Supplementary Material for

"Anti-symmetric medium chirality leading to symmetric field helicity in response to a pair of circularly polarized plane waves in counter-propagating configuration"

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Section S1. Energy conservation laws for any pair of $\{\mathbf{D}, \mathbf{B}\}$

Recall that the set of generic formulas presented in the main text are all derived by processing the Faraday law $\nabla \times \mathbf{E} = \mathrm{i}\mathbf{B}$ and the Ampère's law $\nabla \times \mathbf{H} = -\mathrm{i}\mathbf{D}$ by forming the dot products $\mathbf{H}^* \bullet (\nabla \times \mathbf{E}) = \mathrm{i}\mathbf{H}^* \bullet \mathbf{B}$ and $\mathbf{E}^* \bullet (\nabla \times \mathbf{H}) = -\mathrm{i}\mathbf{E}^* \bullet \mathbf{D}$.

In this section, we will try another approach of forming another pair of dot products $\mathbf{B}^* \bullet (\nabla \times \mathbf{E}) = \mathrm{i}\mathbf{B}^* \bullet \mathbf{B}$ and $\mathbf{D}^* \bullet (\nabla \times \mathbf{H}) = -\mathrm{i}\mathbf{D}^* \bullet \mathbf{D}$. The ensuing set of all pertinent formulas are no less complicated than the ones presented in the main text. This distinction between the two approaches is again linked to the Abraham-Minkowski dilemma, which lies beyond the scope of this study.

Establishing a sum and a difference between the two relations,

$$\begin{cases} \nabla \times \mathbf{E} = \mathrm{i}\mathbf{B} \\ \nabla \times \mathbf{H} = -\mathrm{i}\mathbf{D} \end{cases} \Rightarrow \begin{cases} \mathbf{B}^* \bullet (\nabla \times \mathbf{E}) = \mathrm{i}\mathbf{B}^* \bullet \mathbf{B} \\ \mathbf{D}^* \bullet (\nabla \times \mathbf{H}) = -\mathrm{i}\mathbf{D}^* \mathbf{D} \end{cases} \Rightarrow \begin{cases} \mathbf{H}^* \bullet (\mathbf{FL}) - \mathbf{E}^* \bullet (\mathbf{AL}) : \\ \mathbf{H}^* \bullet (\mathbf{FL}) + \mathbf{E}^* \bullet (\mathbf{AL}) : \end{cases}$$

$$\begin{cases} \mathbf{H}^* \bullet (\mathbf{FL}) - \mathbf{E}^* \bullet (\mathbf{AL}) : \quad \mathbf{B}^* \bullet (\nabla \times \mathbf{E}) - \mathbf{D}^* \bullet (\nabla \times \mathbf{H}) = \mathrm{i}\left(\mathbf{B}^* \bullet \mathbf{B} + \mathbf{D}^* \mathbf{D}\right) = 2\mathrm{i}\tfrac{1}{2}\left(|\mathbf{B}|^2 + |\mathbf{D}|^2\right) \\ \mathbf{H}^* \bullet (\mathbf{FL}) + \mathbf{E}^* \bullet (\mathbf{AL}) : \quad \mathbf{B}^* \bullet (\nabla \times \mathbf{E}) + \mathbf{D}^* \bullet (\nabla \times \mathbf{H}) = \mathrm{i}\left(\mathbf{B}^* \bullet \mathbf{B} - \mathbf{D}^* \mathbf{D}\right) = 2\mathrm{i}\tfrac{1}{2}\left(|\mathbf{B}|^2 - |\mathbf{D}|^2\right) \end{cases} \Rightarrow \quad . \quad \text{(S1.1)}$$

$$\begin{cases} I_{\mathbf{BD}} \equiv \tfrac{1}{2}\left(|\mathbf{D}|^2 + |\mathbf{B}|^2\right) \geq 0 \\ J_{\mathbf{BD}} \equiv \tfrac{1}{2}\left(|\mathbf{D}|^2 - |\mathbf{B}|^2\right) \geq 0 \end{cases} \Rightarrow \begin{cases} \mathbf{B}^* \bullet (\nabla \times \mathbf{E}) - \mathbf{D}^* \bullet (\nabla \times \mathbf{H}) = 2iI_{\mathbf{BD}} \\ \mathbf{B}^* \bullet (\nabla \times \mathbf{E}) + \mathbf{D}^* \bullet (\nabla \times \mathbf{H}) = -2iJ_{\mathbf{BD}} \end{cases}$$

