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Controllability and Observability Analysis of a Fractional-Order Neutral Pantograph System

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Abstract: In the recent past, a number of research articles have explored the stability, existence, and uniqueness of the solutions and controllability of dynamical systems with a fractional order (FO). Nevertheless, aside from the controllability and other dynamical aspects, very little attention has been given to the observability of FO dynamical systems. This paper formulates a novel type of FO delay system of the Pantograph type in the Caputo sense and explores its controllability and observability results. This research endeavor begins with the conversion of the proposed dynamical system into a fixed-point problem by utilizing Laplace transforms, the convolution of Laplace functions, and the Mittag–Leffler function (MLF). We then set out Gramian matrices for both the controllability and observability of the linear parts of our proposed dynamical system and prove that both the Gramian matrices are invertible, thus confirming the controllability and observability in a given domain. Considering the controllability and observability results of the linear part along with some other assumptions, we investigate the controllability and observability results related to the nonlinear system. The Banach contraction result, the fixed-point result of Schaefer, the MLF, and the Caputo FO derivative are used as the main tools for establishing these results. To establish the authenticity of the established results, we add two examples at the end of the manuscript.

Keywords: controllability; observability; Gramian matrix; fractional differential equations; fixed-point theorem



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1. Introduction

The history of fractional calculus (FC) had its origin in a well-known correspondence that took place in 1665 between two renowned mathematicians, namely, Leibnitz and L'Hospital. L'Hospital wrote a letter to Leibnitz to ask him what the derivative would be if n was a fraction, and he introduced the notations $d^n y / dx^n$ for the n th-order derivative. In a response letter, Leibnitz wrote to L'Hospital that “this would be an apparent paradox, and one day some vital consequences may be derived from it.” This historical debate drew many brilliant researchers, including Laplace, Fourier, Abel, Riemann, Liouville, Grunwald, Hadamard, Letnikov, Heaviside, and others, and it laid the groundwork for FC. Like different methodologies, e.g., those of Grunwald–Letnikov, Riemann–Liouville, and Hadamard, towards fractional-order (FO) derivatives, the Caputo FO operator is also useful in the analysis of initial and boundary value problems [1,2].

There are many applications of FC in the applied sciences. For instance, FC has been used to study hidden chaotic structures in a 4D dynamical system [3]. The oscillatory and chaotic dynamics of an HIV-1 model were studied through FC in the literature [4]. FC also has applications in mathematical physics [5], bioengineering [6], and agriculture [7]. A number of research articles have studied the existence and uniqueness of the solution, stability [8–10], and controllability of FO dynamical systems [11,12].

The theory of inequalities represents a long-standing topic in many mathematical areas and remains an attractive research domain with many applications. In the evolution of complex mathematical control theory, controllability and observability are the two most important aspects [13,14]. Designing feedback controllers and predictive observer models for networked systems requires an understanding of the notions of observability and controllability. The symmetry analysis of observability and controllability has been discussed in the literature [15,16]. For a system to be controllable, we mean that the system state can be driven to any desirable state by applying an input control function within a determined time duration. More specifically, a system is said to be controllable if there exists an admissible control input signal $u(t)$ that steers the system state y_0 at $t = t_0$ to any advisable state y_f at time $t = t_f$. However, the system is said to be observable if the system state $y(t)$ at any time t can be figured out from the system output $z(t)$. The research work done on state controllability and state observability includes the work of Balachandran et al. [17], where the authors explored the controllability of FO systems. In their work, Younus et al. [18] reported the observability of a conformable time-invariant linear control system. In [19], Buedo et al. established similar results for the controllability of the fractional-order systems that Kalman obtained for an integer-order linear system. In [20], Baleanu et al. established the approximate controllability criterion of an FO degenerate control system. Li et al. [21] investigated FO models and established controllability and observability results. Balachandran et al. [22] investigated a fractional-order class of FO systems of order $0 < j < 1$ and determined their observability conditions. They provided proof for the observability of the linear part of the system with the help of a Gramian matrix and set out conditions for the nonlinear case by using fixed-point techniques. Their proposed dynamical system was described by

$$\begin{aligned} {}^c\mathcal{D}^j M(t) &= PM(t) + G(t, M(t)), \quad t \in [0, T], \\ M(0) &= M_0, \end{aligned} \quad (1)$$

with the linear observation

$$z(t) = FM(t), \quad (2)$$

that in Equations (1) and (2), $j \in (0, 1)$, P and F represent $n \times n$ and $m \times n$ matrices, respectively, $M(t) \in \mathcal{R}^n$, and $G : [0, T] \rightarrow \mathcal{R}^n$ denotes a nonlinear continuous function.

In everyday life, the delay phenomenon is a significant problem that causes system instability and degrades performance. The influence of delay on the dynamic behavior of a dynamical system is essential to consider. It results in obtaining the desired outcomes properly, effectively, and correctly. A variety of contributions have been made in the literature on this scenario. For instance, in [23], Yan studied FO neutral-state-delay integro-differential (ID) inclusion. Later on, Mutukumar et al. [24] explored the controllability of an FO neutral stochastic ID system with infinite delay. In the paper [25], the researchers established the controllability results for FO neutral ID delay equations with nonlocal conditions. Recently, in [11], Arthi et al. considered FO damped systems with distributed delay and investigated their controllability. Even more recently, Ahmad et al. [26] analyzed the controllability of a damped FO nonlinear ID system with an input delay. Nirmala et al. [27] investigated a nonlinear delay integro-differential system with FO, and they established the controllability conditions by using the concept of the fixed-point results of Schauder. In the recent past, Yapeng et al. [28] utilized the combined techniques of the fixed-point results of Schaefer with the theorem of Arzela–Ascoli [29] and established the controllability results for an FO nonlinear ID system:

$$\begin{cases} {}^c\mathcal{D}^j y(t) = Py(t) + Qu(t) + Ru(t - \omega) + h(t, y(t)) \\ \quad + g(t, y(t), \int_0^t f(t, m, y(m)) dm), \quad t \in [0, T], \\ y(0) = y_0, u(t) = \psi(t), \quad -\omega \leq t \leq 0, \end{cases} \quad (3)$$

with an input delay. The symbols in the above system are described in Table 1.

Table 1. Symbols and their descriptions.

Symbols	Description
j	$(0, 1)$
${}^c\mathcal{D}^j$	Caputo derivative operator of the j th order
$y(t)$	n -dimensional vector
P, Q	Square matrices of order n
R	n by m matrix, with $m < n$
g	$[0, T] \times \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$, a continuous function
f	$[0, T] \times [0, T] \times \mathcal{R}^n \rightarrow \mathcal{R}^n$, a continuous function
h	$[0, T] \times \mathcal{R}^n \rightarrow \mathcal{R}^n$, a continuous function

Inspired by the research work in [22,28], in this manuscript, we analyze the controllability and observability of a nonlinear neutral FO system in the Caputo sense with proportional delays given by

$$\begin{cases} {}^c\mathcal{D}^j y(t) = Py(t) + Qu(t) + g(t, y(t), y(Yt), {}^c\mathcal{D}^j y(Yt)), & t \in [0, T] = \mathcal{I}, \\ y(0) = y_0. \end{cases} \quad (4)$$

The symbols in our proposed system are described in Table 2.

Table 2. Symbols and their descriptions.

Symbols	Description
j, Y	The fractional order and delay, respectively, $\in (0, 1)$
${}^c\mathcal{D}^j$	Caputo derivative operator of the j th order
$y(t)$	n -dimensional vector
P	Square matrix of order n
Q	n by m matrix, with $m < n$
g	$\mathcal{I} \times \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$, a continuous function

In mathematics, a type of differential equation in which the derivative of an unknown function at a certain time is given in terms of the values of the function at previous times is termed a delay differential equation (DDE). The terms systems with an aftereffect or dead time, time-delay systems, and hereditary systems can also be used instead of DDEs. There are reasons for why DDEs are so popular. (i) The aftereffect is an applied problem—it is well known that, with rising expectations for dynamic performance, engineers need their models to behave more like the actual process. (ii) Numerous processes involve aftereffect phenomena in their internal dynamics. (iii) Moreover, the actuators, sensors, and communication networks that are now involved in feedback control loops introduce such delays. (iv) In addition, interest in DDEs continues to grow in all scientific domains and, in particular, in control engineering. (v) Lastly, in addition to actual delays, delays are frequently used to simplify very-high-order models.

