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On Some Dynamic $(\Delta\Delta)^{\nabla}-$Gronwall–Bellman–Pachpatte-Type Inequalities on Time Scales and Its Applications

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Abstract: In the present paper, some new generalizations of dynamic inequalities of Gronwall–Bellman–Pachpatte-type on time scales are established. Some integral and discrete Gronwall–Bellman–Pachpatte-type inequalities that are given as special cases of main results are original. The main results are proved by using the dynamic Leibniz integral rule on time scales. To highlight our research advantages, several implementations of these findings are presented. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Gronwall's inequality; dynamic inequality; time scales; Leibniz integral rule on time scales

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1. Introduction

Time scales calculus with the objective to unify discrete and continuous analysis was introduced by S. Hilger [1]. We assume that the reader has a good background on time scales calculus. For additional subtleties on time scales, we allude the peruser to the books by Bohner and Peterson [2,3].

Theorem 1 ([4], Leibniz Integral Rule on Time Scales)**.** *In the following by $\phi^{\Delta}(\varrho,\varsigma)$ we mean the delta derivative of $\phi(\varrho,\varsigma)$ with respect to ϱ. Similarly, $\phi^{\nabla}(\varrho,\varsigma)$ is understood. If ϕ , ϕ^{Δ} and ϕ^{∇} are continuous, and $u,h:\mathbb{T}\to\mathbb{T}$ are delta differentiable functions, then the following formulas holds $\forall\varrho\in\mathbb{T}^{\kappa}$.*

(i) $$\left[\int_{u(\varrho)}^{h(\varrho)}\phi(\varrho,\varsigma)\Delta\varsigma\right]^{\Delta}=\int_{u(\varrho)}^{h(\varrho)}\phi^{\Delta}(\varrho,\varsigma)\Delta\varsigma+h^{\Delta}(\varrho)\phi(\sigma(\varrho),h(\varrho))-u^{\Delta}(\varrho)\phi(\sigma(\varrho),u(\varrho));$$

(ii) $$\left[\int_{u(\varrho)}^{h(\varrho)}\phi(\varrho,\varsigma)\Delta\varsigma\right]^{\nabla}=\int_{u(\varrho)}^{h(\varrho)}\phi^{\nabla}(\varrho,\varsigma)\Delta\varsigma+h^{\nabla}(\varrho)\phi(\rho(\varrho),h(\varrho))-u^{\nabla}(\varrho)\phi(\rho(\varrho),u(\varrho));$$

(iii) $$\left[\int_{u(\varrho)}^{h(\varrho)}\phi(\varrho,\varsigma)\nabla\varsigma\right]^{\Delta}=\int_{u(\varrho)}^{h(\varrho)}\phi^{\Delta}(\varrho,\varsigma)\nabla\varsigma+h^{\Delta}(\varrho)\phi(\sigma(\varrho),h(\varrho))-u^{\Delta}(\varrho)\phi(\sigma(\varrho),u(\varrho));$$

(iv) $$\left[\int_{u(\varrho)}^{h(\varrho)}\phi(\varrho,\varsigma)\nabla\varsigma\right]^{\nabla}=\int_{u(\varrho)}^{h(\varrho)}\phi^{\nabla}(\varrho,\varsigma)\nabla\varsigma+h^{\nabla}(\varrho)\phi(\rho(\varrho),h(\varrho))-u^{\nabla}(\varrho)\phi(\rho(\varrho),u(\varrho)).$$

Recently, Gronwall–Bellman-type inequalities, that have several applications in qualitative and quantitative behavior, have been developed by many mathematicians, and several refinements and extensions have been made to the previous results, such as boundedness, stability, existence, uniqueness and oscillation behavior, we refer the reader to the works [5–19].

Gronwall–Bellman's inequality [13] in the integral form stated: Let v and f be continuous and non-negative functions defined on $[a,b]$, and let v_0 be non-negative constant. Then the inequality

$$v(t) \leq v_0 + \int_a^t f(s)v(s)ds, \quad \text{for all} \quad t \in [a,b], \tag{1}$$

implies that

$$v(t) \leq v_0 \exp\left(\int_a^t f(s)ds\right), \quad \text{for all} \quad t \in [a,b].$$

Baburao G. Pachpatte [20] proved the discrete version of (1). In particular, he proved that: if $v(n)$, $a(n)$, $\gamma(n)$ are non-negative sequences defined for $n \in \mathbb{N}_0$ and $a(n)$ is non-decreasing for $n \in \mathbb{N}_0$, and if

$$v(n) \leq a(n) + \sum_{s=0}^{n-1} \gamma(n)v(n), n \in \mathbb{N}_0, \tag{2}$$

then

$$v(n) \leq a(n) \prod_{s=0}^{n-1} [1 + \gamma(n)], n \in \mathbb{N}_0.$$

Bohner and Peterson [2] unify the integral form (2) and the discrete form (1) by introducing a dynamic inequality on a time scale $\mathbb{T}$ stated: If v, ζ are right dense continuous functions and $\gamma \geq 0$ is regressive and right dense continuous functions, then

$$v(t) \leq \zeta(t) + \int_{t_0}^t v(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all} \quad t \in \mathbb{T},$$

implies

$$v(t) \leq \zeta(t) + \int_{t_0}^t e_\gamma(t, \sigma(\eta))\zeta(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all} \quad t \in \mathbb{T},$$