This pair of the EM and reactive energy densities are positive for any pair $\{\mathbf{D}, \mathbf{B}\}$ for the electric displacement and the magnetic induction. Via vector identities,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \quad \Rightarrow$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A} \quad \Rightarrow \quad \mathbf{B} \cdot (\nabla \times \mathbf{A}) = \nabla \cdot (\mathbf{A} \times \mathbf{B}) + (\nabla \times \mathbf{B}) \cdot \mathbf{A} \quad \Rightarrow$$

$$\begin{cases} \mathbf{B}^* \cdot (\nabla \times \mathbf{E}) = \nabla \cdot (\mathbf{E} \times \mathbf{B}^*) + (\nabla \times \mathbf{B}^*) \cdot \mathbf{E} \\ \mathbf{D}^* \cdot (\nabla \times \mathbf{H}) = \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) + (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \end{cases} \Rightarrow \begin{cases} \mathbf{B}^* \cdot (\nabla \times \mathbf{E}) - \mathbf{D}^* \cdot (\nabla \times \mathbf{H}) = 2i I_{BD} \\ \mathbf{B}^* \cdot (\nabla \times \mathbf{E}) + \mathbf{D}^* \cdot (\nabla \times \mathbf{H}) = -2i J_{BD} \end{cases} \Rightarrow \qquad (S1.2)$$

$$\begin{cases} \nabla \cdot (\mathbf{E} \times \mathbf{B}^*) + (\nabla \times \mathbf{B}^*) \cdot \mathbf{E} - \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) - (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} = 2i I_{BD} \\ \nabla \cdot (\mathbf{E} \times \mathbf{B}^*) + (\nabla \times \mathbf{B}^*) \cdot \mathbf{E} + \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) + (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} = -2i J_{BD} \end{cases}$$

Let us take the respective real and imaginary parts.

$$\begin{cases} \tfrac{1}{2} \mathrm{Re}\left[\nabla \cdot (\mathbf{E} \times \mathbf{B}^*) - \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Re}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} - (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = 0 \\ \tfrac{1}{2} \mathrm{Im}\left[\nabla \cdot (\mathbf{E} \times \mathbf{B}^*) - \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Im}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} - (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = I_{BD} \end{cases}$$

$$\begin{cases} \tfrac{1}{2} \mathrm{Re}\left[\nabla \cdot (\mathbf{E} \times \mathbf{B}^*) + \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Re}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} + (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = 0 \\ \tfrac{1}{2} \mathrm{Im}\left[\nabla \cdot (\mathbf{E} \times \mathbf{B}^*) + \nabla \cdot (\mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Im}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} + (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = -J_{BD} \end{cases}$$

$\mathbf{H}^* \cdot (\textcolor{red}{\mathbf{FL}}) - \mathbf{E}^* \cdot (\textcolor{blue}{\mathbf{AL}}):$ $\qquad (S1.3)$

$$\begin{cases} \nabla \cdot \left[\tfrac{1}{2} \mathrm{Re}(\mathbf{E} \times \mathbf{B}^* - \mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Re}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} - (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = 0 \\ \nabla \cdot \left[\tfrac{1}{2} \mathrm{Im}(\mathbf{E} \times \mathbf{B}^* - \mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Im}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} - (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = I_{BD} \end{cases}$$

$\mathbf{H}^* \cdot (\textcolor{red}{\mathbf{FL}}) + \mathbf{E}^* \cdot (\textcolor{blue}{\mathbf{AL}}):$

$$\begin{cases} \nabla \cdot \left[\tfrac{1}{2} \mathrm{Re}(\mathbf{E} \times \mathbf{B}^* + \mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Re}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} + (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = 0 \\ \nabla \cdot \left[\tfrac{1}{2} \mathrm{Im}(\mathbf{E} \times \mathbf{B}^* + \mathbf{H} \times \mathbf{D}^*) \right] + \tfrac{1}{2} \mathrm{Im}\left[(\nabla \times \mathbf{B}^*) \cdot \mathbf{E} + (\nabla \times \mathbf{D}^*) \cdot \mathbf{H} \right] = -J_{BD} \end{cases}$$

