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New Quantum Mercer Estimates of Simpson–Newton-like Inequalities via Convexity

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Abstract: Recently, developments and extensions of quadrature inequalities in quantum calculus have been extensively studied. As a result, several quantum extensions of Simpson's and Newton's estimates are examined in order to explore different directions in quantum studies. The main motivation of this article is the development of variants of Simpson–Newton-like inequalities by employing Mercer's convexity in the context of quantum calculus. The results also give new quantum bounds for Simpson–Newton-like inequalities through Hölder's inequality and the power mean inequality by employing the Mercer scheme. The validity of our main results is justified by providing examples with graphical representations thereof. The obtained results recapture the discoveries of numerous authors in quantum and classical calculus. Hence, the results of these inequalities lead us to the development of new perspectives and extensions of prior results.

Keywords: Simpson's inequality; Jesnen–Mercer inequality; convex functions; quantum calculus

MSC: 26A33; 26D15; 26E60

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1. Introduction

Integral inequalities have historically been viewed as a classical field of research. From classical to contemporary applications, inequalities have been used in mathematical analysis. In 1934, Polya and Hardy introduced classical work on inequalities. Integral inequalities play a vital role in differential equation theory. Many researchers have studied integral inequalities in classical calculus, along with their applications (see [1,2]). Because the value of mathematical inequalities was well established in the past, inequalities such as the Hermite–Hadamard, Hölder, Ostrowski, Jensen, Hardy, and Cauchy–Schwartz inequalities have played an essential role in the theory of classical calculus and quantum calculus [3].

Convexity is a growing area of research that has applications in complex analysis, number theory, and many other fields. Convexity also has a significant impact on people's lives with its numerous applications [1,3]. Convex functions are defined as follows.

Definition 1 ([1]). *Let $\mathfrak{g} : [\rho_1, \rho_2] \subseteq \Re \to \Re$ be convex if, for every $\varkappa, \mathrm{y} \in [\rho_1, \rho_2]$ and every $\lambda \in [0, 1]$, we have:*

$$\mathfrak{g}(\lambda\, \mathrm{y} + (1-\lambda)\varkappa) \;\leq\; \lambda\, \mathfrak{g}(\mathrm{y}) + (1-\lambda)\, \mathfrak{g}(\varkappa).$$

which holds for every $\varkappa$, $\mathrm{y} \in [\rho_1, \rho_2]$ and every $\lambda \in [0, 1]$.

Convexity theory also provides us with an amazing framework for initiating and developing numerical tools to tackle and study complicated problems in mathematics. Due to the number of expedient properties, they are magical, especially in optimization theory. The theories of mathematical inequalities and convex functions have a wonderful relationship. One of the most remarkable inequalities, which, we may say, is the natural extension of the convex function $\mathfrak{g} : [\wp_1, \wp_2] \to \Re$, is Jensen's inequality [1], which is given as:

$$\mathfrak{g}\left(\sum_{\jmath=1}^{\mathcal{N}} \omega_\jmath \varkappa_\jmath\right) \leq \left(\sum_{\jmath=1}^{\mathcal{N}} \omega_\jmath \mathfrak{g}(\varkappa_\jmath)\right), \tag{1}$$

for all $\varkappa_\jmath \in [\wp_1, \wp_2]$, $\omega_\jmath \in [0,1]$ satisfying $\sum \omega_\jmath = 1$ for $(\jmath = 1, 2, \ldots, \mathcal{N})$. For $\mathcal{N} = 2$, Jensen's inequality (1) recaptures the definition of a convex function. There are several important applications of Jensen's inequality in economics, finance, statistics, and optimization, but it is also the most effective inequality for predicting the estimations of the bounds of distance functions in information theory (see [4,5]).

In the year 2003, Mercer [6] introduced an interesting variant of Jensen's inequality, which is called the Jensen–Mercer inequality:

$$\mathfrak{g}\left(\wp_1 + \wp_2 - \sum_{\jmath=1}^{\mathcal{N}} \omega_\jmath \varkappa_\jmath\right) \leq \mathfrak{g}(\wp_1) + \mathfrak{g}(\wp_2) - \sum_{\jmath=1}^{\mathcal{N}} \omega_\jmath \mathfrak{g}(\varkappa_\jmath), \tag{2}$$

which holds for all finite positive increasing sequences $\varkappa_\jmath \in [\wp_1, \wp_2]$, for $(\jmath = 1, 2, \ldots, \jmath)$, along with the weights $\omega_\jmath \in [0,1]$ defined in (1). Many scholars have investigated and studied the Jensen–Mercer inequality over the years, and they have even brought it to higher dimensions, using it for convex operators along with several purifications, operator variants for super-quadratic functions, improvements, and many generalizations with applications in information theory (see [7–10]).

In the applied sciences, there are a number of significant inequalities due to their important perspectives. However, the present study revolves around the renowned Simpson inequality [11], whose error estimates are written as:

$$\begin{aligned}\left|\frac{1}{3}\left\{\frac{\mathfrak{g}(\wp_1) + \mathfrak{g}(\wp_2)}{2} + 2\mathfrak{g}\left(\frac{\wp_1 + \wp_2}{2}\right) - \frac{1}{\wp_2 - \wp_1}\int_{\wp_1}^{\wp_2} \mathfrak{g}(\varkappa)d\varkappa\right\}\right| \\ \leq \frac{1}{2880}\left|\left|\mathfrak{g}^4\right|\right|_\infty (\wp_2 - \wp_1)^4,\end{aligned}$$

where $\mathfrak{g} : [\wp_1, \wp_2] \to \Re$ is a four-time continuously differentiable mapping on $(\wp_1, \wp_2)$ and $||\mathfrak{g}^4||_\infty = \sup_{\varkappa \in (\wp_1, \wp_2)} |\mathfrak{g}^4(\varkappa)| < \infty$.

Simpson-type inequalities are a topic of great interest for many scholars, and they have been explored and studied for various classes of functions. Some fractional Simpson results were established with applications in [12,13].

One of the trickiest math problems to comprehend is in the history of quantum calculus, which is sometimes known as q̂-calculus or q̂-disease and dates back 300 years to Bernoulli and Euler. When there is no limit in calculus, it is referred as q̂-calculus. Euler was the creator of q̂-calculus and the inventor of the q̂-parameter by using the q̂-parameter in Newton's work on infinite series. Jackson was the first to present the developing ideas about q̂-calculus in a symmetrical manner in the nineteenth century by introducing q̂-definite integrals [14]. Al-Salam presented a fractional integral operator and a q̂-analog of the Riemann–Liouville fractional integral operator from 1966 to 1967 [15]. Due to its applicability in a wide range of domains, including mathematics, mechanics, and physics, there is now a tremendous rise in the area of q̂-calculus. One can see this in [15–17] and the references therein. q̂-difference operators are of huge importance because of their applications in a variety of mathematical disciplines, including orthogonal polynomials, basic hypergeometric functions, relativity, combinatorics, and mechanics. Many essential concepts of quantum calculus are covered in Kac and Cheung's [18] book. These ideas help us to develop new inequalities, which can be useful in the discovery of new boundaries. The following are some of the fundamental definitions of quantum calculus.

2. Preliminaries

We first present the definitions of quantum derivatives and quantum integrals.

$$[n]_{\hat{\mathfrak{q}}} = \frac{\hat{\mathfrak{q}}^n - 1}{\hat{\mathfrak{q}} - 1}.$$

The ${}_{\rho_1}D_{\hat{\mathfrak{q}}}$-difference operator and $\hat{\mathfrak{q}}_{\rho_1}$-integral were first presented by Tariboon in [19]. Later, Bermudo et al. [20] intervened with the concept of the $\hat{\mathfrak{q}}^{\rho_2}$-derivative and $\hat{\mathfrak{q}}^{\rho_2}$-integral.

Definition 2 ([20])**.** *The $\hat{\mathfrak{q}}^{\rho_2}$-derivative of mapping $\mathfrak{g} : [\rho_1, \rho_2] \to \mathbb{R}$ is defined as:*

$$^{\rho_2}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa) = \frac{\mathfrak{g}(\hat{\mathfrak{q}}\varkappa + (1-\hat{\mathfrak{q}})\rho_2) - \mathfrak{g}(\varkappa)}{(1-\hat{\mathfrak{q}})(\rho_2 - \varkappa)},\ \textit{if}\ \varkappa \neq \rho_2.$$

$$\textit{If}\ \ \varkappa = \rho_2,\ {}^{\rho_2}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2) = \lim_{\varkappa \to \rho_2} {}^{\rho_2}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa),$$

which is a $\hat{\mathfrak{q}}$-Jackson derivative.

Definition 3 ([20])**.** *The $\hat{\mathfrak{q}}^{\rho_2}$-integral of mapping $\mathfrak{g} : [\rho_1, \rho_2] \to \mathbb{R}$ is defined as:*

$$\int_{\rho_1}^{\rho_2} \mathfrak{g}(\varkappa)\ {}^{\rho_2}d_{\hat{\mathfrak{q}}}\varkappa = (1-\hat{\mathfrak{q}})(\rho_2 - \rho_1)\sum_{n=0}^{\infty} \hat{\mathfrak{q}}^n \mathfrak{g}(\hat{\mathfrak{q}}^n\rho_1 + (1-\hat{\mathfrak{q}}^n)\rho_2),$$

which is a $\hat{\mathfrak{q}}$-Jackson integral.

Theorem 1 ([20])**.** *If $\mathfrak{g} : [\rho_1, \rho_2] \to \mathbb{R}$ is a convex mapping that is differentiable on $[\rho_1, \rho_2]$, then the following inequality is true:*

$$\mathfrak{g}\left(\frac{\rho_1 + \hat{\mathfrak{q}}\rho_2}{[2]_{\hat{\mathfrak{q}}}}\right) \leq \frac{1}{\rho_2 - \rho_1}\int_{\rho_1}^{\rho_2} \mathfrak{g}(\varkappa)\ {}^{\rho_2}d_{\hat{\mathfrak{q}}}\varkappa \leq \frac{\mathfrak{g}(\rho_1) + \hat{\mathfrak{q}}\mathfrak{g}(\rho_2)}{[2]_{\hat{\mathfrak{q}}}},$$

where $\hat{\mathfrak{q}} \in (0,1)$.

Lemma 1. *We have the equality*

$$\int_{\rho_1}^{\rho_2} (\varkappa - \rho_1)^{\alpha}\ \ {}_{\rho_1}d_{\hat{\mathfrak{q}}}\varkappa = \frac{(\rho_2 - \rho_1)^{\alpha+1}}{[\alpha+1]_{\hat{\mathfrak{q}}}},$$

for $\alpha \in \mathbb{R}\backslash\{-1\}$.

It is pertinent to mention an important lemma of interest.

