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Properties of q-Symmetric Starlike Functions of Janowski Type

Afis Saliu [1,2], Isra Al-Shbeil [3], Jianhua Gong [4,*], Sarfraz Nawaz Malik [5] and Najla Aloraini [6]

1 Department of Mathematics, University of the Gambia, Birkama Campus, MDI Road, Kanifing Serrekunda P.O. Box 3530, The Gambia
2 Department of Mathematics, Gombe State University, P.M.B 127, Tudun Wada, Gombe 760253, Gombe State, Nigeria
3 Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
4 Department of Mathematical Sciences, United Arab Emirates University, Al Ain 18006, United Arab Emirates
5 Department of Mathematics, COMSATS University Islamabad, Wah Campus, Wah Cantt 47040, Pakistan
6 Department of Mathematics, Qassim University, Buraydah 51452, Saudi Arabia
* Correspondence: j.gong@uaeu.ac.ae

Abstract: The word "symmetry" is a Greek word that originated from "symmetria". It means an agreement in dimensions, due proportion, and arrangement; however, in complex analysis, it means objects remaining invariant under some transformation. This idea has now been recently used in geometric function theory to modify the earlier classical q-derivative introduced by Ismail et al. due to its better convergence properties. Consequently, we introduce a new class of analytic functions by using the notion of q-symmetric derivative. The investigation in this paper obtains a number of the latest important results in q-theory, including coefficient inequalities and convolution characterization of q-symmetric starlike functions related to Janowski mappings.

Keywords: univalent functions; subordination; analytic functions; q-symmetric derivative; Janowski function

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1. Introduction and Preliminaries

Let $\mathcal{H}$ be the class of analytic functions $f(\nu)$ having the series form

$$f(\nu) = \nu + \sum_{n=2}^{\infty} a_n \nu^n, \quad \nu \in \mho := \{\nu \in \mathbb{C} : |\nu| < 1\}. \tag{1}$$

Let $\mathcal{S}$ denote the subclasses of $\mathcal{H}$ consisting of functions that are univalent in $\mho$. We say $f(\nu) \in \mathcal{H}$ is subordinate to $g(\nu) \in \mathcal{H}$ (written as $f \prec g$ or $f(\nu) \prec g(\nu)$) if there exists a Schwarz function $w(\nu)$ such that $f(\nu) = g(w(\nu))$ for all $\nu \in \mho$ [1]. For $f, g \in \mathcal{H}$ with $f(\nu) = \nu + \sum\limits_{n=2}^{\infty} a_n \nu^n$ and $g(\nu) = \nu + \sum\limits_{n=2}^{\infty} b_n \nu^n$, the convolution of f and g depicted by $f(\nu) * g(\nu)$ is defined as

$$f(\nu) * g(\nu) = \nu + \sum_{n=2}^{\infty} a_n b_n \nu^n.$$

Let $\mathcal{P}(\widetilde{\Phi}, \widetilde{\Psi})$ denote the class of all functions $p(\nu)$ such that the following subordination condition is satisfied:

$$p(\nu) \prec \frac{1 + \widetilde{\Phi}\nu}{1 + \widetilde{\Psi}\nu}, \quad \nu \in \mho. \tag{2}$$

If we choose $p(\nu) = \nu f'(\nu)/f(\nu)$ and $(\nu f'(\nu))'/f'(\nu)$ in (2), then $f \in \mathcal{S}^*(\widetilde{\Phi}, \widetilde{\Psi})$ and $f \in \mathcal{C}(\widetilde{\Phi}, \widetilde{\Psi})$, respectively [2]. In particular, if $\widetilde{\Phi} = 1$, $\widetilde{\Psi} = -1$, the class $\mathcal{P}(\widetilde{\Phi}, \widetilde{\Psi})$ reduces to the usual class $\mathcal{P}$ of functions with positive real part, and $\mathcal{S}^*(\widetilde{\Phi}, \widetilde{\Psi}) \equiv \mathcal{S}^*$ and $\mathcal{C}(\widetilde{\Phi}, \widetilde{\Psi}) \equiv \mathcal{C}$ of starlike and convex functions, respectively.

q-calculus is a significant concept in modern mathematics. It also plays a crucial role in many fields of physics such as cosmic strings and black holes, nuclear and high energy physics [3]. This idea of q-calculus was developed by Jackson [4] and its calculus is based on q-derivative

$$\frac{f(q\nu) - f(\nu)}{(q-1)\nu}, \quad 0 < q < 1, \, \nu \in \mho.$$

We observed that several results in the area of q-theory are analogs of the important results from the classical analysis.

In geometric function theory (GFT), Ismail et al. [5] first utilized the q-derivative to define the class of starlike functions. As a result, numerous articles (which contain new ideas or nice extensions of the classical classes in GFT) are scattered in the literature. We refer the reader to [6–11] and the references cited therein, for the most recent work; therefore, the generalization of q-calculus popped up in different subjects, such as complex analysis, hypergeometric series, statistics and particle physics. Alb Lupaş [12] used the techniques of differential subordination to study the geometric properties of q-Sălăgean differential operator. Altintaş and Mustafa [13] introduced new classes of analytic functions defined by q-operator and gave the necessary condition for analytic functions to be members of those classes. In addition, they established the growth and distortion results related with these families of functions. Closely related to the classes of Altintaş and Mustafa, Orhan et al. [14] studied the Fekete–Szegö problem connected to a new class of analytic functions.

However, in the "Survey-cum-expository" by Srivastava [10] , it was noted that the so-called (p, q)-calculus extension is a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant.

For a fixed $q \in (0, 1)$ and $\nu \neq 0$, the q-symmetric derivative of a function $f \in \mathcal{H}$ at a point ν is defined by

$$\frac{f(q\nu) - f(q^{-1}\nu)}{(q - q^{-1})\nu}, \quad 0 < q < 1, \, \nu \in \mho.$$

The q-symmetric quantum calculus has been resourceful in many areas of study; for instance, in quantum mechanics. It was noted in [3] that the q-symmetric derivative has, in general, better convergence properties than the classical q-derivative.