The authors [21] studied the following result:

$$\begin{aligned} \Xi(v(\ell,t)) \quad \leq \quad & a(\ell,t) + \int_0^{\theta(\ell)} \int_0^{\vartheta(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(v(\varsigma,\eta))\varpi(v(\varsigma,\eta)) \\ & + \int_0^{\varsigma} \Im_2(\chi,\eta)\zeta(v(\chi,\eta))\varpi(v(\chi,\eta))d\chi\Big] d\eta d\varsigma, \end{aligned}$$

where v, f, $\Im \in C(I_1 \times I_2, \mathbb{R}_+)$, $a \in C(\zeta, \mathbb{R}_+)$ are nondecreasing functions, $I_1, I_2 \in \mathbb{R}$, $\theta \in C^1(I_1, I_1), \vartheta \in C^1(I_2, I_2)$ are nondecreasing with $\theta(\ell) \leq \ell$ on I_1, $\vartheta(t) \leq t$ on I_2, $\Im_1$, $\Im_2 \in C(\zeta, \mathbb{R}_+)$, and Ξ, ζ, $\varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\{\Xi, \zeta, \varpi\}(v) > 0$ for $v > 0$, and $\lim\limits_{v \to +\infty} \Xi(v) = +\infty$.

Additionally, Anderson [22] studied the following result.

$$\omega(v(t,s)) \leq a(t,s) + c(t,s) \int_{t_0}^t \int_s^\infty \omega'(v(\tau,\eta))[d(\tau,\eta)w(v(\tau,\eta)) + b(\tau,\eta)]\nabla\eta\Delta\tau, \tag{3}$$

where v, a, c, d are non-negative continuous functions defined for $(t,s) \in \mathbb{T} \times \mathbb{T}$, and b is a non-negative continuous function for $(t,s) \in [t_0, \infty)_\mathbb{T} \times [t_0, \infty)_\mathbb{T}$ and $\omega \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\omega' > 0$ for $v > 0$.

We will start with the following basic lemma:

Lemma 1. *Suppose $\mathbb{T}_1$, $\mathbb{T}_2$ are two times scales and $a \in C(\Omega = \mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R}_+)$ is nondecreasing with respect to $(\ell, t) \in \Omega$. Assume that $\Im$, v, $f \in C_{rd}(\Omega, \mathbb{R}_+)$, $\lambda_1 \in C^1_{rd}(\mathbb{T}_1, \mathbb{T}_1)$ and $\lambda_2 \in C^1_{rd}(\mathbb{T}_2, \mathbb{T}_2)$ be nondecreasing functions with $\lambda_1(\ell) \leq \ell$ on $\mathbb{T}_1$, $\lambda_2(t) \leq t$ on $\mathbb{T}_2$. Furthermore,* *suppose* Ξ, $\zeta \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are nondecreasing functions with* $\{\Xi, \Omega\}(v) > 0$ *for* $v > 0$, *and* $\lim_{v \to +\infty} \Xi(v) = +\infty$. *If* $v(\ell, t)$ *satisfies*

$$\Xi(v(\ell,t)) \leq a(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \zeta(v(\varsigma,\eta)) \Delta\eta \Delta\varsigma \tag{4}$$

for $(\ell, t) \in \Omega$, *then*

$$v(\ell,t) \leq \Xi^{-1}\left\{G^{-1}G(a(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma\right] \tag{5}$$

for $0 \leq \ell \leq \ell_1, 0 \leq t \leq t_1$, *where*

$$G(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\zeta(\Xi^{-1}(\varsigma))}, v \geq v_0 > 0,\ G(+\infty) = \int_{v_0}^{+\infty} \frac{\nabla\varsigma}{\zeta(\Xi^{-1}(\varsigma))} = +\infty \tag{6}$$

and $(\ell_1, t_1) \in \Omega$ *is chosen so that*

$$\left(G(a(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma\right) \in \mathrm{Dom}\left(G^{-1}\right).$$

Proof. First we assume that $a(\ell, t) > 0$. Fixing an arbitrary $(\ell_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\ell, t)$ by

$$\psi(\ell,t) = a(\ell_0,t_0) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \zeta(v(\varsigma,\eta)) \Delta\eta \Delta\varsigma \tag{7}$$

for $0 \leq \ell \leq \ell_0 \leq \ell_1, 0 \leq t \leq t_0 \leq t_1$, then $\psi(\ell_0, t) = \psi(\ell, t_0) = a(\ell_0, t_0)$ and

$$v(\ell,t) \leq \Xi^{-1}(\psi(\ell,t)) \tag{8}$$

Taking ∇-derivative for (7) with employing Theorem 1(ii), we have

$$\begin{aligned}
\psi^{\nabla_\ell}(\ell,t) &= \lambda_1^{\nabla}(\ell) \int_{t_0}^{\lambda_2(t)} \Im(\lambda_1(\ell),\eta) f(\lambda_1(\ell),\eta) \zeta(v(\lambda_1(\ell),\eta)) \Delta\eta \\
&\leq \lambda_1^{\nabla}(\ell) \int_{t_0}^{\lambda_2(t)} \Im(\lambda_1(\ell),\eta) f(\lambda_1(\ell),\eta) \zeta\left(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\right) \Delta\eta \\
&\leq \zeta\left(\Xi^{-1}(\psi(\lambda_1(\ell),\lambda_2(t)))\right) \lambda_1^{\nabla}(\ell) \int_{t_0}^{\lambda_2(t)} \Im(\lambda_1(\ell),\eta) f(\lambda_1(\ell),\eta) \Delta\eta
\end{aligned} \tag{9}$$

