symmetry

MDPI

Article

Graphs of Wajsberg Algebras via Complement Annihilating

Necla Kırcalı Gürsoy

Ege Vocational School, Ege University, Bornova, İzmir 35100, Turkey; necla.kircali.gursoy@ege.edu.tr

Abstract: In this paper, W-graph, called the notion of graphs on Wajsberg algebras, is introduced such that the vertices of the graph are the elements of Wajsberg algebra and the edges are the association of two vertices. In addition to this, commutative W-graphs are also symmetric graphs. Moreover, a graph of equivalence classes of Wajsberg algebra is constructed. Meanwhile, new definitions as complement annihilator and $\triangle$-connection operator on Wajsberg algebras are presented. Lemmas and theorems on these notions are proved, and some associated results depending on the graph's algebraic properties are presented, and supporting examples are given. Furthermore, the algorithms for determining and constructing all these new notions in each section are generated.

Keywords: Wajsberg algebras; graphs of Wajsberg algebras; algorithms on Wajsberg algebras; complement annihilator; MV-algebras; graphs of equivalence classes

MSC: 03G25; 05C25; 68W30

1. Introduction

Generally, because different algebraic structures are important for mathematics, logic algebraic structures are also very significant. In particular, algebraic structures related to non-classical logic are suitable structures for many-valued reasoning under uncertainty and vagueness. For these reasons, many researchers have presented new and important algebras for mathematics and logic [1–7]. Moreover, they have studied several areas related to these algebraic structures for years [8–14]. These structures have been used to solve issues in a variety of disciplines of mathematics and computer sciences, including fuzzy information with rough and soft ideas.

Firstly, the notion of Wajsberg algebras (shortly, we use W-algebras) was introduced in 1958 by Rose et al. [15]. Then, A. J. Rodriguez presented Wajsberg algebras functioning as the algebraic counterpart of infinite-valued propositional calculi of Lukasiewicz [16]. C. Chang introduced the fundamental operations, which are called implication and negation, considered by Lukasiewicz, in Wajsberg algebras [17,18]. Furthermore, the axioms match those employed by M. Wajsberg to axiomatize the infinite-valued propositional calculi following a Lukasiewicz conjecture. The definition of implicative filters and the family of implicative filters in lattice W-algebra were examined and some of their properties were investigated by Font et al. [19]. Basheer et al. presented the concepts of a fuzzy implicative filter and an anti-fuzzy implicative filter of lattice Wajsberg algebras and examined some of its features [20,21].

Using graphs to investigate algebraic structures has grown into a significant topic in recent times. As a result, many academics assign a graph to a ring or other algebraic structures and then investigate the algebraic features of these structures using the associated graphs [22–28]. Rings and algebras were first studied in this way in 1988 by Beck [25]. Beck establishes a link between graph theory and commutative ring theory in the paper. The zero-divisor graph has been extended to other algebraic structures in [26–28]. Many academics have examined graphs with classical structures such as commutative rings [29], commutative semirings [30], commutative semigroups [31], near-rings [32], and Cayley vague graphs [33]. In 1999, Anderson and Livingston gave another definition that is unlike

check for updates

Citation: Kırcalı Gürsoy, N. Graphs of Wajsberg Algebras via Complement Annihilating. *Symmetry* **2023**, *15*, 121. https://doi.org/10.3390/sym15010121

Academic Editors: Ismail Naci Cangul, Kinkar Chandra Das and Ahmet Sinan Cevik

Received: 21 November 2022
Revised: 26 December 2022
Accepted: 27 December 2022
Published: 1 January 2023

the earlier work of Anderson and Naseer in [22] and Beck in [25]. In other words, this definition excludes zero and all zero divisors of the ring. Mulay is interested in the new zero divisor graph [34]. The author classifies the cycle-structure of this graph. Moreover, Mulay establishes some group-theoretic properties of graph-automorphisms $\Gamma(R)$. Redmond [35] introduces zero divisor graphs for noncommutative rings. Also, Bozic and Petrovic study the relationship between the diameter of the zero-divisor graph of a commutative ring and that of the matrix ring [36]. Moreover, many researchers labor over commuting graphs of some rings [37]. For example, in [38–40], the authors analyze the diameters and connectivity of the commuting graphs of matrix rings. The orthogonality graphs $\Gamma O(R)$ of matrix rings were studied in [41–43], almost at the same time.

To date, the notion of annihilator has been applied to graphs built on algebraic structures [23,44,45]. Gursoy et al. present a new notion of complement annihilator on MV-algebras [46]. We give the structure of graphs in this study by connecting them with an algorithmic technique on W-algebras. For researchers taking action in different areas of science using W-algebras, introducing alternative constructions of graphs and developing related fundamental algorithms are the main contributions of this paper. Section 2 contains some fundamental definitions, lemmas, notions for W-algebras and graphs. In Section 3, $\triangle$-connection operator is defined. Then, the graph of W-algebras is constructed by using this notion and algorithms are presented. In Section 4, the complement annihilator on W-algebras is defined and some fundamental properties are examined. Section 5 constructs the equivalence class of W-algebras using the $\triangle$-connection operator and complement annihilator. The graph of equivalence classes of W- algebras is proposed as a construction. Moreover, the notions given in each section are supported by algorithms. Another feature that makes this paper different and novel is that, because the W-algebra structure has an upper bound, the notion of complement annihilator was established and W-graphs and graphs of equivalence classes were formed using this notion.

2. Preliminaries

We offer fundamental lemmas, definitions, theorems, and results of Wajsberg algebras and graphs in this section. Firstly, we recall some properties of Wajsberg algebras.

Definition 1 ([47])**.** *Let A be a nonempty set such that the binary operation $\to$, the unary operation $\neg$ and the distinguished element 1. If a system $\mathcal{A} = \langle A, \to, \neg, 1\rangle$ satisfies the following, then it is called a Wajsberg algebra (for short, a W-algebra).*

$(W1)$ $1 \to x = x$
$(W2)$ $(x \to y) \to ((y \to z) \to (x \to z)) = 1$
$(W3)$ $(x \to y) \to y = (y \to x) \to x,$
$(W4)$ $(\neg x \to \neg y) \to (y \to x) = 1.$

A binary relation $\leq$ in a W-algebra $\mathcal{A}$ is defined by

$$x \leq y \Leftrightarrow x \to y = 1.$$

We imagine that the binary relation $\leq$ has a boundary POS$(A, \leq)$ where zero is the smallest element of A such that $\neg 1 = 0$.

Lemma 1 ([47])**.** *Let $\mathcal{A} = \langle A, \to, 1\rangle$ be a W-algebra. Then the following properties hold:*

$(W5)$ $x \to x = 1,$
$(W6)$ *If* $x \to y = y \to x = 1$ *then* $x = y,$
$(W7)$ $x \to 1 = 1,$
$(W8)$ $x \to (y \to x) = 1,$
$(W9)$ *If* $x \to y = y \to z = 1$ *then* $x \to z = 1,$
$(W10)$ $(x \to y) \to ((z \to x) \to (z \to y)) = 1,$
$(W11)$ $x \to (y \to z) = y \to (x \to z)$

where x, y and z are in A.

Lemma 2 ([47]). *The following equations are satisfied in every W-algebra $\mathcal{A}$:*

(i) $\neg 1 \to x = 1$,
(ii) $\neg x = x \to \neg 1$,
(iii) $\neg\neg x = x$,
(iv) $x \to y = \neg y \to \neg x$.

Chang is the first person to introduce MV-algebra as a generalized boolean algebra [48]. Cignoli et al. [47] show the categorical equivalence between MV-algebras and ℓ-groups using a strong unit. Mostly, rather than a set-theoretic significance, the operations $\odot$ and $\oplus$ are distinctly arithmetic. Furthermore, there is no fundamental difference of expressive power between the additive connectives and the implication connective, according to the identities $x \to y = \neg x \oplus y$ and $x \oplus y = \neg x \to y$. So, replacing the $\odot$ and $\oplus$ operations with the implication connective $\to$ is more convenient. The following theorem is produced in this manner.

Theorem 1 ([47]). *Let $\langle A, \to, \neg, 1\rangle$ be a W-algebra. Upon defining $x \oplus y = \neg x \to y$ and $0 = \neg 1$, the system $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra.*

Now, we give some fundamental notions in graph theory. A graph is indicated by $G = (V(G), E(G))$ in which $V(G)$ demonstrates the vertex set and $E(G)$ demonstrates the edge set. If $u, v \in V(G^{'})$ while $(u, v) \in E(G^{'})$, $V(G^{'}) \subseteq V(G)$ and $E(G^{'}) \subseteq E(G)$, then $G^{'}$ is called a subgraph of G. Furthermore, if there is a path in G between any two distinct vertices u and v for $u, v \in V(G)$, that is, there are no isolated vertices, then G is said to be connected [49,50].