A pantograph is equipment that is commonly installed on the roof of an electric train to gather current from an overhead line. The dynamics of this device were modeled by Ockendon and Taylor [30] in the form of a delay differential equation. This particular type of delay differential equation is called a pantograph differential equation. Because of its widespread applications in diverse scientific fields, e.g., electrodynamics, quantum mechanics, control systems, etc. [31], researchers have expressed this equation in diverse forms and have addressed its solvability aspects [32,33]. For the existence and stability

results of such systems in the framework of fractional calculus, we refer interested readers to [34,35].

2. Basic Concepts

In this section of our manuscript, we give some necessary definitions, lemmas, and theorems to be used throughout the paper; see, e.g., [36–39] for more details.

Definition 1. [37] For a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$, where i is a positive integer, the Caputo operator of FO j is described as

$${}^c D^j f(t) = \frac{1}{\Gamma(i-j)} \int_0^t (t-m)^{i-j-1} f^{(i)}(m) dm, \quad j \in (i-1, i).$$

Here, $i = [j] + 1$, $0 < j \leq 1$. In particular, for $i = 1$,

$${}^c D^j f(t) = \frac{1}{\Gamma(1-j)} \int_0^t (t-m)^{-j} f'(m) dm.$$

Definition 2. [37] For a continuous function $f : [0, \infty) \rightarrow \mathbb{R}$, the FO integral is expressed by

$$I_0^j f(t) = \frac{1}{\Gamma(j)} \int_0^t (t-m)^{j-1} f(m) dm.$$

$\Gamma(j)$ represents the gamma function of j .

Definition 3. The MLF in two parameters is defined as

$$\mathcal{G}_{j,\mu}(P\dot{t}^j) = \sum_{k=0}^{\infty} \frac{P^k t^{kj}}{\Gamma(kj + \mu)}, \quad j, \mu > 0.$$

Lemma 1. In the two-parameter MLF, the q th-order derivative is defined as

$$\frac{d^q}{dt^q} [t^{\mu-1} \mathcal{G}_{j,\mu}(P\dot{t}^j) = \dot{t}^{j-q-1} \mathcal{G}_{j,\mu-q}(P\dot{t}^j)], q \in \mathbb{N}.$$

Schaefer's fixed-point theorem:

Definition 4. [40] Assume that $f : X \rightarrow X$ is a continuous and compact mapping, where X is a Banach space into itself. Moreover, consider that the set $S = \{d \in X : d = \gamma f(d)\}$ for some $\gamma \in [0, 1]$ is bounded. Then, the set S possesses a fixed point.

The Arzela–Ascoli theorem:

Definition 5. [29] Let $S \subset C(J, \mathcal{R})$; S is relatively compact (that is, the set $\bar{S}$ is compact) if the following hold.

1. The set S is uniformly bounded, that is, $\exists M > 0$ such that $\|f(d)\| < M$ for all $f \in S$ and $d \in J$.
2. S is equicontinuous, that is, for each $\epsilon > 0$, one can associate a positive δ , however small, such that for all $d, \bar{d} \in J$, $\|d - \bar{d}\| \leq \delta$ implies that, for every $f \in S$, $\|f(d) - f(\bar{d})\| \leq \epsilon$.

3. Main Results

We convert the problem under consideration to a fixed-point problem as a first step toward the controllability results. We divide the solution into two parts, namely, linear and nonlinear parts. We show the controllability of the linear part in the beginning, and with

the help of that, we derive the necessary conditions for the controllability of the nonlinear parts in the next sections. Consider the fractional-order nonlinear system

$$\begin{cases} {}^c\mathcal{D}^j y(t) = Py(t) + Qu(t) + g(t, y(t), y(Yt), {}^c\mathcal{D}^j y(Yt)), y(t) \in \mathbb{R}^n, t \in [0, T] = \mathcal{I}, \\ y(0) = y_0, \end{cases} \quad (5)$$

where $0 < j < 1$, $Y \in (0, 1)$, $P \in \mathbb{R}^{n \times n}$, $Q \in \mathbb{R}^{n \times m}$, $u(t) \in \mathbb{R}^m$ is the input control function, and $g : \mathcal{I} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear continuous function. Applying the Laplace transform, we have

$$m^j Y(m) + m^{j-1} y(0) = PY(m) + QU(m) + G(m),$$

where $G(m) = \mathcal{L}g(t, y(t), y(Yt), {}^c\mathcal{D}^j y(Yt)) = \mathcal{L}G_y(t)$. Simplifying and using the convolution property and inverse Laplace transform, one has

$$y(t) = \mathcal{G}_j(Pt^j)y_0 + \int_0^t (t-m)^{j-1} \mathcal{G}_{j,j}(P(t-m)^j)[Qu(m) + G_y(t)]dm, \quad (6)$$

where we apply the following property:

$$f(t)*g(t) = \int_0^t f(t-m)g(m)dm, \quad \mathcal{G}_{j,\mu}(Pt^j) = \mathcal{L}^{-1}\left(\frac{m^{j-\mu}}{mj-P}\right).$$

Controllability of the Linear Part

This section explores the necessary and sufficient conditions of controllability for the system (5) in the absence of the nonlinear function. In the absence of the nonlinear function g , the linear system will look like

$$\begin{cases} {}^c\mathcal{D}^j y(t) = Py(t) + Qu(t), \quad y(t) \in \mathbb{R}^n, t \in [0, T] = \mathcal{I}, \\ y(0) = y_0. \end{cases} \quad (7)$$

The solution to Equation (7) in the light of the solution to Equation (5) is given by

$$y(t) = \mathcal{G}_j(Pt^j)y_0 + \int_0^t (t-m)^{j-1} \mathcal{G}_{j,j}(P(t-m)^j)Qu(m)dm. \quad (8)$$

Definition 6. The system in Equation (7) is called controllable on $\mathcal{I}$ if, for all $y_0, y_1 \in \mathbb{R}^n$, we can associate an input control signal $u \in \mathbb{R}^m$ such that the solution to the system in Equation (7) fulfills $y(0) = y_0$ and $y(T) = y_s$.

We define the controllability Gramian matrix $\mathcal{G}_c(0, T)$ for the system in Equation (7) by

$$\mathcal{G}_c(0, T) = \int_0^T (T-m)^{2j-2} \mathcal{G}_{j,j}(P(T-m)^j)QQ^* \mathcal{G}_{j,j}(P^*(T-m)^j)dm, \quad (9)$$

where the symbol $*$ is used to represent the matrix transpose.

Theorem 1. The linear system described by Equation (7) is said to be controllable in $\mathcal{I}$ iff the controllability Gramian matrix $\mathcal{G}_c(0, T)$ is invertible.

Proof. Assume that $\mathcal{G}_c(0, T)$ is invertible; then, the control signal $u \in \mathbb{R}^m$ is defined as given by

$$u(t) = (T-t)^{j-1} Q^* \mathcal{G}_{j,j}(P^*(T-t)^j) \mathcal{G}_c^{-1}(0, T)(y_s - \mathcal{G}_j(PT^j)y_0), t \in \mathcal{I}. \quad (10)$$

Substituting $t = T$ into Equation (8) and then plugging Equation (10) into the resultant equation, we have

$$\begin{aligned} y(T) &= \mathcal{G}_j(PT^j)y_0 + \int_0^T (T-m)^{2j-2}\mathcal{G}_{jj}(P(T-m)^j)QQ^*\mathcal{G}_{jj}(P^*(T-t)^j) \\ &\quad \times \mathcal{G}_c^{-1}(0,T)(y_s - \mathcal{G}_j(PT^j)y_0)m)dm, \\ &= y_s. \end{aligned} \quad (11)$$

