le_m) + i(Le_n Le_{m+1} + Le_{n+1} Le_m) \\ &\quad + j(Le_n Le_{m+2} + Le_{n+2} Le_m) + k(Le_n Le_{m+3} + Le_{n+3} Le_m). \end{aligned} \tag{6}$$

The scalar and vector part of the n-th term Q_n of the ordered Leonardo quadruple number sequence are given as

$$S_{Q_n} = Le_n \quad \text{and} \quad V_{Q_n} = iLe_{n+1} + jLe_{n+2} + kLe_{n+3}. \tag{7}$$

So, Q_n can be given as $Q_n = S_{Q_n} + V_{Q_n}$, and the multiplication in Equation (6) is denoted by

$$Q_n Q_m = S_{Q_n} S_{Q_m} + S_{Q_n} V_{Q_m} + S_{Q_m} V_{Q_n}.$$

Moreover, it can easily be seen that $V_{Q_n} \times V_{Q_m} = 0$.

The conjugate of Q_n is given by

$$\overline{Q_n} = Le_n - iLe_{n+1} - jLe_{n+2} - kLe_{n+3}, \tag{8}$$

and the norm of Q_n is given by

$$N_{Q_n} = \|Q_n\| = Q_n\overline{Q_n} = (Le_n)^2. \tag{9}$$

Now, we state the following theorems including the properties of ordered Leonardo quadruple numbers.

Remark 2. *The ordered quadruple $Q = 1 + i + j + k$ will be represented in this paper as "Q".*

Theorem 1. *Let the n-th terms of the Leonardo sequence (Le_n) and ordered Leonardo quadruple number sequence (Q_n) be Le_n and Q_n, respectively. For $n \geq 0$ and $m \geq 1$, the following equations hold:*

$$\begin{aligned} Q_n + Q_{n+1} + Q &= Q_{n+2} && (10)\\ Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} &= Le_n && (11)\\ Q_nQ_m + Q_{n+1}Q_{m+1} &= 2(2Q_{n+m+2} + 1) - Q_{n+2} - Q_{m+2} && (12)\\ &\quad -2Le_{n+m+2} - (i + j + k)(Le_{n+2} + Le_{m+2} - 4). \end{aligned}$$

Proof. (10): By using the n-th and $(n+1) - th$ terms in Equations (3) and (1), we have

$$\begin{aligned} Q_n + Q_{n+1} &= (Le_{n+2} - 1) + i(Le_{n+3} - 1) + j(Le_{n+4} - 1) + k(Le_{n+5} - 1)\\ &= Le_{n+2} + iLe_{n+3} + jLe_{n+4} + kLe_{n+5} - (1 + i + j + k)\\ &= Q_{n+2} - Q. \end{aligned}$$

(11): By Equation (3), we directly obtain

$$\begin{aligned} Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} &= Le_n + iLe_{n+1} + jLe_{n+2} + kLe_{n+3}\\ &\quad -i(Le_{n+1} + iLe_{n+2} + jLe_{n+3} + kLe_{n+4})\\ &\quad -j(Le_{n+2} + iLe_{n+3} + jLe_{n+4} + kLe_{n+5})\\ &\quad -k(Le_{n+3} + iLe_{n+4} + jLe_{n+5} + kLe_{n+6})\\ &= Le_n. \end{aligned}$$

(12): By Equation (3), we first obtain

$$\begin{aligned} Q_nQ_m &= Le_nLe_m + i(Le_nLe_{m+1} + Le_{n+1}Le_m)\\ &\quad +j(Le_nLe_{m+2} + Le_{n+2}Le_m) + k(Le_nLe_{m+3} + Le_{n+3}Le_m) \end{aligned} \tag{13}$$

and

$$\begin{aligned} Q_{n+1}Q_{m+1} &= Le_{n+1}Le_{m+1} + i(Le_{n+1}Le_{m+2} + Le_{n+2}Le_{m+1})\\ &\quad +j(Le_{n+1}Le_{m+3} + Le_{n+3}Le_{m+1}) + k(Le_{n+1}Le_{m+4} + Le_{n+4}Le_{m+1}). \end{aligned} \tag{14}$$