Lemma 2 ([21])**.** *For the continuous functions $\mathfrak{g}, \mathfrak{h} : [\rho_1, \rho_2] \to \mathbb{R}$, the following inequality is true:*

$$\begin{aligned}\int_0^c \mathfrak{h}(\eta)\ {}^{\rho_2}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\eta\rho_1 + (1-\eta)\rho_2)\ d_{\hat{\mathfrak{q}}}\eta = {}& \frac{1}{\rho_2 - \rho_1}\int_0^c D_{\hat{\mathfrak{q}}}\ \mathfrak{h}(\eta)\ \mathfrak{g}(\hat{\mathfrak{q}}\eta\rho_1 + (1-\hat{\mathfrak{q}}\eta)\rho_2)\ d_{\hat{\mathfrak{q}}}\eta \\ & - \frac{\mathfrak{h}(\eta)\mathfrak{g}(\eta\rho_1 + (1-\eta)\rho_2)}{\rho_2 - \rho_1}\Bigg|_0^c.\end{aligned}$$

Despite our concentration on the thrill and romance of the development of quantum calculus and its applications in specific areas of the mathematical sciences and physics, $\hat{\mathfrak{q}}$-analogs of integral inequalities are a topic of supreme interest. Using the ${}_{\rho_1}D_{\hat{\mathfrak{q}}}$-derivative, $\hat{\mathfrak{q}}_{\rho_1}$-integral, $\hat{\mathfrak{q}}^{\rho_2}$-derivative, and $\hat{\mathfrak{q}}^{\rho_2}$-integral, several integral inequalities—namely, those of Hölder, Grüss, Ostrwoski, Hermite–Hadamrd, the trapezoid, and the midpoint —have been investigated with respect to various kinds of functions (see [20,22–26]). Some quantum integral inequalities in co-ordinates can be observed in [27,28]. New quantum versions of Simpson- and Newton-type inequalities relevant to convex functions were developed in [29–31], and their significance was established. Recently, Budak et al. introduced a variant of the quantum Hermite–Jensen–Mercer inequalities [32,33], while in [34], Bohner et al. provided their post-quantum extensions.

Motivated by the ongoing trend, the purpose of this study is to give an analysis of quantum Simpon-like inequalities under Mercer's concept. We formulated new quantum auxiliary results. By applying convexity and the related Jensen–Mercer inequality, we give a variety of new estimates of Simpson's quantum inequality.

3. Auxiliary Results

In this section, we describe our discovery of novel quantum Simpson–Mercer and quantum Newton–Mercer-like identities using $\hat{\mathfrak{q}}^{\rho_2}$-integrals.

Lemma 3. *Let a mapping $\mathfrak{g} : \mathcal{I} = [\rho_1, \rho_2] \subseteq \Re^+ \to \Re$ be differentiable on (ρ_1, ρ_2). If ${}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g} \in \mathcal{L}[\rho_1, \rho_2]$, then, for all $\varkappa, \mathrm{y} \in [\rho_1, \rho_2]$ with $\varkappa \leq \mathrm{y}$, the following identity holds:*

$$\begin{aligned}
&\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa} \mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y}) + \hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right) + \hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right] \\
&= \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\int_0^1 \wp(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\, d_{\hat{\mathfrak{q}}}\lambda,
\end{aligned}$$

where

$$\wp(\lambda) = \begin{cases} \lambda - \frac{1}{[6]_{\hat{\mathfrak{q}}}}, & \lambda \in \left[0, \frac{1}{[2]_{\hat{\mathfrak{q}}}}\right) \\ \lambda - \frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}, & \lambda \in \left[\frac{1}{[2]_{\hat{\mathfrak{q}}}}, 1\right]. \end{cases}$$

Proof. Taking into account the fundamental properties of quantum integrals, we have

$$\begin{aligned}
&\int_0^1 \wp(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\ d_{\hat{\mathfrak{q}}}\lambda \\
&= \int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}} {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\left(\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right) d_{\hat{\mathfrak{q}}}\lambda \\
&+ \int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1 {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\left(\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right) d_{\hat{\mathfrak{q}}}\lambda, \\
&= \frac{\hat{\mathfrak{q}}[4]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}} {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\, d_{\hat{\mathfrak{q}}}\lambda \\
&+ \int_0^1 \left(\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right) {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\, d_{\hat{\mathfrak{q}}}\lambda \\
&= I_1 + I_2.
\end{aligned}$$

Now, as a consequence of Lemma 2, we get

$$\begin{aligned}
I_1 &= \frac{\hat{\mathfrak{q}}[4]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}} {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\, d_{\hat{\mathfrak{q}}}\lambda \\
&= -\frac{\hat{\mathfrak{q}}[4]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\frac{\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))}{\mathrm{y}-\varkappa}\Bigg|_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\hat{\mathrm{q}}[4]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\left[\mathfrak{g}\left(\rho_1+\rho_2-\left(\frac{1}{[2]_{\hat{\mathrm{q}}}}\mathrm{y}+\left(1-\frac{1}{[2]_{\hat{\mathrm{q}}}}\right)\varkappa\right)\right)-\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right] \\
&= -\frac{\hat{\mathrm{q}}[4]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\left[\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathrm{q}}\varkappa}{[2]_{\hat{\mathrm{q}}}}\right)-\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right].
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 &= \int_0^1 {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\left(\lambda-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}\right)d_{\hat{\mathrm{q}}}\lambda \\
&= -\left(\lambda-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}\right)\frac{\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))}{\mathrm{y}-\varkappa}\Bigg|_0^1 \\
&+\frac{1}{\mathrm{y}-\varkappa}\int_0^1 D_{\hat{\mathrm{q}}}(\lambda-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}})\mathfrak{g}(\rho_1+\rho_2-(\hat{\mathrm{q}}\lambda\mathrm{y}+(1-\hat{\mathrm{q}}\lambda)\varkappa))\,d_{\hat{\mathrm{q}}}\lambda \\
&= -\left(1-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}\right)\frac{\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})}{\mathrm{y}-\varkappa}-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}\frac{\mathfrak{g}(\rho_1+\rho_2-\varkappa)}{\mathrm{y}-\varkappa} \\
&+\frac{1}{\varkappa-\mathrm{y}}\int_0^1 \mathfrak{g}(\rho_1+\rho_2-(\hat{\mathrm{q}}\lambda\mathrm{y}+(1-\hat{\mathrm{q}}\lambda)\varkappa))\,d_{\hat{\mathrm{q}}}\lambda \\
&= -\frac{[6]_{\hat{\mathrm{q}}}-[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}\frac{\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})}{\mathrm{y}-\varkappa}-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}\frac{\mathfrak{g}(\rho_1+\rho_2-\varkappa)}{\mathrm{y}-\varkappa} \\
&+\frac{(1-\hat{\mathrm{q}})\Sigma_{n=0}^{\infty}\,\hat{\mathrm{q}}^n\mathfrak{g}(\rho_1+\rho_2-(\hat{\mathrm{q}}^{n+1}\mathrm{y}+(1-\hat{\mathrm{q}}^{n+1})\varkappa))}{\mathrm{y}-\varkappa} \\
&= -\frac{\hat{\mathrm{q}}^5}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\mathfrak{g}(\rho_1+\rho_2-\varkappa) \\
&+\frac{(1-\hat{\mathrm{q}})\Sigma_{n=0}^{\infty}\,\hat{\mathrm{q}}^{n+1}\mathfrak{g}(\rho_1+\rho_2-(\hat{\mathrm{q}}^{n+1}\mathrm{y}+(1-\hat{\mathrm{q}}^{n+1})\varkappa))}{\hat{\mathrm{q}}(\mathrm{y}-\varkappa)} \\
&= -\frac{\hat{\mathrm{q}}^5}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\mathfrak{g}(\rho_1+\rho_2-\varkappa) \\
&-\frac{(1-\hat{\mathrm{q}})}{\hat{\mathrm{q}}(\varkappa-\mathrm{y})}\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\frac{1}{\hat{\mathrm{q}}(\varkappa-\mathrm{y})^2}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda \\
&= -\frac{1}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}(\mathrm{y}-\varkappa)}\mathfrak{g}(\rho_1+\rho_2-\varkappa) \\
&+\frac{1}{\hat{\mathrm{q}}(\varkappa-\mathrm{y})^2}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
\hat{\mathrm{q}}(\mathrm{y}-\varkappa)(I_1+I_2) &= \frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathrm{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathrm{q}}^2[4]_{\hat{\mathrm{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathrm{q}}\varkappa}{[2]_{\hat{\mathrm{q}}}}\right)+\hat{\mathrm{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right].
\end{aligned}
$$

Thus, the required equality is captured here. □

Remark 1. *If we set $\varkappa=\rho_1$ and $\mathrm{y}=\rho_2$ in Lemma 3, it can be reduced to the following equality:*

$$
\begin{aligned}
&\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathfrak{g}(\lambda)\ {}^{\rho_2}d_{\hat{\mathrm{q}}}\lambda-\frac{1}{[6]_{\hat{\mathrm{q}}}}\left[\mathfrak{g}(\rho_1)+\hat{\mathrm{q}}^2[4]_{\hat{\mathrm{q}}}\mathfrak{g}\left(\frac{\rho_1+\hat{\mathrm{q}}\rho_2}{[2]_{\hat{\mathrm{q}}}}\right)+\hat{\mathrm{q}}\mathfrak{g}(\rho_2)\right] \\
= \quad &\hat{\mathrm{q}}(\rho_2-\rho_1)\int_0^1 \wp(\lambda)\ {}^{\rho_2}D_{\hat{\mathrm{q}}}\mathfrak{g}(\lambda\rho_1+(1-\lambda)\rho_2)\ d_{\hat{\mathrm{q}}}\lambda,
\end{aligned}
$$

which was proven in [31].

Lemma 4. *With the suppositions of Lemma 3, we have the following identity:*

$$
\begin{aligned}
&\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[8]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\frac{\hat{\mathfrak{q}}^3[6]_{\hat{\mathfrak{q}}}}{1+\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+(1+\hat{\mathfrak{q}})\hat{\mathfrak{q}}\varkappa}{[3]_{\hat{\mathfrak{q}}}}\right)\right. \\
&\left.+\frac{\hat{\mathfrak{q}}^2[6]_{\hat{\mathfrak{q}}}}{(1+\hat{\mathfrak{q}})}\mathfrak{g}\left(\rho_1+\rho_2-\frac{(1+\hat{\mathfrak{q}})\mathrm{y}+\hat{\mathfrak{q}}^2\varkappa}{[3]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\ \mathfrak{g}(\rho_1+\rho_2-\varkappa)\right] \\
&=\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\int_0^1\psi(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)\ d_{\hat{\mathfrak{q}}}\lambda,
\end{aligned}
\tag{3}
$$

where

$$
\psi(\lambda)=\begin{cases}\lambda-\frac{1}{[8]_{\hat{\mathfrak{q}}}}, & \lambda\in\left[0,\frac{1}{[3]_{\hat{\mathfrak{q}}}}\right)\\ \lambda-\frac{1}{[2]_{\hat{\mathfrak{q}}}}, & \lambda\in\left[\frac{1}{[3]_{\hat{\mathfrak{q}}}},\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}\right)\\ \lambda-\frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}}, & \lambda\in\left[\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}},1\right].\end{cases}
$$