Recently, this concept of the derivative has been used to introduce and study different classes of univalent functions. In this direction, Kanas et al. [15], using the notion of the symmetric operator of q-derivative, defined and studied a new family of univalent functions in a conic region. Khan et al. [16,17] slightly modified this Kanas class and investigated certain properties associated with the class, which include structural formula, necessary and sufficient conditions, coefficient estimates, Fekete–Szegö problem, distortion inequalities, closure theorem and subordination results. It is worthy of note that results presented by Khan et al. in [16,17] have no significant difference. Moreover, Seoudy [18] introduced certain classes of symmetric q-starlike and symmetric q-convex functions. For these classes, he obtained convolution properties and coefficient inequalities. Zhang et al. [19] initiated symmetric Salagean q-differential operator and then used it to introduce the class of harmonic univalent functions. Then, they examined many interesting properties associated with the defined class. Furthermore, very recently, Khan et al. [20] extended the notion of q-symmetric derivative to multivalent functions. They introduced multivalent q-symmetric starlike functions and obtained its geometric characterizations.

Motivated by these current developments, we initiate the class of q-symmetric starlike functions of the Janowski type and examine many coefficient inequalities and sufficient conditions for this class. In addition, a convolution property for it is established.

Next, we present some fundamental preliminaries which are necessary for our findings.

Definition 1 ([21]). *Let $0 < q < 1$, $n \in \mathbb{N}$. Then, the symmetric q-number denoted by $\widetilde{[n]}_q$ is defined as*

$$\widetilde{[n]}_q = \begin{cases} \frac{q^n - q^{-n}}{q - q^{-1}}, & n \in \mathbb{N}, \\ n, & \text{as} \quad q \to 1^-, \end{cases} \tag{3}$$

and the symmetric q-derivative of a function $f \in \mathcal{H}$ in $\mho$ is given by

$$\widetilde{D_q} f(\nu) = \begin{cases} \frac{f(q\nu) - f(q^{-1}\nu)}{(q - q^{-1})\nu}, & \nu \neq 0 \\ f'(0), & \nu = 0, \\ f'(\nu), & \text{as} \quad q \to 1^-. \end{cases} \tag{4}$$

We note that the symmetric q- number is not reducible to the classical q-number. It is cleared from the above definition that for $f \in \mathcal{H}$ given by (1), we have

$$\widetilde{D_q} f(\nu) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n \nu^n.$$

Let $f, g \in \mathcal{H}$, we have the following rules for q-symmetric difference operator.

Theorem 1 ([3]). *Let $f, g \in \mathcal{H}$ be q-symmetric differentiable and $\alpha, \beta \in \mathbb{C}$. Then*
(a) $\widetilde{D_q} f(\nu) = 0$ *if and only $f(\nu)$ is a constant;*
(b) $\widetilde{D_q}(\alpha f + \beta g)(\nu) = \alpha \widetilde{D_q} f(\nu) + \beta \widetilde{D_q} g(\nu)$;
(c) $\widetilde{D_q}(fg)(\nu) = g(q\nu)\widetilde{D_q} f(\nu) + f(q^{-1}\nu)\widetilde{D_q} g(\nu)$;
(d) $\widetilde{D_q}\left(\frac{f}{g}\right)(\nu) = \frac{g(q^{-1}\nu)\widetilde{D_q} f(\nu) - f(q^{-1}\nu)\widetilde{D_q} g(\nu)}{g(q\nu)g(q^{-1}\nu)}, \quad g(q\nu)g(q^{-1}\nu) \neq 0.$

Definition 2 ([17]). *Let $f \in \mathcal{H}$ and $0 < q < 1$. Then $f \in \widetilde{\mathcal{ST}}_q$ if and only if*

$$\left| \frac{\nu \widetilde{D_q} f(\nu)}{f(\nu)} - \frac{1}{1 - \frac{q}{q^{-1}}} \right| \leq \frac{1}{1 - \frac{q}{q^{-1}}}, \quad \nu \in \mho. \tag{5}$$

By the principle of subordination, $f \in \widetilde{\mathcal{ST}}_q$ if and only if

$$\frac{\nu \widetilde{D_q} f(\nu)}{f(\nu)} \prec \frac{1 + \nu}{1 - \frac{q}{q^{-1}}\nu}, \quad \nu \in \mho.$$

Definition 3 ([2]). *Let $f \in \mathcal{H}$ and $-1 \leq \widetilde{\Psi} < \widetilde{\Phi} \leq 1$. Then $f \in \mathcal{ST}(\widetilde{\Phi}, \widetilde{\Psi})$ if and only if*

$$\frac{\nu f'(\nu)}{f(\nu)} = \frac{\left(1 + \widetilde{\Phi}\right)p(\nu) + \left(1 - \widetilde{\Phi}\right)}{\left(1 + \widetilde{\Psi}\right)p(\nu) + \left(1 - \widetilde{\Psi}\right)},$$

where

$$p(\nu) \prec \frac{1 + \nu}{1 - \nu}, \quad \nu \in \mho.$$

One way to extend the class $\mathcal{ST}(\widetilde{\Phi}, \widetilde{\Psi})$ is to assume that the function

$$p(\nu) \prec \frac{1 + \nu}{1 - \frac{q}{q^{-1}}\nu}, \quad \nu \in \mho.$$

Then, the appropriate definition of the corresponding class $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ is given as:

Definition 4. *Let $f \in \mathcal{H}$, $0 < q < 1$ and $-1 \le \widetilde{\Psi} < \widetilde{\Phi} \le 1$. Then $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ if and only if*

$$\frac{\nu \widetilde{D_q} f(\nu)}{f(\nu)} \prec \varphi(\nu), \quad \nu \in \mho,$$

where

$$\varphi(\nu) = \frac{2q^{-1} + (1+\widetilde{\Phi})\nu + (\widetilde{\Phi}-1)q\nu}{2q^{-1} + (1+\widetilde{\Psi})\nu + (\widetilde{\Psi}-1)q\nu}.$$

Equivalently, $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ if and only if

$$\left| \frac{(\widetilde{\Psi}-1)\frac{\nu \widetilde{D_q} f(\nu)}{f(\nu)} - (\widetilde{\Phi}-1)}{(\widetilde{\Psi}+1)\frac{\nu \widetilde{D_q} f(\nu)}{f(\nu)} - (\widetilde{\Phi}+1)} - \frac{1}{1-\frac{q}{q^{-1}}} \right| \le \frac{1}{1-\frac{q}{q^{-1}}}, \quad \nu \in \mho.$$

Remark 1.