Taking into account Equations (13) and (14) and the property $Le_nLe_m + Le_{n+1}Le_{m+1} = 2(Le_{n+m+2} + 1) - Le_{n+2} - Le_{m+2}$ given in [13], we obtain

$$\begin{aligned}
Q_nQ_m + Q_{n+1}Q_{m+1} &= 4(Le_{n+m+2} + iLe_{n+m+3} + jLe_{n+m+4} + kLe_{n+m+5}) \\
&\quad -2Le_{n+m+2} + 2 + 4i + 4j + 4k \\
&\quad -(Le_{n+2} + iLe_{n+3} + jLe_{n+4} + kLe_{n+5}) \\
&\quad -(Le_{m+2} + iLe_{m+3} + jLe_{m+4} + kLe_{m+5}) \\
&\quad -(i+j+k)Le_{n+2} - (i+j+k)Le_{m+2} \\
&= 4Q_{n+m+2} - 2Le_{n+m+2} + 2 + 4i + 4j + 4k \\
&\quad -Q_{n+2} - Q_{m+2} - (i+j+k)(Le_{n+2} + Le_{m+2}) \\
&= 2(2Q_{n+m+2} + 1) - Q_{n+2} - Q_{m+2} - 2Le_{n+m+2} \\
&\quad -(i+j+k)(Le_{n+2} + Le_{m+2} - 4).
\end{aligned}$$

□

Theorem 2. *Let the n-th terms of the dual Lucas quaternion sequence (T_n) and ordered Leonardo quadruple number sequence (Q_n) be T_n and Q_n, respectively. For Equation (15), $n \geq 1$, and for Equation (16), $n \geq 2$, the following equations hold:*

$$Q_{n-1} + Q_{n+1} = 2(T_{n+1} - Q) \tag{15}$$
$$Q_{n+2} - Q_{n-2} = 2T_{n+1}. \tag{16}$$

Proof. From Equation (3), the property $Le_{n-1} + Le_{n+1} = 2L_{n+1} - 2$ given in [13], and the dual Lucas quaternion $T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$ in [28], we obtain

$$\begin{aligned}
Q_{n-1} + Q_{n+1} &= (Le_{n-1} + Le_{n+1}) + i(Le_n + Le_{n+2}) \\
&\quad +j(Le_{n+1} + Le_{n+3}) + k(Le_{n+2} + Le_{n+4}) \\
&= 2(L_{n+1} + iL_{n+2} + jL_{n+3} + kL_{n+4}) - 2 - 2i - 2j - 2k \\
&= 2(T_{n+1} - Q).
\end{aligned}$$

Taking $m = 2$ in the equation $Le_{n+m} - (-1)^m Le_{n-m} = L_{n+1}(Le_{m-1} + 1) - 1 + (-1)^m$ given in [13], we have

$$Le_{n+2} - Le_{n-2} = 2L_{n+1}. \tag{17}$$

By using Equations (3) and (17), we obtain

$$\begin{aligned}
Q_{n+2} - Q_{n-2} &= (Le_{n+2} - Le_{n-2}) + i(Le_{n+3} - Le_{n-1}) \\
&\quad +j(Le_{n+4} - Le_n) + k(Le_{n+5} - Le_{n+1}) \\
&= 2(L_{n+1} + iL_{n+2} + jL_{n+3} + kL_{n+4}) \\
&= 2T_{n+1}.
\end{aligned}$$