Set $N_G(v)$ contains all neighbors of vertex v in G such that $N_G(v) = \{u \mid (u, v) \in E(G)\}$. $d(v)$ demonstrates the degree of vertex v, that is, the number of edges connected to v, where $d(v) = |N_G(v)|$.

If G has a one-edge path between every pair of distinct vertices, then G is named a complete graph. If G has a vertex sequence as $v_1, v_2, \ldots, v_{n-1}, v_n$ where $d(v_1) = d(v_n) = 1$, $d(v_i) = 2$ for $i = 2, 3, \ldots, n-1$ and $E(G) = \bigcup_{j=1,2,\ldots,n-1}(v_j, v_{j+1})$, then G becomes a path graph. If $E(G) = \bigcup_{i=2,3,\ldots,n}(v_1, v_i)$ and $d(v_1) = n-1$, $d(v_i) = 1$ for $i = 2, 3, \ldots, n$, then G with n vertices is called a star graph [49,50].

G is called a bipartite graph if $V(G)$ can be separated into two subsets as V_1 and V_2, where $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V(G)$, for $u \in V_1, v \in V_2$ and $(u, v) \in E(G)$. Furthermore, G is called a complete bipartite graph if each vertex in V_1 is connected to each vertex in V_2 .

$d(u, v)$, the distance between u and v, points out the length of the shortest path between u and v for $u, v \in V(G)$. Besides, the diameter of G, namely $diam(G)$, denotes the maximum of the shortest paths in G, $diam(G) = max\{d(u, v) \mid u, v \in V(G)\}$.

Considering any two graphs G and H, if $f : V(G) \longrightarrow V(H)$ is a bijective mapping where $(f(u), f(v)) \in E(H)$ when $(u, v) \in E(G)$, then G and H are isomorphic and represented by $G \cong H$. Otherwise, graphs G and H are not isomorphic.

3. $\triangle$-Connection Operator and Graphs on W-Algebras

In this section, we first present an algorithm for determining if a structure is a W-algebra or not. Following that, we describe $\triangle$-connection operator and demonstrate some of its characteristics. Lastly, we give a notion of the graphs on W-algebra and present Algorithms 1 and 2.

3.1. Determining W-Algebras Using an Algorithm

Using Definition 1, we give an algorithm for determining whether or not an algebraic structure is a W-algebra in this subsection. This algorithm is the initial stage of an algorithmic outlook.

Algorithm 1: Determining W-algebra

```
Data: Set A, → operation, ¬ negation table
Result: A is a W-algebra or not.
W-ALGEBRA(A, →, ¬)
1  if A = ∅ OR 0 ∉ A OR 1 ∉ A then
2      Return false
3  foreach x in A do
4      if 1 → x ≠ x then
5          Return false
6      foreach y in A do
7          if (x → y) → y ≠ (y → x) → x then
8              Return false
9          if (¬x → ¬y) → (y → x) ≠ 1 then
10             Return false
11         foreach z in A do
12             if (x → y) → ((y → z) → (x → z)) ≠ 1 then
13                 Return false
14 Return true
```

W-ALGEBRA $(A, \to, \neg)$ (Algorithm 1) decides whether a given set A is a W-algebra with $\neg$ operator and $\to$ operation. If set A is a W-algebra, then the algorithm returns *true*, otherwise *false*.

Lines 1–2 of Algorithm 1 investigate whether or not the set A is empty, and whether or not 0 and 1 are elements of A. Then, axioms W1, W3, W4, and W2 are scrutinized in lines 4–5, 7–8, 9–10, and 12–13, respectively. Finally, in line 14, set A, satisfying all of the conditions in lines 1–13, is a W-algebra and returns to the value *true*.

3.2. The Notion of $\triangle$-Connection Operator on W-Algebras

In this subsection, we describe a $\triangle$-connection operator on W-algebras. After that, we look into some of this operator's fundamental characteristics and develop a method for acquiring graphs using $\triangle$-connection operator on W-algebras.

Definition 2. *Let $\mathcal{A}$ be a W-algebra. The $\triangle$-connection operator is described as $a\triangle b = (a \to b) \to b$ for each $a, b \in A$.*

Proposition 1. *The following properties are satisfied on each W-algebra:*

(i) $1\triangle a = 1$,
(ii) $a \leq a$,
(iii) *If* $a \leq b$ *then* $(b \to c) \leq (a \to c)$,
(iv) *If* $a \leq b$ *then* $(c \to a) \leq (c \to b)$,
(v) $a \leq (b \to a)$,
(vi) *If* $a \leq b$ *and* $b \leq c$ *then* $a \leq c$,
(vii) $a \leq (a\triangle b)$ *and* $b \leq (a\triangle b)$,
$(viii)$ $(a\triangle b) \to (b\triangle a) = 1$

for all $a, b, c \in A$.

Proof. (i) Let a be an element of A. Then, we obtain

$$1\triangle a = (1 \to a) \to a = a \to a = 1.$$

(ii) It is clear from $W5$.
(iii) Since $a \leq b$, we have $a \to b = 1$ from the definition of relation. So, we get the following equality from $W2$

$$\begin{aligned}(b \to c) \to (a \to c) &= 1 \to ((b \to c) \to (a \to c)) \\ &= (a \to b) \to ((b \to c) \to (a \to c)) \\ &= 1.\end{aligned}$$

Hence, we conclude that $(b \to c) \leq (a \to c)$.
(iv) Since $a \leq b$, we have $a \to b = 1$ from definition of relation. Then, we have from $W10$

$$\begin{aligned}(c \to a) \to (c \to b) &= 1 \to ((c \to a) \to (c \to b)) \\ &= (a \to b) \to ((c \to a) \to (c \to b)) \\ &= 1.\end{aligned}$$

Hence, we attain $(c \to a) \leq (c \to b)$.
(v) We obtain the following equality from $W11$

$$\begin{aligned}a \to (b \to a) &= b \to (a \to a) \\ &= b \to 1 \\ &= 1.\end{aligned}$$

Therefore, $a \leq (b \to a)$.
(vi) We have $a \to b = 1$ and $b \to c = 1$ by assumption. Hence, we obtain $a \to c = 1$, i.e., $a \leq c$.
(vii) By definition $\triangle$, we have

$$\begin{aligned}a \to ((a \to b) \to b) &= a \to ((b \to a) \to a) && (by\ \ W3) \\ &= 1. && (by\ \ W8)\end{aligned}$$

Hence, we conclude that $a \leq (a \triangle b)$. The rest of the proof is similar.
$(viii)$ The following equality is taken from the definition of $\triangle$. By using $W3$ and $W5$, we obtain that

$$\begin{aligned}(a \triangle b) \to (b \triangle a) &= ((a \to b) \to b) \to ((b \to a) \to a) \\ &= 1.\end{aligned}$$

□

Lemma 3. *The $\triangle$-connection operator is commutative.*

Proof. It is obvious from W3. □

Lemma 4. *Let $\mathcal{A}$ be a W-algebra and $a, b, x, y \in A$ such that $a \leq (x \triangle y)$ and $b \leq (x \triangle y)$. Then, the equality $(a \triangle b) = (x \triangle y)$ is verified.*

Proof. We have $a \leq (x \triangle y)$ and $b \leq (x \triangle y)$ for all $a, b, x, y \in A$. Since $a \leq (x \to y) \to y$, we get

$$\begin{aligned}((x \to y) \to y) \to b &\leq a \to b && \text{(by Proposition 1 (iv))} \\ ((a \to b) \to b) &\leq (((x \to y) \to y) \to b) \to b && \text{(by Proposition 1 (iv))} \\ ((a \to b) \to b) &\leq (b \to ((x \to y) \to y)) \to ((x \to y) \to y) && \text{(by W3)} \\ ((a \to b) \to b) &\leq 1 \to ((x \to y) \to y) && \text{(by assumption)} \\ ((a \to b) \to b) &\leq ((x \to y) \to y)\end{aligned}$$

Hence, we obtain $a \triangle b = x \triangle y$. $\square$

Because of its commutativity, $\triangle$-connection operator provides various advantages for the algebraic technique. The following example indicates an analysis of $\triangle$-connection operator.