(a) $\lim\limits_{q \to 1^-} \widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi}) = \widetilde{\mathcal{ST}}(\widetilde{\Phi},\widetilde{\Psi})$.

(b) *For $\widetilde{\Phi} = 1$ and $\widetilde{\Psi} = -1$, then $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ reduces to $\widetilde{\mathcal{ST}}_q$.*

(c) *For $\widetilde{\Phi} = 1, \widetilde{\Psi} = -1$ and as $q \to 1$, then $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ is equivalent to the usual class $\mathcal{S}^*$ of starlike functions.*

Definition 5 (Subordinating Factor Sequence)**.** *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex number is called a subordinating factor sequence if, whenever $f(\nu)$ of the form (1) is analytic, univalent and convex in $\mho$, we have the subornation given by*

$$\sum_{n=1}^{\infty} a_n b_n \nu^n \prec f(\nu) \quad \nu \in \mho, \, a_1 := 1).$$

The following results are required for our findings.

Lemma 1 ([22])**.** *Let $h(\nu) = 1 + \sum\limits_{n=1}^{\infty} c_n \nu^n \in \mathcal{P}$. Then for a real σ,*

$$|c_2 - \sigma c_1^2| \le \begin{cases} -4\sigma + 2, & \text{for } \sigma \le 0, \\ 2, & \text{for } 0 \le \sigma \le 1, \\ 4\sigma + 2 & \text{for } \sigma \ge 1. \end{cases}$$

When $\sigma < 0$ or $\sigma > 1$, equality holds if and only if $h(\nu) = (1+\nu)/(1-\nu)$ or one of its rotations. If $0 < \sigma < 1$, then equality holds if and only if $h(\nu) = (1+\nu^2)/(1-\nu^2)$ or one of its rotations. Equality holds for $\sigma = 0$ if and only if

$$h(\nu) = \left(\frac{1+v}{2}\right)\left(\frac{1+\nu}{1-\nu}\right) + \left(\frac{1-v}{2}\right)\left(\frac{1-\nu}{1+\nu}\right), \quad 0 \le v \le 1, \, \nu \in \mho$$

or one of its rotations while for $\sigma = 1$, equality holds if and only if $h(\nu)$ is the reciprocal of one of the functions such that the equality holds true in the case when $v = 0$.

In addition, the sharp upper bound above can be improved as follows when $-1 \le \sigma \le 1$:

$$|c_2 - \sigma c_1^2| + \sigma |c_1|^2 \le 2, \quad 0 < \sigma \le \frac{1}{2}$$

and

$$|c_2 - \sigma c_1^2| + (1-\sigma)|c_1|^2 \leq 2, \quad \frac{1}{2} \leq \sigma < 1.$$

Lemma 2 ([23]). *The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\mathrm{Re}\left(1 + 2\sum_{n=1}^{\infty} b_n \nu^n\right) > 0, \quad \nu \in \mho.$$

2. Main Results

In this section, we present our main findings and assume $0 < q < 1$ and $-1 \leq \widetilde{\Psi} < \widetilde{\Phi} \leq 1$ in the entire presentation.

Theorem 2. *Let $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi}, \widetilde{\Psi})$ be of the form, as given in (1). Then for $n \geq 2$,*

$$4q^{-1}\left(\widetilde{[n]}_q - 1\right)^2 |a_2|^2 \leq q(q+q^{-1})^2(\widetilde{\Phi} - \widetilde{\Psi})^2 + \sum_{k=2}^{n-1}\left[\left(\widetilde{[k]}_q \mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)^2 q - 4q^{-1}\left(\widetilde{[k]}_q - 1\right)^2\right]|a_k|^2,$$

where

$$\mathcal{L}_q(x) = (1+x)q^{-1} + (x-1)q. \tag{6}$$

Proof. From the definition of $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi}, \widetilde{\Psi})$, we have

$$\frac{\nu \widetilde{D}_q f(\nu)}{f(\nu)} = \frac{2q^{-1} + (1+\widetilde{\Phi})w(\nu) + (\widetilde{\Phi} - 1)qw(\nu)}{2q^{-1} + (1+\widetilde{\Psi})w(\nu) + (\widetilde{\Psi} - 1)qw(\nu)},$$

where $w(0) = 0$ with $|w(\nu)| < 1$ $(\nu \in \mho)$, and $w(\nu) = \sum\limits_{k=1}^{\infty} c_k \nu^k$. A computation gives

$$\left(\mathcal{L}_q(\widetilde{\Phi})\nu \widetilde{D}_q f(\nu) - \mathcal{L}_q(\widetilde{\Psi})\right)w(\nu) = 2q^{-1}\left(\widetilde{D}_q f(\nu) - f(\nu)\right), \tag{7}$$

where $\mathcal{L}_q(\widetilde{\Phi})$ and $\mathcal{L}_q(\widetilde{\Psi})$ are defined by (6). From (7), we have

$$\left[\left(\mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)\nu + \sum_{k=2}^{\infty}\left(\widetilde{[k]}_q \mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)a_k \nu^k\right]\sum_{k=1}^{\infty} c_k \nu^k = 2q^{-1}\sum_{k=2}^{\infty}\left(\widetilde{[k]}_q - 1\right)a_k \nu^k. \tag{8}$$

Comparing coefficients for $n \geq 2$, we have

$$\begin{aligned}
&\left(\widetilde{[n-1]}_q \mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)a_{n-1}c_1 + \left(\widetilde{[n-2]}_q \mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)a_{n-2}c_2 \\
&\quad + \left(\widetilde{[n-3]}_q \mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)a_{n-3}c_3 + \cdots + \left(\mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)a_1 c_{n-1} \\
&\quad = 2q^{-1}\left(\widetilde{[n]}_q - 1\right)a_n, \quad a_1 = 1.
\end{aligned} \tag{9}$$