□

Theorem 3. *Let the n-th term of the ordered Leonardo quadruple number sequence (Q_n) be Q_n and the conjugate of Q_n be $\overline{Q_n}$. For $n \geq 1$, the following equations hold:*

$$\begin{aligned}
Q_n + \overline{Q_n} &= 2Le_n \\
Q_n^2 &= 2Q_n Le_n - Le_n^2 \\
Q_n\overline{Q_n} + Q_{n-1}\overline{Q}_{n-1} &= 2(1 + Le_{2n} - Le_{n+1}) \\
Q_n\overline{Q_n} + Q_{n+1}\overline{Q}_{n+1} &= 2(1 + Le_{2n+2} - Le_{n+2}) \\
Q_{n+1}\overline{Q}_{n+1} - Q_{n-1}\overline{Q}_{n-1} &= 2(Le_{2n+1} - Le_n) \\
Q_n^2 + Q_{n-1}^2 &= 4Q_{2n} - 2Q_{n+1} + Q(4 - 2Le_{n+1}) \\
&\quad -2(1 + Le_{2n} - Le_{n+1}).
\end{aligned} \tag{18}$$

Proof. The first four properties in Equation (18) can be easily proved by using Equations (3) and (8).

Considering Equation (9) and $Le_{n+1}^2 - Le_{n-1}^2 = 2(Le_{2n+1} - Le_n)$ given in [13], we obtain the fifth equation as

$$\begin{aligned}
Q_{n+1}\overline{Q}_{n+1} - Q_{n-1}\overline{Q}_{n-1} &= Le_{n+1}^2 - Le_{n-1}^2 \\
&= 2(Le_{2n+1} - Le_n).
\end{aligned}$$

Lastly for the sixth equation, by using the second equation in (18) and Equation (3),

$$\begin{aligned}
Q_n^2 + Q_{n-1}^2 &= 2Q_n Le_n - Le_n^2 + 2Q_{n-1}Le_{n-1} - Le_{n-1}^2 \\
&= Le_n^2 + Le_{n-1}^2 + 2i(Le_n Le_{n-1} + Le_{n+1}Le_n) \\
&\quad +2j(Le_{n-1}Le_{n+1} + Le_n Le_{n+2}) + 2k(Le_{n-1}Le_{n+2} + Le_n Le_{n+3}).
\end{aligned}$$

Here, by using $Le_{n+1}^2 + Le_n^2 = 2(Le_{2n+2} - Le_{n+2} + 1)$ and $Le_n Le_m + Le_{n+1}Le_{m+1} = 2(Le_{n+m+2} + 1) - Le_{n+2} - Le_{m+2}$ given in [13], we have the result

$$\begin{aligned}
Q_n^2 + Q_{n-1}^2 &= 2(Le_{2n} - Le_{n+1} + 1) \\
&\quad +2i(2(Le_{2n+1} + 1) - Le_{n+1} - Le_{n+2}) \\
&\quad +2j(2(Le_{2n+2} + 1) - Le_{n+1} - Le_{n+3}) \\
&\quad +2k(2(Le_{2n+3} + 1) - Le_{n+1} - Le_{n+4}) \\
&\quad +2Le_{2n} - 2Le_{2n} \\
&= 4(Le_{2n} + iLe_{2n+1} + jLe_{2n+2} + kLe_{2n+3}) \\
&\quad +4(1 + i + j + k) \\
&\quad -2(Le_{n+1} + iLe_{n+2} + jLe_{n+3} + kLe_{n+4}) \\
&\quad -2Le_{n+1}(1 + i + j + k) - 2 - 2Le_{2n} + 2Le_{n+1} \\
&= 4Q_{2n} - 2Q_{n+1} + (1 + i + j + k)(4 - 2Le_{n+1}) \\
&\quad -2(1 + Le_{2n} - Le_{n+1}).
\end{aligned}$$