Proof. By using the fundamental properties of quantum integrals, we have

$$
\begin{aligned}
&\int_0^1\psi(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)\ \ d_{\hat{\mathfrak{q}}}\lambda \\
=&\int_0^{\frac{1}{[3]_{\hat{\mathfrak{q}}}}}{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)\left(\lambda-\frac{1}{[8]_{\hat{\mathfrak{q}}}}\right)\ d_{\hat{\mathfrak{q}}}\lambda \\
&+\int_{\frac{1}{[3]_{\hat{\mathfrak{q}}}}}^{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}}{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)\left(\lambda-\frac{1}{[2]_{\hat{\mathfrak{q}}}}\right)\ d_{\hat{\mathfrak{q}}}\lambda \\
&+\int_{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}}^1{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)\left(\lambda-\frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}}\right)\ d_{\hat{\mathfrak{q}}}\lambda \\
=&I_1+I_2+I_3.
\end{aligned}
$$

The desired results may be attained if steps similar to those in Lemma 3 are used for the rest of this proof. □

Remark 2. *If we set $\varkappa=\rho_1$ and $\mathrm{y}=\rho_2$ in Lemma 4, it can be reduced to the following equality:*

$$
\begin{aligned}
&\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathfrak{g}(\lambda)\ {}^{\rho_2}d_{\hat{\mathfrak{q}}}\lambda-\frac{1}{[8]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1)+\frac{\hat{\mathfrak{q}}^3[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}}\mathfrak{g}\left(\frac{\rho_1+\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}\rho_2}{[3]_{\hat{\mathfrak{q}}}}\right)\right. \\
&\left.+\frac{\hat{\mathfrak{q}}^2[6]_{\hat{\mathfrak{q}}}}{(1+\hat{\mathfrak{q}})}\mathfrak{g}\left(\frac{(1+\hat{\mathfrak{q}})\rho_1+\hat{\mathfrak{q}}^2\rho_2}{[3]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\ \mathfrak{g}(\rho_2)\right] \\
&=\hat{\mathfrak{q}}(\rho_2-\rho_1)\int_0^1\psi(\lambda)\ {}^{\rho_2}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\lambda\rho_1+(1-\lambda)\rho_2)\ d_{\hat{\mathfrak{q}}}\lambda,
\end{aligned}
$$

which was proven in [31].

3.1. Simpson–Mercer 1/3 Formula-Type Inequalities

Theorem 2. *Let the assumptions of Lemma 3 hold. If $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|$ is convex on $[\rho_1,\rho_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda\right.\\
&\left.-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\right|\\
&\leq\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left\{\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|\left(\frac{\hat{\mathfrak{q}}^3[3]_{\hat{\mathfrak{q}}}-\hat{\mathfrak{q}}}{[6]_{\hat{\mathfrak{q}}}[2]^3_{\hat{\mathfrak{q}}}}+\frac{1}{[2]_{\hat{\mathfrak{q}}}}+2\hat{\mathfrak{q}}\frac{(1+[5]^2_{\hat{\mathfrak{q}}})}{[2]_{\hat{\mathfrak{q}}}[6]^2_{\hat{\mathfrak{q}}}}\right.\right.\\
&\left.\left.-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}-\frac{[5]_{\hat{\mathfrak{q}}}[2]^2_{\hat{\mathfrak{q}}}-[6]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}[2]^3_{\hat{\mathfrak{q}}}}\right)-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|\left[\mathfrak{c}_{1\hat{\mathfrak{q}}}+\mathfrak{c}_{2\hat{\mathfrak{q}}}\right]-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|\left[\mathrm{d}_{1\hat{\mathfrak{q}}}+\mathrm{d}_{2\hat{\mathfrak{q}}}\right]\right\},
\end{aligned}\tag{4}$$

where

$$\omega_{1\hat{\mathfrak{q}}}=[3]_{\hat{\mathfrak{q}}}[6]^3_{\hat{\mathfrak{q}}}[2]^3_{\hat{\mathfrak{q}}},$$
$$\omega_{2\hat{\mathfrak{q}}}=[3]_{\hat{\mathfrak{q}}}[6]^3_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}},$$

$$\begin{aligned}
\mathfrak{c}_{1\hat{\mathfrak{q}}}&=\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\lambda\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda=\frac{2\hat{\mathfrak{q}}^2[2]^2_{\hat{\mathfrak{q}}}+\hat{\mathfrak{q}}^3[6]^3_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}}{\omega_{1\hat{\mathfrak{q}}}},\\
\mathfrak{c}_{2\hat{\mathfrak{q}}}&=\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}(1-\lambda)\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda=\frac{2\hat{\mathfrak{q}}^2[5]^3_{\hat{\mathfrak{q}}}[2]^2_{\hat{\mathfrak{q}}}+[6]^3_{\hat{\mathfrak{q}}}(1+[2]^3_{\hat{\mathfrak{q}}})-[3]_{\hat{\mathfrak{q}}}[5]_{\hat{\mathfrak{q}}}[6]^2_{\hat{\mathfrak{q}}}(1+[2]^2_{\hat{\mathfrak{q}}})}{\omega_{1\hat{\mathfrak{q}}}},\\
\mathrm{d}_{1\hat{\mathfrak{q}}}&=\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\lambda\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda=\left(\frac{\hat{\mathfrak{q}}}{[3]_{\hat{\mathfrak{q}}}[2]^2_{\hat{\mathfrak{q}}}}-\frac{\hat{\mathfrak{q}}(\hat{\mathfrak{q}}+2)}{[6]_{\hat{\mathfrak{q}}}[2]^3_{\hat{\mathfrak{q}}}}+2\frac{\hat{\mathfrak{q}}[3]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}-\hat{\mathfrak{q}}^2}{\omega_{2\hat{\mathfrak{q}}}}\right),\\
\mathrm{d}_{2\hat{\mathfrak{q}}}&=\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1(1-\lambda)\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda=2\frac{\hat{\mathfrak{q}}[5]^2_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}-\hat{\mathfrak{q}}^2[5]^3_{\hat{\mathfrak{q}}}}{\omega_{2\hat{\mathfrak{q}}}}+\frac{\hat{\mathfrak{q}}^2[6]_{\hat{\mathfrak{q}}}-\hat{\mathfrak{q}}[5]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}}\\
&\quad-\left[\frac{[5]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}(2\hat{\mathfrak{q}}+\hat{\mathfrak{q}}^2)-\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}}{[2]^3_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}}\right],
\end{aligned}$$

and $0<\hat{\mathfrak{q}}<1$.

Proof. By taking the modulus in Lemma 3, we have

$$\begin{aligned}
&\left|\frac{1}{(\mathrm{y}-\varkappa)}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda\right.\\
&\left.-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\right|\\
&\leq\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\right|d_{\hat{\mathfrak{q}}}\lambda\\
&+\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\right|d_{\hat{\mathfrak{q}}}\lambda.
\end{aligned}$$

By using the convexity of $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|$, we get

$$
\begin{aligned}
&\leq \int_0^{\frac{1}{[2]_{\hat{q}}}} \left|\lambda - \frac{1}{[6]_{\hat{q}}}\right| \left[\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_1)\right| + \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_2)\right|\right] - \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\mathrm{y})\right| \int_0^{\frac{1}{[2]_{\hat{q}}}} \lambda \left|\lambda - \frac{1}{[6]_{\hat{q}}}\right| d_{\hat{q}}\lambda \\
&- \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\varkappa)\right| \int_0^{\frac{1}{[2]_{\hat{q}}}} (1-\lambda) \left|\lambda - \frac{1}{[6]_{\hat{q}}}\right| d_{\hat{q}}\lambda \\
&+ \int_{\frac{1}{[2]_{\hat{q}}}}^1 \left|\lambda - \frac{[5]_{\hat{q}}}{[6]_{\hat{q}}}\right| d_{\hat{q}}\lambda \left[\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_1)\right| + \left|{}^{\rho_1+\rho_2-\varkappa}D_{q}\mathfrak{g}(\rho_2)\right|\right] - \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\mathrm{y})\right| \int_{\frac{1}{[2]_{\hat{q}}}}^1 \left|\lambda - \frac{[5]_{\hat{q}}}{[6]_{\hat{q}}}\right| \lambda \, d_{\hat{q}}\lambda \\
&- \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\varkappa)\right| \int_{\frac{1}{[2]_{\hat{q}}}}^1 (1-\lambda) \left|\lambda - \frac{1}{[6]_{\hat{q}}}\right| \, d_{\hat{q}}\lambda.
\end{aligned}
$$

Here, we have

$$
\int_0^{\frac{1}{[2]_{\hat{q}}}} \left|\lambda - \frac{1}{[6]_{\hat{q}}}\right| d_{\hat{q}}\lambda = \frac{2\hat{q}}{[2]_{\hat{q}}[6]_{\hat{q}}^2} + \frac{\hat{q}^3[3]_{\hat{q}} - \hat{q}}{[6]_{\hat{q}}[2]_{\hat{q}}^3},
$$

and by using similar operations, we have

$$
\begin{aligned}
&\int_{\frac{1}{[2]_{\hat{q}}}}^1 \left|\lambda - \frac{[5]_{\hat{q}}}{[6]_{\hat{q}}}\right| d_{\hat{q}}\lambda = -\frac{[5]_{\hat{q}}}{[6]_{\hat{q}}} + \frac{1}{[2]_{\hat{q}}} - \frac{[5]_{\hat{q}}[2]_{\hat{q}}^2 - [6]_{\hat{q}}}{[6]_{\hat{q}}[2]_{\hat{q}}^3} + 2\hat{q}\frac{[5]_{\hat{q}}^2}{[2]_{\hat{q}}[6]_{\hat{q}}^2} \\
&= \hat{q}(\mathrm{y}-\varkappa)\Bigg\{\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_1)\right| + \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_2)\right| \Bigg(\frac{\hat{q}^3[3]_{\hat{q}} - \hat{q}}{[6]_{\hat{q}}[2]_{\hat{q}}^3} + \frac{1}{[2]_{\hat{q}}} \\
&- \frac{[5]_{\hat{q}}}{[6]_{\hat{q}}} - \frac{[5]_{\hat{q}}[2]_{\hat{q}}^2 - [6]_{\hat{q}}}{[6]_{\hat{q}}[2]_{\hat{q}}^3} + 2\hat{q}\frac{(1+[5]_{\hat{q}}^2)}{[2]_{\hat{q}}[6]_{\hat{q}}^2}\Bigg) - \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\mathrm{y})\right| \left[\mathfrak{c}_{1\hat{q}} + \mathfrak{c}_{2\hat{q}}\right] - \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\varkappa)\right| \left[\mathrm{d}_{1\hat{q}} + \mathrm{d}_{2\hat{q}}\right]\Bigg\}.
\end{aligned}
$$

Hence, the proof is completed. □

We give the following example to show the validity of the results given in Theorem 2.