Recall the two types of constitutive relations.

$$C_\beta^{curl} : \begin{cases} \mathbf{D} = \varepsilon(\mathbf{E} + \beta \nabla \times \mathbf{E}) \\ \mathbf{B} = \mu(\mathbf{H} + \beta \nabla \times \mathbf{H}) \end{cases} ; \quad C_\kappa^{field} : \begin{cases} \mathbf{D} = \varepsilon \mathbf{E} + i\kappa \mathbf{H} \\ \mathbf{B} = \mu \mathbf{H} - i\kappa \mathbf{E} \end{cases} \Rightarrow \begin{cases} I_{BD} \equiv \tfrac{1}{2}\left(|\mathbf{D}|^2 + |\mathbf{B}|^2 \right) \geq 0 \\ J_{BD} \equiv \tfrac{1}{2}\left(|\mathbf{D}|^2 - |\mathbf{B}|^2 \right) \geq 0 \end{cases} . \qquad (S1.4)$$

For the curl-based pair of constitutive relations,

$\varepsilon, \mu \in \mathbb{R}$

$C_\beta^{curl}:$

$$2I_{\mathbf{BD}} = |\mathbf{D}|^2 - |\mathbf{B}|^2 = |\varepsilon\mathbf{E} + i\kappa\mathbf{H}|^2 + |\mu\mathbf{H} - i\kappa\mathbf{E}|^2$$

$$= (\varepsilon\mathbf{E} + i\kappa\mathbf{H})^* \cdot (\varepsilon\mathbf{E} + i\kappa\mathbf{H}) + (\mu\mathbf{H} - i\kappa\mathbf{E})^* \cdot (\mu\mathbf{H} - i\kappa\mathbf{E})$$

$$= \varepsilon^2 |\mathbf{E}|^2 + \kappa^2 |\mathbf{H}|^2 + \mu^2 |\mathbf{H}|^2 + \kappa^2 |\mathbf{E}|^2 + i\kappa\varepsilon\mathbf{E}^* \cdot \mathbf{H} - i\kappa\varepsilon\mathbf{E} \cdot \mathbf{H}^* + i\kappa\mu\mathbf{E}^* \cdot \mathbf{H} - i\kappa\mu\mathbf{E} \cdot \mathbf{H}^*$$

$$= (\varepsilon^2 + \kappa^2)|\mathbf{E}|^2 + (\mu^2 + \kappa^2)|\mathbf{H}|^2 + \mathrm{Re}\left[i\kappa(\varepsilon + \mu)\mathbf{E}^* \cdot \mathbf{H}\right]$$

$$= (\varepsilon^2 + \kappa^2)|\mathbf{E}|^2 + (\mu^2 + \kappa^2)|\mathbf{H}|^2 - \mathrm{Im}\left[\kappa(\varepsilon + \mu)\mathbf{E}^* \cdot \mathbf{H}\right]$$

$$= (\varepsilon^2 + \kappa^2)|\mathbf{E}|^2 + (\mu^2 + \kappa^2)|\mathbf{H}|^2 + \kappa(\varepsilon + \mu)\mathrm{Im}\left(\mathbf{E} \cdot \mathbf{H}^*\right)$$

$$2J_{\mathbf{BD}} = |\mathbf{D}|^2 - |\mathbf{B}|^2 = |\varepsilon\mathbf{E} + i\kappa\mathbf{H}|^2 - |\mu\mathbf{H} - i\kappa\mathbf{E}|^2$$

$$= (\varepsilon\mathbf{E} + i\kappa\mathbf{H})^* \cdot (\varepsilon\mathbf{E} + i\kappa\mathbf{H}) - (\mu\mathbf{H} - i\kappa\mathbf{E})^* \cdot (\mu\mathbf{H} - i\kappa\mathbf{E})$$

$$= \varepsilon^2 |\mathbf{E}|^2 + \kappa^2 |\mathbf{H}|^2 - \mu^2 |\mathbf{H}|^2 - \kappa^2 |\mathbf{E}|^2 + i\kappa\varepsilon\mathbf{E}^* \cdot \mathbf{H} - i\kappa\varepsilon\mathbf{E} \cdot \mathbf{H}^* - i\kappa\mu\mathbf{E}^* \cdot \mathbf{H} + i\kappa\mu\mathbf{E} \cdot \mathbf{H}^*$$