□

Theorem 4. *Let the n-th term of the ordered Leonardo quadruple number sequence (Q_n) be Q_n. Then, the following equations hold:*

$$\begin{aligned}
\sum_{s=0}^{n} Q_s &= Q_{n+2} - (n+2)Q - (2i+4j+8k) && (19)\\
\sum_{s=0}^{n} Q_{2s} &= Q_{2n+1} - nQ - (2i+2j+4k) && (20)\\
\sum_{s=0}^{n} Q_{2s+1} &= Q_{2n+2} - nQ - (2+2i+4j+6k) && (21)\\
\left(\sum_{s=0}^{p} Q_{n+s}\right) + Q_{n+1} &= Q_{n+p+2} - (p+1)Q. && (22)
\end{aligned}$$

Proof. (19): From Equation (3) and $\sum_{s=0}^{n} Le_s = Le_{n+2} - (n+2)$ given in [12], we have

$$\begin{aligned}
\sum_{s=0}^{n} Q_s &= \sum_{s=0}^{n} (Le_s + iLe_{s+1} + jLe_{s+2} + kLe_{s+3})\\
&= \sum_{s=0}^{n} Le_s + i\left(Le_{n+1} - Le_0 + \sum_{s=0}^{n} Le_s\right)\\
&\quad + j\left(Le_{n+2} + Le_{n+1} - Le_0 - Le_1 + \sum_{s=0}^{n} Le_s\right)\\
&\quad + k\left(Le_{n+3} + Le_{n+2} + Le_{n+1} - Le_0 - Le_1 - Le_2 + \sum_{s=0}^{n} Le_s\right)\\
&= Le_{n+2} - (n+2) + i(Le_{n+1} + Le_{n+2} - Le_0 - (n+2))\\
&\quad j(Le_{n+2} + Le_{n+1} - Le_0 - Le_1 + Le_{n+2} - (n+2))\\
&\quad + k(Le_{n+3} + Le_{n+2} + Le_{n+1} - Le_0 - Le_1 - Le_2 + Le_{n+2} - (n+2)).
\end{aligned}$$

In this stage, by Equation (1), we obtain the result simply as

$$\begin{aligned}
\sum_{s=0}^{n} Q_s &= Le_{n+2} + iLe_{n+3} + jLe_{n+4} + kLe_{n+5}\\
&\quad -(n+2)(1+i+j+k) - (i+2j+3k)\\
&\quad -Le_0(i+j+k) - Le_1(j+k) - Le_2 k\\
&= Q_{n+2} - (n+2)Q - (2i+4j+8k).
\end{aligned}$$

(20): By using Equations (1) and (3), and the summation formulas $\sum_{s=0}^{n} Le_{2s} = Le_{2n+1} - n$ and $\sum_{s=0}^{n} Le_{2s+1} = Le_{2n+2} - (n+2)$ given in [12], we obtain

$$\begin{aligned}
\sum_{s=0}^{n} Q_{2s} &= \sum_{s=0}^{n} Le_{2s} + i\left(\sum_{s=0}^{n} Le_{2s+1}\right)\\
&\quad + j\left(\sum_{s=0}^{n} Le_{2s+1} + Le_{2s} + 1\right)\\
&\quad + k\left(\sum_{s=0}^{n} 2Le_{2s+1} + Le_{2s} + 2\right)\\
&= Le_{2n+1} + iLe_{2n+2} + jLe_{2n+3} + kLe_{2n+4}\\
&\quad -n - i(n+2) - j(n+2) - k(n+4)\\
&= Q_{2n+1} - n(1+i+j+k) - (2i+2j+4k).
\end{aligned}$$

(21): The proof is similiar to (20).
(22): Firstly, we have

$$\sum_{s=0}^{p} Q_{n+s} = Q_n + Q_{n+1} + ... + Q_{n+p} \tag{23}$$

and

$$\begin{aligned} \sum_{r=0}^{n+p} Q_r - \sum_{r=0}^{n-1} Q_r &= Q_0 + Q_1 + ... + Q_n + Q_{n+1} + ...Q_{n+p} \\ &\quad -Q_0 - Q_1 - ... - Q_{n-1} \\ &= Q_n + Q_{n+1} + ... + Q_{n+p}. \end{aligned} \tag{24}$$