It is observed that the coefficient a_n on the right side of (9) depends only on $a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_2$ on the left side; therefore, we can write (8) as

$$\left[\left(\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)\nu+\sum_{k=2}^{n-1}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k+\sum_{k=n}^{\infty}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k\right]w(\nu)$$
$$=2q^{-1}\left[\sum_{k=2}^{n}\left(\widetilde{[k]_q}-1\right)a_k\nu^k+\sum_{k=n+1}^{\infty}\left(\widetilde{[k]_q}-1\right)a_k\nu^k\right].$$

That is

$$\left[\left(\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)\nu+\sum_{k=2}^{n-1}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k\right]w(\nu)$$
$$=2q^{-1}\left[\sum_{k=2}^{n}\left(\widetilde{[k]_q}-1\right)a_k\nu^k+\sum_{k=n+1}^{\infty}\left(\widetilde{[k]_q}-1\right)a_k\nu^k\right]-\left[\sum_{k=n}^{\infty}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k\right]w(\nu).$$

Applying the method of Clunie and Keogh [24], we arrive at

$$2q^{-1}\sum_{k=2}^{n}\left(\widetilde{[k]_q}-1\right)a_k\nu^k+\sum_{k=n+1}^{\infty}d_k\nu^k=\left[\left(\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)\nu\right.$$
$$\left.+\sum_{k=2}^{n-1}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k\right]w(\nu),$$

where

$$d_k=2q^{-1}\left(\widetilde{[k]_q}-1\right)a_k-\sum_{j=2}^{k-n}\left(\widetilde{[k-j]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_{k-j}c_j,\quad n+1\leq k<\infty.$$

This means that

$$\left|2q^{-1}\sum_{k=2}^{n}\left(\widetilde{[k]_q}-1\right)a_k\nu^k+\sum_{k=n+1}^{\infty}d_k\nu^k\right|^2$$
$$=\left|\left(\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)\nu+\sum_{k=2}^{n-1}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k\right|^2|w(\nu)|^2$$
$$<\left|\left(\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)\nu+\sum_{k=2}^{n-1}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)a_k\nu^k\right|^2.$$

Integrating around the circle $|\nu|<r\,(0<r<1)$ and on the account of Parseval's identity ([25], p. 100), we have

$$4q^{-1}\sum_{k=2}^{n}\left(\widetilde{[k]_q}-1\right)^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|d_k|^2r^{2k}$$
$$<q\left(\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)^2r^2+q\sum_{k=2}^{n-1}\left(\widetilde{[k]_q}\mathcal{L}_q(\widetilde{\Phi})-\mathcal{L}_q(\widetilde{\Psi})\right)^2|a_k|^2r^{2k}.$$

Letting $r \to 1$, we have

$$4q^{-1}\sum_{k=2}^{n}\left(\widetilde{[k]}_q - 1\right)^2 |a_k|^2 + \sum_{k=n+1}^{\infty}|d_k|^2$$
$$\leq q\left(\mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)^2 + q\sum_{k=2}^{n-1}\left(\widetilde{[k]}_q\mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)^2 |a_k|^2,$$

where we obtain

$$4q^{-1}\left(\widetilde{[n]}_q - 1\right)^2 |a_n|^2$$
$$\leq q(q+q^{-1})^2(\widetilde{\Phi} - \widetilde{\Psi})^2 + \sum_{k=2}^{n-1}\left[\left(\widetilde{[k]}_q\mathcal{L}_q(\widetilde{\Phi}) - \mathcal{L}_q(\widetilde{\Psi})\right)^2 q\right.$$
$$\left. - 4q^{-1}\left(\widetilde{[k]}_q - 1\right)^2\right]|a_k|^2 \quad n \geq 2.$$

□

As $(a)\, q \to 1^-$, $(b)\, \widetilde{\Phi} = 1, \widetilde{\Psi} = -1$ and $(c)\, q \to 1^-, \widetilde{\Phi} = 1, \widetilde{\Psi} = -1$ in Theorem 2, respectively, we are led to the following results.

Corollary 1. *Let $f \in \mathcal{H}$ be of the form (1). If*

(a) $f \in \widetilde{\mathcal{ST}}(\widetilde{\Phi}, \widetilde{\Psi})$, *then for* $n \geq 2$,

$$(n-1)^2|a_n|^2 \leq \left(\widetilde{\Phi} - \widetilde{\Psi}\right)^2 + \sum_{k=2}^{n-1}\left[k(\widetilde{\Phi}+1) - (\widetilde{\Psi}+1)\right]\left[k(\widetilde{\Phi}-1) - (\widetilde{\Psi}-1)\right];$$

(b) $f \in \widetilde{\mathcal{ST}}_q$, *then*

$$q^{-1}(\widetilde{[n]} - 1)^2|a_2|^2 \leq q(q+q^{-1})\left[(q+q^{-1}) + \sum_{k=2}^{n-1}(2k + (q - q^{-1}))|a_k|^2\right], \quad n \geq 2;$$

(c) *([26], Theorem 4)* $f \in \mathcal{S}^*$, *then*

$$(n+1)^2|a_n|^2 \leq 4\left(1 + \sum_{k=2}^{n} k|a_k|^2\right), \quad n \geq 2.$$

Theorem 3. *Let $f \in \mathcal{H}$. then $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi}, \widetilde{\Psi})$ if*

$$\sum_{n=2}^{\infty}\left(2\left|\widetilde{[n]}_q - 1\right| + \left|\widetilde{\Phi} + 1 - \left(\widetilde{\Psi} - 1\right)\widetilde{[n]}_q\right|\right)|a_n| < \widetilde{\Phi} - \widetilde{\Psi}. \tag{10}$$

The inequality is sharp for the function

$$f(\nu) = \nu - \frac{\widetilde{\Phi} - \widetilde{\Psi}}{2\left|\widetilde{[n]}_q - 1\right| + \left|\widetilde{\Phi} + 1 - \left(\widetilde{\Psi} - 1\right)\widetilde{[n]}_q\right|}\nu^n.$$