Next, we assume that $\mathcal{G}_c(0, T)$ is not invertible. Then, $\exists$ a vector $\underline{v} \neq 0$ such that

$$\begin{aligned} w^*\mathcal{G}_c(0, T)w &= 0, \\ \Rightarrow \int_0^T \underline{v}^*(T-m)^{2j-2}\mathcal{G}_{jj}(P(T-m)^j)QQ^*\mathcal{G}_{jj}(P^*(T-m)^j)\underline{v}dm &= 0, \\ \Rightarrow \int_0^T \|v^*(T-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)Q\|^2dm &= 0, \end{aligned}$$

so $v^*(T-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)Q = 0$ for $s \in \mathcal{J}$. Let $y_0 = 0$. In addition, the controllability of the system gives that one can find a control signal $u(t), t \in \mathcal{J}$ that will steer the system to $\underline{v} = y_s$ at $t = T$, i.e., $y(T) = y_s = \underline{v}$. In view of these assumptions, Equation (8) looks like

$$\underline{v} = \int_0^T (T-m)^{j-1}\mathcal{G}_{jj}(P(T-m)^j)Qu(m)dm,$$

which further implies that

$$\underline{v}^*\underline{v} = \int_0^T (T-m)^{j-1}\underline{v}^*\mathcal{G}_{jj}(P(T-m)^j)Qu(m)dm = 0.$$

Hence, our supposition that $\underline{v} \neq 0$ is wrong and the statement that $\mathcal{G}_c(0, T)$ is invertible is true. $\square$

4. Controllability Results of the Proposed System

In this section, we discuss the controllability results regarding the nonlinear system in Equation (5). In integral form, the system in Equation (5) can be expressed as

$$\begin{aligned} y(t) &= \mathcal{G}_j(Pt^j)y_0 + \int_0^t (t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)Qu(m)dm \\ &\quad + \int_0^t (t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)G_y(tm)dm. \end{aligned} \quad (12)$$

Corollary 1. (Schaefer's theorem [40]): Assume that $f : X \rightarrow X$ is a continuous and compact mapping, where X is a Banach space into itself. Let $d = \gamma f(d)$ have a solution for $\gamma = 1$, and all such solutions for $\gamma \in (0, 1)$ are unbounded.

To derive the desired results of the controllability for the nonlinear system in Equation (5), we impose some assumptions here:

$A_1 : g : \mathcal{J} \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ is continuous and measurable, and

$$\|g(t, y(t), y(Yt), {}^c\mathcal{D}^j y(Yt))\| \leq M, \forall t \in \mathcal{J}.$$

A_2 : For the sake of brevity, we also assume that

$$\begin{aligned} k_1 &= \sup \|\mathcal{G}_j(Pt^j)\|, t \in \mathcal{J}, \\ k_2 &= \sup \|(t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)\|, t \in \mathcal{J}, \\ k_3 &= \sup \|P\mathcal{G}_{j,1-q}(Pt^j)\|, t \in \mathcal{J}, \\ k_4 &= \sup \|(t-m)^{j-q-1}\mathcal{G}_{jj-q}(P(t-m)^j)\|, t \in \mathcal{J}, \\ k_5 &= \sup \|(t-m)^{q-j-1}\|, t \in \mathcal{J}. \end{aligned}$$

Theorem 2. Presume that the assumptions (A_1 & A_2) hold, and the linear system described by Equation (7) is controllable on $\mathcal{J}$; then, the nonlinear system in Equation (5) is also controllable on $\mathcal{J}$.

Proof. Consider the Banach space $Y = \{y : y^{(n)}, {}^c D^j y(t) \in \mathcal{C}(\mathcal{J}, \mathbb{R}^n)\}$, with a norm $\|y\| = \sup\{\|y^{(n)}\|, \|{}^c D^j y(t)\|, \|u\|\}$. Then, for an arbitrary solution $y(\cdot)$ to (5), an admissible control input signal $u(t)$ is expressed as

$$u(t) = (T-t)^{j-1}Q^*\mathcal{G}_{jj}(P^*(T-t)^j)\mathcal{G}_c^{-1}(0,T)\Theta, t \in \mathcal{J}, \quad (13)$$

where

$$\Theta = y_s - \mathcal{G}_j(PT^j)y_0 - \int_0^T (T-m)^{j-1}\mathcal{G}_{jj}(P(T-m)^j)G_y(m)dm.$$

In order to move forward, we here define an operator $\mathcal{T} : Y \rightarrow Y$ given by

$$\begin{aligned} (\mathcal{T}y)(t) &= \mathcal{G}_j(Pt^j)y_0 + \int_0^t (t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)Qu(m)dm \\ &\quad + \int_0^t (t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)G_y(m)dm. \end{aligned} \quad (14)$$

A fixed point of the operator $\mathcal{T}$ is a specific solution of the system (5). By utilizing (13) in the resultant equation obtained by substituting $t = T$ in (14), one reach

$$\begin{aligned} (\mathcal{T}y)(t) &= \mathcal{G}_j(PT^j)y_0 + \int_0^T (T-m)^{2j-2}\mathcal{G}_{jj}(P(T-m)^j)QQ^*\mathcal{G}_{jj}(P^*(T-t)^j)m dm \mathcal{G}_c^{-1}(0,T) \\ &\quad \times (y_s - \mathcal{G}_j(PT^j)y_0 - \int_0^T (T-m)^{j-1}\mathcal{G}_{jj}(P(T-m)^j)G_y(m)dm) \\ &\quad + \int_0^T (T-m)^{j-1}\mathcal{G}_{jj}(P(T-m)^j)G_y(m)dm, \\ &= y_s. \end{aligned}$$

The above equation demonstrates that if one can find a fixed point of the nonlinear operator $\mathcal{T}$, then the system is transferrable from some primary state y_0 to any ultimate state y_s in a determinate time T by applying the input signal $u(t)$. From here onwards, we verify that $\mathcal{T}$ satisfies the conditions of the theorem of the Schaefer fixed point. The proofs comprise several steps:

Step I: First, we demonstrate the boundedness of $\ell(\mathcal{T}) = \{y \in Y : y = \gamma\mathcal{T}y, 0 \leq \gamma \leq 1\}$ on $\mathcal{J}$. For each $y \in \ell(\mathcal{T})$ and $t \in \mathcal{J}$, one has

$$\begin{aligned} y(t) &= \gamma\mathcal{G}_j(Pt^j)y_0 + \gamma \int_0^t (t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)Qu(m)dm \\ &\quad + \gamma \int_0^t (t-m)^{j-1}\mathcal{G}_{jj}(P(t-m)^j)G_y(m)dm, \end{aligned} \quad (15)$$

which gives

$$\begin{aligned}\|y(t)\| &\leq k_1\|y_0\| + Tk_2\|Q\|\|u\| + Tk_2\mathcal{M}, \\ &= k_1\|y_0\| + k_2T[\|Q\|\|u\| + \mathcal{M}], \\ &= \mu_1,\end{aligned}\tag{16}$$

where

$$\begin{aligned}\|u(t)\| &= \|(T-t)^{j-1}Q^*\mathcal{G}_{jj}(P^*(T-t)^j)\mathcal{G}_c^{-1}(0,T)\Theta\|, t \in \mathcal{J}, \\ &\leq k_2\|Q^*\|\|\mathcal{G}_c^{-1}\|\|\Theta\|,\end{aligned}$$

and

$$\begin{aligned}\|\Theta\| &= \|(y_s - \mathcal{G}_j(PT^j)y_0 - \int_0^T (T-m)^{j-1}\mathcal{G}_{jj}(P(T-m)^j)G_y(m)dm), \| \\ &\leq \|y_s\| + k_1\|y_0\| + k_2T\mathcal{M}.\end{aligned}$$

Now, by Lemma 1, one can get

$$\begin{aligned}(y)^{(q)}(t) &= \gamma P\mathcal{G}_{j,1-q}(Pt^j)y_0 + \gamma \int_0^t (t-m)^{j-q-1}\mathcal{G}_{jj-q}(P(t-m)^j)QQ^* \\ &\quad \times (T-t)^{j-1}\mathcal{G}_{jj}(P^*(T-t)^j)\mathcal{G}_c^{-1}(0,T)\Theta dm \\ &\quad + \gamma \int_0^t (t-m)^{j-q-1}\mathcal{G}_{jj-q}(P(t-m)^j)G_y(m)dm,\end{aligned}$$

and as above, this gives

$$\begin{aligned}\|(y)^{(q)}(t)\| &= k_3\|y_0\| + Tk_2k_4\|Q\|\|Q^*\|\|\mathcal{G}_c^{-1}\|\|\Theta\| + Tk_4\mathcal{M}, \\ &= \mu_2.\end{aligned}$$

Further, we can estimate the norm of the operator ${}^cD^j(\cdot)$ as

$$\begin{aligned}\|{}^cD^jy(t)\| &= \left\| \frac{1}{\Gamma(q-j)} \int_0^t (t-m)^{q-j-1}y^{(q)}(m)dm \right\|, \\ &\leq \left\| \frac{1}{\Gamma(q-j)} \right\| \int_0^t \|(t-m)^{q-j-1}\| \|y^{(q)}(m)\| dm, \\ &\leq \frac{1}{\Gamma(q-j)} Tk_5\mu_2.\end{aligned}$$

This implies that ${}^cD^jy(t)$ is bounded. As $\|y\| = \sup\{\|y\|, \|{}^cD^jy(t)\|, \|u\|\}$, hence, it is concluded that $\ell(\mathcal{T})$ is bounded as well.