By putting (24) into Equation (23), we obtain

$$\sum_{s=0}^{p} Q_{n+s} = \sum_{r=0}^{n+p} Q_r - \sum_{r=0}^{n-1} Q_r. \tag{25}$$

If we take $n \longrightarrow n-1$ and $n \longrightarrow n+p$ in Equation (19), then

$$\sum_{r=0}^{n-1} Q_r = Q_{n+1} - (n+1)(1+i+j+k) - (2i+4j+8k) \tag{26}$$

$$\sum_{r=0}^{n+p} Q_r = Q_{n+p+2} - (n+p+2)(1+i+j+k) - (2i+4j+8k). \tag{27}$$

Considering Equations (25)–(27), we find

$$\sum_{s=0}^{p} Q_{n+s} = Q_{n+p+2} - Q_{n+1} - (p+1)(1+i+j+k). \tag{28}$$

Finally, by using Equation (28), the result is obvious. □

Theorem 5. *(Binet's Formula): Let the n-th term of the ordered Leonardo quadruple number sequence* (Q_n) *be* Q_n*. For* $n \geq 0$*, Binet's formula is given by:*

$$Q_n = \frac{2}{\alpha - \beta}\left(\underline{\alpha}\alpha^{n+1} - \underline{\beta}\beta^{n+1}\right) - Q,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$*,* $\beta = \frac{1-\sqrt{5}}{2}$*,* $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$*, and* $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$*.*

Proof. Binet's formula for Leonardo numbers was given in [12] as

$$Le_n = 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - 1. \tag{29}$$

By Equations (3) and (29) and making the appropriate calculations, we obtain

$$\begin{aligned} Q_n &= \frac{2}{\alpha - \beta}(\alpha^{n+1}(1 + i\alpha + j\alpha^2 + k\alpha^3) - \beta^{n+1}(1 + i\beta + j\beta^2 + k\beta^3)) \\ &\quad -1 - i - j - k. \end{aligned}$$

If we let $\underline{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\underline{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$ in the last equation, we obtain the result.
□

Theorem 6. *(Cassini Identity): Let the n-th term of the ordered Leonardo quadruple number sequence* (Q_n) *be* Q_n*. For* $n \geq 1$*, the Cassini identity is given by:*

$$Q_{n-1}Q_{n+1} - Q_n^2 = 4(-1)^{n+1}(1+i+3j+4k) + (1+i+2j+k)Le_{n-2} + kLe_n + (j+k) - Q_{n-1}.$$

Proof. By using Equation (3) and arranging the calculations, we have

$$\begin{aligned} Q_{n-1}Q_{n+1} - Q_n^2 &= Le_{n-1}Le_{n+1} - Le_n^2 \\ &\quad +i(Le_{n-1}Le_{n+2} - Le_nLe_{n+1}) \\ &\quad +j(Le_{n-1}Le_{n+3} - Le_nLe_{n+2} + Le_{n+1}Le_{n+1} - Le_{n+2}Le_n) \\ &\quad +k(Le_{n-1}Le_{n+4} - Le_nLe_{n+3} + Le_{n+2}Le_{n+1} - Le_{n+3}Le_n). \end{aligned} \tag{30}$$

The Cassini identity and the d'Ocagnes identity for the Leonardo numbers were given in [12] as

$$Le_n^2 - Le_{n-1}Le_{n+1} = Le_{n-1} - Le_{n-2} + 4(-1)^n \tag{31}$$
$$Le_mLe_{n+1} - Le_{m+1}Le_n = 2(-1)^{n+1}(Le_{m-n-1}+1) + Le_{m-1} - Le_{n-1}. \tag{32}$$

Now, we use Equation (32) and replace the following indices as follows.
For $m \longrightarrow n-1$ and $n \longrightarrow n+1$,

$$Le_{n-1}Le_{n+2} - Le_nLe_{n+1} = 2(-1)^{n+2}(Le_{-3}+1) + Le_{n-2} - Le_n. \tag{33}$$