Example 1. *Let us consider the function* $\mathfrak{g} : [1,2] \to \mathbb{R}$ *defined by* $\mathfrak{g}(\zeta) = \zeta^2$*, and let* $x = \frac{4}{3}$ *and* $\mathrm{y} = \frac{5}{3}$*. Then, we have* ${}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\zeta) = {}^{\frac{5}{3}}D_{\hat{q}}\mathfrak{g}(\zeta) = (1+\hat{q})\zeta + (1-\hat{q})\frac{5}{3}$*, and* $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}|$ *is convex on* $[1,2]$*. Under these assumptions, we have*

$$
\mathfrak{g}(\rho_1 + \rho_2 - \mathrm{y}) = \frac{16}{9},
$$

$$
\mathfrak{g}\left(\rho_1 + \rho_2 - \frac{\mathrm{y} + \hat{q}\varkappa}{[2]_{\hat{q}}}\right) = \left(\frac{4+5\hat{q}}{3[2]_{\hat{q}}}\right)^2
$$

and

$$
\mathfrak{g}(\rho_1 + \rho_2 - \varkappa) = \frac{25}{9}.
$$

From Definition 3, we get

$$
\begin{aligned}
\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa} \mathfrak{g}(\lambda) \ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{q}}\lambda &= \int_{\frac{4}{3}}^{\frac{5}{3}} \lambda^2 \ {}^{\frac{5}{3}}d_{\hat{q}}\lambda \\
&= \frac{(1-\hat{q})}{3}\sum_{n=0}^{\infty} \hat{q}^n \left(\hat{q}^n \frac{4}{3} + (1-\hat{q}^n)\frac{5}{3}\right)^2 \\
&= \frac{(1-\hat{q})}{27}\sum_{n=0}^{\infty} \hat{q}^n \left(25 - 10\hat{q}^n + \hat{q}^{2n}\right) \\
&= \frac{1}{27}\left[25 - \frac{10}{[2]_{\hat{q}}} + \frac{1}{[3]_{\hat{q}}}\right].
\end{aligned}
$$

Hence, the left-hand side of the inequality (4) reduces to

$$\begin{aligned}
&\left|\frac{1}{(\mathrm{y}-\varkappa)}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda\right.\\
&\left.-\frac{1}{[6]_{\hat{\mathrm{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathrm{q}}^2[4]_{\hat{\mathrm{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathrm{q}}\varkappa}{[2]_{\hat{\mathrm{q}}}}\right)+\hat{\mathrm{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\right|\\
&=\left|\frac{1}{9}\left[25-\frac{10}{[2]_{\hat{\mathrm{q}}}}+\frac{1}{[3]_{\hat{\mathrm{q}}}}\right]-\frac{1}{[6]_{\hat{\mathrm{q}}}}\left[\frac{16+25\hat{\mathrm{q}}}{9}+\hat{\mathrm{q}}^2[4]_{\hat{\mathrm{q}}}\left(\frac{4+5\hat{\mathrm{q}}}{3[2]_{\hat{\mathrm{q}}}}\right)^2\right]\right|.
\end{aligned}$$

Since

$$\begin{aligned}
\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1)\right| &= \frac{8-2\hat{\mathrm{q}}}{3},\ \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_2)\right|=\frac{11+\hat{\mathrm{q}}}{3}\\
\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\mathrm{y})\right| &= \frac{10}{3},\ \left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\varkappa)\right|=\frac{9-\hat{\mathrm{q}}}{3},
\end{aligned}\tag{5}$$

we can write the right-hand side of the inequality (4) as

$$\begin{aligned}
&\hat{\mathrm{q}}(\mathrm{y}-\varkappa)\left\{\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1)\right|+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_2)\right|\left(\frac{\hat{\mathrm{q}}^3[3]_{\hat{\mathrm{q}}}-\hat{\mathrm{q}}}{[6]_{\hat{\mathrm{q}}}[2]^3_{\hat{\mathrm{q}}}}+\frac{1}{[2]_{\hat{\mathrm{q}}}}+2\hat{\mathrm{q}}\frac{(1+[5]^2_{\hat{\mathrm{q}}})}{[2]_{\hat{\mathrm{q}}}[6]^2_{\hat{\mathrm{q}}}}\right.\right.\\
&\left.\left.-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}-\frac{[5]_{\hat{\mathrm{q}}}[2]^2_{\hat{\mathrm{q}}}-[6]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}[2]^3_{\hat{\mathrm{q}}}}\right)-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\mathrm{y})\right|\left[\mathfrak{c}_{1\hat{\mathrm{q}}}+\mathfrak{c}_{2\hat{\mathrm{q}}}\right]-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\varkappa)\right|\left[\mathrm{d}_{1\hat{\mathrm{q}}}+\mathrm{d}_{2\hat{\mathrm{q}}}\right]\right\}\\
&=\frac{\hat{\mathrm{q}}}{3}\left\{\frac{8-2\hat{\mathrm{q}}}{3}+\frac{11+\hat{\mathrm{q}}}{3}\left(\frac{\hat{\mathrm{q}}^3[3]_{\hat{\mathrm{q}}}-\hat{\mathrm{q}}}{[6]_{\hat{\mathrm{q}}}[2]^3_{\hat{\mathrm{q}}}}+\frac{1}{[2]_{\hat{\mathrm{q}}}}+2\hat{\mathrm{q}}\frac{(1+[5]^2_{\hat{\mathrm{q}}})}{[2]_{\hat{\mathrm{q}}}[6]^2_{\hat{\mathrm{q}}}}\right.\right.\\
&\left.\left.-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}-\frac{[5]_{\hat{\mathrm{q}}}[2]^2_{\hat{\mathrm{q}}}-[6]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}[2]^3_{\hat{\mathrm{q}}}}\right)-\frac{10}{3}\left[\mathfrak{c}_{1\hat{\mathrm{q}}}+\mathfrak{c}_{2\hat{\mathrm{q}}}\right]-\frac{9-\hat{\mathrm{q}}}{3}\left[\mathrm{d}_{1\hat{\mathrm{q}}}+\mathrm{d}_{2\hat{\mathrm{q}}}\right]\right\}.
\end{aligned}$$

Then, by using the inequality (4), we can write

$$\begin{aligned}
&\left|\frac{1}{9}\left[25-\frac{10}{[2]_{\hat{\mathrm{q}}}}+\frac{1}{[3]_{\hat{\mathrm{q}}}}\right]-\frac{1}{[6]_{\hat{\mathrm{q}}}}\left[\frac{41}{9}+\hat{\mathrm{q}}^2[4]_{\hat{\mathrm{q}}}\left(\frac{4+5\hat{\mathrm{q}}}{3[2]_{\hat{\mathrm{q}}}}\right)^2\right]\right|\\
&\leq\frac{\hat{\mathrm{q}}}{3}\left\{\frac{8-2\hat{\mathrm{q}}}{3}+\frac{11+\hat{\mathrm{q}}}{3}\left(\frac{\hat{\mathrm{q}}^3[3]_{\hat{\mathrm{q}}}-\hat{\mathrm{q}}}{[6]_{\hat{\mathrm{q}}}[2]^3_{\hat{\mathrm{q}}}}+\frac{1}{[2]_{\hat{\mathrm{q}}}}+2\hat{\mathrm{q}}\frac{(1+[5]^2_{\hat{\mathrm{q}}})}{[2]_{\hat{\mathrm{q}}}[6]^2_{\hat{\mathrm{q}}}}\right.\right.\\
&\left.\left.-\frac{[5]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}}-\frac{[5]_{\hat{\mathrm{q}}}[2]^2_{\hat{\mathrm{q}}}-[6]_{\hat{\mathrm{q}}}}{[6]_{\hat{\mathrm{q}}}[2]^3_{\hat{\mathrm{q}}}}\right)-\frac{10}{3}\left[\mathfrak{c}_{1\hat{\mathrm{q}}}+\mathfrak{c}_{2\hat{\mathrm{q}}}\right]-\frac{9-\hat{\mathrm{q}}}{3}\left[\mathrm{d}_{1\hat{\mathrm{q}}}+\mathrm{d}_{2\hat{\mathrm{q}}}\right]\right\}.
\end{aligned}\tag{6}$$

One can see the validity of the inequality (6) in Figure 1.

Remark 3. *If we choose $\varkappa=\rho_1$ and $\mathrm{y}=\rho_2$ in Theorem 2, we get Theorem 4 of [31].*

Corollary 1. *With the assumptions of Theorem 2 with $\varkappa=\rho_1$, $\mathrm{y}=\rho_2$, and $\hat{\mathrm{q}}\to 1^-$, we have (see [35]):*

$$\begin{aligned}
&\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathfrak{g}(\lambda)d\lambda-\frac{1}{6}\left[\mathfrak{g}(\rho_1)+4\mathfrak{g}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathfrak{g}(\rho_2)\right]\right|\\
&\leq\frac{5(\rho_2-\rho_1)}{36}\left(\frac{|\mathfrak{g}'(\rho_1)|+|\mathfrak{g}'(\rho_2)|}{2}\right).
\end{aligned}\tag{7}$$

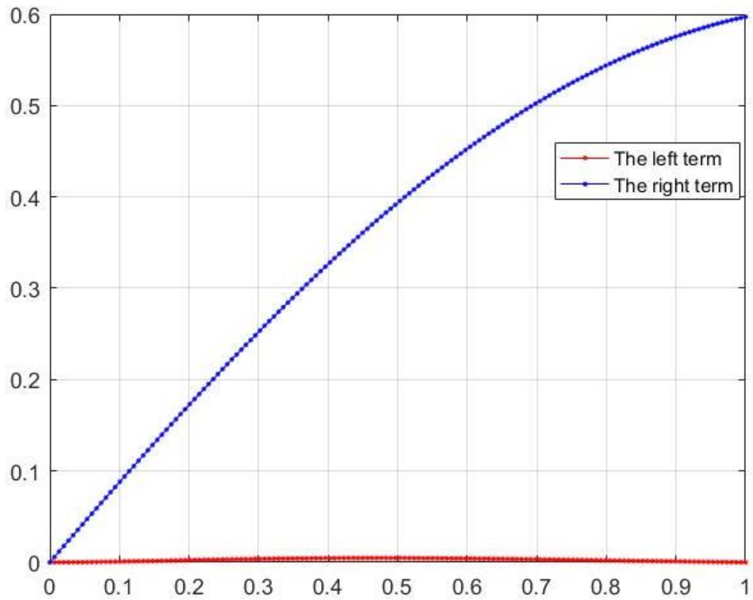

Figure 1. An example of the dependence of the inequality (6) on $\hat{\mathfrak{q}}$ was computed and plotted with MATLAB.

Theorem 3. *Let the assumptions of Lemma 3 hold. If $|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|^{\wp_1}$ is convex on $[\rho_1,\rho_2]$ for some fixed $\wp_1 > 1$, then the following inequality holds:*

$$\begin{aligned}
&\Bigg|\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\Bigg| \\
&\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\frac{\hat{\mathfrak{q}}^{2\wp_2}[4]_{\hat{\mathfrak{q}}}^{\wp_2}}{[2]_{\hat{\mathfrak{q}}}^{\wp_2+1}[6]_{\hat{\mathfrak{q}}}^{\wp_2}}\right)^{\frac{1}{\wp_2}}\times\left(\frac{1}{[2]_{\hat{\mathfrak{q}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}+\frac{1}{[2]_{\hat{\mathfrak{q}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}\right. \\
&\left.-\frac{1}{[2]_{\hat{\mathfrak{q}}}^3}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}-\frac{\hat{\mathfrak{q}}^2+2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}} \\
&+\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\frac{[2]_{\hat{\mathfrak{q}}}^{\wp_2+1}\hat{\mathfrak{q}}^{5\wp_2}-\hat{\mathfrak{q}}^{\wp_2}[4]_{\hat{\mathfrak{q}}}^{\wp_2}}{[2]_{\hat{\mathfrak{q}}}^{\wp_2+1}[6]_{\hat{\mathfrak{q}}}^{\wp_2}}\right)^{\frac{1}{\wp_2}}\times\left(\frac{\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}+\frac{\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}\right. \\
&\left.-\frac{\hat{\mathfrak{q}}^2+2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}-\frac{\hat{\mathfrak{q}}^3+\hat{\mathfrak{q}}^2-\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}},
\end{aligned}$$

where $0 < \hat{\mathfrak{q}} < 1$ and $\frac{1}{\wp_1}+\frac{1}{\wp_2}=1$.