Inequality (9) can be written in the form

$$\frac{\psi^{\nabla_\ell}(\ell,t)}{\zeta(\Xi^{-1}(\psi(\ell,t)))} \leq \lambda_1^{\nabla}(\ell) \int_{t_0}^{\lambda_2(t)} \Im(\lambda_1(\ell),\eta) f(\lambda_1(\ell),\eta) \Delta\eta. \tag{10}$$

Taking ∇-integral for Inequality (10), obtains

$$\begin{aligned}
G(\psi(\ell,t)) &\leq G(\psi(\ell_0,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma \\
&\leq G(a(\ell_0,t_0)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma.
\end{aligned}$$

Since $(\ell_0, t_0) \in \Omega$ is chosen arbitrary,

$$\psi(\ell,t) \leq G^{-1}\left[G(a(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma\right]. \tag{11}$$

From (11) and (8) we obtain the desired result (5). We carry out the above procedure with $\epsilon > 0$ instead of $a(\ell,t)$ when $a(\ell,t) = 0$ and subsequently let $\epsilon \to 0$. □

Remark 1. *If we take $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$ and $t_0 = 0$ in Lemma 1, then, inequality (4) becomes the inequality obtained in [21] (Lemma 2.1).*

In this article, by employing the results of Theorems 1, we prove the delayed time scale versions of the inequalities proved in [21]. Further, these results are proved here extend some known results in [23–25]. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

2. Main Results

Theorem 2. *Let v, a, f, λ_1 and λ_2 be as in Lemma 1. Let $\Im_1, \Im_2 \in C_{rd}(\Omega, \mathbb{R}_+)$. If $v(\ell,t)$ satisfies*

$$\begin{aligned}\Xi(v(\ell,t)) \quad \leq \quad & a(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(v(\varsigma,\eta)) \\ & + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(v(\chi,\eta))\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{12}$$

for $(\ell,t) \in \Omega$, then

$$v(\ell,t) \leq \Xi^{-1}\left\{G^{-1}\left(p(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right)\right\} \tag{13}$$

for $0 \leq \ell \leq \ell_1, 0 \leq t \leq t_1$, where G is defined by (6) and

$$p(\ell,t) = G(a(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma \tag{14}$$

and $(\ell_1,t_1) \in \Omega$ is chosen so that

$$\left(p(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right) \in \mathrm{Dom}\left(G^{-1}\right).$$

Proof. By the same steps of the proof of Lemma 1 we can obtain (13), with suitable changes. □

Remark 2. *If we take $\Im_2(\ell,t) = 0$, then Theorem 2 reduces to Lemma 1.*

Corollary 1. *Let the functions v, f, $\Im_1$, $\Im_2$, a, λ_1 and λ_2 be as in Theorem 2. Further suppose that $q > p > 0$ are constants. If $v(\ell,t)$ satisfies*

$$\begin{aligned}v^q(\ell,t) \quad \leq \quad & a(\ell,t) + \frac{q}{q-p}\int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)v^p(\varsigma,\eta) \\ & + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)v^p(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{15}$$

for $(\ell,t) \in \Omega$, then

$$v(\ell,t) \leq \left\{p(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right\}^{\frac{1}{q-p}} \tag{16}$$

where

$$p(\ell,t) = (a(\ell,t))^{\frac{q-p}{q}} + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma.$$

Proof. In Theorem 2, by letting $\Xi(v) = v^q, \zeta(v) = v^p$ we have

$$G(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\zeta(\Xi^{-1}(\varsigma))} = \int_{v_0}^{v} \frac{\nabla\varsigma}{\varsigma^{\frac{p}{q}}} \geq \frac{q}{q-p}\left(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}}\right), v \geq v_0 > 0$$

and

$$G^{-1}(v) \geq \left\{v_0^{\frac{q-p}{q}} + \frac{q-p}{q}v\right\}^{\frac{1}{q-p}}$$

we obtain the inequality (16). □

Theorem 3. *Under the hypotheses of Theorem 2. Suppose Ξ, $\zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\{\Xi, \Omega, \varpi\}(v) > 0$ for $v > 0$ and $v(\ell, t)$ satisfies*

$$\begin{aligned} \Xi(v(\ell,t)) \quad &\leq \quad a(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(v(\varsigma,\eta))\varpi(v(\varsigma,\eta)) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(v(\chi,\eta))\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{17}$$

for $(\ell, t) \in \Omega$, then

$$v(\ell,t) \leq \Xi^{-1}\left\{G^{-1}\left(F^{-1}\left[F(p(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right)\right\} \tag{18}$$

for $0 \leq \ell \leq \ell_1, 0 \leq t \leq t_1$, where G and p are as in (A_1) and

$$F(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{\varpi(\Xi^{-1}(G^{-1}(\varsigma)))}, v \geq v_0 > 0, \qquad F(+\infty) = +\infty \tag{19}$$

and $(\ell_1, t_1) \in \Omega$ is chosen so that

$$\left[F(p(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\ell, t) > 0$. Fixing an arbitrary $(\ell_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\ell, t)$ by

$$\begin{aligned} \psi(\ell,t) \quad &= \quad a(\ell_0,t_0) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(v(\varsigma,\eta))\varpi(v(\varsigma,\eta)) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(v(\chi,\eta))\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{20}$$

for $0 \leq \ell \leq \ell_0 \leq \ell_1, 0 \leq t \leq t_0 \leq t_1$, then $\psi(\ell_0, t) = \psi(\ell, t_0) = a(\ell_0, t_0)$ and