For $m \longrightarrow n-1$ and $n \longrightarrow n+2$,

$$Le_{n-1}Le_{n+3} - Le_nLe_{n+2} = 2(-1)^{n+3}(Le_{-4}+1) + Le_{n-2} - Le_{n+1}. \tag{34}$$

For $m \longrightarrow n+1$ and $n \longrightarrow n$,

$$Le_{n+1}Le_{n+1} - Le_{n+2}Le_n = 2(-1)^{n+1}(Le_0+1) + Le_n - Le_{n-1}. \tag{35}$$

For $m \longrightarrow n-1$ and $n \longrightarrow n+3$,

$$Le_{n-1}Le_{n+4} - Le_nLe_{n+3} = 2(-1)^{n+4}(Le_{-5}+1) + Le_{n-2} - Le_{n+2}. \tag{36}$$

For $m \longrightarrow n+2$ and $n \longrightarrow n$,

$$Le_{n+2}Le_{n+1} - Le_{n+3}Le_n = 2(-1)^{n+1}(Le_1+1) + Le_{n+1} - Le_{n-1}. \tag{37}$$

Here, by using negaleonardo numbers, which are Leonardo numbers with a negative index given in [13] and by taking into account Equations (31) and (33)–(37) in Equation (30), we find

$$\begin{aligned} Q_{n-1}Q_{n+1} - Q_n^2 &= 4(-1)^{n+1}(1+i+3j+4k) + (1+i+j+k)Le_{n-2} \\ &\quad +j(Le_n - Le_{n-1}) + k(Le_{n+1} - Le_{n-1}) \\ &\quad -(Le_{n-1} + iLe_n + jLe_{n+1} + kLe_{n+2}). \end{aligned}$$

Considering Equations (1) and (3) in last equation, we obtain the result as

$$\begin{aligned} Q_{n-1}Q_{n+1} - Q_n^2 &= 4(-1)^{n+1}(1+i+3j+4k) + (1+i+j+k)Le_{n-2} \\ &\quad +j(Le_{n-2}+1) + k(Le_n+1) - Q_{n-1} \\ &= 4(-1)^{n+1}(1+i+3j+4k) + (1+i+2j+k)Le_{n-2} \\ &\quad +kLe_n + (j+k) - Q_{n-1}. \end{aligned}$$

□

4. Generating Function

In this part, generating functions for the ordered Leonardo quadruple number sequence are attained. To correlate every number sequence via a function, we use "generating functions".

Let $\{f_n\}_{n=0}^{\infty}$ be a real numbers sequence. The power series

$$F(x) = \sum_{n=0}^{\infty} f_n x^n \tag{38}$$

is called the generating function of the sequence $\{f_n\}_{n=0}^{\infty}$.

In [12], the generating function of the Leonardo sequence is obtained as

$$gLe(t) = \frac{1 - t + t^2}{1 - 2t + t^3}.$$

Theorem 7. *The generating function for the ordered Leonardo quadruple number sequence* $\{Q_n\}_{n=0}^{\infty}$ *is given as*

$$gQ(t) = \frac{(1 - t + t^2) + i(1 + t - t^2) + j(3 - t - t^2) + k(5 - t - 3t^2)}{1 - 2t + t^3},$$

where $1 - 2t + t^3 \neq 0$.

Proof. First, we indicate the property

$$Q_n = 2Q_{n-1} - Q_{n-3}. \tag{39}$$

In Equation (10), if we replace $n \to n - 1$, then we have

$$Q_{n+1} = Q_{n-1} + Q_n + (1 + i + j + k). \tag{40}$$

By subtracting Equation (10) from Equation (40), we obtain $Q_{n+2} = 2Q_{n+1} - Q_{n-1}$. If we take n as $n - 2$ in this last equality, we find Equation (39).