Proof. Applying the well-known Hölder inequality for the $\hat{\mathfrak{q}}^{\rho_2}$-integral on the right-hand side of Lemma 3, we have

$$\begin{aligned}
&\Bigg|\frac{1}{(\mathrm{y}-\varkappa)}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\Bigg| \\
&\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_2}} \\
&\qquad\times\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_1}} \\
&\qquad+\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_2}} \\
&\qquad\times\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_1}}.
\end{aligned}$$

By using the convexity of $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|^{\wp_1}$, we obtain:

$$\begin{aligned}
&\Bigg|\frac{1}{(\mathrm{y}-\varkappa)}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\Bigg| \\
&\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_2}}\times\Bigg(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda \\
&+\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda-\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\lambda\ d_{\hat{\mathfrak{q}}}\lambda\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}-\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}(1-\lambda)\ d_{\hat{\mathfrak{q}}}\lambda\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\Bigg)^{\frac{1}{\wp_1}} \\
&+\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_2}}\times\Bigg(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda \\
&+\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda-\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\lambda\ d_{\hat{\mathfrak{q}}}\lambda\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}-\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1(1-\lambda)d_{\hat{\mathfrak{q}}}\lambda\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\Bigg)^{\frac{1}{\wp_1}}.
\end{aligned}$$

To calculate the integrals,

$$\begin{aligned}
\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2}d_{\hat{\mathfrak{q}}}\lambda &= \frac{(1-\hat{\mathfrak{q}})}{[2]_{\hat{\mathfrak{q}}}}\sum_{n=0}^{\infty}\hat{\mathfrak{q}}^n\left|\frac{\hat{\mathfrak{q}}^n}{[2]_{\hat{\mathfrak{q}}}}-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2} \\
&\leq \frac{(1-\hat{\mathfrak{q}})}{[2]_{\hat{\mathfrak{q}}}}\sum_{n=0}^{\infty}\hat{\mathfrak{q}}^n\left|\frac{1}{[2]_{\hat{\mathfrak{q}}}}-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2} \\
&= \frac{\hat{\mathfrak{q}}^{2\wp_2}[4]_{\hat{\mathfrak{q}}}^{\wp_2}}{[2]_{\hat{\mathfrak{q}}}^{\wp_2+1}[6]_{\hat{\mathfrak{q}}}^{\wp_2}}.
\end{aligned}$$

Similarly,

$$\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|^{\wp_2}d_{\hat{\mathfrak{q}}}\lambda=\frac{[2]_{\hat{\mathfrak{q}}}^{\wp_2+1}\hat{\mathfrak{q}}^{5\wp_2}-\hat{\mathfrak{q}}^{\wp_2}[4]_{\hat{\mathfrak{q}}}^{\wp_2}}{[2]_{\hat{\mathfrak{q}}}^{\wp_2+1}[6]_{\hat{\mathfrak{q}}}^{\wp_2}}.$$

Here, we also have

$$\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}} \lambda \, d_{\hat{\mathfrak{q}}}\lambda = \frac{1}{[2]_{\hat{\mathfrak{q}}}^3},$$

$$\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}} (1-\lambda) d_{\hat{\mathfrak{q}}}\lambda = \frac{\hat{\mathfrak{q}}^2 + 2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3},$$

$$\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1 \lambda \, d_{\hat{\mathfrak{q}}}\lambda = \frac{\hat{\mathfrak{q}}^2 + 2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3},$$

and

$$\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1 (1-\lambda) d_{\hat{\mathfrak{q}}}\lambda = \frac{\hat{\mathfrak{q}}^3 + \hat{\mathfrak{q}}^2 - \hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3}.$$

By substituting these integrals, we get the required results. □

Remark 4. *If we put $\varkappa = \rho_1$ and $\mathrm{y} = \rho_2$ into Theorem 3, then we get Theorem 5 of [31].*

Theorem 4. *Let the assumptions of Lemma 3 hold. If $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|^{\wp_1}$ is convex on $[\rho_1, \rho_2]$ for some fixed $\wp_1 \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\Bigg| \frac{1}{(\mathrm{y}-\varkappa)} \int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa} \mathfrak{g}(\lambda) \ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&- \frac{1}{[6]_{\hat{\mathfrak{q}}}} \left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y}) + \hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2 - \frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right) + \hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa) \right] \Bigg| \\
&\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa) \left(\frac{\hat{\mathfrak{q}}^3[3]_{\hat{\mathfrak{q}}} - \hat{\mathfrak{q}}}{[6]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}^3} + \frac{2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}^2} \right)^{1-\frac{1}{\wp_1}} \\
&\times \Bigg(\left(\frac{\hat{\mathfrak{q}}^3[3]_{\hat{\mathfrak{q}}} - \hat{\mathfrak{q}}}{[6]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}^3} + \frac{2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}^2} \right) \left\{ \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1) \right|^{\wp_1} + \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2) \right|^{\wp_1} \right\} \\
&- \mathfrak{c}_{1\hat{\mathfrak{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y}) \right|^{\wp_1} - \mathrm{d}_{1\hat{\mathfrak{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa) \right|^{\wp_1} \Bigg)^{\frac{1}{\wp_1}} \\
&+ \left(\frac{1}{[2]_{\hat{\mathfrak{q}}}} - \frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}} - \frac{[5]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}^2 - [6]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}^3} + 2\hat{\mathfrak{q}}\frac{[5]_{\hat{\mathfrak{q}}}^2}{[2]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}^2} \right)^{1-\frac{1}{\wp_1}} \\
&\times \Bigg(\left(-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}} - \frac{[5]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}^2 - [6]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}^3} + 2\hat{\mathfrak{q}}\frac{[5]_{\hat{\mathfrak{q}}}^2}{[2]_{\hat{\mathfrak{q}}}[6]_{\hat{\mathfrak{q}}}^2} + \frac{1}{[2]_{\hat{\mathfrak{q}}}} \right) \left\{ \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1) \right|^{\wp_1} + \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2) \right|^{\wp_1} \right\} \\
&\quad - \mathfrak{c}_{2\hat{\mathfrak{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y}) \right|^{\wp_1} - \mathrm{d}_{2\hat{\mathfrak{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}(\varkappa) \right|^{\wp_1} \Bigg)^{\frac{1}{\wp_1}},
\end{aligned}$$

where $0 < \hat{\mathfrak{q}} < 1$, $\mathfrak{c}_{1\hat{\mathfrak{q}}}, \mathfrak{c}_{2\hat{\mathfrak{q}}}, \mathrm{d}_{1\hat{\mathfrak{q}}}$, and $\mathrm{d}_{2\hat{\mathfrak{q}}}$ are defined in 2.

Proof. By applying the power mean inequality on the right-hand side of Lemma 3, we have

$$\begin{aligned}
&\Bigg|\frac{1}{(\mathrm{y}-\varkappa)}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}(\rho_1+\rho_2-\varkappa)\right]\Bigg| \\
&\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\right)^{1-\frac{1}{\wp_1}} \\
&\times\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}-(1-\lambda)\varkappa))\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_1}} \\
&+\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\right)^{1-\frac{1}{\wp_1}} \\
&\times\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa))\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_1}}.
\end{aligned}$$

Now, applying the convexity of $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|^{\wp_1}$, we have

$$\begin{aligned}
&\Bigg|\frac{1}{(\mathrm{y}-\varkappa)}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda \\
&-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\hat{\mathfrak{q}}^2[4]_{\hat{\mathfrak{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}\varkappa}{[2]_{\hat{\mathfrak{q}}}}\right)+\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\Bigg| \\
&\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\right)^{1-\frac{1}{\wp_1}} \\
&\Bigg(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\left\{\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}\right\} \\
&-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}(1-\lambda)\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\Bigg)^{\frac{1}{\wp_1}} \\
&+\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\right)^{1-\frac{1}{\wp_1}} \\
&\times\Bigg(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\left\{\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}\right\} \\
&-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\lambda\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda-\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|(1-\lambda)d_{\hat{\mathfrak{q}}}\lambda\Bigg)^{\frac{1}{\wp_1}}
\end{aligned}$$

$$= \hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_0^{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}\left|\lambda-\frac{1}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\right)^{1-\frac{1}{\wp_1}} \times \left(\mathfrak{c}_{3\hat{\mathfrak{q}}}\left\{\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}+\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}\right\}\right.$$
$$\left.-\,\mathfrak{c}_{1\hat{\mathfrak{q}}}(\hat{\mathfrak{q}})\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda-\mathrm{d}_{1\hat{\mathfrak{q}}}\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_1}}$$
$$+\,\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left(\int_{\frac{1}{[2]_{\hat{\mathfrak{q}}}}}^1\left|\lambda-\frac{[5]_{\hat{\mathfrak{q}}}}{[6]_{\hat{\mathfrak{q}}}}\right|d_{\hat{\mathfrak{q}}}\lambda\right)^{1-\frac{1}{\wp_1}} \times \left(\mathrm{d}_{3\hat{\mathfrak{q}}}\left\{\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|^{\wp_1}+\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|^{\wp_1}\right\}\right.$$
$$\left.-\,\mathfrak{c}_{2\hat{\mathfrak{q}}}\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda-\mathrm{d}_{2\hat{\mathfrak{q}}}\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|^{\wp_1}d_{\hat{\mathfrak{q}}}\lambda\right)^{\frac{1}{\wp_1}}.$$

By putting in the values of the integrals, we get required results. □

Remark 5. *If we put $\varkappa=\rho_1$ and $\mathrm{y}=\rho_2$ into Theorem 4, we get Theorem 5 of [31].*

Corollary 2. *With the suppositions of Theorem 4 with $\varkappa=\rho_1$, $\mathrm{y}=\rho_2$, and $\hat{\mathfrak{q}}\to 1^-$, we have (see [35]):*

$$\left|\frac{1}{\rho_2-\rho_1}\int_{\rho_1}^{\rho_2}\mathfrak{g}(\lambda)d\lambda-\frac{1}{6}\left[\mathfrak{g}(\rho_1)+4\mathfrak{g}\left(\frac{\rho_1+\rho_2}{2}\right)+\mathfrak{g}(\rho_2)\right]\right|$$
$$\leq(\rho_2-\rho_1)\frac{1}{(1296)^{\frac{1}{\wp_1}}}\left(\frac{5}{72}\right)^{1-\frac{1}{\wp_1}}$$
$$\times\left\{\left(61|\mathfrak{g}'(\rho_1)|^{\wp_1}+29|\mathfrak{g}'(\rho_2)|^{\wp_1}\right)^{\frac{1}{\wp_1}}+\left(61|\mathfrak{g}'(\rho_1)|^{\wp_1}+29|\mathfrak{g}'(\rho_2)|^{\wp_1}\right)^{\frac{1}{\wp_1}}\right\}. \tag{8}$$