$$v(\ell,t) \leq \Xi^{-1}(\psi(\ell,t)) \tag{21}$$

Taking ∇-derivative for (20) with employing Theorem 1 (*i*), gives

$$\begin{aligned}\psi^{\nabla_\ell}(\ell,t) &= \lambda_1^{\nabla}(\ell)\int_{t_0}^{\lambda_2(t)}\Im_1(\lambda_1(\ell),\eta)[f(\lambda_1(\ell),\eta)\zeta(v(\lambda_1(\ell),\eta))\varpi(v(\lambda_1(\ell),\eta))\\ &\qquad +\int_{\ell_0}^{\lambda_1(\ell)}\Im_2(\chi,\eta)\zeta(v(\chi,\eta))\Delta\chi\Big]\Delta\eta\\ &\le \lambda_1^{\nabla}(\ell)\int_{t_0}^{\lambda_2(t)}\Im_1(\lambda_1(\ell),\eta)\Big[f(\lambda_1(\ell),\eta)\zeta\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big)\varpi\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big)\\ &\quad +\int_{\ell_0}^{\lambda_1(\ell)}\Im_2(\chi,\eta)\zeta\Big(\Xi^{-1}(\psi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta \\ &\le \lambda_1^{\nabla}(\ell).\zeta\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\lambda_2(t)))\Big)\times\\ &\quad \int_{t_0}^{\lambda_2(t)}\Im_1(\lambda_1(\ell),\eta)\Big[f(\lambda_1(\ell),\eta)\varpi\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big)+\int_{\ell_0}^{\lambda_1(\ell)}\Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\end{aligned}\tag{22}$$

From (22), we have

$$\begin{aligned}\frac{\psi^{\nabla_\ell}(\ell,t)}{\zeta(\Xi^{-1}(\psi(\ell,t)))} &\le \lambda_1^{\nabla}(\ell)\int_{t_0}^{\lambda_2(t)}\Im_1(\lambda_1(\ell),\eta)\Big[f(\lambda_1(\ell),\eta)\varpi\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big)\\ &\qquad +\int_{\ell_0}^{\lambda_1(\ell)}\Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta.\end{aligned}\tag{23}$$

Taking ∇-integral for (23), gives

$$\begin{aligned}G(\psi(\ell,t)) &\le G(\psi(\ell_0,t))+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\\ &\qquad +\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma\\ &\le G(a(\ell_0,t_0))+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\\ &\qquad +\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma.\end{aligned}$$

Since $(\ell_0,t_0)\in\Omega$ is chosen arbitrarily, the last inequality can be rewritten as

$$G(\psi(\ell,t))\le p(\ell,t)+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\Delta\eta\nabla\varsigma.\tag{24}$$

Since $p(\ell,t)$ is a nondecreasing function, an application of Lemma 1 to (24) gives us

$$\psi(\ell,t)\le G^{-1}\Big(F^{-1}\Big[F(p(\ell,t))+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\Big]\Big).\tag{25}$$

From (21) and (25) we obtain the desired inequality (18).

Now we take the case $a(\ell,t)=0$ for some $(\ell,t)\in\Omega$. Let $a_\epsilon(\ell,t)=a(\ell,t)+\epsilon$, for all $(\ell,t)\in\Omega$, where $\epsilon>0$ is arbitrary, then $a_\epsilon(\ell,t)>0$ and $a_\epsilon(\ell,t)\in C(\Omega,\mathbb{R}_+)$ be nondecreasing with respect to $(\ell,t)\in\Omega$. We carry out the above procedure with $a_\epsilon(\ell,t)>0$ instead of $a(\ell,t)$, and we obtain

$$v(\ell,t)\le\Xi^{-1}\Big\{G^{-1}\Big(F^{-1}\Big[F(p_\epsilon(\ell,t))+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\Big]\Big)\Big\}$$

where

$$p_\epsilon(\ell,t)=G(a_\epsilon(\ell,t))+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma.$$

Letting $\epsilon\to 0^+$, we obtain (18). The proof is complete. □

Remark 3. *If we take $\mathbb{T}=\mathbb{R}$, $\ell_0=0$ and $t_0=0$ in Theorem 3, then, inequality (17) becomes the inequality obtained in [21] (Theorem 2.2(A_2)).*

Corollary 2. *Let the functions v, a, f, $\Im_1$, $\Im_2$, λ_1 and λ_2 be as in Theorem 2. Further suppose that q, p and r are constants with $p>0$, $r>0$ and $q>p+r$. If $v(\ell,t)$ satisfies*

$$\begin{aligned} v^q(\ell,t) \quad &\leq \quad a(\ell,t)+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta)v^p(\varsigma,\eta)v^r(\varsigma,\eta) \\ &\quad +\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)v^p(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{26}$$

for $(\ell,t)\in\Omega$, then

$$v(\ell,t)\leq\left\{[p(\ell,t)]^{\frac{q-p-r}{q-p}}+\frac{q-p-r}{q}\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right\}^{\frac{1}{q-p-r}} \tag{27}$$

where

$$p(\ell,t)=(a(\ell,t))^{\frac{q-p}{q}}+\frac{q-p}{q}\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma$$

Proof. An application of Theorem 3 with $\Xi(v)=v^q,\zeta(v)=v^p$ and $\varpi(v)=v^r$ yields the desired inequality (27). □