Now by the help of Equation (38), the generating function for the ordered Leonardo quadruple number sequence $\{Q_n\}_{n=0}^{\infty}$ is written by

$$gQ(t) = \sum_{n=0}^{\infty} Q_n t^n. \tag{41}$$

Let us open the sum in Equation (41) and calculate $gQ(t)$.

$$gQ(t) = Q_0 + Q_1 t + Q_2 t^2 + \sum_{n=3}^{\infty} Q_n t^n.$$

In this part, by using Equation (39), we have

$$
\begin{aligned}
gQ(t) &= Q_0 + Q_1 t + Q_2 t^2 + \sum_{n=3}^{\infty} (2Q_{n-1} - Q_{n-3}) t^n \\
&= Q_0 + Q_1 t + Q_2 t^2 + 2\sum_{n=3}^{\infty} Q_{n-1} t^n - \sum_{n=3}^{\infty} Q_{n-3} t^n \\
&= Q_0 + Q_1 t + Q_2 t^2 + 2t\sum_{n=3}^{\infty} Q_{n-1} t^{n-1} - t^3 \sum_{n=3}^{\infty} Q_{n-3} t^{n-3} \\
&= Q_0 + Q_1 t + Q_2 t^2 + 2t\sum_{n=2}^{\infty} Q_n t^n - t^3 \sum_{n=0}^{\infty} Q_n t^n \\
&= Q_0 + Q_1 t + Q_2 t^2 \\
&\quad +2t\left(\sum_{n=2}^{\infty} Q_n t^n + Q_0 + Q_1 t - Q_0 - Q_1 t\right) - t^3 \sum_{n=0}^{\infty} Q_n t^n \\
&= Q_0 + Q_1 t + Q_2 t^2 - 2Q_0 t - 2Q_1 t^2 \\
&\quad +2t\sum_{n=0}^{\infty} Q_n t^n - t^3 \sum_{n=0}^{\infty} Q_n t^n
\end{aligned}
$$

By using Equation (41) and arranging the last equation, we find

$$
gQ(t) - 2tgQ(t) + t^3 gQ(t) = Q_0 + (Q_1 - 2Q_0)t + (Q_2 - 2Q_1)t^2. \tag{42}
$$

On the other hand, we have

$$
\begin{aligned}
Q_0 &= 1 + i + 3j + 5k \quad (43) \\
Q_1 - 2Q_0 &= -1 + i - j - k \quad (44) \\
Q_2 - 2Q_1 &= 1 - i - j - 3k. \quad (45)
\end{aligned}
$$

Finally, by putting Equations (43)–(45) into Equation (42), we obtain the generating function as

$$
gQ(t) = \frac{(1 - t + t^2) + i(1 + t - t^2) + j(3 - t - t^2) + k(5 - t - 3t^2)}{1 - 2t + t^3}.
$$

□

In the last part, some examples are given.

Example 1. ***1-****If $n = 0$ and $m = 1$ for Equations (11) and (12), then the following expressions are calculated, and they satisfy Theorem 1:*

$$
Q_0 - iQ_1 - jQ_2 - kQ_3 = 1 \quad \textit{and} \quad Le_0 = 1
$$

and

$$
\begin{aligned}
Q_0 Q_1 + Q_1 Q_2 &= 4 + 18i + 32j + 56k \\
2(2Q_3 + 1) - Q_2 - Q_3 - 2Le_3 - (i + j + k)(Le_2 + Le_3 - 4) &= 4 + 18i + 32j + 56k.
\end{aligned}
$$

2-*If $n = 1$ and $n = 2$ for Equations (15) and (16) in Theorem 2, respectively, then*

$$
\begin{aligned}
Q_0 + Q_2 &= 4 + 6i + 12j + 20k \\
2(T_2 - 1 - i - j - k) &= 4 + 6i + 12j + 20k
\end{aligned}
$$

and

$$\begin{aligned} Q_4 - Q_0 &= 8 + 14i + 22j + 36k \\ 2T_3 &= 8 + 14i + 22j + 36k \end{aligned}$$