$$= (\varepsilon^2 - \kappa^2)|\mathbf{E}|^2 - (\mu^2 - \kappa^2)|\mathbf{H}|^2 + \mathrm{Re}\left[i\kappa(\varepsilon - \mu)\mathbf{E}^* \cdot \mathbf{H}\right]$$

$$= (\varepsilon^2 - \kappa^2)|\mathbf{E}|^2 - (\mu^2 - \kappa^2)|\mathbf{H}|^2 - \mathrm{Im}\left[\kappa(\varepsilon - \mu)\mathbf{E}^* \cdot \mathbf{H}\right]$$

$$= (\varepsilon^2 - \kappa^2)|\mathbf{E}|^2 - (\mu^2 - \kappa^2)|\mathbf{H}|^2 + \kappa(\varepsilon - \mu)\mathrm{Im}\left(\mathbf{E} \cdot \mathbf{H}^*\right) \tag{S1.5}$$

For the field-based constitutive relations,

$\varepsilon, \mu \in \mathbb{R}$

$C_\beta^{curl}:$

$$2I_{\mathbf{BD}} = |\mathbf{D}|^2 - |\mathbf{B}|^2 = |\varepsilon(\mathbf{E} + \beta\nabla\times\mathbf{E})|^2 + |\mu(\mathbf{H} + \beta\nabla\times\mathbf{H})|^2$$

$$= \varepsilon^2 (\mathbf{E}^* + \beta\nabla\times\mathbf{E}^*) \cdot (\mathbf{E} + \beta\nabla\times\mathbf{E}) + \mu^2 (\mathbf{H}^* + \beta\nabla\times\mathbf{H}^*) \cdot (\mathbf{H} + \beta\nabla\times\mathbf{H})$$

$$= \varepsilon^2 |\mathbf{E}|^2 + \varepsilon^2\beta^2 |\nabla\times\mathbf{E}|^2 + \mu^2 |\mathbf{H}|^2 + \mu^2\beta^2 |\nabla\times\mathbf{H}|^2$$

$$+ 2\varepsilon^2\beta\,\mathrm{Re}\left[\mathbf{E}^* \cdot (\nabla\times\mathbf{E})\right] + 2\mu^2\beta\,\mathrm{Re}\left[\mathbf{H}^* \cdot (\nabla\times\mathbf{H})\right] \tag{S1.6}$$

$$2J_{\mathbf{BD}} = |\mathbf{D}|^2 - |\mathbf{B}|^2 = |\varepsilon(\mathbf{E} + \beta\nabla\times\mathbf{E})|^2 - |\mu(\mathbf{H} + \beta\nabla\times\mathbf{H})|^2$$

$$= \varepsilon^2 (\mathbf{E}^* + \beta\nabla\times\mathbf{E}^*) \cdot (\mathbf{E} + \beta\nabla\times\mathbf{E}) - \mu^2 (\mathbf{H}^* + \beta\nabla\times\mathbf{H}^*) \cdot (\mathbf{H} + \beta\nabla\times\mathbf{H})$$

$$= \varepsilon^2 |\mathbf{E}|^2 + \varepsilon^2\beta^2 |\nabla\times\mathbf{E}|^2 - \mu^2 |\mathbf{H}|^2 - \mu^2\beta^2 |\nabla\times\mathbf{H}|^2$$

$$+ 2\varepsilon^2\beta\,\mathrm{Re}\left[\mathbf{E}^* \cdot (\nabla\times\mathbf{E})\right] - 2\mu^2\beta\,\mathrm{Re}\left[\mathbf{H}^* \cdot (\nabla\times\mathbf{H})\right]$$

In addition to the divergence (potential-like) terms, we still obtain non-divergent terms. Consequently, the approach to obtaining the energy conservations laws based on the pair $\{\mathbf{D}, \mathbf{B}\}$ does not pay off.

Meanwhile, we have the following set of dot products and cross products that use quite useful in the following two sections.