Proof. Suppose (10) holds. We need to show that

$$\left|\frac{(\widetilde{\Psi}-1)\frac{\nu\widetilde{D}_q f(\nu)}{f(\nu)} - (\widetilde{\Phi}-1)}{(\widetilde{\Psi}+1)\frac{\nu\widetilde{D}_q f(\nu)}{f(\nu)} - (\widetilde{\Phi}+1)} - \frac{1}{1-\frac{q}{q^{-1}}}\right| \leq \frac{1}{1-\frac{q}{q^{-1}}}, \quad \nu \in \mho.$$

Now,

$$
\begin{aligned}
&\left|\frac{(\widetilde{\Psi}-1)\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)}-(\widetilde{\Phi}-1)}{(\widetilde{\Psi}+1)\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)}-(\widetilde{\Phi}+1)}-\frac{1}{1-\frac{q}{q^{-1}}}\right| \\
&\quad\leq\left|\frac{(\widetilde{\Psi}-1)\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)}-(\widetilde{\Phi}-1)}{(\widetilde{\Psi}+1)\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)}-(\widetilde{\Phi}+1)}-1\right|+\frac{q}{q^{-1}-q} \\
&\quad=\left|\frac{(\widetilde{\Psi}-1)\nu\widetilde{D_q}f(\nu)-(\widetilde{\Phi}-1)f(\nu)}{(\widetilde{\Psi}+1)\nu\widetilde{D_q}f(\nu)-(\widetilde{\Phi}+1)f(\nu)}\right|+\frac{q}{q^{-1}-q} \\
&\quad=2\left|\frac{f(\nu)-\nu\widetilde{D_q}f(\nu)}{(\widetilde{\Psi}+1)\nu\widetilde{D_q}f(\nu)-(\widetilde{\Phi}+1)f(\nu)}\right|+\frac{q}{q^{-1}-q} \\
&\quad=2\left|\frac{\sum_{n=2}^{\infty}\left(\widetilde{[n]}_q-1\right)a_n\nu^{n-1}}{\left(\widetilde{\Phi}-\widetilde{\Psi}\right)+\sum_{n=2}^{\infty}\left(\widetilde{\Phi}+1-\left(\widetilde{\Psi}+1\right)\widetilde{[n]}_q\right)a_n\nu^{n-1}}\right|+\frac{q}{q^{-1}-q} \\
&\quad\leq 2\frac{\sum_{n=2}^{\infty}\left|\widetilde{[n]}_q-1\right||a_n|}{\left(\widetilde{\Phi}-\widetilde{\Psi}\right)+\sum_{n=2}^{\infty}\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}+1\right)\widetilde{[n]}_q\right||a_n|}+\frac{q}{q^{-1}-q}.
\end{aligned}
$$

This last inequality is bounded by $\frac{1}{1-\frac{q}{q^{-1}}}$ provided (10) is satisfied. Thus, $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$. $\square$

Corollary 2. *Let $f \in \mathcal{H}$. then $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ if*

$$
|a_n| < \frac{\widetilde{\Phi}-\widetilde{\Psi}}{2\left|\widetilde{[n]}_q-1\right|+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)\widetilde{[n]}_q\right|}, \quad n \geq 2.
$$

Setting (a) $\widetilde{\Phi}=1$, $\widetilde{\Psi}=-1$ and (b) $q\to 1^-$, $\widetilde{\Phi}=1$, $\widetilde{\Psi}=-1$ in Theorem 3, respectively, we have the following results.

Corollary 3. *Let $f \in \mathcal{H}$.*

(a) *If*

$$
\sum_{n=2}^{\infty}\left(\left|\widetilde{[n]}_q-1\right|+1\right)|a_n| < 1,
$$

then $f \in \widetilde{\mathcal{ST}}_q$.

(b) *If*

$$
\sum_{n=2}^{\infty} n|a_n| < 1,
$$

then $f \in \mathcal{S}^*$.

Theorem 4. *Let $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi}, \widetilde{\Psi})$ be of the form (1). Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{q(q^{-1}+q)(\widetilde{\Phi}-\widetilde{\Psi})}{4(q^{-2}+q^2)} \rho_q\left(\widetilde{\Phi}, \widetilde{\Psi}\right), & \mu < \sigma_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right), \\ \frac{q(q^{-1}+q)(\widetilde{\Phi}-\widetilde{\Psi})}{4(q^{-2}+q^2)}, & \sigma_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right) \leq \mu \leq \sigma_q^2\left(\widetilde{\Phi}, \widetilde{\Psi}\right), \\ \frac{q(q^{-1}+q)(\widetilde{\Psi}-\widetilde{\Phi})}{4(q^{-2}+q^2)} \rho_q\left(\widetilde{\Phi}, \widetilde{\Psi}\right), & \mu > \sigma_q^2\left(\widetilde{\Phi}, \widetilde{\Psi}\right), \end{cases}$$

where

$$\begin{aligned} \rho_q\left(\widetilde{\Phi}, \widetilde{\Psi}\right) &= \frac{1}{(q^{-1}+q-1)^2}\Big\{q(q^{-1}+q)\left(\widetilde{\Phi}-\widetilde{\Psi}\right) + \big[2(q^{-1}+q-1) \\ &\quad - \big(3+\widetilde{\Psi}+q^2(\widetilde{\Psi}-1)\big)\big](q^{-1}+q-1) - \mu q(q^{-1}+q)(q^{-2}+q^2)(\widetilde{\Phi}-\widetilde{\Psi})\Big\}, \\ \sigma_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right) &= \frac{q(q^{-1}+q)(\widetilde{\Phi}-\widetilde{\Psi}) - \big(3+\widetilde{\Psi}+q^2(\widetilde{\Psi}-1)\big)(q^{-1}+q-1)}{q(q^{-1}+q)(q^{-2}+q^2)(\widetilde{\Phi}-\widetilde{\Psi})}, \\ \sigma_q^2\left(\widetilde{\Phi}, \widetilde{\Psi}\right) &= \frac{q(q^{-1}+q)(\widetilde{\Phi}-\widetilde{\Psi}) + \Big[4(q^{-1}+q-1) - \big(3+\widetilde{\Psi}+q^2(\widetilde{\Psi}-1)\big)\Big](q^{-1}+q-1)}{q(q^{-1}+q)(q^{-2}+q^2)(\widetilde{\Phi}-\widetilde{\Psi})}. \end{aligned}$$