Step II: Our next step is to verify that the operator $\mathcal{T}$ is completely continuous. We start with the following assumptions:

1. The operator $\mathcal{T}$ is uniformly bounded.
2. The operator $\mathcal{T}$ is supposed to be compact.

Assertion 1. Let the bounded set $B_r = \{y \in Y : \|y\| \leq r\}$ be mapped into an equicontinuous family by $\mathcal{T}$. Then, for any $y \in B_r$ and $t_1, t_2 \in \mathcal{J}$ with $0 < t_1 < t_2 < T$, one gets

$$\begin{aligned}
& \|(\mathcal{T}y)(t_2) - (\mathcal{T}y)(t_1)\| \\
& \leq \|(\mathcal{G}_j(Pt_2^j) - \mathcal{G}_j(Pt_1^j))\| \|y_0\| \\
& + \int_0^{t_1} \|(t_2 - m)^{j-1} \mathcal{G}_{jj}(P(t_2 - m)^j) - (t_1 - m)^{j-1} \mathcal{G}_{jj}(P(t_1 - m)^j)\| \\
& \times \|Q\| \|Q^*\| \|(T - m)^{j-1} \mathcal{G}_{jj}(P^*(T - m)^j)\| \|\mathcal{G}_c^{-1}\| \|\Theta\| dm \\
& + \int_0^{t_1} \|(t_2 - m)^{j-1} \mathcal{G}_{jj}(P(t_2 - m)^j) - (t_1 - m)^{j-1} \mathcal{G}_{jj}(P(t_1 - m)^j)\| G_y(m) dm \quad (17) \\
& + \int_{t_1}^{t_2} \|(t_2 - m)^{j-1} \mathcal{G}_{jj}(P(t_2 - m)^j)\| \\
& \times \|Q\| \|Q^*\| \|(T - m)^{j-1} \mathcal{G}_{jj}(P^*(T - m)^j)\| \|\mathcal{G}_c^{-1}\| \|\Theta\| dm \\
& + \int_{t_1}^{t_2} \|(t_2 - m)^{j-1} \mathcal{G}_{jj}(P(t_2 - m)^j)\| G_y(m) dm.
\end{aligned}$$

This also gives

$$\begin{aligned}
& \|{}^c D^j(\mathcal{T}y)(t_2) - {}^c D^j(\mathcal{T}y)(t_1)\| \\
& \leq \left\| \frac{1}{\Gamma(q-j)} \int_{t_1}^{t_2} (t-m)^{q-j-1} (\mathcal{T}y)^{(q)}(m) dm \right\| \quad (18) \\
& + \left\| \frac{1}{\Gamma(q-j)} \int_0^{t_1} [(t_2-m)^{q-j-1} - (t_1-m)^{q-j-1}] (\mathcal{T}y)^{(q)}(m) dm \right\|.
\end{aligned}$$

Clearly, Equations (17) and (18) both tend to zero as $t_2 \rightarrow t_1$. Hence, $\{(\mathcal{T}y) : y \in B_r\}$ is a class of functions that are equicontinuous and fulfill the condition of uniform boundedness.

Assertion 2. To demonstrate that the operator $\mathcal{T}$ is compact, take a real number ϵ that belongs to the interval $(0, t)$. Then, for every $y \in B_r$, one obtains

$$\begin{aligned}
(\mathcal{T}_\epsilon y)(t) &= \mathcal{G}_j(Pt^j)y_0 + \int_0^{t-\epsilon} (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j) QQ^* \\
&\times (T-t)^{j-1} \mathcal{G}_{jj}(P^*(T-t)^j) \mathcal{G}_c^{-1}(0, T) \Theta dm \quad (19) \\
&+ \int_0^{t-\epsilon} (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j) G_y(m) dm.
\end{aligned}$$

Here, the aim is to show that $\{(\mathcal{T}_\epsilon y) : y \in B_r\}$ is a class of functions that are equicontinuous and satisfy the requirement of uniform boundedness. One can then write

$$\begin{aligned}
& \|(\mathcal{T}y)(t) - (\mathcal{T}_\epsilon y)(t)\| \\
& = \left\| \int_{t-\epsilon}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j) QQ^* \right. \\
& \times (T-t)^{j-1} \mathcal{G}_{jj}(P^*(T-t)^j) \mathcal{G}_c^{-1}(0, T) \Theta dm \\
& + \int_{t-\epsilon}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j) G_y(m) dm \Big\|, \quad (20) \\
& \leq \int_{t-\epsilon}^t \|(t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)\| \|Q\| \|Q^*\| \\
& \times \|(T-t)^{j-1} \mathcal{G}_{jj}(P^*(T-t)^j)\| \|\mathcal{G}_c^{-1}(0, T)\| \|\Theta\| dm \\
& + \int_{t-\epsilon}^t \|(t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)\| \|G_y(m)\| dm, \\
& \leq \epsilon k_2 [k_2 \|Q\| \|Q^*\| \|\mathcal{G}_c^{-1}\| + \mathcal{M}].
\end{aligned}$$

From this, one achieves

$$\begin{aligned}
 & \|(\mathcal{T}y)^{(q)}(t) - (\mathcal{T}_\epsilon y)^{(q)}(t)\| \\
 & \leq \int_{t-\epsilon}^t \|(t-m)^{j-q-1} \mathcal{G}_{j,j-q}(P(t-m)^j)\| \|Q\| Q^* \| \\
 & \times \|(T-t)^{j-1} \mathcal{G}_{j,j}(P^*(T-t)^j)\| \|\mathcal{G}_c^{-1}(0, T)\| \|\Theta\| dm \\
 & + \int_{t-\epsilon}^t \|(t-m)^{j-q-1} \mathcal{G}_{j,j-q}(P(t-m)^j)\| \|G_y(m)\| dm, \\
 & \leq \epsilon k_4 [k_2 \|Q\| \|Q^*\| \|\mathcal{G}_c^{-1}\| + \mathcal{M}].
 \end{aligned} \tag{21}$$

Now, consider

$$\begin{aligned}
 & \|{}^c D^j(\mathcal{T}y)(t) - {}^c D^j(\mathcal{T}_\epsilon y)(t)\| \\
 & \leq \left\| \frac{1}{\Gamma(q-j)} \right\| \left\| \int_0^t (t-m)^{q-j-1} [(\mathcal{T}y)^{(q)}(t) - (\mathcal{T}_\epsilon y)^{(q)}(t)] dm \right\|.
 \end{aligned} \tag{22}$$

Distinctly,

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} \|(\mathcal{T}y)(t) - (\mathcal{T}_\epsilon y)(t)\| \rightarrow 0, \\
 & \lim_{\epsilon \rightarrow 0} \|(\mathcal{T}y)^{(q)}(t) - (\mathcal{T}_\epsilon y)^{(q)}(t)\| \rightarrow 0, \\
 & \lim_{\epsilon \rightarrow 0} \|{}^c D^j(\mathcal{T}y)(t) - {}^c D^j(\mathcal{T}_\epsilon y)(t)\| \rightarrow 0.
 \end{aligned}$$

By applying the Arzela–Ascoli theorem, one can conclude upon the compactness of $\{(\mathcal{T}y) : \in B_r\}$ in Y .

Step III: Next, we verify the continuity of $\mathcal{T}$. Here, we impose two more assumptions.

A_3 : Let $Y = \{y_1, y_2, \dots, y_n\}$, $\lim_{n \rightarrow \infty} \|y_n - y(t)\| = 0$.

A_4 : $\exists \rho > 0$ such that $\rho = \sup\{\|y_n\|, \|{}^c D^j y_n\|, \|u_n\|\}$, $\forall n$ and $t \in \mathcal{J}$.