3-*If $n = 1$ for Theorem 3, then the third, fourth, and fifth Equations in (18) are calculated as:*

$$\begin{aligned} Q_1\overline{Q_1} + Q_0\overline{Q}_0 &= 2 \\ 2(1 + Le_2 - Le_2) &= 2 \end{aligned}$$

$$\begin{aligned} Q_1\overline{Q_1} + Q_2\overline{Q}_2 &= 10 \\ 2(1 + Le_4 - Le_3) &= 10 \end{aligned}$$

$$\begin{aligned} Q_2\overline{Q}_2 - Q_0\overline{Q}_0 &= 8 \\ 2(Le_3 - Le_1) &= 8 \end{aligned}$$

and if $n = 2$, for the sixth equation,

$$\begin{aligned} Q_2^2 + Q_1^2 &= 10 + 36i + 64j + 108k \\ 4Q_4 - 2Q_3 + (1 + i + j + k)(4 - 2Le_3) - 2(1 + Le_4 - Le_3) &= 10 + 36i + 64j + 108k. \end{aligned}$$

4-*If $n = 2$ for Equation (20) in Theorem 4, then*

$$\begin{aligned} \sum_{s=0}^{2} Q_{2s} &= Q_0 + Q_2 + Q_4 = 13 + 21i + 37j + 61k \\ Q_5 - 2(1 + i + j + k) - (2i + 2j + 4k) &= 13 + 21i + 37j + 61k, \end{aligned}$$

$n = 3$ for Equation (21) in Theorem 4;

$$\sum_{s=0}^{3} Q_{2s+1} = Q_1 + Q_3 + Q_5 + Q_7 = 62 + 104i + 170j + 278k$$

$$Q_8 - 3(1 + i + j + k) - (2 + 2i + 4j + 6k) = 62 + 104i + 170j + 278k,$$

and $n = 2$ and $p = 1$ for Equation (22) in Theorem 4,

$$\begin{aligned} \left(\sum_{s=0}^{1} Q_{2+s}\right) + Q_3 &= Q_2 + 2Q_3 = 13 + 23i + 39j + 65k \\ Q_5 - 2(1 + i + j + k) &= 13 + 23i + 39j + 65k. \end{aligned}$$

Example 2. ***1-****If $n = 1$ for the Cassini identity in Theorem 6, then*

$$Q_0Q_2 - Q_1^2 = 2 + 2i + 8j + 12k$$

$$4(-1)^2(1 + i + 3j + 4k) + (1 + i + 2j + k)Le_{-1} + kLe_1 + (j + k) - Q_0 = 2 + 2i + 8j + 12k$$

and if $n = 2$ for the Cassini identity,

$$Q_1Q_3 - Q_2^2 = -4 - 6i - 14j - 20k$$

$$4(-1)^3(1 + i + 3j + 4k) + (1 + i + 2j + k)Le_0 + kLe_2 + (j + k) - Q_1 = -4 - 6i - 14j - 20k.$$

5. Conclusions and Remarks

Inspired by the dual quaternions given in [25], we took the coefficients of the dual quaternions as Leonardo numbers. We did not call these new notions "quaternions", because the set of numbers given as the definition in (3) was not a ring, as this set was not closed under multiplication. To not suffer the lack of closure under multiplication, we named these numbers the "ordered Leonardo quadruple numbers" in the set $Q_{\mathbb{L}}$ which was isomorphic to the set of ordered quadruple numbers.

Further study and applications of ordered Leonardo quadruple numbers will include matrices and their theories. In addition, new number systems can be studied in hypercomplex numbers and quaternion-like structures, whose components are Leonardo numbers. The results presented here have the potential to motivate further research into the subject of the higher-order or generalized-ordered Leonardo quadruple numbers, which may also be worth further study.

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