Example 1. *Let $\rightarrow$ binary operation and $\neg$ unary operation describe on $A = \{0, a_1, a_2, a_3, a_4, 1\}$ as indicated in the following Cayley tables:*

$\rightarrow$	0	a_1	a_2	a_3	a_4	1
0	1	1	a_1	1	a_1	1
a_1	a_3	1	0	a_3	a_1	1
a_2	1	1	1	1	1	1
a_3	a_1	a_4	a_4	1	a_4	1
a_4	a_3	1	a_3	a_3	1	1
1	0	a_1	a_2	a_3	a_4	1

$\neg$	0	a_1	a_2	a_3	a_4	1
	a_1	a_2	0	1	a_4	a_3

Then the structure $\mathcal{A} = (A, \rightarrow, \neg, 0)$ is a W-algebra. In addition, the following Cayley table is obtained for $\triangle-$connection operator by using the definition of $\triangle$ as below:

$\triangle$	0	a_1	a_2	a_3	a_4	1
0	0	a_1	0	a_3	a_1	1
a_1	a_1	a_1	a_1	1	a_1	1
a_2	0	a_1	a_2	a_3	a_4	1
a_3	a_3	1	a_3	a_3	1	1
a_4	a_1	a_1	a_4	1	a_4	1
1	1	1	1	1	1	1

The $\triangle-$connection operator's commutativity property is obviously achieved from the Cayley table above on this W-algebra.

3.3. Graphs on W-Algebras

This section introduces the W-graph and provides examples. Then, we prove lemmas and theorems. First, Algorithm 2 is presented, which constructs the $\triangle$-operation Cayley table. We offer a new approach named Algorithm 3 that uses this algorithm to build the graph of a given W-algebra.

Definition 3. *A W-algebra $\mathcal{A} = (A, \rightarrow, \neg, 0)$ corresponds to an undirected graph $G(\mathcal{A})$, where $V(G(\mathcal{A}))$ consists of the elements of A and two distinct elements $a, b \in A$ are called adjacent if and only if $a \triangle b = 1$. If G provides these conditions, then it is said to be a W-graph.*

Example 2. *Let $\mathcal{A} = (A, \rightarrow, \neg, 0)$ be a W-algebra on $A = \{0, a_1, a_2, a_3, a_4, a_5, a_6, 1\}$. The Cayley table for $\triangle-$connection operator is as follows:*

$\triangle$	0	a_1	a_2	a_3	a_4	a_5	a_6	1
0	0	a_1	a_2	a_3	a_4	a_5	a_6	1
a_1	a_1	a_1	1	a_3	a_5	a_5	a_3	1
a_2	a_2	1	a_2	1	a_2	1	a_2	1
a_3	a_3	a_3	1	a_3	a_3	a_3	a_3	1
a_4	a_4	a_5	a_2	a_3	a_4	a_5	a_6	1
a_5	a_5	a_5	1	a_3	a_5	a_5	a_3	1
a_6	a_6	a_3	a_2	a_3	a_6	a_3	a_6	1
1	1	1	1	1	1	1	1	1

Using Definition 3, the adjacency matrix of the graph of $\mathcal{A}$ can be obtained as follows:

$$Adj(G(\mathcal{A})) = \begin{array}{cc} & \begin{array}{cccccccc} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & 1 \end{array} \\ \begin{array}{c} 0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ 1 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

The elements of A form the vertex set of the graph in Figure 1. Hence, the graph of $\mathcal{A}$, namely $G(A)$, is a W-graph.

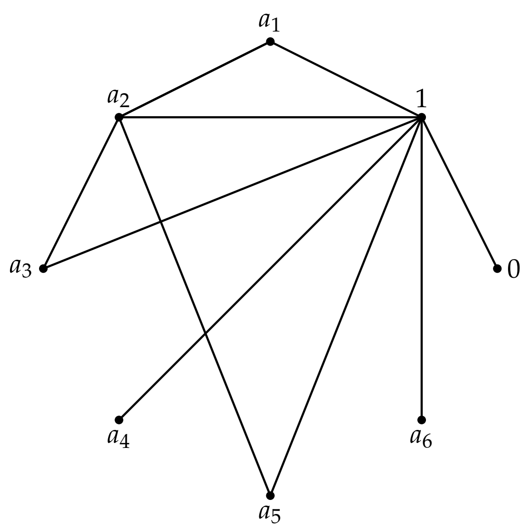

Figure 1. $G(\mathcal{A})$ of Example 2.

Lemma 5. *In a W-graph, vertex 1 is adjacent to all vertices in $G(\mathcal{A})$.*

Proof. It is obtained that $a \triangle 1 = (a \to 1) \to 1 = 1 \to 1 = 1$ for each $a \in A$ by the Definition 2. Therefore, vertex 1 and vertex a are connected with an edge in every W-graph. □

Theorem 2. *W-graph $G(\mathcal{A})$ is connected by having $diam(G(\mathcal{A})) \leq 2$.*

Proof. Let $a, b \in A$ be any two distinct vertices of $G(\mathcal{A})$. Firstly, we suppose $a \triangle b = 1$. Thereby, we have $d(a, b) = 1$, so this provides $diam(G(\mathcal{A})) \leq 2$. Then, we suppose $a \triangle b \neq 1$. By Lemma 5, we have $a \triangle 1 = (a \to 1) \to 1 = 1 \to 1 = 1$ and $b \triangle 1 = (b \to 1) \to 1 = 1 \to 1 = 1$. So, a is adjacent to 1 and also b is adjacent to 1. Therefore, we get $d(a, b) = 2$, which means that $diam(G(\mathcal{A})) \leq 2$.

Accordingly, we acquire $diam(G(\mathcal{A})) \leq 2$ in a W-graph $G(\mathcal{A})$. □

Example 3. *Let $\mathcal{A} = (A, \to, \neg, 0)$ be a W-algebra on $A = \{0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, 1\}$. The Cayley table for $\triangle-$connection operator is as follows:*

$\triangle$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	1
0	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	1
a_1	a_1	a_1	a_2	a_1	a_4	a_1	a_4	a_2	1
a_2	a_2	a_2	a_2	a_2	1	a_2	1	a_2	1
a_3	a_3	a_1	a_2	a_3	a_4	a_1	a_6	a_2	1
a_4	a_4	a_4	1	a_4	a_4	a_4	a_4	1	1
a_5	a_5	a_1	a_2	a_1	a_4	a_5	a_4	a_7	1
a_6	a_6	a_4	1	a_6	a_4	a_4	a_6	1	1
a_7	a_7	a_2	a_2	a_2	1	a_7	1	a_7	1
1	1	1	1	1	1	1	1	1	1

The $V(G(\mathcal{A}))$ in Figure 2 occurs from the elements of A. As a result, the vertices of the W-graph below correspond to the elements of the W-algebra above.

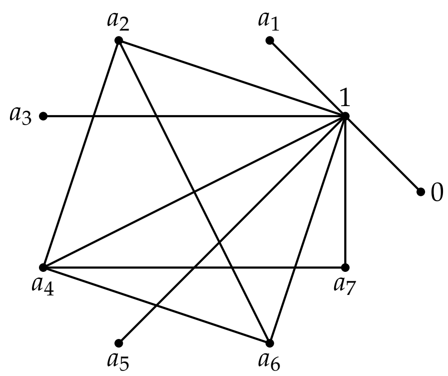

Figure 2. $G(\mathcal{A})$ of Example 3.

$G(\mathcal{A})$ is a connected graph because it has no isolated vertices. Furthermore, because the longest length of the shortest path between a_1 and a_2 is 2 for all $a_1, a_2 \in V(G(\mathcal{A}))$, $diam(G(\mathcal{A})) \leq 2$.

Definition 4. *Let $\mathcal{A}$ be a W-algebra. If $a \to b = b \to a$ for all $a, b \in A$, then it is called commutative W-algebra.*

Lemma 6. *Let $\mathcal{A}$ be a commutative W-algebra. Then $a \triangle b = 1$ for all $a, b \in A$.*

Corollary 1. *If $\mathcal{A}$ is a commutative W-algebra then its graph is complete.*

Corollary 2. *If $\mathcal{A}$ is a commutative W-algebra then $G(\mathcal{A})$ is a symmetric graph.*

Now, Algorithm 2 is presented for $\triangle$-operation's Cayley table by using Algorithm 1.

Algorithm 2: $\triangle$-operation's Cayley table

Data: Set A, operation $\to$, and operation $\neg$
Result: $\triangle$-operation's Cayley table
$\triangle$-OPERATION$(A, \to, \neg)$
$\triangle$-*Table*$[\,,\,] = null$ `// initializing all elements to null`
foreach x **in** A **do**
  foreach y **in** A **do**
    $\triangle$-*Table*$[x, y] = (x \to y) \to y$
Return $\triangle$-*Table*

Using a given set A, $\to$ binary operation and $\neg$ unary operation, $\triangle$-OPERATION$(A, \to, \neg)$ (Algorithm 2) creates the $\triangle$ operation's Cayley table.

In line 1, $\triangle$-Table is declared as a two-dimensional array with all of its elements set to null. $(a \to b) \to b$ operation, namely $a \triangle b$, is executed in line 4 and the result of the $\triangle$ operation is placed in the cell of row a and column b in the Cayley table. In line 5, the $\triangle$ operation's Cayley table is eventually returned.