By the assumptions (A_3 and A_4), one has

$$g(t, y_n(t), y_n(Yt), {}^c D^j y_n(Yt)) \leq g(t, y(t), y(Yt), {}^c D^j(Yt)).$$

Then by the Fatou–Lebesgue theorem [41], one has

$$\begin{aligned}
 & \|(\mathcal{T}y)(t_n) - (\mathcal{T}y)(t)\| \\
 & \leq \int_0^t \|(t-m)^{j-1} \mathcal{G}_{j,j}(P(t-m)^j)Q\| \|Q^*(T-m)^{j-1} \mathcal{G}_{j,j}(P^*(T-t)^j) \mathcal{G}_c^{-1}(0, T)\| \\
 & \times \int_0^T \|(T-m)^{j-1} \mathcal{G}_{j,j}(P(T-m)^j)\| \|G_{y_n}(m) - G_y(m)\| dm dm \\
 & + \int_0^t \|(t-m)^{j-1} \mathcal{G}_{j,j}(P(t-m)^j)\| \|G_{y_n}(m) - G_y(m)\| dm, \\
 & \leq \int_0^t k_2 \|Q\| \|Q^*\| k_2^2 \|\mathcal{G}_c^{-1}\| \left(\int_0^T \|G_{y_n}(m) - G_y(m)\| dm \right) dm \\
 & + \int_0^t k_2 \|G_{y_n}(m) - G_y(m)\| dm,
 \end{aligned}$$

where $G_{y_n}(m) = g(t, y_n(t), y_n(Yt), {}^c D^j y_n(Yt))$. In the same way as above, we also have

$$\begin{aligned} & \| {}^c D^j(\mathcal{T}y)(t_n) - {}^c D^j(\mathcal{T}y)(t) \| \\ & \leq \int_0^t \|(t-m)^{j-q-1} \mathcal{G}_{j,j-q}(P(t-m)^j) Q\| \|Q^*(T-m)^{j-1} \mathcal{G}_{j,j}(P^*(T-t)^j) \mathcal{G}_c^{-1}(0, T)\| \\ & \times \int_0^T (\|(T-m)^{j-1} \mathcal{G}_{j,j}(P(T-m)^j)\| \|G_{y_n}(m) - G_y(m)\| dm) dm \\ & + \int_0^t \|(t-m)^{j-q-1} \mathcal{G}_{j,j-q}(P(t-m)^j)\| \|G_{y_n}(m) - G_y(m)\| dm, \\ & \leq \int_0^t k_4 \|Q\| \|Q^*\| k_2^2 \|\mathcal{G}_c^{-1}\| \left(\int_0^T \|G_{y_n}(m) - G_y(m)\| dm \right) dm \\ & + \int_0^t k_4 \|G_{y_n}(m) - G_y(m)\| dm. \end{aligned}$$

By utilizing the definition of Caputo's fractional-order derivative, we may achieve

$$\begin{aligned} & \| {}^c D^j(\mathcal{T}y_n)(t) - {}^c D^j(\mathcal{T}y)(t) \| \\ & \leq \left\| \frac{1}{\Gamma(q-j)} \right\| \left\| \int_0^t (t-m)^{q-j-1} [(\mathcal{T}y_n)^{(q)}(t) - (\mathcal{T}y)^{(q)}(t)] dm \right\|. \end{aligned}$$

Evidently,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|(\mathcal{T}y_n)(t) - (\mathcal{T}y)(t)\| & \rightarrow 0, \\ \lim_{n \rightarrow \infty} \|(\mathcal{T}y_n)^{(q)}(t) - (\mathcal{T}y)^{(q)}(t)\| & \rightarrow 0, \\ \lim_{n \rightarrow \infty} \| {}^c D^j(\mathcal{T}y_n)(t) - {}^c D^j(\mathcal{T}y)(t) \| & \rightarrow 0. \end{aligned}$$

Thus, the operator $\mathcal{T}$ is continuous. It is, therefore, deduced with the aid of the theorems of Arzela–Ascoli and Schaefer's fixed point that $\mathcal{T}$ possesses a fixed point in B_r and is a solution to the system in Equation (5). Overall, the system in Equation (5) is controllable on $\mathcal{I}$.

□

5. Observability

This section of the manuscript explores the observability results of the system in Equation (5). The results regarding the linear and nonlinear cases can be presented in separate sections.

5.1. Observability of the Linear System

We take into account the underlying linear system

$$\begin{cases} {}^c \mathcal{D}^j y(t) = Py(t), y(t) \in \mathcal{R}^n, 0 < j < 1, t \in [0, T] = \mathcal{I}, \\ z(t) = Fy(t), \\ y(0) = y_0, \end{cases} \quad (23)$$

where P and F are $n \times n$ and $m \times n$ matrices with $m < n$, $z \in \mathcal{R}^m$, and $z(t) = Fy(t)$ is a linear observation. As the observability is independent of the control function $u(t)$, from here onward, we will have $u(t) = 0$.

Definition 7. The system in Equation (23) is known to be observable on the interval $\mathcal{I}$ whenever $z(t) = Fy(t) = 0 \Rightarrow y(t) = 0, t \in \mathcal{I}$.

Theorem 3. The system in Equation (23) with the given linear observation is observable on an interval $\mathcal{J}$ if and only if the observability Gramian matrix

$$\mathcal{G}_o(0, T) = \int_0^T \mathcal{G}_j(P^* t) F^* F \mathcal{G}_j(P t) dt, \quad (24)$$

is nonsingular. Here, $*$ represents the matrix transpose.

Proof. In view of Equation (8), the solution to Equation (23) can be written as

$$y(t) = \mathcal{G}_j(P t) y_0, \quad (25)$$

while in view of Equation (25), the linear observation can be expressed as

$$z(t) = F \mathcal{G}_j(P t) y_0. \quad (26)$$

By thorough pre-multiplication of Equation (26) by $\mathcal{G}_j(P^* t)$ and then integration of the resultant equation between 0 to T , we may obtain

$$\int_0^T \mathcal{G}_j(P^* t) z(t) dt = \int_0^T \mathcal{G}_j(P^* t) F \mathcal{G}_j(P t) y_0 dt,$$

which, by using Equation (24), gives

$$\int_0^T \mathcal{G}_j(P^* t) z(t) dt = \mathcal{G}_o(0, T) y_0. \quad (27)$$

The last equation, Equation (27), clearly suggests that the initial state $y(0) = y_0$ can be uniquely achieved by the system output $z(t) \forall t \in [0, T]$ if $\mathcal{G}_o(0, T)$ is invertible, that is, the system in Equation (23) is observable on $\mathcal{J} = [0, T]$.

Now, we have to verify that if $\mathcal{G}_o(0, T)$ is singular, then the system in Equation (23) with the output $z(t) = Fy(t)$ is not observable. Assume that $\mathcal{G}_o(0, T)$ is singular; then, one can find such an $n \times n$ constant vector e that satisfies

$$\begin{aligned} e^* \mathcal{G}_o(0, T) e &= \int_0^T e^* \mathcal{G}_j(P^* t) F \mathcal{G}_j(P t) e dt, \\ &= \int_0^T \|F \mathcal{G}_j(P t) e\|^2 dt = 0. \end{aligned}$$

This implies that $F \mathcal{G}_j(P t) e = 0, \forall t \in \mathcal{J}$. Let $y(0) = y_0 = e$; then, $y(t)$, the output, is described as

$$y(t) = F \mathcal{G}_j(P t) y_0 = F \mathcal{G}_j(P t) e = 0.$$

Since the output $z(t)$ cannot determine the initial state y_0 uniquely, the system in Equation (23) with the output $z(t) = Fy(t)$ is not observable. The proof is now complete. $\square$

Definition 8. A matrix function $\zeta(t) \in \mathcal{R}^{n \times n}$, defined on $\mathcal{J}$, is a reconstructed kernel iff

$$\int_0^T \zeta(t) z(t) dt = I.$$

Theorem 4. A reconstructed kernel $\zeta(t) \in \mathcal{R}^{n \times n}$ exists on an interval $\mathcal{J}$ if and only if the system in Equation (23) is observable on $\mathcal{J}$.