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A} \implies (\nabla\times\mathbf{A})\cdot\mathbf{B} = \nabla\cdot(\mathbf{A}\times\mathbf{B}) + (\nabla\times\mathbf{B})\cdot\mathbf{A}$$

$$\begin{cases} \mathbf{H}^* \cdot (\nabla\times\mathbf{E}) = (\nabla\times\mathbf{E})\cdot\mathbf{H}^* = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*) + (\nabla\times\mathbf{H}^*)\cdot\mathbf{E} \\ \mathbf{E}^* \cdot (\nabla\times\mathbf{H}) = (\nabla\times\mathbf{H})\cdot\mathbf{E}^* = \nabla\cdot(\mathbf{H}\times\mathbf{E}^*) + (\nabla\times\mathbf{E}^*)\cdot\mathbf{H} \end{cases} \tag{S1.7}$$

$$\begin{cases} \mathbf{E}^* \cdot (\nabla\times\mathbf{E}) = (\nabla\times\mathbf{E})\cdot\mathbf{E}^* = \nabla\cdot(\mathbf{E}\times\mathbf{E}^*) + (\nabla\times\mathbf{E}^*)\cdot\mathbf{E} \\ \mathbf{H}^* \cdot (\nabla\times\mathbf{H}) = (\nabla\times\mathbf{H})\cdot\mathbf{H}^* = \nabla\cdot(\mathbf{H}\times\mathbf{H}^*) + (\nabla\times\mathbf{H}^*)\cdot\mathbf{H} \end{cases}$$

Based on the preceding set, we make a difference and a sum of the above pair.

$$
\begin{aligned}
dif: \quad & \mathbf{H}^*\cdot(\nabla\times\mathbf{E})+\mathbf{E}^*\cdot(\nabla\times\mathbf{H}) = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+(\nabla\times\mathbf{H}^*)\cdot\mathbf{E}+\nabla\cdot(\mathbf{H}\times\mathbf{E}^*)+(\nabla\times\mathbf{E}^*)\cdot\mathbf{H} \\
& = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+\nabla\cdot(\mathbf{H}\times\mathbf{E}^*)+(\nabla\times\mathbf{H}^*)\cdot\mathbf{E}+(\nabla\times\mathbf{E}^*)\cdot\mathbf{H} \\
& = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)-\nabla\cdot(\mathbf{E}^*\times\mathbf{H})+(\nabla\times\mathbf{H}^*)\cdot\mathbf{E}+(\nabla\times\mathbf{E}^*)\cdot\mathbf{H} \\
sum: \quad & \mathbf{H}^*\cdot(\nabla\times\mathbf{E})-\mathbf{E}^*\cdot(\nabla\times\mathbf{H}) = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+(\nabla\times\mathbf{H}^*)\cdot\mathbf{E}-\nabla\cdot(\mathbf{H}\times\mathbf{E}^*)-(\nabla\times\mathbf{E}^*)\cdot\mathbf{H} \\
& = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)-\nabla\cdot(\mathbf{H}\times\mathbf{E}^*)+(\nabla\times\mathbf{H}^*)\cdot\mathbf{E}-(\nabla\times\mathbf{E}^*)\cdot\mathbf{H} \\
& = \nabla\cdot(\mathbf{E}\times\mathbf{H}^*)+\nabla\cdot(\mathbf{E}^*\times\mathbf{H})+(\nabla\times\mathbf{H}^*)\cdot\mathbf{E}-(\nabla\times\mathbf{E}^*)\cdot\mathbf{H}
\end{aligned}
\tag{S1.8}
$$

It is useful to introduce the following pairs of auxiliary vectors.

$$
\begin{cases}
\mathbf{A}_{\pm} \equiv \left[\mathbf{E}^*\cdot(\nabla\times\mathbf{H})\mp\mathbf{H}^*\cdot(\nabla\times\mathbf{E})\right]^* \\
\mathbf{B}_{\pm} \equiv \mu\mathbf{H}^*\cdot(\nabla\times\mathbf{H})\mp\varepsilon\mathbf{E}^*\cdot(\nabla\times\mathbf{E})
\end{cases}
\tag{S1.9}
$$

Section S2. Energy conservation laws for curl-based constitutive relations

For a chiral media with the curl-β-based constitutive relations, let us examine the Maxwell equations.