3.2. Simpson–Mercer 3/8 Formula-Type Inequalities

Theorem 5. *With the suppositions of Theorem 2, we have*

$$\left|\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathfrak{q}}}\lambda\right. \tag{9}$$
$$-\frac{1}{[8]_{\hat{\mathfrak{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\frac{\hat{\mathfrak{q}}^3[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}\varkappa}{[3]_{\hat{\mathfrak{q}}}}\right)+\frac{\hat{\mathfrak{q}}^2[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{[2]_{\hat{\mathfrak{q}}}\mathrm{y}+\varkappa\hat{\mathfrak{q}}^2}{[3]_{\hat{\mathfrak{q}}}}\right)\right.$$
$$\left.\left.+\,\hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\right|$$
$$\leq\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\left[\left|\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\right|+\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\right|\right|\left(-\frac{\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2}\frac{[8]_{\hat{\mathfrak{q}}}-[3]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{1}{[2]_{\hat{\mathfrak{q}}}}\right.\right.$$
$$\left.-\frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}}-\frac{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}[7]_{\hat{\mathfrak{q}}}-[2]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{2\hat{\mathfrak{q}}\left[[8]_{\hat{\mathfrak{q}}}^2+[2]_{\hat{\mathfrak{q}}}^2(1+[7]_{\hat{\mathfrak{q}}}^2)\right]}{[2]_{\hat{\mathfrak{q}}}^3[8]_{\hat{\mathfrak{q}}}^2}\right)$$
$$\left.-\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\right|\left[\mathfrak{c}_{3\hat{\mathfrak{q}}}+\mathfrak{c}_{4\hat{\mathfrak{q}}}+\mathfrak{c}_{5\hat{\mathfrak{q}}}\right]-\left|^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\right|\left[\mathrm{d}_{3\hat{\mathfrak{q}}}+\mathrm{d}_{4\hat{\mathfrak{q}}}+\mathrm{d}_{5\hat{\mathfrak{q}}}\right]\right]$$

where

$$\omega_{3\hat{\mathfrak{q}}}=[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}^3,$$

$$\mathfrak{c}_{3\hat{\mathfrak{q}}} = \int_0^{\frac{1}{[3]_{\hat{\mathfrak{q}}}}} \lambda \left| \lambda - \frac{1}{[8]_{\hat{\mathfrak{q}}}} \right| d_{\hat{\mathfrak{q}}}\lambda = \frac{[8]_{\hat{\mathfrak{q}}}^3[2]_{\hat{\mathfrak{q}}} - [3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}^2 + 2\hat{\mathfrak{q}}^2[3]_{\hat{\mathfrak{q}}}^3}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^4[8]_{\hat{\mathfrak{q}}}^3},$$

$$\mathfrak{c}_{4\hat{\mathfrak{q}}} = \left| \lambda - \frac{1}{[8]_{\hat{\mathfrak{q}}}} \right| (1-\lambda) d_{\hat{\mathfrak{q}}}\lambda = \frac{[2]_{\hat{\mathfrak{q}}}^2(1+[2]_{\hat{\mathfrak{q}}}^3) - [3]_{\hat{\mathfrak{q}}}^2(1+[2]_{\hat{\mathfrak{q}}}^2)}{[2]_{\hat{\mathfrak{q}}}^2[3]_{\hat{\mathfrak{q}}}^4} + \frac{2\hat{\mathfrak{q}}^2}{[2]_{\hat{\mathfrak{q}}}^4[3]_{\hat{\mathfrak{q}}}},$$

$$\mathfrak{c}_{5\hat{\mathfrak{q}}} = \int_{\frac{1}{[3]_{\hat{\mathfrak{q}}}}}^{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}} \lambda \left| \lambda - \frac{1}{[8]_{\hat{\mathfrak{q}}}} \right| d_{\hat{\mathfrak{q}}}\lambda = \frac{2\hat{\mathfrak{q}}^2[7]_{\hat{\mathfrak{q}}}^3}{\omega_{3\hat{\mathfrak{q}}}} + \frac{[2]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}([2]_{\hat{\mathfrak{q}}}^3 + [3]_{\hat{\mathfrak{q}}}^3) - ([2]_{\hat{\mathfrak{q}}}^3 + [3]_{\hat{\mathfrak{q}}}^3)[3]_{\hat{\mathfrak{q}}}^2[7]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^4[8]_{\hat{\mathfrak{q}}}},$$

$$\mathfrak{d}_{3\hat{\mathfrak{q}}} = \int_{\frac{1}{[3]_{\hat{\mathfrak{q}}}}}^{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}} (1-\lambda) \left| \lambda - \frac{1}{[8]_{\hat{\mathfrak{q}}}} \right| d_{\hat{\mathfrak{q}}}\lambda = 2\frac{\hat{\mathfrak{q}}[3]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}} - \hat{\mathfrak{q}}^2}{\omega_{3\hat{\mathfrak{q}}}} + \frac{1 - [2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}} + \frac{1}{[3]_{\hat{\mathfrak{q}}}^2},$$

$$\mathfrak{d}_{4\hat{\mathfrak{q}}} = \int_{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}}^{1} \lambda \left| \lambda - \frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}} \right| d_{\hat{\mathfrak{q}}}\lambda = \frac{2\hat{\mathfrak{q}}}{[2]_{\hat{\mathfrak{q}}}^3} - \hat{\mathfrak{q}}\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2} - \mathfrak{c}_{4\hat{\mathfrak{q}}},$$

$$\mathfrak{d}_{5\hat{\mathfrak{q}}} = \int_{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}}^{1} (1-\lambda) \left| \lambda - \frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}} \right| d_{\hat{\mathfrak{q}}}\lambda = 2\frac{\hat{\mathfrak{q}}[3]_{\hat{\mathfrak{q}}}[7]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}} - \hat{\mathfrak{q}}^2[7]_{\hat{\mathfrak{q}}}^3}{\omega_{3\hat{\mathfrak{q}}}} + \frac{\hat{\mathfrak{q}}^2}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}} - \frac{\hat{\mathfrak{q}}[7]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}}$$
$$+ \frac{[2]_{\hat{\mathfrak{q}}}([3]_{\hat{\mathfrak{q}}}^2 - [2]_{\hat{\mathfrak{q}}}^2)}{[3]_{\hat{\mathfrak{q}}}^4} - \frac{\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}^2[7]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}},$$

and $0 < \hat{\mathfrak{q}} < 1$.

Proof. Applying the modulus to Lemma 4, we attain

$$\left| \frac{1}{\mathrm{y} - \varkappa} \int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa} \mathfrak{g}(\lambda) \ {}^{\rho_1+\rho_2-\varkappa} d_{\hat{\mathfrak{q}}}\lambda \right.$$
$$- \frac{1}{[8]_{\hat{\mathfrak{q}}}} \left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y}) + \frac{\hat{\mathfrak{q}}^3[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}} \mathfrak{g}\left(\rho_1+\rho_2 - \frac{\mathrm{y} + \hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}\varkappa}{[3]_{\hat{\mathfrak{q}}}}\right) + \frac{\hat{\mathfrak{q}}^2[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}} \mathfrak{g}\left(\rho_1+\rho_2 - \frac{[2]_{\hat{\mathfrak{q}}}\mathrm{y} + \varkappa\hat{\mathfrak{q}}^2}{[3]_{\hat{\mathfrak{q}}}}\right) \right.$$
$$\left.\left. + \hat{\mathfrak{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa) \right] \right|$$
$$\leq \hat{\mathfrak{q}}(\mathrm{y}-\varkappa) \int_0^{\frac{1}{[3]_{\hat{\mathfrak{q}}}}} \left| \lambda - \frac{1}{[8]_{\hat{\mathfrak{q}}}} \right| \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)) \right| d_{\hat{\mathfrak{q}}}\lambda$$
$$+ \hat{\mathfrak{q}}(\mathrm{y}-\varkappa) \int_{\frac{1}{[3]_{\hat{\mathfrak{q}}}}}^{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}} \left| \lambda - \frac{1}{[8]_{\hat{\mathfrak{q}}}} \right| \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)) \right| d_{\hat{\mathfrak{q}}}\lambda$$
$$+ \hat{\mathfrak{q}}(\mathrm{y}-\varkappa) \int_{\frac{[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}}}^{1} \left| \lambda - \frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}} \right| \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)) \right| d_{\hat{\mathfrak{q}}}\lambda.$$

It is now sufficient to use similar steps to those in the results of Theorem 5 to find the desired outcomes. □

We give the following example to show the validity of the results given in Theorem 5.

Example 2. *Let* x, y, *and the function* $\mathfrak{g}$ *be as in Example 1. Then,* $|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}|$ *is convex, and we get*

$$\left|\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda\right.$$
$$-\frac{1}{[8]_{\hat{\mathrm{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\frac{\hat{\mathrm{q}}^3[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}}\varkappa}{[3]_{\hat{\mathrm{q}}}}\right)+\frac{\hat{\mathrm{q}}^2[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{[2]_{\hat{\mathrm{q}}}\mathrm{y}+\varkappa\hat{\mathrm{q}}^2}{[3]_{\hat{\mathrm{q}}}}\right)\right.$$
$$\left.\left.+\hat{\mathrm{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\right|$$

$$\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})=\frac{16}{9},$$
$$\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}}\varkappa}{[3]_{\hat{\mathrm{q}}}}\right)=\left(\frac{5[3]_{\hat{\mathrm{q}}}-1}{3[3]_{\hat{\mathrm{q}}}}\right)^2,$$
$$\mathfrak{g}\left(\rho_1+\rho_2-\frac{[2]_{\hat{\mathrm{q}}}\mathrm{y}+\varkappa\hat{\mathrm{q}}^2}{[3]_{\hat{\mathrm{q}}}}\right)=\left(\frac{4[3]_{\hat{\mathrm{q}}}+\hat{\mathrm{q}}^2}{3[3]_{\hat{\mathrm{q}}}}\right)^2,$$
$$\mathfrak{g}(\rho_1+\rho_2-\varkappa)=\frac{25}{9},$$

and

$$\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda=\frac{1}{27}\left[25-\frac{10}{[2]_{\hat{\mathrm{q}}}}+\frac{1}{[3]_{\hat{\mathrm{q}}}}\right].$$