Proof. Let $\zeta(t) \in \mathcal{R}^{n \times n}$ be a reconstructed kernel; then, from Equation (26), one may write

$$\begin{aligned} \int_0^T \zeta(t) z(t) dt &= \int_0^T \zeta(t) F \mathcal{G}_j(P t) y_0 dt, \\ &= y_0. \end{aligned}$$

Now, if $z(t) = 0$, then $y_0 = 0$, which implies that $y(t) = 0$, which, in turn, implies that the system in Equation (23) is observable on $\mathcal{I}$. Conversely, suppose that the system in Equation (23) is observable on $\mathcal{I}$; then, $\mathcal{G}^{-1}(0, T)$ must exist. Then, from

$$\mathcal{G}_0(0, T) = \int_0^T \mathcal{G}_j(P^* \dot{t}) F^* F \mathcal{G}_j(P \dot{t}) dt,$$

we have

$$\zeta(t) = \mathcal{G}^{-1}(0, T) \mathcal{G}_j(P^* \dot{t}) F^*, t \in \mathcal{I},$$

which is the required reconstructed kernel on $\mathcal{I}$, as

$$\begin{aligned} \int_0^T \zeta(t) F \mathcal{G}_j(P \dot{t}) dt &= \mathcal{G}^{-1}(0, T) \int_0^T \mathcal{G}_j(P^* \dot{t}) F^* F \mathcal{G}_j(P \dot{t}) dt, \\ &= I. \end{aligned}$$

Hence, the the required proof is provided. $\square$

5.2. Observability of the Nonlinear System

We take into account the following system:

$$\begin{cases} {}^c \mathcal{D}^j y(t) = Py(t) + g(t, y(t), y(Yt), {}^c D^j y(Yt)), y(t) \in \mathbb{R}^n, j \in (0, 1), \text{ and } t \in [0, T] = \mathcal{I}, \\ z(t) = Fy(t), \\ y(0) = y_0, \end{cases} \quad (28)$$

where P and F are $n \times n$ and $m \times n$ matrices with $m < n$, $g: \mathcal{I} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear continuous function, and $z \in \mathbb{R}^m$, while $z(t) = Fy(t)$ is a linear observation, $u(t) = 0$, as the observability is independent of the control function.

Definition 9. The system in Equation (28) is said to be observable at time t if, for some past time $t_p < t$, one can estimate the system state $y(t)$ at any time t from the system output $z(t)$ in $[t_p, t]$. The system in Equation (28) is said to be completely observable if it is observable for all $t \in \mathcal{I}$.

Let t_p be some past time. Then, the aim is to estimate the unknown state $y(t)$ from the system output $z(t)$ at the present time t in the interval $[t_p, t]$. We also suppose that for any initial condition, the system in Equation (28) possesses a unique solution. This solution to Equation (28) with the initial condition $y = y(\omega)$, $\omega \in [t_p, t]$ is described by

$$y(t) = \mathcal{G}_j(P(t - \omega)^j) y(\omega) + \int_{\omega}^t (t - m)^{j-1} \mathcal{G}_{j,j}(P(t - m)^j) G_y(tm) dm, \quad (29)$$

where $G_y(t) = g(t, y(t), y(Yt), {}^c D^j y(Yt))$. This can be rearranged to give

$$y(\omega) = [\mathcal{G}_j(P(t - \omega)^j)]^{-1} \times [y(t) - \int_{\omega}^t (t - m)^{j-1} \mathcal{G}_{j,j}(P(t - m)^j) G_y(m) dm]. \quad (30)$$

In addition, $z(t) = Fy(t)$ in the terms of Equation (30) is given by

$$\mathcal{G}_j(P(t - \omega)^j) z(\omega) = [Fy(t) - F \int_{\omega}^t (t - m)^{j-1} \mathcal{G}_{j,j}(P(t - m)^j) G_y(m) dm], \quad (31)$$

which further implies that

$$\begin{aligned} &[\mathcal{G}_j(P(t - \omega)^j)]^2 z(\omega) \\ &= F \mathcal{G}_j(P(t - \omega)^j) y(t) - F \mathcal{G}_j(P(t - \omega)^j) \int_{\omega}^t (t - m)^{j-1} \mathcal{G}_{j,j}(P(t - m)^j) G_y(m) dm. \end{aligned}$$

Multiplying both sides by $\mathcal{G}_j(P^*(t-\omega)^j)F^*$ and integrating between t_p to t will yield

$$\begin{aligned} & \int_{t_p}^t [\mathcal{G}_j(P^*(t-\omega)^j)F^*\mathcal{G}_j(P(t-\omega)^j)]^2 z(\omega) d\omega \\ &= \int_{t_p}^t \mathcal{G}_j(P^*(t-\omega)^j)F^*F\mathcal{G}_j(P(t-\omega)^j)y(t) d\omega \\ & - \int_{t_p}^t \mathcal{G}_j(P^*(t-\omega)^j)F^*F\mathcal{G}_j(P(t-\omega)^j) \left[\int_{\omega}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)G_y(m) dm \right] d\omega, \\ &= \int_{t_p}^t \mathcal{G}_j(P^*(t-\omega)^j)F^*F\mathcal{G}_j(P(t-\omega)^j)y(t) d\omega \\ & - \int_{t_p}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)G_y(m) \left[\int_{t_p}^s \mathcal{G}_j(P^*(t-\omega)^j)F^*F\mathcal{G}_j(P(t-\omega)^j) d\omega \right] dm, \\ &= \mathcal{G}_o(t_p, t)y(t) - \int_{t_p}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)G_y(m)\mathcal{G}_o(t_p, sm) dm. \end{aligned}$$

If the linear system Equation (23) is observable, i.e., $\mathcal{G}_o(t_p, t)$ is nonsingular, we will have

$$\begin{aligned} y(t) &= \mathcal{G}_o^{-1}(t_p, t) \int_{t_p}^t [\mathcal{G}_j(P^*(t-m)^j)F^*\mathcal{G}_j(P(t-m)^j)]^2 z(sm) dm \\ & + \mathcal{G}_o^{-1}(t_p, t) \int_{t_p}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)G_y(m)\mathcal{G}_o(t_p, sm) dm, \end{aligned}$$

which gives

$$y(t) = \int_{t_p}^t \psi_1(t, t_p, m)z(m)dm + \int_{t_p}^t (t-m)^{j-1} \psi_2(t, t_p, m)G_y(m)dm, \quad (32)$$

where

$$\begin{aligned} \psi_1(t, t_p, m) &= \mathcal{G}_o^{-1}(t_p, t) [\mathcal{G}_j(P^*(t-m)^j)F^*\mathcal{G}_j(P(t-m)^j)]^2, \\ \psi_2(t, t_p, m) &= \mathcal{G}_o^{-1}(t_p, t) \mathcal{G}_{jj}(P(t-m)^j)G_y(m)\mathcal{G}_o(t_p, m). \end{aligned} \quad (33)$$

Over the interval $[t_p, t]$, this equation reflects the relationship between the unknown state $y(t)$ and the observed output z . Consequently, we now have the underlying immediate result.

Theorem 5. *If the assumptions given below hold, then, the system in Equation (28) is (a) observable globally at time t and (b) completely observable.*

- I. There corresponds a positive constant ℓ in such a way that $|\mathcal{G}_o(t_p, t)| \geq \ell$.
- II. For any z that is continuous on $[t_p, t]$, Equation (32) has a unique solution (1) for some $t_p < t$ in the case of an observable system at time t and (2) $\forall t$, with $t_p < t$, in the case of being completely observable.