Using Algorithm 2, we achieve Algorithm 3 to construct the W-graph.

Algorithm 3: Constructing of a W-graph

Data: W-algebra $\mathcal{A}$, Cayley tables of $\to$ and $\neg$ operations
Result: Graph of W-algebra $\mathcal{A}$
GRAPH$(\mathcal{A}, \to, \neg)$
1 $V = A$
2 $E = \emptyset$
3 $\triangle = \triangle$-OPERATION$(A, \to, \neg)$
4 **foreach** x **in** A **do**
5   **foreach** y **in** A **do** where $y > x$
6   **if** $x \triangle y = 1$ **then**
7             $E = E \cup (x, y)$
8 $G(\mathcal{A}) = (V, E)$
9 Return $G(\mathcal{A})$

For a given W-algebra $\mathcal{A}$, $\oplus$ and $\neg$ operations, the preceding method (Algorithm 3) generates the W-graph $G(\mathcal{A})$. In line 1, the vertex set is initialized as the set A and the edge set is initialized to an empty set for $G(\mathcal{A})$ in line 2. The Cayley table of $\triangle$ is then described in line 3 by Algorithm 2, $\triangle$-OPERATION, using $\to$ binary and $\neg$ unary operations over A. In addition, in lines 6-7, if $x \triangle y = 1$ where $x, y \in A$, the (x, y) becomes an edge and it is added to the set of edges ($E = E \cup (x, y)$). As a result, in lines 8-9, V and G determine $G(\mathcal{A})$, and it is returned as the graph of the specified W-algebra $\mathcal{A}$.

4. Complement Annihilating in W-Algebras

This section introduces the notion of the complement annihilator on W-algebras and investigates some important properties of complement annihilating on the algebraic construction. Besides, we initiate a relationship between complement annihilator sets and filters on W-algebras.

Definition 5. *Let $\mathcal{A}$ be a W-algebra. The complement annihilator of M is described as*

$$ann^c(M) = \{x \in A \mid a \triangle x = 1, \forall a \in M\}$$

such that M is a subset of A.

Lemma 7. *If $a \leq b$ for $a, b \in A$, then $ann^c(a) \subseteq ann^c(b)$ for a $W-$algebra $\mathcal{A} = (A, \to, \neg, 0)$.*

Proof. Suppose that $x \in ann^c(a)$. So, we get $x \triangle a = 1$ and the following equality:

$$(x \to a) \to a = (a \to x) \to x = 1.$$

We have

$$a \leq b$$
$$(b \to x) \leq (a \to x)$$
$$(a \to x) \to x \leq (b \to x) \to x$$

using Proposition 1(iii). Hence, we attain that $(b \to x) \to x = 1$ since $(a \to x) \to x = 1$. That is, $x \in ann^c(b)$. $\square$

Lemma 8. *If $M, N \subseteq A$, then the following results are obtained on a W-algebra $\mathcal{A} = (A, \rightarrow, \neg, 0)$.*

(i) *If $M \subseteq N$ then $ann^c(N) \subseteq ann^c(M)$,*
(ii) *$ann^c(M \cup N) = ann^c(M) \cap ann^c(N)$,*
(iii) *$ann^c(M) \cup ann^c(N) \subseteq ann^c(M \cap N)$.*

Proof. (i) We assume that $x \in ann^c(N)$. From Definition 5, we have

$$c \triangle x = 1$$

for all $c \in N$. As $M \subseteq N$, we attain $b \triangle x = 1$ for all $b \in M$. That is, $x \in ann^c(M)$, i.e., $ann^c(N) \subseteq ann^c(M)$.
(ii) $M \subseteq M \cup N$ and $N \subseteq M \cup N$. By using (i), we have

$$ann^c(M \cup N) \subseteq ann^c(M) \quad \text{and} \quad ann^c(M \cup N) \subseteq ann^c(N).$$

So, we get

$$ann^c(M \cup N) \subseteq ann^c(M) \cap ann^c(N). \tag{1}$$

Conversely, suppose that $x \in ann^c(M) \cap ann^c(N)$. Then, $b \triangle x = 1$ for all $b \in M$ and $c \triangle x = 1$ for all $c \in N$. For any $a \in M \cup N \Rightarrow a \in M$ or $a \in N$ and so $a \triangle x = 1$ for all $a \in M \cup N$. If $x \in ann^c(M \cup N)$ then this implies:

$$ann^c(M) \cap ann^c(N) \subseteq ann^c(M \cup N). \tag{2}$$

We attain $ann^c(M \cup N) = ann^c(M) \cap ann^c(N)$ using the relations (1) and (2).
(iii) We have $M \cap N \subseteq M$ and $M \cap N \subseteq N$. We get

$$ann^c(M) \subseteq ann^c(M \cap N)$$

and

$$ann^c(N) \subseteq ann^c(M \cap N)$$

from (i). So, $ann^c(M) \cup ann^c(N) \subseteq ann^c(M \cap N)$ is obtained. □

Lemma 9. *$ann^c(M) = \bigcap_{b \in M} ann^c(b)$ if M is a non-empty subset of A.*

Proof. By the help of Lemma 8(ii), we deduce that

$$ann^c(M) = ann^c\{\bigcup_{b \in M} \{b\}\} = \bigcap_{b \in M} ann^c(b).$$

□

Lemma 10. *$a \beta b \Leftrightarrow ann^c(a) = ann^c(b)$, β is a relation on W-algebra.*

Definition 6. *Suppose that F_W is a subset of A and $\mathcal{A} = (A, \rightarrow, \neg, 0)$ be a W-algebra. Then, F_W is a filter of $\mathcal{A}$ if and only if*

(i) *$1 \in F_W$,*
(ii) *If $x \in F_W$ and $x \rightarrow y \in F_W$ then $y \in F_W$ for all $x, y \in A$.*

Theorem 3. *$ann^c(M)$ is a filter of $\mathcal{A}$ for each $M \subseteq A$.*

Proof. (i) We obtain that $1 \triangle b = (1 \rightarrow b) \rightarrow b = b \rightarrow b = 1$ using the definition of $ann^c(M)$, for all $b \in M$. Hence, $1 \in ann^c(M)$.
(ii) By assumption, we have $m_1 \in ann^c(M)$ and $m_1 \rightarrow m_2 \in ann^c(M)$. Therefore $m_1 \triangle b =$

$(m_1 \to b) \to b = 1$ and $(m_1 \to m_2) \triangle b = ((m_1 \to m_2) \to b) \to b = 1$ for all $b \in M$. From the properties of W-algebra and our assumptions, we derive the following.

$$(b \to m_1) \to m_1 = (x \to y) \to (z \to ((z \to x) \to y)) \quad \text{(by W10, W11, and W3)} \tag{3}$$
$$(b \to m_1) \to m_1 = x \to (y \to x) = 1 \quad \text{(by W8)} \tag{4}$$
$$(b \to m_1) \to m_1 = x \to x = 1 \quad \text{(by W5)} \tag{5}$$
$$((b \to m_1) \to m_1) \to x = x \quad \text{(by W1)} \tag{6}$$
$$((x \to y) \to z) \to z = x \to ((z \to (x \to y)) \to y) \quad \text{(by W3 and W11)} \tag{7}$$

We get equation (8) by using (3), W3, (4) and W11, respectively.

$$x \to ((x \to ((y \to ((y \to x) \to z)) \to z)) \to z) = y \to ((y \to x) \to z) \tag{8}$$
$$((b \to m_1) \to m_1) = x \to (y \to (z \to x)) \quad \text{(by W9 and (4))} \tag{9}$$

We suppose that $((b \to m_1) \to m_1) \neq ((b \to m_2) \to m_2)$. Then, we have

$$((b \to m_2) \to m_2) \neq x \to (y \to x) \quad \text{(from (4))} \tag{10}$$
$$m_1 \to b = b \quad \text{(by W6, (4), and W3)} \tag{11}$$

and we have

$$b \to m_1 = m_1 \tag{12}$$

using W6 and (4).

$$(m_1 \to m_1) = x \to (y \to (z \to x)) \quad \text{(by (9) and (12))} \tag{13}$$
$$(m_1 \to m_1) \to x = x \quad \text{(by (6) and (12))} \tag{14}$$

and

$$(m_1 \to m_1) = x \to x \quad \text{(by (5) and (12))} \tag{15}$$

Since we have $((m_1 \to m_2) \to b) \to b = 1$, we obtain

$$m_1 \to ((b \to (m_1 \to m_2)) \to m_2) = (b \to m_1) \to m_1 \tag{16}$$

by using W3 and W11.

$$m_1 \to ((b \to (m_1 \to m_2)) \to m_2) = m_1 \to m_1 \quad \text{(by (16) and (12))} \tag{17}$$
$$m_1 \to (x \to b) = x \to b \quad \text{(by (11) and W11)} \tag{18}$$
$$(b \to m_2) \to m_2 \neq x \to x \quad \text{(by (14) and (10))} \tag{19}$$
$$(x \to x) \to y = y \quad \text{(by (15) and (14))} \tag{20}$$

When applying the properties of W3 and W11 using equations (18) and (8), we obtain

$$x \to ((x \to (m_1 \to ((b \to (m_1 \to x)) \to x))) \to b) = (m_1 \to x) \to b \tag{21}$$
$$(x \to (y \to (z \to x))) \to u = u \quad \text{(by (13) and (20))} \tag{22}$$

for $u \in A$.

$$(m_1 \to x) \to b = x \to b \quad \text{(by (21) and (22))} \tag{23}$$
$$(b \to m_2) \to m_2 = m_1 \to t_1 \quad \text{(by (17), (7), (23), and W3)} \tag{24}$$

A contradiction results from (19) and (24). □

To obtain complement annihilator sets of A, we construct Algorithm 4 by using the $\triangle$ Cayley table on A.