Theorem 4. *Under the hypotheses of Theorem 3. If $v(\ell,t)$ satisfies*

$$\begin{aligned} \Xi(v(\ell,t)) \quad &\leq \quad a(\ell,t)+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(v(\varsigma,\eta))\varpi(v(\varsigma,\eta)) \\ &\quad +\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\zeta(v(\chi,\eta))\varpi(v(\chi,\eta))\Delta\chi\Big]\Delta\eta\Delta\varsigma \end{aligned} \tag{28}$$

for $(\ell,t)\in\Omega$, then

$$v(\ell,t)\leq\Xi^{-1}\left\{G^{-1}\left(F^{-1}\left[p_0(\ell,t)+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right)\right\} \tag{29}$$

for $0\leq\ell\leq\ell_1$, $0\leq t\leq t_1$ where

$$p_0(\ell,t)=F(G(a(\ell,t)))+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma$$

and $(\ell_1,t_1)\in\Omega$ is chosen so that

$$\left[p_0(\ell,t)+\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\in\operatorname{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\ell,t) > 0$. Fixing an arbitrary $(\ell_0,t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\ell,t)$ by

$$\begin{aligned}\psi(\ell,t) &= a(\ell_0,t_0) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(v(\varsigma,\eta))\varpi(v(\varsigma,\eta)) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\zeta(v(\chi,\eta))\varpi(v(\chi,\eta))\Delta\chi\Big]\Delta\eta\Delta\varsigma\end{aligned}$$

for $0 \le \ell \le \ell_0 \le \ell_1$, $0 \le t \le t_0 \le t_1$, then $\psi(\ell_0,t) = \psi(\ell,t_0) = a(\ell_0,t_0)$ and

$$v(\ell,t) \le \Xi^{-1}(\psi(\ell,t)). \tag{30}$$

By the same steps as the proof of Theorem 3, we obtain

$$\begin{aligned}\psi(\ell,t) &\le G^{-1}\Big\{G(a(\ell_0,t_0)) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\varpi\Big(\Xi^{-1}(\psi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta\nabla\varsigma\Big\}.\end{aligned}$$

We define a non-negative and nondecreasing function $v(\ell,t)$ by

$$\begin{aligned}v(\ell,t) &= G(a(\ell_0,t_0)) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\Big[\Big[f(\varsigma,\eta)\varpi\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\Big] \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\varpi\Big(\Xi^{-1}(\psi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta\nabla\varsigma\end{aligned}$$

then $v(\ell_0,t) = v(\ell,t_0) = G(a(\ell_0,t_0))$,

$$\psi(\ell,t) \le G^{-1}[v(\ell,t)] \tag{31}$$

and then

$$\begin{aligned}v^{\nabla\ell}(\ell,t) &\le \lambda_1^{\nabla}(\ell)\int_{t_0}^{\lambda_2(t)} \Im_1(\lambda_1(\ell),\eta)\Big[f(\lambda_1(\ell),\eta)\varpi\Big(\Xi^{-1}\Big(G^{-1}(v(\lambda_1(\ell),t))\Big)\Big) \\ &\quad + \int_{\ell_0}^{\lambda_1(\ell)} \Im_2(\chi,\eta)\varpi\Big(\Xi^{-1}\Big(G^{-1}(v(\chi,t))\Big)\Big)\Delta\chi\Big]\Delta\eta \\ &\le \lambda_1^{\nabla}(\ell)\varpi\Big(\Xi^{-1}\Big(G^{-1}(v(\lambda_1(\ell),\lambda_2(t)))\Big)\Big)\int_{t_0}^{\lambda_2(t)} \Im_1(\lambda_1(\ell),\eta)[f(\lambda_1(\ell),\eta) \\ &\quad + \int_{\ell_0}^{\lambda_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\end{aligned}$$

or

$$\begin{aligned}\frac{v^{\nabla\ell}(\ell,t)}{\varpi(\Xi^{-1}(G^{-1}(v(\ell,t))))} &\le \lambda_1^{\nabla}(\ell)\int_{t_0}^{\lambda_2(t)} \Im_1(\lambda_1(\ell),\eta)[f(\lambda_1(\ell),\eta) \\ &\quad + \int_{\ell_0}^{\lambda_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta.\end{aligned}$$

Taking ∇-integral for the above inequality, gives

$$F(v(\ell,t)) \le F(v(\ell_0,t)) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma$$

or

$$\begin{aligned} v(\ell,t) &\leq F^{-1}\bigg\{F(G(a(\ell_0,t_0))) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta) \\ &\quad + \int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\bigg]\Delta\eta\nabla\varsigma\bigg\}. \end{aligned} \tag{32}$$

From (30)–(32), and since $(\ell_0,t_0)\in\Omega$ is chosen arbitrarily, we obtain the desired inequality (29). If $a(\ell,t)=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(\ell,t)$ and subsequently let $\epsilon\to 0$. The proof is complete. □

Remark 4. *If we take $\mathbb{T}=\mathbb{R}$ and $\ell_0=0$ and $t_0=0$ in Theorem 4, then, inequality (28) becomes the inequality obtained in [21] (Theorem 2.2(A_3)).*

Corollary 3. *Under the hypothesise of Corollary 2. If $v(\ell,t)$ satisfies*

$$\begin{aligned} v^q(\ell,t) &\leq a(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)[f(\varsigma,\eta)v^p(\varsigma,\eta)v^r(\varsigma,\eta) \\ &\quad + \int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)v^p(\chi,\eta)v^r(\chi,\eta)\Delta\chi\bigg]\Delta\eta\Delta\varsigma \end{aligned} \tag{33}$$

for $(\ell,t)\in\Omega$, then

$$v(\ell,t) \leq \left\{p_0(\ell,t) + \frac{q-p-r}{q}\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right\}^{\frac{1}{q-p-r}} \tag{34}$$

where

$$p_0(\ell,t) = (a(\ell,t))^{\frac{q-p-r}{q}} + \frac{q-p-r}{q}\int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma$$