It is also asserted that

$$|a_3 - \mu a_2^2| + \left(\mu - \sigma_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right)\right)|a_2|^2 \leq \frac{q(q^{-1}+q)(\widetilde{\Phi}-\widetilde{\Psi})}{2(q^{-2}+q^2)}, \quad \sigma_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right) < \mu \leq \sigma_q^3\left(\widetilde{\Phi}, \widetilde{\Psi}\right)$$

and

$$|a_3 - \mu a_2^2| - \left(\mu - \sigma_q^2\left(\widetilde{\Phi}, \widetilde{\Psi}\right)\right)|a_2|^2 \leq \frac{q(q^{-1}+q)(\widetilde{\Phi}-\widetilde{\Psi})}{2(q^{-2}+q^2)}, \quad \sigma_q^3\left(\widetilde{\Phi}, \widetilde{\Psi}\right) < \mu \leq \sigma_q^2\left(\widetilde{\Phi}, \widetilde{\Psi}\right),$$

where

$$\sigma_q^3\left(\widetilde{\Phi}, \widetilde{\Psi}\right) = \frac{2(q^{-1}+q-1)^2}{q(q^{-1}+q)(q^{-2}+q^2)(\widetilde{\Phi}-\widetilde{\Psi})} - \sigma_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right).$$

Each of these inequalities is sharp.

Proof. By the definition of $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi}, \widetilde{\Psi})$, we have that

$$\frac{\nu \widetilde{D}_q f(\nu)}{f(\nu)} = \varphi(w(\nu)), \quad \nu \in \mho,$$

where $w(\nu)$ is a Schwarz function. Using the relationship between $w(\nu)$ and $h \in \mathcal{P}$, we have

$$h(\nu) = \frac{1+w(\nu)}{1-w(\nu)} = 1 + c_1\nu + c_2\nu^2 + c_3\nu^3 + \dots.$$

Therefore,

$$\begin{aligned} \varphi(w(\nu)) &= \frac{\left[\left(3+\widetilde{\Phi}\right)q^{-1} + \left(\widetilde{\Phi}-1\right)q\right]h(\nu) - (q^{-1}+q)\left(\widetilde{\Phi}-1\right)}{\left[\left(3+\widetilde{\Psi}\right)q^{-1} + \left(\widetilde{\Psi}-1\right)q\right]h(\nu) - (q^{-1}+q)\left(\widetilde{\Psi}-1\right)} \\ &= 1 + \frac{q(q^{-1}+q)\left(\widetilde{\Phi}-\widetilde{\Psi}\right)c_1}{4}\nu \\ &\quad + \frac{q\left[4c_2 - \left(3+\widetilde{\Psi}+q^2\left(\widetilde{\Psi}-1\right)\right)c_1^2\right](q^{-1}+q)\left(\widetilde{\Phi}-\widetilde{\Psi}\right)}{16}\nu^2 + \dots. \end{aligned} \tag{11}$$

Similarly,

$$\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)} = 1 + (q^{-1} + q - 1)a_2\nu + \left[(q^{-2} + q^2)a_3 - (q^{-1} + q - 1)a_2^2\right]\nu^2 + \dots. \tag{12}$$

On comparing (11) and (12), we arrive at

$$a_2 = \frac{q(q^{-1} + q)\left(\widetilde{\Phi} - \widetilde{\Psi}\right)c_1}{4(q^{-1} + q - 1)}$$

and

$$a_3 = \frac{q(q^{-1} + q)\left(\widetilde{\Phi} - \widetilde{\Psi}\right)c_1}{4(q^{-2} + q^2)}\left(c_2 - \psi_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right)c_1^2\right),$$

where

$$\psi_q^1\left(\widetilde{\Phi}, \widetilde{\Psi}\right) = \frac{\left(3 + \widetilde{\Psi} + q^2\left(\widetilde{\Psi} - 1\right)\right)(q^{-1} + q - 1) - q(q^{-1} + q)\left(\widetilde{\Phi}, \widetilde{\Psi}\right)}{4(q^{-1} + q - 1)}.$$

Consequently, for $\mu \in \mathbb{R}$,

$$|a_3 - \mu a_2^2| = \frac{q(q^{-1} + q)\left(\widetilde{\Phi} - \widetilde{\Psi}\right)}{4(q^{-1} + q - 1)}\,|c_2 - \psi_q^2\left(\widetilde{\Phi}, \widetilde{\Psi}\right)c_1^2|.$$

Thus, by applying Lemma 1, we obtain the required result. □

In particular, when $\widetilde{\Phi} = 1$, $\widetilde{\Psi} = -1$, Theorem 4 produces the following result.

Corollary 4. *Let $f \in \widetilde{\mathcal{ST}}_q$ be of the form* (1)*. Then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{q(q^{-1}+q)}{(q^{-2}+q^2)}\rho_q, & \mu < \sigma_q^1, \\ \frac{q(q^{-1}+q)}{2(q^{-2}+q^2)}, & \sigma_q^1 \le \mu \le \sigma_q^2, \\ -\frac{q(q^{-1}+q)}{(q^{-2}+q^2)}\rho_q, & \mu > \sigma_q^2, \end{cases}$$

where

$$\rho_q = \frac{q(q^{-1} + q) + \left[q^{-1} + q - 1 - q(q^{-1} - q)\right](q^{-1} + q - 1) - \mu q(q^{-1} + q)(q^{-2} + q^2)}{(q^{-1} + q - 1)^2},$$

$$\sigma_q^1 = \frac{q(q^{-1} + q) - q(q^{-1} - q)(q^{-1} + q - 1)}{q(q^{-1} + q)(q^{-2} + q^2)},$$

$$\sigma_q^2 = \frac{q(q^{-1} + q) - \left[2(q^{-1} + q - 1) - q(q^{-1} - q)\right](q^{-1} + q - 1)}{q(q^{-1} + q)(q^{-2} + q^2)}.$$

It is also asserted that

$$|a_3 - \mu a_2^2| + \left(\mu - \sigma_q^1\right)|a_2|^2 \le \frac{q(q^{-1} + q)}{(q^{-2} + q^2)}, \quad \sigma_q^1 < \mu \le \sigma_q^3$$

and

$$|a_3 - \mu a_2^2| - \left(\mu - \sigma_q^2\right)|a_2|^2 \le \frac{q(q^{-1} + q)}{(q^{-2} + q^2)}, \quad \sigma_q^3 < \mu \le \sigma_q^2,$$

where

$$\sigma_q^3 = \frac{(q^{-1}+q-1)^2}{q(q^{-1}+q)(q^{-2}+q^2)} - \sigma_q^1.$$

Each of these inequalities is sharp.