Algorithm 4: Determination of complement annihilator sets

Data: Set A and the $\triangle$-operation
Result: Sets of complement annihilators of A
ann^cSETS$(A, \triangle)$

1 $ann^c[\,] = \emptyset$ `// initializing all elements to` $\emptyset$
2 **foreach** x **in** A **do**
3   **foreach** y **in** A **do**
4   **if** $x \triangle y = 1$ **then**
5             $ann^c[x] = ann^c[x] \cup y$
6 Return $\underset{\forall a \in A}{ann^c[a]}$

By $\triangle$ operation, this method (Algorithm 4) concludes a set of complement annihilators of each element in A.

ann^c is determined as a one-dimensional array initializing all elements to an empty set in line 1. The set of $ann^c[x]$ is then generated in lines 4-5 using x which produces the $x \triangle y = 1$. In line 6, the array of complement annihilator sets is returned for each element of A.

Lemma 11. *The $\triangle-$connection operator provides associativity property on W-algebras.*

Proof. Since $y \leq y \triangle z$, we get

$$x \triangle y \leq x \triangle (y \triangle z). \tag{25}$$

Similarly, we have $z \leq y \triangle z$ and $y \triangle z \leq x \triangle (y \triangle z)$. Therefore,

$$z \leq x \triangle (y \triangle z). \tag{26}$$

Using inequalities (25),(26) and Lemma 4, we attain

$$(x \triangle y) \triangle z \leq x \triangle (y \triangle z). \tag{27}$$

On the other hand, since $y \leq x \triangle y$, we get $y \leq (x \triangle y) \triangle z$ and $z \leq (x \triangle y) \triangle z$. Therefore, we have

$$y \triangle z \leq (x \triangle y) \triangle z. \tag{28}$$

Since $x \leq x \triangle y$ and $x \triangle y \leq (x \triangle y) \triangle z$, we get

$$x \leq (x \triangle y) \triangle z. \tag{29}$$

From (28) and (29), we attain

$$x \triangle (y \triangle z) \leq (x \triangle y) \triangle z. \tag{30}$$

We achieve associativity from (27) and (30). □

Lemma 12. *If $ann^c(a) = ann^c(x)$ and $ann^c(b) = ann^c(y)$ then $ann^c(a \triangle b) = ann^c(x \triangle y)$.*

Proof. Suppose that $ann^c(a) = ann^c(x)$ and $ann^c(b) = ann^c(y)$. If $m \in ann^c(a) = ann^c(x)$ then we have $m \triangle a = 1$ and $m \triangle x = 1$. Similarly, if $t \in ann^c(b) = ann^c(y)$, then we have $t \triangle b = 1$ and $t \triangle y = 1$. Assume that $m \in ann^c(a \triangle b)$; and by using associative, we get $m \in ann^c(x \triangle y)$ since

$$m \triangle (x \triangle y) = (m \triangle x) \triangle y = 1 \triangle y = 1.$$

Therefore, $ann^c(a \triangle b) \subset ann^c(x \triangle y)$. A similar procedure is carried out for the other direction. Thus, we attain $ann^c(a \triangle b) = ann^c(x \triangle y)$. $\square$

5. Equivalence Classes on W-Algebras and Their Graphs

In this section, we first establish the equivalence classes of W-algebras using the $\triangle$-connection operator and the complement annihilator notion. The graphs of equivalence classes are developed as a result of this foundation. We also investigate their properties and their connection to W-graphs.

Now, with the help of equivalence classes of $\mathcal{A}$, we set a graph, which is indicated by $G_E(\mathcal{A})$. The vertex set of $G_E(\mathcal{A})$ corresponds to the equivalence classes of $\mathcal{A}$, $\{[a] : a \in \mathcal{A}\}$.

Definition 7. *Vertices $[a]$ and $[b]$ are called adjacent in $G_E(\mathcal{A})$ if and only if $[a] \triangle [b] = \{1\}$ where $[a]$ and $[b]$ are two distinct equivalence classes of $\mathcal{A}$.*

Example 4. *Let $A = \{0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, 1\}$ be a set. The $\triangle$-operation is defined on A and the Cayley table is obtained as follows:*

$\triangle$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	1
0	0	0	0	1	0	0	1	0	1	0	0	1
a_1	0	a_1	a_1	a_3	0	a_7	a_6	a_7	a_8	a_9	a_9	1
a_2	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	1
a_3	1	a_3	a_3	a_3	1	a_8	a_6	a_8	a_8	a_6	a_6	1
a_4	0	0	a_4	1	a_4	a_4	1	0	1	0	a_4	1
a_5	0	a_7	a_5	a_8	a_4	a_5	1	a_7	a_8	0	a_4	1
a_6	1	a_6	a_6	a_6	1	1	a_6	1	1	a_6	a_6	1
a_7	0	a_7	a_7	a_8	0	a_7	1	a_7	a_8	0	0	1
a_8	1	a_8	a_8	a_8	1	a_8	1	a_8	a_8	1	1	1
a_9	0	a_9	a_9	a_6	0	0	a_6	0	1	a_9	a_9	1
a_{10}	0	a_9	a_{10}	a_6	a_4	a_4	a_6	0	1	a_9	a_{10}	1
1	1	1	1	1	1	1	1	1	1	1	1	1

Then, the structure $\mathcal{A} = (A, \rightarrow, \neg, 0)$ is a W-algebra. From this Cayley table, we have a vertex and edge sets such as $V(G(\mathcal{A})) = \{0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, 1\}$ and $E(G(\mathcal{A})) = \{(0, a_3), (0, a_6), (0, a_8), (0, 1), (a_3, a_4), (a_4, a_6), (a_4, a_8), (a_5, a_6), (a_6, a_7), (a_6, a_8), (a_8, a_9), (a_8, a_{10}), (1, a_1), (1, a_2), (1, a_3), (1, a_4), (1, a_5), (1, a_6), (1, a_7), (1, a_8), (1, a_9), (1, a_{10})\}$, respectively.

In addition, we get $V(G_E(\mathcal{A})) = \{[0], [a_1], [a_3], [a_5], [a_6], [a_8], [a_9], [1]\}$ for the reason that

$$ann^c(0) = \{1\}, \quad ann^c(1) = A, \quad ann^c(a_1) = \{1\},$$
$$ann^c(a_2) = \{1\}, \quad ann^c(a_3) = \{0, 1, a_4\}, \quad ann^c(a_4) = \{1, a_3, a_6, a_8\},$$
$$ann^c(a_5) = \{1, a_6\}, \quad ann^c(a_6) = \{0, 1, a_4, a_5, a_7, a_8\}, \quad ann^c(a_7) = \{1, a_6\},$$
$$ann^c(a_8) = \{0, 1, a_4, a_6, a_9, a_{10}\}, \quad ann^c(a_9) = \{1, a_8\}, \quad ann^c(a_{10}) = \{1, a_8\}.$$

Also, we obtain the edge set of $G_E(\mathcal{A})$ as $E(G_E(\mathcal{A})) = \{([0], [1]), ([0], [a_3]), ([0], [a_6]), ([0], [a_8]), ([a_5], [a_6]), ([a_6], [a_8]), ([a_8], [a_9]), ([1], [a_1]), ([1], [a_3]), ([1], [a_5]), ([1], [a_6]), ([1], [a_8]), ([1], [a_9])\}$. Therefore, the adjacency matrix for the graph of equivalence classes of $\mathcal{A}$ is as follows.

$$Adj(G_E(\mathcal{A})) = \begin{array}{c} \\ [0] \\ [a_1] \\ [a_3] \\ [a_5] \\ [a_6] \\ [a_8] \\ [a_9] \\ [1] \end{array} \begin{array}{c} \begin{array}{cccccccc} [0] & [a_1] & [a_3] & [a_5] & [a_6] & [a_8] & [a_9] & [1] \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Then, the following graph corresponds to the graph of equivalence classes of W-algebra in Figure 3.