In Equation (32), the time t_p can be replaced by ω and is not fixed. Substituting Equation (32) into Equation (30) after this change is made, one obtains

$$\begin{aligned} y(\omega) &= [\mathcal{G}_j(P(t-\omega)^j)]^{-1} \times \left[\int_{\omega}^t \psi_1(t, \omega, m)z(m)dm + \int_{\omega}^t (t-m)^{j-1} \psi_2(t, \omega, m)G_y(m)dm \right. \\ & \left. - \int_{\omega}^t (t-m)^{j-1} \mathcal{G}_{jj}(P(t-m)^j)G_y(m)dm \right], \end{aligned}$$

$$y(\omega) = \int_{\omega}^t \psi_3(t, \omega, m) z(m) dm + \int_{\omega}^t (t - m)^{j-1} \psi_4(t, \omega, m) G_y(m) dm, \omega < t, \quad (34)$$

where

$$\begin{aligned} \psi_3(t, \omega, m) &= [\mathcal{G}_j(P(t - \omega)^j)]^{-1} \psi_1(t, \omega, m), \\ \psi_4(t, \omega, m) &= [\mathcal{G}_j(P(t - \omega)^j)]^{-1} [\psi_2(t, \omega, m) - \mathcal{G}_{jj}(P(t - m)^j)]. \end{aligned}$$

The results given in Equation (32) are also valid if they are replaced by (34) with a simple change of variables. Now, we utilize the fixed-point result of Banach for the analysis of the observability of the system:

$$\begin{cases} {}^c D^j y(t) = Py(t) + \epsilon g(t, y(t), y(Yt), {}^c D^j y(Yt)), y(t) \in \mathbb{R}^n, 0 < j < 1, t \in [0, T] = \mathcal{I}, \\ z(t) = Fy(t), \end{cases} \quad (35)$$

with the initial condition $y = y(\omega)$ and a positive constant ϵ . Let the continuous function $g : \mathcal{I} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfy the assumptions given below.

A_5 : $\exists K > 0$ and $L > 0$ such that

$$\|g(t, u_1, u_2, u_3) - g(t, \bar{u}_1, \bar{u}_2, \bar{u}_3)\| \leq K(\|u_1 - \bar{u}_1\| + \|u_2 - \bar{u}_2\|) + L(\|u_3 - \bar{u}_3\|),$$

$$\forall u_1, u_2, u_3, \bar{u}_1, \bar{u}_2, \bar{u}_3 \in \mathbb{R}^n \text{ and } t \in \mathcal{I}.$$

where $G_y(t) \in \mathcal{C}(\mathcal{I}, \mathbb{R}^n)$, and the following is satisfied.

Theorem 6. *If the assumptions given below hold, then the system (28) is (a) globally observable at time t and (b) completely observable.*

- I. There corresponds a positive constant ℓ in such a manner that $\det \mathcal{G}_o(t_p, t) \geq \ell$.
- II. In addition,

$$\epsilon < \frac{j}{(t - t_p)^j \psi_7(t, t_p)},$$

(1) for some $t_p < t$ in the case of an observable system at time t and (2) $\forall t$, with $t_p < t$, in the case of being completely observable.

Proof. As we did earlier, a general solution to Equation (35) with the initial condition $y = y(\omega)$ is expressed as

$$y(t) = \mathcal{G}_j(P(t - \omega)^j) y(\omega) + \epsilon \int_{\omega}^t (t - m)^{j-1} \mathcal{G}_{jj}(P(t - m)^j) G_y(m) dm, \quad (36)$$

where $G_y(t) = g(t, y(t), y(Yt), {}^c D^j y(Yt))$. This can be rearranged to get the form

$$y(\omega) = [\mathcal{G}_j(P(t - \omega)^j)]^{-1} \times [y(t) - \epsilon \int_{\omega}^t (t - m)^{j-1} \mathcal{G}_{jj}(P(t - m)^j) G_y(m) dm]. \quad (37)$$

A similar result, Equation (32), which we obtained from Equation (30), can also be obtained for Equation (37):

$$y(t) = \int_{t_p}^t \psi_1(t, t_p, m) z(m) dm + \epsilon \int_{t_p}^t (t - m)^{j-1} \psi_2(t, t_p, m) G_y(m) dm, \quad (38)$$

where

$$\begin{aligned} \psi_1(t, t_p, m) &= \mathcal{G}_o^{-1}(t_p, t) [\mathcal{G}_j(P^*(t - m)^j) F^* \mathcal{G}_j(P(t - m)^j)]^2, \\ \psi_2(t, t_p, m) &= \mathcal{G}_o^{-1}(t_p, t) \mathcal{G}_{jj}(P(t - m)^j) G_y(m) \mathcal{G}_o(t_p, m). \end{aligned}$$

Plugging Equation (38) into Equation (37) will yield

$$\begin{aligned} y(\omega) &= [\mathcal{G}_j(P(t-\omega)^j)]^{-1} \times [\int_{t_p}^t \psi_1(t, t_p, m) z(m) dm \\ &\quad + \epsilon \int_{t_p}^t (t-m)^{j-1} \psi_2(t, t_p, m) G_y(m) dm \\ &\quad - \epsilon \int_{\omega}^t (t-m)^{j-1} \mathcal{G}_{j,j}(P(t-m)^j) G_y(m) dm]. \end{aligned} \quad (39)$$

This equation clearly suggests that for the system (35) to be observable, it is sufficient that $\mathcal{G}_o^{-1}(t_p, t)$ exists and Equation (39) has a unique solution. Assume that there are two solutions y_1 and y_2 to Equation (39) with $y_1 \neq y_2$. Then, one can write

$$\begin{aligned} \|y_1(\omega) - y_2(\omega)\| &\leq \|(\mathcal{G}_j(P(t-\omega)^j))^{-1}\| \\ &\quad \times [\epsilon \int_{t_p}^t (t-m)^{j-1} \|\psi_2(t, t_p, m)\| \|G_{y_1}(m) - G_{y_2}(m)\| dm \\ &\quad - \epsilon \int_{\omega}^t \|(t-m)^{j-1} \mathcal{G}_{j,j}(P(t-m)^j)\| \|G_{y_1}(m) - G_{y_2}(m)\| dm]. \end{aligned} \quad (40)$$

To simplify the situation, we suppose that

$$\begin{aligned} \gamma_y(t) &= {}^c D^j y(t), \\ &= g(t, y(t), y(Yt), {}^c D^j y(Yt)), \\ &= g(t, y(t), y(Yt), \gamma_y(Yt)), \\ &= G_y(t). \end{aligned}$$

Then

$$\begin{aligned} \|\gamma_{y_1}(t) - \gamma_{y_2}(t)\| &= \|G_{y_1}(t) - G_{y_2}(t)\|, \\ &= \|g(t, y_1(t), y_1(Yt), {}^c D^j y_1(Yt)) - g(t, y_2(t), y_2(Yt), {}^c D^j y_2(Yt))\|, \\ &\leq K[\|y_1(t) - y_2(t)\| + \|y_1(Yt) - y_2(Yt)\|] + L\|G_{y_1}(Yt) - G_{y_2}(Yt)\|, \\ &\leq K[\|y_1(t) - y_2(t)\| + \|y_1(t) - y_2(t)\|] + L\|G_{y_1}(t) - G_{y_2}(t)\|, \end{aligned}$$

which gives

$$\|G_{y_1}(t) - G_{y_2}(t)\| \leq \frac{2K}{1-L} \|y_1(t) - y_2(t)\|. \quad (41)$$

Using Equation (40) in terms of Equation (41) will yield

$$\begin{aligned} \|y_1(\omega) - y_2(\omega)\| &\leq \|(\mathcal{G}_j(P(t-\omega)^j))^{-1}\| \\ &\quad \times [\epsilon \frac{2K}{1-L} \|y_1(t) - y_2(t)\| \int_{t_p}^t (t-m)^{j-1} \|\psi_2(t, t_p, m)\| dm \\ &\quad + \epsilon \frac{2K}{1-L} \|y_1(t) - y_2(t)\| \int_{\omega}^t (t-m)^{j-1} \|\mathcal{G}_{j,j}(P(t-m)^j)\| dm], \\ &\leq \|(\mathcal{G}_j(P(t-\omega)^j))^{-1}\| \\ &\quad \times [\epsilon \frac{2K}{1-L} \|y_1(t) - y_2(t)\| \int_{t_p}^t (t-m)^{j-1} \|\psi_2(t, t_p, m)\| dm \\ &\quad + \epsilon \frac{2K}{1-L} \|y_1(t) - y_2(t)\| \int_{t_p}^t (t-m)^{j-1} \|\mathcal{G}_{j,j}(P(t-m)^j)\| dm], \\ &\leq \frac{\epsilon}{j} (t-t_p)^j \psi_5(t, t_p) \|y_1(t) - y_2(t)\| \\ &\quad + \frac{\epsilon}{j} (t-t_p)^j \psi_6(t, t_p) \|y_1(t) - y_2(t)\|, \end{aligned}$$

where

$$\begin{aligned}\psi_5(t, t_p) &= \sup \left[\frac{2K}{1-L} \|(\mathcal{G}_j(P(t-\omega)^j))^{-1}\| \|\psi_2(t, t_p, \mathbf{m})\| \right], \\ \psi_6(t, t_p) &= \sup \left[\frac{2K}{1-L} \|(\mathcal{G}_j(P(t-\omega)^j))^{-1}\| \|\mathcal{G}_{j,j}(P(t-\mathbf{m})^j)\| \right].\end{aligned}$$