Proof. An application of Theorem 4 with $\Xi(v)=v^q, \zeta(v)=v^p$ and $\varpi(v)=v^r$ yields the desired inequality (16). □

Theorem 5. *Under the hypotheses of Theorem 3. If $v(\ell,t)$ satisfies*

$$\begin{aligned} \Xi(v(\ell,t)) &\leq a(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\varpi(v(\varsigma,\eta))\times \\ &\quad \left[f(\varsigma,\eta)\zeta(v(\varsigma,\eta)) + \int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\Delta\varsigma \end{aligned} \tag{35}$$

for $(\ell,t)\in\Omega$, then

$$v(\ell,t) \leq \Xi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right)\right\} \tag{36}$$

for $0\leq\ell\leq\ell_2, 0\leq t\leq t_2$, where

$$G_1(v) = \int_{v_0}^{v}\frac{\nabla\varsigma}{\varpi(\Xi^{-1}(\varsigma))}, v\geq v_0>0, G_1(+\infty) = \int_{v_0}^{+\infty}\frac{\nabla\varsigma}{\varpi(\Xi^{-1}(\varsigma))} = +\infty \tag{37}$$

$$F_1(v) = \int_{v_0}^{v}\frac{\nabla\varsigma}{\zeta\left[\Xi^{-1}\left(G_1^{-1}(\varsigma)\right)\right]}, v\geq v_0>0, F_1(+\infty) = +\infty \tag{38}$$

$$p_1(\ell,t) = G_1(a(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)}\int_{t_0}^{\lambda_2(t)}\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma}\Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma \tag{39}$$

and $(\ell_2, t_2) \in \Omega$ *is chosen so that*

$$\left[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta) \Delta\eta \nabla\varsigma\right] \in \mathrm{Dom}\Big(F_1^{-1}\Big).$$

Proof. Suppose that $a(\ell,t) > 0$. Fixing an arbitrary $(\ell_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\ell,t)$ by

$$\begin{aligned}\psi(\ell,t) \quad &= \quad a(\ell_0,t_0) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) \varpi(v(\varsigma,\eta)) [f(\varsigma,\eta)\zeta(v(\varsigma,\eta)) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big] \Delta\eta\Delta\varsigma\end{aligned}$$

for $0 \le \ell \le \ell_0 \le \ell_2, 0 \le t \le t_0 \le t_2$, then $\psi(\ell_0,t) = \psi(\ell,t_0) = a(\ell_0,t_0)$,

$$v(\ell,t) \le \Xi^{-1}(\psi(\ell,t)) \tag{40}$$

and

$$\begin{aligned}\psi^{\nabla_\ell}(\ell,t) \quad &\le \quad \lambda_1^{\nabla}(\ell) \int_{t_0}^{\lambda_2(t)} \Im_1(\lambda_1(\ell),\eta)\eta\Big[\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big]\Big[f(\lambda_1(\ell),\eta)\zeta\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big) \\ &\quad + \int_{\ell_0}^{\lambda_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta \\ &\le \quad \lambda_1^{\nabla}(\ell)\eta\Big[\Xi^{-1}(\psi(\lambda_1(\ell),\lambda_2(t)))\Big] \int_{t_0}^{\lambda_2(t)} \Im_1(\lambda_1(\ell),\eta)\Big[f(\lambda_1(\ell),\eta)\zeta\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big) \\ &\quad + \int_{\ell_0}^{\lambda_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\end{aligned}$$

then

$$\begin{aligned}\frac{\psi^{\nabla_\ell}(\ell,t)}{\eta[\Xi^{-1}(\psi(\ell,t))]} \quad &\le \quad \lambda_1^{\nabla}(\ell) \int_{t_0}^{\lambda_2(t)} \Im_1(\lambda_1(\ell),\eta)\Big[f(\lambda_1(\ell),\eta)\zeta\Big(\Xi^{-1}(\psi(\lambda_1(\ell),\eta))\Big) \\ &\quad + \int_{\ell_0}^{\lambda_1(\ell)} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta.\end{aligned}$$

Taking ∇-integral for the above inequality, gives

$$\begin{aligned}G_1(\psi(\ell,t)) \quad &\le \quad G_1(\psi(0,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma\end{aligned}$$

then

$$\begin{aligned}G_1(\psi(\ell,t)) \quad &\le \quad G_1(a(\ell_0,t_0)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma.\end{aligned}$$

Since $(\ell_0, t_0) \in \Omega$ is chosen arbitrary, the last inequality can be restated as

$$G_1(\psi(\ell,t)) \le p_1(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\zeta\Big(\Xi^{-1}(\psi(\varsigma,\eta))\Big)\Delta\eta\Delta\varsigma \tag{41}$$

It is easy to observe that $p_1(\ell,t)$ is positive and nondecreasing function for all $(\ell,t) \in \Omega$, then an application of Lemma 1 to (41) yields the inequality

$$\psi(\ell,t) \le G_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right). \tag{42}$$

From (42) and (40) we obtain the desired inequality (36).