Remark 2. *Corollary 4 reduces to the result of Hayami and Owa ([27], Corollary 3) for the class $\mathcal{S}^*$ as $q \to 1^-$.*

Theorem 5. *Let f defined by (1) be in $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$. In addition, let $g \in \mathcal{C}$. If*

$$\Re(f(\nu)) > -\frac{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}, \tag{13}$$

then

$$\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}(f*g)(\nu) \prec g(\nu), \quad \nu \in \mho. \tag{14}$$

The following constant factor in the subordination (14):

$$\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]} \tag{15}$$

cannot be replaced by a larger one.

Proof. Let $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$. In addition, let $g \in \mathcal{C}$ and assume $g(\nu) = \nu + \sum_{n=2}^{\infty} b_n \nu^n \in \mathcal{C}$. Then we readily have

$$\begin{aligned}&\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}(f*g)(\nu)\\ &= \frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}\left(\nu+\sum_{n=2}^{\infty} a_n b_n \nu^n\right).\end{aligned}$$

Therefore, by Definition 5, (14) will hold if

$$\left\{\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence (with $a_1 = 1$). Appealing to Lemma 2, we arrive at

$$\Re\left(1+\sum_{n=1}^{\infty}\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}a_n\nu^n\right) > 0, \quad \nu \in \mho.$$

Now, since

$$2\left|\widetilde{[n]}_q-1\right|+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)\widetilde{[n]}_q\right|$$

is an increasing function for $n \geq 2$, we have

$$\begin{aligned}
&\Re\left(1+\sum_{n=1}^{\infty}\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}a_n\nu^n\right) \\
&= \Re\left(1+\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}\nu\right. \\
&\quad \left.+\sum_{n=1}^{\infty}\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}a_n\nu^n\right) \\
&\geq 1+\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}r \\
&\quad +\sum_{n=1}^{\infty}\frac{2\left|\widetilde{[n]_q}-1\right|+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)\widetilde{[n]_q}\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}|a_n|r^n \\
&> 1+\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}r \\
&\quad +\frac{\widetilde{\Phi}-\widetilde{\Psi}}{\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}r \\
&> 0,
\end{aligned}$$

where we have used (10). This proves the result. Next, for sharpness, we consider

$$g(\nu)=\frac{\nu}{1-\nu} \quad \text{and} \quad f(\nu)=\nu-\frac{\widetilde{\Phi}-\widetilde{\Psi}}{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}\nu^2.$$

Then by (14), we have

$$\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}f(\nu)\prec\frac{\nu}{1-\nu}, \quad \nu\in\mho.$$

Therefore,

$$\begin{aligned}
&\Re\left(\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}f(\nu)\right) \\
&= \frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}\Re(f(\nu)) \\
&> -\frac{1}{2},
\end{aligned}$$

where we have used (13). Thus, we have

$$\min\left\{\frac{2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|}{2\left[\widetilde{\Phi}-\widetilde{\Psi}+2(q^{-1}+q-1)+\left|\widetilde{\Phi}+1-\left(\widetilde{\Psi}-1\right)(q^{-1}+q)\right|\right]}f(\nu)\right\}=-\frac{1}{2},$$

which establishes that the constant (15) is the best possible. □

In its particular case, when $\widetilde{\Phi} = 1, \widetilde{\Psi} = -1$, Theorem 5 produce the following Corollary.

Corollary 5. *Let f defined by (1) be in $\widetilde{\mathcal{ST}}_q$. In addition, let $g \in \mathcal{C}$. If*

$$\Re(f(\nu)) > -\frac{2(q^{-1}+q)+1}{2(q^{-1}+q)}, \tag{16}$$

then

$$\frac{(q^{-1}+q)}{2(q^{-1}+q)+1}(f*g)(\nu) \prec g(\nu), \quad \nu \in \mho. \tag{17}$$

The following constant factor in the subordination (17):

$$\frac{(q^{-1}+q)}{2(q^{-1}+q)+1}$$

cannot be replaced by a larger one.

Theorem 6. *Let $f \in \mathcal{H}$. Then $f \in \widetilde{\mathcal{ST}}_q(\widetilde{\Phi}, \widetilde{\Psi})$ if and only if*

$$\frac{1}{\nu}\left(f(\nu) * \frac{\nu - \mathcal{N}_q(\widetilde{\Phi}, \widetilde{\Psi}; \theta)\nu^2 + \mathcal{M}_q(\widetilde{\Phi}, \widetilde{\Psi}; \theta)\nu^3}{(1-q\nu)(1-q^{-1}\nu)(1-\nu)}\right) \neq 0, \quad \nu \in \mho,$$

where

$$\mathcal{N}_q(\widetilde{\Phi}, \widetilde{\Psi}; \theta) = \frac{\left[(q+q^{-1})(1+\widetilde{\Phi}) - (1-\widetilde{\Psi})\right]\widetilde{p(e^{i\theta})} + (q+q^{-1})(1-\widetilde{\Phi}) - (1-\widetilde{\Psi})}{(\widetilde{\Phi}-\widetilde{\Psi})(\widetilde{p(e^{i\theta})}-1)} \tag{18}$$

and

$$\mathcal{M}_q(\widetilde{\Phi}, \widetilde{\Psi}; \theta) = \frac{(1+\widetilde{\Phi})\widetilde{p(e^{i\theta})} + (1-\widetilde{\Phi})}{(\widetilde{\Phi}-\widetilde{\Psi})(\widetilde{p(e^{i\theta})}-1)} \tag{19}$$

with

$$\widetilde{p(\nu)} = \frac{1+\nu}{1-\frac{q}{q^{-1}}\nu}, \quad \text{for} \quad \nu = e^{i\theta}.$$