Therefore, we have the equality from (9),

$$\left|\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda\right.$$
$$-\frac{1}{[8]_{\hat{\mathrm{q}}}}\left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\frac{\hat{\mathrm{q}}^3[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}}\varkappa}{[3]_{\hat{\mathrm{q}}}}\right)+\frac{\hat{\mathrm{q}}^2[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{[2]_{\hat{\mathrm{q}}}\mathrm{y}+\varkappa\hat{\mathrm{q}}^2}{[3]_{\hat{\mathrm{q}}}}\right)\right.$$
$$\left.\left.+\hat{\mathrm{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\right]\right|$$
$$=\left|\frac{1}{9}\left[25-\frac{10}{[2]_{\hat{\mathrm{q}}}}+\frac{1}{[3]_{\hat{\mathrm{q}}}}\right]-\frac{1}{[8]_{\hat{\mathrm{q}}}}\left[\frac{16+25\hat{\mathrm{q}}}{9}+\frac{\hat{\mathrm{q}}^3[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}}\left(\frac{5[3]_{\hat{\mathrm{q}}}-1}{3[3]_{\hat{\mathrm{q}}}}\right)^2+\frac{\hat{\mathrm{q}}^2[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}}\left(\frac{4[3]_{\hat{\mathrm{q}}}+\hat{\mathrm{q}}^2}{3[3]_{\hat{\mathrm{q}}}}\right)^2\right]\right|$$

By using the equalities (5), we can write the right-hand side of the inequality (9) as

$$\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\Bigg[\Big|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_1)\Big|+\Big|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\rho_2)\Big|\Bigg(-\frac{\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2}\frac{[8]_{\hat{\mathfrak{q}}}-[3]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{1}{[2]_{\hat{\mathfrak{q}}}}$$
$$-\frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}}-\frac{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}[7]_{\hat{\mathfrak{q}}}-[2]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{2\hat{\mathfrak{q}}\left[[8]_{\hat{\mathfrak{q}}}^2+[2]_{\hat{\mathfrak{q}}}^2(1+[7]_{\hat{\mathfrak{q}}}^2)\right]}{[2]_{\hat{\mathfrak{q}}}^3[8]_{\hat{\mathfrak{q}}}^2}\Bigg)$$
$$-\Big|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\mathrm{y})\Big|[\mathfrak{c}_{3\hat{\mathfrak{q}}}+\mathfrak{c}_{4\hat{\mathfrak{q}}}+\mathfrak{c}_{5\hat{\mathfrak{q}}}]-\Big|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathfrak{q}}}\mathfrak{g}(\varkappa)\Big|[\mathrm{d}_{3\hat{\mathfrak{q}}}+\mathrm{d}_{4\hat{\mathfrak{q}}}+\mathrm{d}_{5\hat{\mathfrak{q}}}]\Bigg]$$
$$=\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\Bigg[\frac{8-2\hat{\mathfrak{q}}}{3}+\frac{11+\hat{\mathfrak{q}}}{3}\Bigg|\Bigg(-\frac{\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2}\frac{[8]_{\hat{\mathfrak{q}}}-[3]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{1}{[2]_{\hat{\mathfrak{q}}}}$$
$$-\frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}}-\frac{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}[7]_{\hat{\mathfrak{q}}}-[2]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{2\hat{\mathfrak{q}}\left[[8]_{\hat{\mathfrak{q}}}^2+[2]_{\hat{\mathfrak{q}}}^2(1+[7]_{\hat{\mathfrak{q}}}^2)\right]}{[2]_{\hat{\mathfrak{q}}}^3[8]_{\hat{\mathfrak{q}}}^2}\Bigg)$$
$$-\frac{10}{3}[\mathfrak{c}_{3\hat{\mathfrak{q}}}+\mathfrak{c}_{4\hat{\mathfrak{q}}}+\mathfrak{c}_{5\hat{\mathfrak{q}}}]-\frac{9-\hat{\mathfrak{q}}}{3}[\mathrm{d}_{3\hat{\mathfrak{q}}}+\mathrm{d}_{4\hat{\mathfrak{q}}}+\mathrm{d}_{5\hat{\mathfrak{q}}}]\Bigg].$$

Finally, one can write the inequality (9) as follows and can see the validity of this inequality in Figure 2.

$$\left|\frac{1}{9}\left[25-\frac{10}{[2]_{\hat{\mathfrak{q}}}}+\frac{1}{[3]_{\hat{\mathfrak{q}}}}\right]-\frac{1}{[8]_{\hat{\mathfrak{q}}}}\left[\frac{16+25\hat{\mathfrak{q}}}{9}+\frac{\hat{\mathfrak{q}}^3[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}}\left(\frac{5[3]_{\hat{\mathfrak{q}}}-1}{3[3]_{\hat{\mathfrak{q}}}}\right)^2+\frac{\hat{\mathfrak{q}}^2[6]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}}\left(\frac{4[3]_{\hat{\mathfrak{q}}}+\hat{\mathfrak{q}}^2}{3[3]_{\hat{\mathfrak{q}}}}\right)^2\right]\right|$$
$$\leq\hat{\mathfrak{q}}(\mathrm{y}-\varkappa)\Bigg[\frac{8-2\hat{\mathfrak{q}}}{3}+\frac{11+\hat{\mathfrak{q}}}{3}\Bigg|\Bigg(-\frac{\hat{\mathfrak{q}}[2]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2}\frac{[8]_{\hat{\mathfrak{q}}}-[3]_{\hat{\mathfrak{q}}}[2]_{\hat{\mathfrak{q}}}}{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{1}{[2]_{\hat{\mathfrak{q}}}}$$
$$-\frac{[7]_{\hat{\mathfrak{q}}}}{[8]_{\hat{\mathfrak{q}}}}-\frac{[2]_{\hat{\mathfrak{q}}}[3]_{\hat{\mathfrak{q}}}[7]_{\hat{\mathfrak{q}}}-[2]_{\hat{\mathfrak{q}}}[8]_{\hat{\mathfrak{q}}}}{[3]_{\hat{\mathfrak{q}}}^2[8]_{\hat{\mathfrak{q}}}}+\frac{2\hat{\mathfrak{q}}\left[[8]_{\hat{\mathfrak{q}}}^2+[2]_{\hat{\mathfrak{q}}}^2(1+[7]_{\hat{\mathfrak{q}}}^2)\right]}{[2]_{\hat{\mathfrak{q}}}^3[8]_{\hat{\mathfrak{q}}}^2}\Bigg)$$
$$-\frac{10}{3}[\mathfrak{c}_{3\hat{\mathfrak{q}}}+\mathfrak{c}_{4\hat{\mathfrak{q}}}+\mathfrak{c}_{5\hat{\mathfrak{q}}}]-\frac{9-\hat{\mathfrak{q}}}{3}[\mathrm{d}_{3\hat{\mathfrak{q}}}+\mathrm{d}_{4\hat{\mathfrak{q}}}+\mathrm{d}_{5\hat{\mathfrak{q}}}]\Bigg].$$

Remark 6. *If we choose $\varkappa=\rho_1$ and $\mathrm{y}=\rho_2$ in Theorem 5, we get Theorem 6 of [31].*

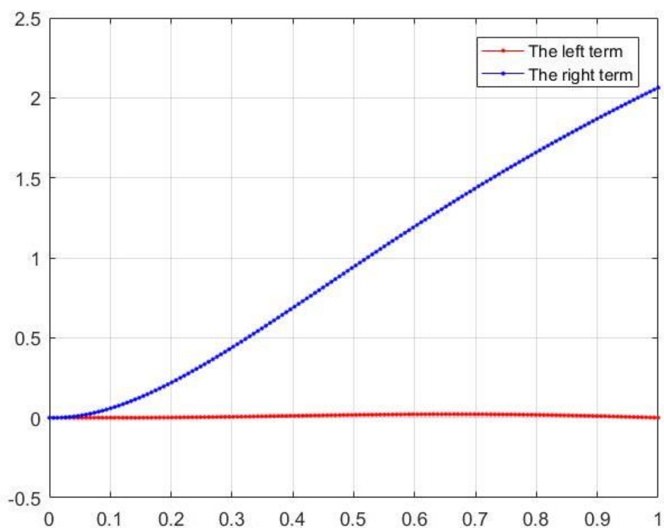

Figure 2. An example of the dependence of the inequality (9) on $\hat{\mathfrak{q}}$ was computed and plotted with MATLAB.

Theorem 6. *With the suppositions of Theorem 3, we have following inequality:*

$$
\begin{aligned}
&\Bigg|\frac{1}{\mathrm{y}-\varkappa}\int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa}\mathfrak{g}(\lambda)\ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{q}}\lambda \\
&-\frac{1}{[8]_{\hat{q}}}\Bigg[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y})+\frac{\hat{q}^3[6]_{\hat{q}}}{[2]_{\hat{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{\mathrm{y}+\hat{q}[2]_{\hat{q}}\varkappa}{[3]_{\hat{q}}}\right)+\frac{\hat{q}^2[6]_{\hat{q}}}{[2]_{\hat{q}}}\mathfrak{g}\left(\rho_1+\rho_2-\frac{[2]_{\hat{q}}\mathrm{y}+\varkappa\hat{q}^2}{[3]_{\hat{q}}}\right) \\
&+\hat{q}\mathfrak{g}(\rho_1+\rho_2-\varkappa)\Bigg]\Bigg| \\
&\leq \hat{q}(\mathrm{y}-\varkappa)\left(\frac{\hat{q}^{3\wp_2}[5]_{\hat{q}}^{\wp_2}}{[3]_{\hat{q}}^{\wp_2+1}[8]_{\hat{q}}^{\wp_2}}\right)^{\frac{1}{\wp_2}} \\
&\times\left(\frac{1}{[3]_{\hat{q}}}\left[\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_1)\right|+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_2)\right|\right]-\frac{\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}}{[3]_{\hat{q}}^2[2]_{\hat{q}}}-\frac{[3]_{\hat{q}}[2]_{\hat{q}}-1}{[3]_{\hat{q}}^2[2]_{\hat{q}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}} \\
&+\hat{q}(\mathrm{y}-\varkappa)\left(\frac{\hat{q}^{\wp_2}[2]_{\hat{q}}-\hat{q}^{2\wp_2}}{[3]_{\hat{q}}^{\wp_2+1}[2]_{\hat{q}}^{\wp_2}}\right)^{\frac{1}{\wp_2}} \\
&\times\left(\frac{\hat{q}}{[3]_{\hat{q}}}\left[\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_1)\right|+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_2)\right|\right]\right. \\
&\left.-\frac{[2]_{\hat{q}}^2-1}{[3]_{\hat{q}}^2[2]_{\hat{q}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}-\frac{[3]_{\hat{q}}[2]_{\hat{q}}\hat{q}-[2]_{\hat{q}}^2+1}{[3]_{\hat{q}}^2[2]_{\hat{q}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}} \\
&+\hat{q}(\mathrm{y}-\varkappa)\left(\frac{\hat{q}^{7\wp_2}}{[8]_{\hat{q}}^{\wp_2}}-\frac{[2]_{\hat{q}}([7]_{\hat{q}}[3]_{\hat{q}}-[8]_{\hat{q}}[2]_{\hat{q}})^{\wp_2}}{[8]_{\hat{q}}^{\wp_2}[3]_{\hat{q}}^{\wp_2+1}}\right)^{\frac{1}{\wp_2}} \\
&\times\left(\frac{\hat{q}^2}{[3]_{\hat{q}}}\left[\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_1)\right|+\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\rho_2)\right|\right]-\frac{[3]_{\hat{q}}^2-[2]_{\hat{q}}^2}{[2]_{\hat{q}}[3]_{\hat{q}}^2}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\mathrm{y})\right|^{\wp_1}\right. \\
&\left.-\frac{[2]_{\hat{q}}[3]_{\hat{q}}\hat{q}^2+[2]_{\hat{q}}^2-[3]_{\hat{q}}^2}{[3]_{\hat{q}}^2[2]_{\hat{q}}}\left|{}^{\rho_1+\rho_2-\varkappa}D_{\hat{q}}\mathfrak{g}(\varkappa)\right|^{\wp_1}\right)^{\frac{1}{\wp_1}},
\end{aligned}
$$

where $\frac{1}{\wp_1}+\frac{1}{\wp_2}=1$.