If $a(\ell,t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\ell,t)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Remark 5. *If we take $\mathbb{T} = \mathbb{R}$ and $\ell_0 = 0$ and $t_0 = 0$ in Theorem 5, then, inequality (36) becomes the inequality obtained in [21] (Theorem 2.7).*

Theorem 6. *Under the hypotheses of Theorem 3 and let p be a non-negative constant. If $v(\ell,t)$ satisfies*

$$\begin{aligned} \Xi(v(\ell,t)) \quad \le \quad & a(\ell,t) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) v^p(\varsigma,\eta) \times \\ & \left[f(\varsigma,\eta)\zeta(v(\varsigma,\eta)) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\Delta\varsigma \end{aligned} \tag{43}$$

for $(\ell,t) \in \Omega$, then

$$v(\ell,t) \le \Xi^{-1}\left\{G_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right)\right\} \tag{44}$$

for $0 \le \ell \le \ell_2, 0 \le t \le t_2$, where

$$G_1(v) = \int_{v_0}^{v} \frac{\nabla\varsigma}{[\Xi^{-1}(\varsigma)]^p}, v \ge v_0 > 0, G_1(+\infty) = \int_{v_0}^{+\infty} \frac{\nabla\varsigma}{[\Xi^{-1}(\varsigma)]^p} = +\infty \tag{45}$$

and F_1, p_1 are as in Theorem 5 and $(\ell_2,t_2) \in \Omega$ is chosen so that

$$\left[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right] \in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. An application of Theorem 5, with $\varpi(v) = v^p$ yields the desired inequality (44). □

Remark 6. *Taking $\mathbb{T} = \mathbb{R}$. The inequality established in Theorem 6 generalizes [25] (Theorem 1) (with $p = 1$, $a(\ell,t) = b(\ell) + c(t)$, $\ell_0 = 0$, $t_0 = 0$, $\Im_1(\varsigma,\eta)f(\varsigma,\eta) = h(\varsigma,\eta)$ and $\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right) = g(\varsigma,\eta)$).*

Corollary 4. *Under the hypotheses of Theorem 6 and $q > p > 0$ be constants. If $v(\ell,t)$ satisfies*

$$\begin{aligned} v^q(\ell,t) \quad \le \quad & a(\ell,t) + \frac{p}{p-q} \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) v^p(\varsigma,\eta) \times \\ & \left[f(\varsigma,\eta)\zeta(v(\varsigma,\eta)) + \int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right]\Delta\eta\Delta\varsigma \end{aligned} \tag{46}$$

for $(\ell,t) \in \Omega$, then

$$v(\ell,t) \le \left\{F_1^{-1}\left[F_1(p_1(\ell,t)) + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right\}^{\frac{1}{q-p}} \tag{47}$$

for $0 \le \ell \le \ell_2$, $0 \le t \le t_2$, *where*

$$p_1(\ell,t) = [a(\ell,t)]^{\frac{q-p}{q}} + \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma$$

and F_1 *is defined in Theorem 5.*

Proof. An application of Theorem 6 with $\Xi(v(\ell,t)) = v^p$ to (46) yields the inequality (47); to save space we omit the details. □

Remark 7. *Taking* $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$, $t_0 = 0$, $a(\ell,t) = b(\ell) + c(t)$, $\Im_1(\varsigma,\eta)f(\varsigma,\eta) = h(\varsigma,\eta)$ *and* $\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right) = g(\varsigma,\eta)$ *in Corollary 4 we obtain [26] (Theorem 1).*

Remark 8. *Taking* $\mathbb{T} = \mathbb{R}$, $\ell_0 = 0$, $t_0 = 0$, $a(\ell,t) = c^{\frac{p}{p-q}}$, $\Im_1(\varsigma,\eta)f(\varsigma,\eta) = h(\eta)$ *and* $\Im_1(\varsigma,\eta)\left(\int_{\ell_0}^{\varsigma} \Im_2(\chi,\eta)\Delta\chi\right) = g(\eta)$ *and keeping t fixed in Corollary 4, we obtain [27] (Theorem 2.1).*

3. Application

Gronwall inequality involving functions of one and more than one independent variables, which provide explicit bounds on unknown functions, plays a fundamental role in the development of qualitative theory and can be used as handy tools in the study of existence, uniqueness, oscillation, stability and other qualitative properties of the solutions of certain dynamic equations on time scales.

In this following, we discus the boundedness of the solutions of the initial boundary value problem for partial delay dynamic equation of the form

$$(\psi^q)^{\Delta_\ell \Delta_t}(\ell,t) = A\left(\ell, t, \psi(\ell - h_1(\ell), t - h_2(t)), \int_{\ell_0}^{\ell} B(\varsigma, t, \psi(\varsigma - h_1(\varsigma), t))\nabla\varsigma\right) \tag{48}$$

$$\psi(\ell,t_0) = a_1(\ell), \psi(\ell_0,t) = a_2(t), a_1(\ell_0) = a_{t_0}(0) = 0$$

for $(\ell,t) \in \Omega$, where $\psi, b \in C(\Omega, \mathbb{R}_+)$, $A \in C(\Omega \times R^2, R)$, $B \in C(\zeta \times R, R)$ and $h_1 \in C^1_{rd}(\mathbb{T}_1, \mathbb{R}_+)$, $h_2 \in C^1_{rd}(\mathbb{T}_2, \mathbb{R}_+)$ are nondecreasing functions such that $h_1(\ell) \le \ell$ on $\mathbb{T}_1$, $h_2(t) \le t$ on $\mathbb{T}_2$ and $h_1^{\nabla}(\ell) < 1$, $h_2^{\nabla}(t) < 1$.