Proof. Let $f \in \mathcal{ST}(\widetilde{\Phi}, \widetilde{\Psi})$. Then $f(\nu)$ is analytic in $\mho$. Therefore, $f(\nu)/\nu \neq 0$ in $\mho$. Thus, there exists $w(\nu)$ analytic in $\mho$ with $w(0) = 0$ and $|w(\nu)|$ in $\mho$ such that

$$\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)} = \frac{\left(1+\widetilde{\Phi}\right)\widetilde{p(w(\nu))} + \left(1-\widetilde{\Phi}\right)}{\left(1+\widetilde{\Psi}\right)\widetilde{p(w(\nu))} + \left(1-\widetilde{\Psi}\right)},$$

which is equivalent to

$$\frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)} \neq \frac{\left(1+\widetilde{\Phi}\right)\widetilde{p(e^{i\theta})} + \left(1-\widetilde{\Phi}\right)}{\left(1+\widetilde{\Psi}\right)\widetilde{p(e^{i\theta})} + \left(1-\widetilde{\Psi}\right)}. \tag{20}$$

That is,

$$0 \neq \frac{1}{\nu}\left[\nu\widetilde{D_q}f(\nu)\left((1+\widetilde{\Psi})\widetilde{p(e^{i\theta})}+(1-\widetilde{\Psi})\right)-f(\nu)\left((1+\widetilde{\Phi})\widetilde{p(e^{i\theta})}+(1-\widetilde{\Phi})\right)\right]$$
$$=\frac{1}{\nu}\left[\left(f(\nu)*\frac{\nu}{(1-q\nu)(1-q^{-1}\nu)}\right)\left((1+\widetilde{\Psi})\widetilde{p(e^{i\theta})}+(1-\widetilde{\Psi})\right)\right.$$
$$\left.-\left(f(\nu)*\frac{\nu}{1-\nu}\right)\left((1+\widetilde{\Phi})\widetilde{p(e^{i\theta})}+(1-\widetilde{\Phi})\right)\right]$$
$$=\frac{1}{\nu}\left[f(\nu)*\left(\frac{\nu\left((1+\widetilde{\Psi})\widetilde{p(e^{i\theta})}+(1-\widetilde{\Psi})\right)}{(1-q\nu)(1-q^{-1}\nu)}-\frac{\nu\left((1+\widetilde{\Phi})\widetilde{p(e^{i\theta})}+(1-\widetilde{\Phi})\right)}{1-\nu}\right)\right]$$
$$=\frac{\left(\widetilde{\Phi}-\widetilde{\Psi}\right)(1-\widetilde{p(e^{i\theta})})}{\nu}\left(f(\nu)*\frac{\nu-\mathcal{N}_q(\widetilde{\Phi},\widetilde{\Psi};\theta)\nu^2+\mathcal{M}_q(\widetilde{\Phi},\widetilde{\Psi};\theta)\nu^3}{(1-q\nu)(1-q^{-1}\nu)(1-\nu)}\right),$$

where $\mathcal{N}_q(\widetilde{\Phi},\widetilde{\Psi};\theta)$ and $\mathcal{M}_q(\widetilde{\Phi},\widetilde{\Psi};\theta)$ are given by (18) and (19).

Conversely, since $f \in \mathcal{H}$, then $f(\nu) \neq 0$ in $\mho$. Therefore, the function $\gamma(\nu) = \frac{\nu\widetilde{D_q}f(\nu)}{f(\nu)}$ is analytic in $\mho$ with $\gamma(0) = 1$. In the first part of the proof, we observe that (20) and

$$\frac{1}{\nu}\left(f(\nu)*\frac{\nu-\mathcal{N}_q(\widetilde{\Phi},\widetilde{\Psi};\theta)\nu^2+\mathcal{M}_q(\widetilde{\Phi},\widetilde{\Psi};\theta)\nu^3}{(1-q\nu)(1-q^{-1}\nu)(1-\nu)}\right)\neq 0, \quad \nu\in\mho,$$

are equivalent. Let

$$\lambda(\nu)=\frac{\left(1+\widetilde{\Phi}\right)\widetilde{p(e^{i\theta})}+\left(1-\widetilde{\Phi}\right)}{\left(1+\widetilde{\Psi}\right)\widetilde{p(e^{i\theta})}+\left(1-\widetilde{\Psi}\right)}, \quad \nu\in\mho.$$

Then

$$\gamma(\mho)\cap\lambda(\partial\mho)=\varnothing.$$

Thus, the connected part of $(\mathbb{C}-\{\lambda(\partial\mho)\})$ contains the simply connected domain $\gamma(\mho)$; therefore, the univalence of the function $\lambda(\nu)$ in $\mho$ and the fact that $\gamma(0)=\lambda(0)=1$ affirm that $\gamma(\nu)\prec\lambda(\nu)$ in $\mho$. Hence, $f\in\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$. □

3. Conclusions

In this findings, we introduced the class $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ of analytic functions by using the notion of q-symmetric derivative, and obtained coefficient related results. Furthermore, some convolution characterization associated with $\widetilde{\mathcal{ST}}_q(\widetilde{\Phi},\widetilde{\Psi})$ were presented. The consequences of our investigation include known and new results.

It is interesting to note that this presented work could be investigated under the context of multivalent functions and some geometric characterizations such as the Fekete–Szegö inequality, Hankel determinant, growth and distortion problems could be explored. In addition, using the theory of differential subordination, Sandwich-type results could be examined for this present class of functions. For more details about the suggested work, one may go through [28,29]. Overall, the results presented here could represent a starting point for full investigations into the study of Janowski functions in the framework of q-symmetric calculus.

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