A little bit more simplification will yield

$$\|y_1(\omega) - y_2(\omega)\| \frac{\epsilon}{j} (t - t_p)^j \psi_7(t, t_p) \|y_1(t) - y_2(t)\|, \quad (42)$$

where

$$\psi_7(t, t_p) = \psi_5(t, t_p) + \psi_6(t, t_p).$$

From the last equation, Equation (42), if

$$\frac{\epsilon}{j} (t - t_p)^j \psi_7(t, t_p) < 1,$$

we have $y_1 = y_2$, which clearly guarantees the existence of a unique solution according to the Banach contraction mapping theorem for Equation (39), which leads to sufficient conditions for the observability of the system in Equation (35). $\square$

Example 1. Consider the fractional-order nonlinear system given by

$$\begin{cases} {}^c \mathcal{D}^j y(t) = Py(t) + Qu(t) + g(t, y(t), y(Yt), {}^c \mathcal{D}^j y(Yt)), y(t) \in \mathbb{R}^n, & t \in [0, T] = \mathcal{I}, \\ y(0) = y_0, \end{cases} \quad (43)$$

where $0 < j < 1$, $Y \in (0, 1)$. And $j = 0.6$, $T = 2$,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix},$$

and

$$g(t, y(t), y(Yt), {}^c \mathcal{D}^j y(Yt)) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} + \frac{1}{9}y(t) + \frac{1}{9}y(t/3) + {}^c \mathcal{D}^j y(t/3) \end{bmatrix}.$$

In addition, we use the Mittag-Leffler function

$$\mathcal{G}_{j,\mu}(Pt^j) = \sum_{j=0}^{\infty} \frac{P^j t^{j\bar{j}}}{\Gamma(j\bar{j} + \mu)}.$$

Then

$$\mathcal{G}_{0.6,0.6}(Pt^{0.6})Q = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix},$$

where

$$\begin{aligned} l_1 &= 1.073671274 (2.0 - 1.0 s)^{6/5} + 0.6715049724 - 2.178248842 (2.0 - 1.0 s)^{3/5} \\ &\quad - 1.610086426 (2.0 - 1.0 s)^{9/5} + \dots, \\ l_2 &= 1.089124421 (2.0 - 1.0 s)^{3/5} + 0.8050432130 (2.0 - 1.0 s)^{9/5} \\ &\quad - 2.147342548 (2.0 - 1.0 s)^{6/5} - 1.343009945 + \dots, \\ l_3 &= 1.089124421 (2.0 - 1.0 s)^{3/5} + 1.073671274 (2.0 - 1.0 s)^{6/5} \\ &\quad + 0.8050432130 (2.0 - 1.0 s)^{9/5} + 0.6715049724 + \dots \end{aligned}$$

Next, from the Gramian matrix

$$\mathcal{G}_c(0, T) = \int_0^T (T - m)^{2j-2} \mathcal{G}_{jj}(P(T - m)^j) Q Q^* \mathcal{G}_{jj}(P^*(T - m)^j m) dm,$$

where the symbol $*$ is used to represent the matrix transpose, we have

$$\begin{aligned} \mathcal{G}_c(0, 2) &= \int_0^2 (2 - m)^{2j-2} [\ell_1 \quad \ell_2 \quad \ell_3]^* [\ell_1 \quad \ell_2 \quad \ell_3] dm, \\ &= \int_0^2 (2 - m)^{2j-2} \begin{bmatrix} \ell_1^2 & \ell_1 \ell_2 & \ell_1 \ell_3 \\ \ell_1 \ell_2 & \ell_2^2 & \ell_2 \ell_3 \\ \ell_1 \ell_3 & \ell_2 \ell_3 & \ell_3^2 \end{bmatrix} dm, \\ &= \begin{bmatrix} 16.46203848 & 7.818919033 & -24.28095751 \\ 7.818919033 & 4.923578081 & -12.74249711 \\ -24.28095751 & -12.74249711 & 37.02345462 \end{bmatrix}, \end{aligned}$$

which is nonsingular. In addition, since the nonlinear map $g(t, y(t), y(Yt, {}^c D^j y(Yt)))$ meets the aforementioned assumptions, thus, using the theorem 2, one may conclude on the controllability of the system in Equation (43) on $\mathcal{J}$.

Example 2. Let us take into account the following system of fractional order:

$$\begin{cases} {}^c \mathcal{D}^j y(t) = P y(t) + g(t, y(t), y(Yt), {}^c D^j y(Yt)), y(t) \in \mathcal{R}^n, j \in (0, 1), \text{ and } t \in [0, T] = \mathcal{J}, \\ z(t) = F y(t), \\ y(0) = y_0, \end{cases} \quad (44)$$

where $0 < j < 1$, $Y \in (0, 1)$. And $j = 0.6$, $T = 2$,

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, F = [1 \quad -2 \quad 2], y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix},$$

and

$$g(t, y(t), y(Yt), {}^c D^j y(Yt)) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3} + \frac{1}{9} y(t) + \frac{1}{9} y(t/3) + {}^c D^j y(t/3) \end{bmatrix}.$$

In addition, we use the Mittag-Leffler function

$$\mathcal{G}_{j,\mu}(Pt^j) = \sum_{j=0}^{\infty} \frac{P^j t^{j\bar{j}}}{\Gamma(j\bar{j} + \mu)}.$$

Then,

$$\mathcal{G}_{0.6,0.6}(Pt^{0.6}) F^* = \begin{bmatrix} d \\ e \\ f \end{bmatrix},$$

where

$$\begin{aligned}d &= 1.073671274 (2.0 - 1.0 s)^{6/5} + 0.6715049724 - 2.178248842 (2.0 - 1.0 s)^{3/5} \\&\quad - 1.610086426 (2.0 - 1.0 s)^{9/5} + \dots, \\e &= 1.089124421 (2.0 - 1.0 s)^{3/5} + 0.8050432130 (2.0 - 1.0 s)^{9/5} \\&\quad - 2.147342548 (2.0 - 1.0 s)^{6/5} - 1.343009945 + \dots, \\f &= 2.178248842 (2.0 - 1.0 s)^{3/5} + 2.147342548 (2.0 - 1.0 s)^{6/5} \\&\quad + 1.610086426 (2.0 - 1.0 s)^{9/5} + 1.343009945 + \dots\end{aligned}$$

From the Gramian matrix

$$\mathcal{G}_o(0, T) = \int_0^T \mathcal{G}_j(P^* \dot{t}) F^* F \mathcal{G}_j(P \dot{t}) dt,$$

where $*$ represents the matrix transpose, we have

$$\begin{aligned}\mathcal{G}_o(0, 2) &= \int_0^2 [d \quad e \quad f]^* [d \quad e \quad f] dt, \\&= \int_0^2 \begin{bmatrix} d^2 & de & df \\ ed & e^2 & ef \\ fd & fe & f^2 \end{bmatrix} dt, \\&= \begin{bmatrix} 7.492447436 & 9.096203141 & 16.58865058 \\ 9.096203141 & 11.33860090 & 20.43480404 \\ 16.58865058 & 20.43480404 & 37.02345462 \end{bmatrix}.\end{aligned}$$

which is nonsingular, and the nonlinear map $g(t, y(t), y(Yt, {}^c D^j y(Yt)))$ meets the aforementioned requirements. As a result, by Theorem 6, the system in Equation (44) is observable on $\mathcal{J}$.

6. Concluding Comments

We formulated a novel type of nonlinear FO neutral pantograph-type system and investigated its dynamic aspects and a qualitative study thereof in this paper. We proved the controllability and observability of the linear system and, with the aid thereof, we established sufficient conditions for the controllability and observability of the corresponding nonlinear system. The Banach contraction principle, Schaefer's fixed-point approach, the MLF, and the theorem of Arzela–Ascoli were the main tools for establishing these results. For the authenticity of the established results, at the end of the paper, we added two examples.

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