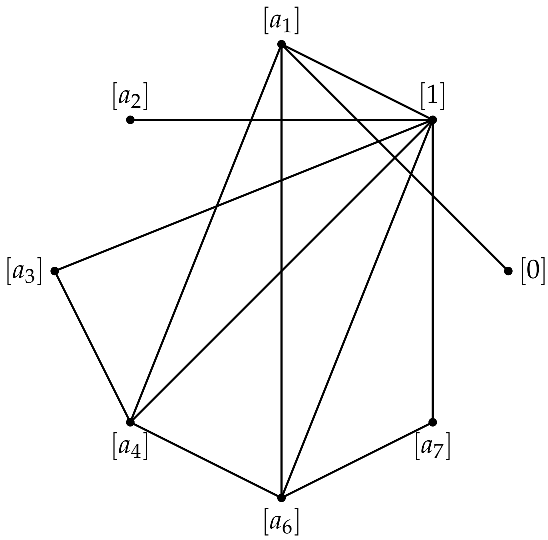

Figure 3. Graph $G_E(\mathcal{A})$ of Example 4.

Theorem 4. *For all $a, b \in \mathcal{A}$, it is satisfied that $[a]\triangle[b] = [a\triangle b]$.*

Proof. Let m be an element of $[a]\triangle[b]$. Then, we attain $m = x\triangle y$ for any $x \in [a]$ and $y \in [b]$. As a result, we have the following equality:

$$ann^c(m) = ann^c(x\triangle y) = ann^c(a\triangle b).$$

So, $m \in [a\triangle b]$, that is, $[a]\triangle[b] \subseteq [a\triangle b]$. Also, we get $[a\triangle b] \subseteq [a]\triangle[b]$, similarly. Hence, it is accomplished that $[a]\triangle[b] = [a\triangle b]$. □

Lemma 13. *If $N([1]) = \{a \mid a \in A \setminus \{1\}\}$, then the graph of W-algebra $\mathcal{A}$ is a star graph.*

Proof. Let $G(\mathcal{A})$ not be a star graph and $N([1]) = A \setminus \{1\}$. So, a and b are adjacent in $G(\mathcal{A})$ where $a, b \in V(G(\mathcal{A}))$ with $a \neq 1$ and $b \neq 1$. Then, $G' = (V', E')$ becomes a subgraph of $G(\mathcal{A})$ as in Figure 4 such that $V' = \{a, b, 1\}$ and $E' = \{(a, b), (a, 1), (b, 1)\}$.

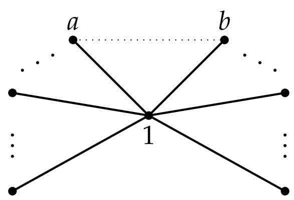

Figure 4. Subgraph of $G(\mathcal{A})$.

We have $a\triangle 1 = 1$ because a and 1 are adjacent. Therefore, it is obtained that $[a] = [1]$. Then, we attain $[b] = [1]$, similarly. Hence, $a, b \in [1]$.

Hence, $N([1])$ does not contain elements a and b. Thus, it is obtained that $N([1]) \neq A \setminus \{1\}$. We get a contradiction with our hypothesis. □

Corollary 3. *The graph of equivalence classes, $G_E(\mathcal{A})$, consists of an edge if and only if $G(\mathcal{A})$ is a star graph.*

Lemma 14. *$G_E(\mathcal{A})$, the graph of equivalence classes, is a subgraph of $G(\mathcal{A})$.*

Proof. Suppose that $[v] \in V(G_E(\mathcal{A}))$. Using Definition 3 and Definition 7, we obtain that $v \in V(G(\mathcal{A}))$. Let $([v_1],[v_2]) \in E(G_E(\mathcal{A}))$. Then, we have $[v_1]\triangle[v_2] = \{1\}$. Since $\{1\} = [v_1]\triangle[v_2] = [v_1 \triangle v_2]$, we obtain $[v_1 \triangle v_2] = \{1\}$, i.e., $v_1 \triangle v_2 = 1$. Hence, we conclude that $(v_1, v_2) \in E(G(\mathcal{A}))$. Then, $G_E(\mathcal{A})$ is a subgraph of $G(\mathcal{A})$. □

Lemma 15. *The elements a and 1 are adjacent in $G(\mathcal{A})$ for each $a \in G(\mathcal{A})$.*

Proof. We have $a\triangle 1 = 1$ for each $a \in G(\mathcal{A})$. Accordingly, $(a,1) \in E(G(\mathcal{A}))$. This means that vertices a and 1 are adjacent. □

Theorem 5. *A star graph with $n-$vertex is a subgraph for every graph $G(\mathcal{A})$.*

Proof. By Lemma 15, we have $a\triangle 1 = 1$ for all $a \in \mathcal{A}$. In other words, a is adjacent to vertex "1" for each $a \in \mathcal{A}$. The degree of the vertex "1" is $n-1$ and the degree of the other $n-1$ vertices is 1 where $n = |A|$. □

Theorem 6. *Let $G_E(\mathcal{A})$ be the graph of equivalence classes of $\mathcal{A}$ having any distinct vertices $[a]$, $[b]$. If vertex $[a]$ and vertex $[b]$ are adjacent, then $ann^c(a)$ and $ann^c(b)$ become separated W-complement annihilators of $\mathcal{A}$.*

Proof. We suppose that $ann^c(a) = ann^c(b)$. This means that a and b are in the same equivalence class. We get $[a] \neq [b]$ since $[a]$ and $[b]$ are adjacent. This implies that a and b are in different equivalence classes. This is a contradiction with our hypothesis. Then, it is obtained that $ann^c(a) \neq ann^c(b)$, that is, $ann^c(a)$ and $ann^c(b)$ are distinct W-complement annihilators of $\mathcal{A}$. □

Theorem 7. *If W-graph $G(\mathcal{A})$ is a complete graph, then $G_E(\mathcal{A})$ is isomorphic to $G(\mathcal{A})$.*

Proof. We assume that $A = \{a_1, a_2, \ldots, a_n\}$. Elements of A correspond to vertices of $G(\mathcal{A})$ by Definition 3. Then, we get $v_i \in V(G(\mathcal{A}))$ for $i \in \{1,2,\ldots,n\}$.

Since $G(\mathcal{A})$ is a complete graph, two distinct vertices v_i and v_j are adjacent where $i \neq j$. Accordingly, we have

$$N(v_i) = \{a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\} = ann^c(v_i)$$

for $i \in \{1,2,\ldots,n\}$. Then, we get

$$[v_i] = ann^c(v_i) \neq ann^c(v_j) = [v_j]$$

for $i, j \in \{1,2,\ldots,n\}$ and $i \neq j$. Consequently, each distinct vertex in $G(\mathcal{A})$ coincides with a distinct equivalence class of $\mathcal{A}$.

Now, we describe a mapping between vertex sets of $G(\mathcal{A})$ and $G_E(\mathcal{A})$ as follows:

$$\begin{aligned} g : V(G(\mathcal{A})) &\longrightarrow V(G_E(\mathcal{A})). \\ v_i &\longmapsto g(v_i) = [v_i] \end{aligned}$$

for $i \in \{1,2,\ldots,n\}$. The mapping is an isomorphism between $G(\mathcal{A})$ and $G_E(\mathcal{A})$.