Proof. By applying Hölder's inequality in Lemma 4, we attain

$$\left| \frac{1}{\mathrm{y}-\varkappa} \int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa} \mathfrak{g}(\lambda) \ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda \right.$$
$$- \frac{1}{[8]_{\hat{\mathrm{q}}}} \left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y}) + \frac{\hat{\mathrm{q}}^3[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}} \mathfrak{g}\left(\rho_1+\rho_2 - \frac{\mathrm{y}+\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}}\varkappa}{[3]_{\hat{\mathrm{q}}}}\right) + \frac{\hat{\mathrm{q}}^2[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}} \mathfrak{g}\left({}_1 + \rho_2 - \frac{[2]_{\hat{\mathrm{q}}}\mathrm{y}+\varkappa\hat{\mathrm{q}}^2}{[3]_{\hat{\mathrm{q}}}} \right) \right.$$
$$\left.\left. + \hat{\mathrm{q}}\mathfrak{g}({}_1+\rho_2-\varkappa) \right] \right|$$
$$\leq \hat{\mathrm{q}}(\mathrm{y}-\varkappa) \left(\int_0^{\frac{1}{[3]_{\hat{\mathrm{q}}}}} \left| \lambda - \frac{1}{[8]_{\hat{\mathrm{q}}}} \right|^{\wp_2} d_{\hat{\mathrm{q}}}\lambda \right)^{\frac{1}{\wp_2}} \left(\int_0^{\frac{1}{[3]_{\hat{\mathrm{q}}}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)) \right|^{\wp_1} d_{\hat{\mathrm{q}}}\lambda \right)^{\frac{1}{\wp_1}}$$
$$+ \hat{\mathrm{q}}(\mathrm{y}-\varkappa) \left(\int_{\frac{1}{[3]_{\hat{\mathrm{q}}}}}^{\frac{[2]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}}} \left| \lambda - \frac{1}{[8]_{\hat{\mathrm{q}}}} \right|^{\wp_2} d_{\hat{\mathrm{q}}}\lambda \right)^{\frac{1}{\wp_2}} \left(\int_{\frac{1}{[3]_{\hat{\mathrm{q}}}}}^{\frac{[2]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}({}_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)) \right|^{\wp_1} d_{\hat{\mathrm{q}}}\lambda \right)^{\frac{1}{\wp_1}}$$
$$+ \hat{\mathrm{q}}(\mathrm{y}-\varkappa) \left(\int_{\frac{[2]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}}}^{1} \left| \lambda - \frac{[7]_{\hat{\mathrm{q}}}}{[8]_{\hat{\mathrm{q}}}} \right|^{\wp_2} d_{\hat{\mathrm{q}}}\lambda \right)^{\frac{1}{\wp_2}} \left(\int_{\frac{[2]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}}}^{1} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1+\rho_2-(\lambda\mathrm{y}+(1-\lambda)\varkappa)) \right|^{\wp_1} d_{\hat{\mathrm{q}}}\lambda \right)^{\frac{1}{\wp_1}}.$$

By applying a similar method to that used in the results of Theorem 3, we find the desired outcomes. □

Remark 7. *If we put* $\varkappa = \rho_1$ *and* $\mathrm{y} = \rho_2$ *into Theorem 6, we get Theorem 8 of [31].*

Theorem 7. *With the suppositions of Theorem 4, we have the inequality*

$$\left| \frac{1}{\mathrm{y}-\varkappa} \int_{\rho_1+\rho_2-\mathrm{y}}^{\rho_1+\rho_2-\varkappa} \mathfrak{g}(\lambda) \ {}^{\rho_1+\rho_2-\varkappa}d_{\hat{\mathrm{q}}}\lambda \right.$$
$$- \frac{1}{[8]_{\hat{\mathrm{q}}}} \left[\mathfrak{g}(\rho_1+\rho_2-\mathrm{y}) + \frac{\hat{\mathrm{q}}^3[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}} \mathfrak{g}\left(\rho_1+\rho_2 - \frac{\mathrm{y}+\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}}\varkappa}{[3]_{\hat{\mathrm{q}}}}\right) + \frac{\hat{\mathrm{q}}^2[6]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}} \mathfrak{g}\left(\rho_1 + \rho_2 - \frac{[2]_{\hat{\mathrm{q}}}\mathrm{y}+\varkappa\hat{\mathrm{q}}^2}{[3]_{\hat{\mathrm{q}}}} \right) \right.$$
$$\left.\left. + \hat{\mathrm{q}}\mathfrak{g}(\rho_1+\rho_2-\varkappa) \right] \right|$$
$$\leq \hat{\mathrm{q}}(\mathrm{y}-\varkappa) \left(\frac{2\hat{\mathrm{q}}}{[8]_{\hat{\mathrm{q}}}^2[2]_{\hat{\mathrm{q}}}} + \frac{[8]_{\hat{\mathrm{q}}} - [3]_{\hat{\mathrm{q}}}[2]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}^2[2]_{\hat{\mathrm{q}}}[8]_{\hat{\mathrm{q}}}} \right)^{1-\frac{1}{\wp_1}}$$
$$\left(\frac{2\hat{\mathrm{q}}}{[2]_{\hat{\mathrm{q}}}[8]_{\hat{\mathrm{q}}}^2} + \frac{[8]_{\hat{\mathrm{q}}} - [3]_{\hat{\mathrm{q}}}[2]_{\hat{\mathrm{q}}}}{[2]_{\hat{\mathrm{q}}}[3]_{\hat{\mathrm{q}}}^2[8]_{\hat{\mathrm{q}}}} \left[\left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1) \right| + \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_2) \right| \right] \right.$$
$$\left. - \mathfrak{c}_{3\hat{\mathrm{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\mathrm{y}) \right|^{\wp_1} - \mathrm{d}_3(\hat{\mathrm{q}}) \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\varkappa) \right|^{\wp_1} \right)^{\frac{1}{\wp_1}}$$
$$+ \left(\frac{2\hat{\mathrm{q}}}{[2]_{\hat{\mathrm{q}}}^3} + \frac{\hat{\mathrm{q}}}{[3]_{\hat{\mathrm{q}}}^2[2]_{\hat{\mathrm{q}}}} + \frac{2\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}} + \hat{\mathrm{q}}^2}{[3]_{\hat{\mathrm{q}}}^2[2]_{\hat{\mathrm{q}}}} \right)^{1-\frac{1}{\wp_1}}$$
$$\left(\frac{2\hat{\mathrm{q}}}{[2]_{\hat{\mathrm{q}}}^3} - \frac{\hat{\mathrm{q}}[2]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}^2} \left[\left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_1) \right| + \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_2) \right| \right] - \mathfrak{c}_{4\hat{\mathrm{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\mathrm{y}) \right|^{\wp_1} - \mathrm{d}_{4\hat{\mathrm{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\varkappa) \right|^{\wp_1} \right)^{\frac{1}{\wp_1}}$$
$$+ \left(2\frac{\hat{\mathrm{q}}[7]_{\hat{\mathrm{q}}}^2}{[8]_{\hat{\mathrm{q}}}^2[2]_{\hat{\mathrm{q}}}} + \frac{[3]_{\hat{\mathrm{q}}}^2 + [2]_{\hat{\mathrm{q}}}^2}{[3]_{\hat{\mathrm{q}}}^2[2]_{\hat{\mathrm{q}}}} - \frac{[7]_{\hat{\mathrm{q}}}([3]_{\hat{\mathrm{q}}} + [2]_{\hat{\mathrm{q}}})}{[3]_{\hat{\mathrm{q}}}[8]_{\hat{\mathrm{q}}}} \right)^{1-\frac{1}{\wp_1}}$$
$$\left(\frac{2\hat{\mathrm{q}}[7]_{\hat{\mathrm{q}}}^2}{[2]_{\hat{\mathrm{q}}}[8]_{\hat{\mathrm{q}}}^2} + \frac{1}{[2]_{\hat{\mathrm{q}}}} - \frac{[7]_{\hat{\mathrm{q}}}}{[8]_{\hat{\mathrm{q}}}} - \frac{[2]_{\hat{\mathrm{q}}}[3]_{\hat{\mathrm{q}}}[7]_{\hat{\mathrm{q}}} - [2]_{\hat{\mathrm{q}}}[8]_{\hat{\mathrm{q}}}}{[3]_{\hat{\mathrm{q}}}^2[8]_{\hat{\mathrm{q}}}} \left[\left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}({}_1) \right| + \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\rho_2) \right| \right] \right.$$
$$\left. - \mathfrak{c}_{5\hat{\mathrm{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\mathrm{y}) \right|^{\wp_1} - \mathrm{d}_{5\hat{\mathrm{q}}} \left| {}^{\rho_1+\rho_2-\varkappa}D_{\hat{\mathrm{q}}}\mathfrak{g}(\varkappa) \right|^{\wp_1} \right)^{\frac{1}{\wp_1}},$$

$\mathfrak{c}_{3\hat{\mathrm{q}}}, \mathfrak{c}_{4\hat{\mathrm{q}}}, \mathfrak{c}_{5\hat{\mathrm{q}}}, \mathrm{d}_{3\hat{\mathrm{q}}}, \mathrm{d}_{4\hat{\mathrm{q}}}$, *and* $\mathrm{d}_{5\hat{\mathrm{q}}}$ *are defined in Theorem 5.*

Proof. The proof of the theorem is analogous to the proof of Theorem 4. □

Remark 8. *If we put* $\varkappa = \rho_1$ *and* $\mathrm{y} = \rho_2$ *into Theorem 7, we get Theorem 9 of [31].*

4. Conclusions

In this article, we developed new analogs of Simpson–Mercer-like identities. Thus, by employing quantum integration and the Jensen–Mercer inequality for convex functions, we computed new quantum bounds. This novel framework is a convolution of the Simpon–Mercer concept and the estimation of the $\hat{q}$-definite integral. Some examples are also provided for the main inequalities. By employing the quantum Hölder and power mean integral inequalities, we analyzed new quantum inequalities that are in connection with previously published results. Little work has been done on quantum Mercer inequalities to date, so it is interesting to extend these findings to other general inequalities and convexities. One of the important problems is to check the validity of such results for their coordinate convexity. We presume that our newly announced concept will be the focus of much research in this fascinating field of quantum inequalities.

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