Theorem 7. *Assume that the functions* a_1, a_2, A, B *in (48) satisfy the conditions*

$$|a_1(\ell) + a_2(t)| \le a(\ell,t) \tag{49}$$

$$|A(\varsigma,\eta,\psi,v)| \le \frac{q}{q-p}\Im_1(\varsigma,\eta)\left[f(\varsigma,\eta)|\psi|^p + |v|\right] \tag{50}$$

$$|B(\chi,\eta,\psi)| \le \Im_2(\chi,\eta)|\psi|^p \tag{51}$$

where $a(\ell,t)$, $\Im_1(\varsigma,\eta)$, $f(\varsigma,\eta)$ *and* $\Im_2(\chi,\eta)$ *are as in Theorem 2,* $q > p > 0$ *are constants. If* $\psi(\ell,t)$ *satisfies (48), then*

$$|\psi(\ell,t)| \le \left\{p(\ell,t) + M_1 M_2 \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \bar{\Im}_1(\varsigma,\eta)\bar{f}(\varsigma,\eta)\Delta\eta\nabla\varsigma\right\}^{\frac{1}{q-p}} \tag{52}$$

where

$$\begin{aligned} p(\ell,t) &= (a(\ell,t))^{\frac{q-p}{q}} \\ &+ M_1 M_2 \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \bar{\Im}_1(\varsigma,\eta)\left(M_1 \int_{\ell_0}^{\varsigma} \bar{\Im}_2(\chi,\eta)\Delta\chi\right)\Delta\eta\Delta\varsigma \end{aligned}$$

and
$$M_1 = \underset{\ell \in I_1}{Max} \frac{1}{1 - h_1^{\nabla}(\ell)}, \qquad M_2 = \underset{t \in I_2}{Max} \frac{1}{1 - h_2^{\nabla}(t)}$$

and $\bar{\Im}_1(\gamma, \xi) = \Im_1(\gamma + h_1(\varsigma), \xi + h_2(\eta)), \bar{\Im}_2(\mu, \xi) = \Im_2(\mu, \xi + h_2(\eta)),$
$\bar{f}(\gamma, \xi) = f(\gamma + h_1(\varsigma), \xi + h_2(\eta)).$

Proof. If $\psi(\ell, t)$ is any solution of (48), then

$$\psi^q(\ell, t) = a_1(\ell) + a_2(t)$$
$$+ \int_{\ell_0}^{\ell} \int_{t_0}^{t} A\left(\varsigma, \eta, \psi(\varsigma - h_1(\varsigma), \eta - h_2(\eta)), \int_{\ell_0}^{\varsigma} B(\chi, \eta, \psi(\chi - h_1(\chi), \eta))\Delta\chi\right)\Delta\eta\Delta\varsigma. \tag{53}$$

Using the conditions (49)–(51) in (53) we obtain

$$|\psi(\ell, t)|^q \le a(\ell, t) + \frac{q-p}{q} \int_{\ell_0}^{\ell} \int_{t_0}^{t} \Im_1(\varsigma, \eta)\Big[f(\varsigma, \eta)|\psi(\varsigma - h_1(\varsigma), \eta - h_2(\eta))|^p$$
$$+ \int_{\ell_0}^{\varsigma} \Im_2(\chi, \eta)|\psi(\chi, \eta)|^p \Delta\chi\Big]\Delta\eta\Delta\varsigma. \tag{54}$$

Now making a change of variables on the right side of (54), $\varsigma - h_1(\varsigma) = \gamma, \eta - h_2(\eta) = \xi, \ell - h_1(\ell) = \lambda_1(\ell)$ for $\ell \in \mathbb{T}_1, t - h_2(t) = \lambda_2(t)$ for $t \in \mathbb{T}_2$ we obtain the inequality

$$|\psi(\ell, t)|^q \le a(\ell, t) + \frac{q-p}{q} M_1 M_2 \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \bar{\Im}_1(\gamma, \xi)\Big[\bar{f}(\gamma, \xi)|\psi(\gamma, \xi)|^p$$
$$+ M_1 \int_{\ell_0}^{\gamma} \bar{\Im}_2(\mu, \xi)|\psi(\mu, \eta)|^p \Delta\mu\Big]\Delta\xi\Delta\gamma. \tag{55}$$

We can rewrite the inequality (55) as follows:

$$|\psi(\ell, t)|^q \le a(\ell, t) + \frac{q-p}{q} M_1 M_2 \int_{\ell_0}^{\lambda_1(\ell)} \int_{t_0}^{\lambda_2(t)} \bar{\Im}_1(\varsigma, \eta)\Big[\bar{f}(\varsigma, \eta)|\psi(\varsigma, \eta)|^p$$
$$+ M_1 \int_{\ell_0}^{\varsigma} \bar{\Im}_2(\chi, \eta)|\psi(\chi, \eta)|^p \Delta\chi\Big]\Delta\eta\Delta\varsigma. \tag{56}$$

As an application of Corollary 1 to (56) with $v(\ell, t) = |\psi(\ell, t)|$ we obtain the desired inequality (52). □

4. Conclusions

In this article, by applying the Leibniz integral rule on time scales, we examined additional generalizations of the integral retarded inequality presented in the literature and generalized a few of those inequalities to a general time scale. We also applied some of our results to study the qualitative behavior of certain dynamic equations' time-scale solutions. In future work, I will ask if it is possible to generalize these results using a q-difference operator. Additionally, we intend to extend these inequalities by using α-conformable calculus and also by employing (γ, a)- nabla calculus on time scales. Moreover, we will try to obtain the diamond alpha version for these results. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

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