Furthermore, we depict a mapping between edge sets of $G(\mathcal{A})$ and $G_E(\mathcal{A})$ as follows:

$$\begin{aligned} \overline{g} : E(G(\mathcal{A})) &\longrightarrow E(G_E(\mathcal{A})). \\ (v_i, v_j) &\longmapsto \overline{g}((v_i, v_j)) = ([v_i],[v_j]) \end{aligned}$$

$\overline{g}$, the mapping of edges, is a well-defined bijection. $G_E(\mathcal{A})$ is also a complete graph because $G(\mathcal{A})$ is complete and $\overline{g}$ is an isomorphic mapping between $G(\mathcal{A})$ and $G_E(\mathcal{A})$. □

Lemma 16. *If $G_E(\mathcal{A})$ consists of an edge, then this edge consists of two vertices such that $[1]$ and $[a_1] = [a_2] = \cdots = [a_n]$ where $v_i \neq 1$ for $i \in \{1, 2, \ldots, n\}$.*

Proof. We suppose that there exists an $v_j \in V(G(\mathcal{A}))$ such that $[1] = [v_j]$ for any $j \in \{1, 2, \ldots, n\}$. Since $ann^c(1) = A$, we get $A = [1]$. For all $a_j \in A$, $ann^c(v_j) = N(v_j) = A \setminus \{v_j\}$. So, $[v_j]$ must be distinct from $[1]$. However, our assumption has led us to a contradiction. Therefore, there are no different equivalence classes like $[v_j]$, which is equivalent to the equivalence class $[1]$. Because the graphs of equivalence classes have one edge and two vertices, one of these vertices corresponds to the equivalence class $[1]$, while the other vertex likewise corresponds to the $[v_j]$ equivalence class, which consists of all elements of the $A \setminus \{1\}$. □

Theorem 8. *W-graph $G(\mathcal{A})$ is a complete bipartite graph if and only if $G_E(\mathcal{A})$ consists of an edge.*

Proof. ($\Rightarrow$:) Since $G(\mathcal{A})$ is a complete bipartite graph, there exist

$$V_1 = \{a_1, a_2, \ldots, a_n\} \subseteq V(G(\mathcal{A})) \text{ and } V_2 = \{b_1, b_2, \ldots, b_m\} \subseteq V(G(\mathcal{A}))$$

such that $V_1 \cap V_2 = \emptyset$ and $A = \{a_1, a_2, \ldots, a_n, b_1, \ldots, b_m\}$. Therefore, we get the following set of edges

$$E(G(\mathcal{A})) = \{(v_i, u_j) \mid i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, m\}\}.$$

We attain the following neighborhoods by using $E(G(\mathcal{A}))$

$$N(v_i) = \{u_j \mid j \in \{1, \ldots, m\}\}, \quad \forall v_i \in V_1$$

and

$$N(u_j) = \{v_i \mid i \in \{1, \ldots, n\}\}, \quad \forall u_j \in V_2.$$

Thus, we have $N(v_i) = V_2$ for all $v_i \in V_1$ and $N(u_j) = V_1$ for all $u_j \in V_2$. This implies that there exist only two adjacent and distinct equivalence classes $[v] = \{u_1, \ldots, u_m\}$ and $[u] = \{v_1, \ldots, v_n\}$ in $G_E(\mathcal{A})$. Hence, $G_E(\mathcal{A})$ consists of only one edge.
($\Leftarrow$:) It is obvious from Lemma 16. □

Lemma 17. *Assume that $h : G_1 \longrightarrow G_2$ is an isomorphism where G_1 and G_2 are two graphs of W-algebra. If $h(v) = u$ for $v \in V(G_1)$ and $u \in V(G_2)$, then $h(N(v)) = N(u)$.*

Proof. It is obvious. □

Theorem 9. *Suppose that $G(\mathcal{A})$ and $G(\mathcal{B})$ are two graphs of W-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. There also exists an isomorphism between $G_E(\mathcal{A})$ and $G_E(\mathcal{B})$ if there exists an isomorphism between $G(\mathcal{A})$ and $G(\mathcal{B})$.*

Proof. We assume that there exists an isomorphism between $G(\mathcal{A})$ and $G(\mathcal{B})$. As a result, an isomorphism exists as follows

$$\begin{array}{rcl} h : G(\mathcal{A}) & \longrightarrow & G(\mathcal{B}) \\ v_i & \longmapsto & h(v_i) = u_i \end{array}$$

such that $|V(G(\mathcal{A}))| = |V(G(\mathcal{B}))|$, for all $i \in \{1, \ldots, n\}$. We obtain $h(N(v_i)) = N(u_i)$, i.e., $h(ann^c(v_i)) = ann^c(u_i)$ with the help of Lemma 17.
Herein, we describe a mapping between the edges of $E(G_E(\mathcal{A}))$ and $E(G_E(\mathcal{B}))$ such that

$$\begin{array}{rcl} \overline{h} : E(G_E(\mathcal{A})) & \longrightarrow & E(G_E(\mathcal{B})) \\ ([v_i], [v_j]) & \longmapsto & \overline{h}([v_i], [v_j]) = ([u_i], [u_j]). \end{array}$$

Hence, the mapping $\overline{h}$ is a well-defined bijection. As a consequence, we attain that $G_E(\mathcal{A}) \cong G_E(\mathcal{B})$. $\square$

Now, Algorithm 5 is developed to represent the graph of equivalence classes of $\mathcal{A}$.

Algorithm 5: Determination of graph of equivalence classes of $\mathcal{A}$

Data: Set A and the $\rightarrow$ operation, the $\neg$ operation
Result: Graph of equivalence classes of $\mathcal{A}$
EQUIVALENCEGRAPH$(A, \rightarrow, \neg)$
1 **if** W-ALGEBRA$(A, \rightarrow, \neg) = false$ **then**
2   Print *A is not a W-algebra*
3   Return
4 $\triangle = \triangle$-OPERATION$(A, \rightarrow, \neg)$
5 $G =$GRAPH$(A, \rightarrow, \neg)$ // $G = (V, E)$
6 $N_G = ann^c$SETS$(A, \triangle)$
7 $V^{'} = V, E^{'} = \emptyset$
8 $mark[\,] = false$ // Initializing all elements to false
9 **foreach** u **in** V **do**
10   **if** $mark[u] = true$ **then**
11   continue
12   **foreach** v **in** V **do** where $v > u$
13   **if** $N_G(u) = N_G(v)$ **then**
14             $mark[v] = true$
15             $V^{'} = V^{'} \setminus \{v\}$
16 **foreach** u **in** $V^{'}$ **do**
17   **foreach** v **in** $V^{'}$ **do** where $v > u$
18   **if** $v \in N_G(u)$ **then**
19             $E^{'} = E^{'} \cup (u, v)$
20 $G_E(\mathcal{A}) = (V^{'}, E^{'})$
21 Print $G_E(\mathcal{A})$

The algorithm of determining the graph of equivalence classes of $\mathcal{A}$ establishes the graph of equivalence classes of a W-algebra where the inputs are set A, $\rightarrow$ operation and $\neg$ operation (Algorithm 5, or EQUIVALENCEGRAPH$(A, \rightarrow, \neg)$).

The algorithm investigates whether set A produces a W-algebra in lines 1–3. If A does not become a W-algebra, the algorithm is stopped. $\triangle$ Cayley table is described with the $\triangle$-OPERATION algorithm by using binary operation $\rightarrow$ and unary operation $\neg$ on set A in line 4. Then, in line 5, $G(\mathcal{A}) = (V, E)$, the graph of $\mathcal{A}$, is specified using GRAPH$(A, \rightarrow, \neg)$ algorithm. Following that, the ann^cSETS$(A, \triangle)$ algorithm constructs a set of complement annihilators of each element in A and returns the set of complement annihilators to N_G as neighborhood sets of each vertex in $G(\mathcal{A})$ in line 6.

In accordance with this, line 7 initializes the vertex set of $G^{'}$ to V and the edge set to an empty set. Additionally, in lines 9–15, it is assessed that $v \in V$ vertices have the same neighborhood sets for each $u \in V$. Such v vertices are demonstrated by u in the graph $G^{'}$, and v is removed from $V^{'}$. After that, in lines 16–19, it is checked whether $v \in N_G(u)$. If $v \in N_G(u)$, then the edge (u, v) is added to $E^{'}$. Finally, the EQUIVALENCEGRAPH$(A, \rightarrow, \neg)$ algorithm achieves $G_E(\mathcal{A}) = (V^{'}, E^{'})$, the graph of the equivalence classes of $\mathcal{A}$, in line 20 and prints this graph.

6. Conclusions

In this paper, we present an algorithmic strategy for building graphs by associating with W-algebras. On W-algebras, we introduce concepts such as $\triangle$-connection operator and complement annihilator to achieve this goal. So far, the concept of annihilator has been used in graph structures created on algebras. The novelty of this work is that, because the W-algebra structure is limited with an upper bound, the concept of complement annihilator was introduced and W-graphs and graphs of equivalence classes were produced using this notion. Simultaneously, we provide algorithms to support our results gained throughout the paper. As a result, this paper attempts to build algorithms for researchers working in various fields of science who use W-algebras and graphs by employing a novel notion known as complement annihilating.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: I am grateful to Ege University Planning and Monitoring Coordination of Organizational Development and Directorate of Library and Documentation for their support in editing and proofreading service of this study.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Iampan, A. A new branch of the logical algebra: UP-algebras. *J. Algebra Relat. Top.* **2017**, *5*, 35–54.
2. Kim, C.B.; Kim, H.S. On BG-algebras. *Demonstr. Math.* **2008**, *41*, 497–506. [CrossRef]
3. Oner, T.; Kalkan, T.; Gursoy, N.K. Sheffer stroke BG-algebras. *Int. J. Maps Math.-IJMM* **2021**, *4*, 27–39.
4. Rump, W. L-algebras, self-similarity, and l-groups. *J. Algebra* **2008**, *320*, 2328–2348. [CrossRef]
5. Rump, W.; Yang, Y. Intervals in l-groups as L-algebras. *Algebra Universalis* **2012**, *67*, 121–130. [CrossRef]
6. Traczyk, T. On the structure of BCK-algebras with zx· yx= zy· xy. *Math. Japon* **1988**, *33*, 319–324.
7. Wang, J.; Wu, Y.; Yang, Y. Basic algebras and L-algebras. *Soft Computing* **2020**, *24*, 14327–14332. [CrossRef]
8. Basnet, D.K.; Singh, L.B. fuzzy ideal of BG-algebra. *Int. J. Algebra* **2011**, *5*, 703–708.
9. Smarandache, F.; Bordbar, H.; Borzooei, R.A.; Jun, Y.B. A general model of neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points. *Bull. Sect. Log.* **2020**, *50*, 355–371. [CrossRef]
10. Muhiuddin, G.; Al-Kadi, D.; Shum, K.P.; Alanazi, A.M. Generalized Ideals of BCK/BCI-Algebras Based on Fuzzy Soft Set Theory. *Adv. Fuzzy Syst.* **2021**, *2021*, 8869931. [CrossRef]
11. Senapati, T.; Muhiuddin, G.; Shum, K.P. Representation of up-algebras in interval-valued intuitionistic fuzzy environment. *Ital. J. Pure Appl. Math.* **2017**, *38*, 497–517.
12. Senapati, T.; Jun, Y.B.; Shum, K.P. Cubic set structure applied in UP -algebras. *Discret. Math. Algorithms Appl.* **2018**, *10*, 18500490. [CrossRef]
13. Subha, V.S.; Dhanalakshmi, P. New type of fuzzy ideals in BCK/BCI algebras. *World Sci. News* **2021**, *153*, 80–92.
14. Wu, Y.; Wang, J.; Yang, Y. Lattice-ordered effect algebras and L-algebras. *Fuzzy Sets Syst.* **2019**, *369*, 103–113. [CrossRef]
15. Rose, A.; Rosser, J.B. Fragments of Many-Valued Statement Calculi. *Trans. Am. Math. Soc.* **1958**, *87*, 1–53. [CrossRef]
16. Salas, A.J.R. Un Estudio algebraico de los cálculos proposicionales de Lukasiewicz. Ph.D. Thesis, Universitat de Barcelona, Barcelona, Spain, 1980.
17. Chang, C.C. Algebraic Analysis of Many Valued Logics. *Source Trans. Am. Math. Soc.* **1958**, *88*, 467–490. [CrossRef]
18. Chang, C.C. A New Proof of the Completeness of the Lukasiewicz Axioms. *Trans. Am. Math. Soc.* **1959**, *93*, 74. [CrossRef]
19. Font, J.M.; Rodríguez, A.J.; Torrens, A. Wajsberg algebras. *Stochastica* **1984**, *8*, 5–31.
20. Ahamed, M.B.; Ibrahim, A. Fuzzy implicative filters of lattice Wajsberg Algebras. *Adv. Fuzzy Math.* **2011**, *6*, 235–243.
21. Ahamed, M.B.; Ibrahim, A. Anti fuzzy implicative filters in lattice W-algebras. *Int. J. Comput. Sci. Math.* **2012**, *4*, 49–56.
22. Anderson, D.D.; Naseer, M. Beck's coloring of a commutative ring. *J. Algebra* **1993**, *159*, 500–514. [CrossRef]
23. Anderson, D.F.; Livingston, P.S. The zero-divisor graph of a commutative ring. *J. Algebra* **1999**, *217*, 434–447. [CrossRef]
24. Atani, S.E.; Saraei, F.E.K. The total graph of a commutative semiring. *Analele Stiintifice Ale Univ. Ovidius Constanta Ser. Mat.* **2013**, *21*, 21–33. [CrossRef]
25. Beck, I. Coloring of commutative rings. *J. Algebra* **1988**, *116*, 208–226. [CrossRef]
26. Birch, L.M.; Thibodeaux, J.J.; Tucci, R.P. Zero Divisor Graphs of Finite Direct Products of Finite Rings. *Commun. Algebra* **2014**, *42*, 3852–3860. [CrossRef]
27. DeMeyer, F.R.; McKenzie, T.; Schneider, K. The Zero-divisor Graph of a Commutative Semigroup. *Semigroup Forum* **2002**, *65*, 206–214. [CrossRef]
28. DeMeyer, F.; DeMeyer, L. Zero divisor graphs of semigroups. *J. Algebra* **2005**, *283*, 190–198. [CrossRef]

29. Anderson, D.F.; Badawi, A. The total graph of a commutative ring. *J. Algebra* **2008**, *320*, 2706–2719. [CrossRef]
30. Talebi, Y.; Darzi, A. On graph associated to co-ideals of commutative semirings. *Comment. Math. Univ. Carol.* **2017**, *58*, 293–305. [CrossRef]
31. Maimani, H.R.; Yassemi, S. On the zero-divisor graphs of commutative semigroups. *Houst. J. Math.* **2011**, *37*, 733–740.
32. Chelvam, T.T.; Nithya, S. Zero-divisor graph of an ideal of a near-ring. *Discret. Math. Algorithms Appl.* **2013**, *5*, 1350007. [CrossRef]
33. Akram, M.; Samanta, S.; Pal, M. Cayley vague graphs. *J. Fuzzy Math* **2017**, *25*, 449–462.
34. Mulay, S.B. Cycles and symmetries of zero-divisors. *Commun. Algebra* **2002**, *30*, 3533–3558. [CrossRef]
35. Redmond, S.P. The zero-divisor graph of a non-commutative ring. *Int. J. Commut. Rings* **2002**, *1*, 203–211.
36. Božić, I.; Petrović, Z. Zero-divisor graphs of matrices over commutative rings. *Commun. Algebra* **2009**, *37*, 1186–1192. [CrossRef]
37. Akbari, S.; Ghandehari, M.; Hadian, M.; Mohammadian, A. On commuting graphs of semisimple rings. *Linear Algebra Its Appl.* **2004**, *390*, 345–355. [CrossRef]
38. Akbari, S.; Bidkhori, H.; Mohammadian, A. Commuting graphs of matrix algebras. *Commun. Algebra* **2008**, *36*, 4020–4031. [CrossRef]
39. Akbari, S.; Mohammadian, A.; Radjavi, H.; Raja, P. On the diameters of commuting graphs. *Linear Algebra Its Appl.* **2006**, *418*, 161–176. [CrossRef]
40. Dolinar, G.; Guterman, A.; Kuzma, B.; Oblak, P. Commuting graphs and extremal centralizers. *Ars Math. Contemp.* **2014**, *7*, 453–459. [CrossRef]
41. Bakhadly, B.R. Orthogonality graph of the algebra of upper triangular matrices. *Oper. Matrices* **2017**, *11*. [CrossRef]
42. Bakhadly, B.R.; Guterman, A.E.; Markova, O.V. Graphs defined by orthogonality. *J. Math. Sci. (United States)* **2015**, *207*, 698–717. [CrossRef]
43. Guterman, A.E.; Markova, O.V. Orthogonality Graphs of Matrices Over Skew Fields. *J. Math. Sci.* **2018**, *232*, 797–804. [CrossRef]
44. Afkhami, M.; Khashyarmanesh, K.; Rajabi, Z. Some results on the annihilator graph of a commutative ring. *Czechoslov. Math. J.* **2017**, *67*, 151–169. [CrossRef]
45. Ebrahimi, S.; Nikandish, R.; Tehranian, A.; Rasouli, H. On the Strong Metric Dimension of Annihilator Graphs of Commutative Rings. *Bull. Malays. Math. Sci. Soc.* **2021**, *44*, 2507–2517. [CrossRef]
46. Gursoy, A.; Kircali Gursoy, N.; Oner, T.; Senturk, I. An alternative construction of graphs by associating with algorithmic approach on MV-algebras. *Soft Comput.* **2021**, *25*, 13201–13212. [CrossRef]
47. Cignoli, R.L.; d'Ottaviano, I.M.; Mundici, D. *Algebraic Foundations of Many-Valued Reasoning*; Springer Science & Business Media: Dordrecht, The Netherlands, 2013; Volume 7.
48. Chang, C.C. The writing of the MV-algebras. *Stud. Log. Int. J. Symb. Log.* **1998**, *61*, 3–6.
49. Chartrand, G.; Lesniak, L.; Zhang, P. *Graphs & Digraphs*; Chapman and Hall/CRC: New York, USA, 2010. [CrossRef]
50. Harary, F. *Graph Theory*; Addison-Wesley Publishing Company: Reading, MA, USA, 1969; p. 274.