$$
\begin{cases}
\nabla\times\mathbf{E}=i\mu(\mathbf{H}+\beta\nabla\times\mathbf{H}) \\
\nabla\times\mathbf{H}=-i\varepsilon(\mathbf{E}+\beta\nabla\times\mathbf{E})
\end{cases},
\quad
\begin{cases}
\nabla\cdot\mathbf{E}=0 \\
\nabla\cdot\mathbf{H}=0
\end{cases}
\Rightarrow
$$

$$
K\equiv\frac{i}{1-\varepsilon\mu\beta^2}\begin{pmatrix}-i\varepsilon\mu\beta & \mu \\ -\varepsilon & -i\mu\varepsilon\beta\end{pmatrix},\quad
K^{-1}=\begin{pmatrix}-\beta & i\dfrac{1}{\varepsilon} \\ -i\dfrac{1}{\mu} & -\beta\end{pmatrix}
$$

$$
\nabla\times\begin{Bmatrix}\mathbf{E}\\\mathbf{H}\end{Bmatrix}=K\begin{Bmatrix}\mathbf{E}\\\mathbf{H}\end{Bmatrix}
\Rightarrow
\begin{cases}
\nabla\times\mathbf{E}=\dfrac{1}{1-\varepsilon\mu\beta^2}(\varepsilon\mu\beta\mathbf{E}+i\mu\mathbf{H}) \\
\nabla\times\mathbf{H}=\dfrac{1}{1-\varepsilon\mu\beta^2}(-i\varepsilon\mathbf{E}+\varepsilon\mu\beta\mathbf{H})
\end{cases}
\tag{S2.1}
$$

$$
\begin{Bmatrix}\mathbf{E}\\\mathbf{H}\end{Bmatrix}=\begin{pmatrix}-\beta & i\dfrac{1}{\varepsilon} \\ -i\dfrac{1}{\mu} & -\beta\end{pmatrix}\begin{Bmatrix}\nabla\times\mathbf{E}\\\nabla\times\mathbf{H}\end{Bmatrix}=\begin{Bmatrix}-\beta(\nabla\times\mathbf{E})+i\dfrac{1}{\varepsilon}(\nabla\times\mathbf{H}) \\ -i\dfrac{1}{\mu}(\nabla\times\mathbf{E})-\beta(\nabla\times\mathbf{H})\end{Bmatrix}
$$

Let the Faraday law $\nabla\times\mathbf{E}=i\mu(\mathbf{H}+\beta\nabla\times\mathbf{H})$ and the Ampère's law $\nabla\times\mathbf{H}=-i\varepsilon(\mathbf{E}+\beta\nabla\times\mathbf{E})$ be denoted respectively by $(\mathbf{FL})$ and $(\mathbf{AL})$. We then take the difference $\mathbf{H}^*\cdot(\mathbf{FL})-\mathbf{E}^*\cdot(\mathbf{AL})$ and the sum $\mathbf{H}^*\cdot(\mathbf{FL})+\mathbf{E}^*\cdot(\mathbf{AL})$ so that the following two pairs are obtained when taking the respective real and imaginary parts.

$$\begin{cases} (\mathbf{FL}): \ \nabla\times\mathbf{E}=i\mu\left(\mathbf{H}+\beta\nabla\times\mathbf{H}\right) \\ (\mathbf{AL}): \ \nabla\times\mathbf{H}=-i\varepsilon\left(\mathbf{E}+\beta\nabla\times\mathbf{E}\right) \end{cases} \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\nabla\times\mathbf{E})=i\mu\mathbf{H}^*\bullet\mathbf{H}+i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H}) \\ \mathbf{E}^*\bullet(\nabla\times\mathbf{H})=-i\varepsilon\mathbf{E}^*\bullet\mathbf{E}-i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ \mathbf{H}^*\bullet(\nabla\times\mathbf{E})-\mathbf{E}^*\bullet(\nabla\times\mathbf{H}) \\ \quad =i\varepsilon\mathbf{E}^*\bullet\mathbf{E}+i\mu\mathbf{H}^*\bullet\mathbf{H}+i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H})+i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ \mathbf{H}^*\bullet(\nabla\times\mathbf{E})+\mathbf{E}^*\bullet(\nabla\times\mathbf{H}) \\ \quad =-i\varepsilon\mathbf{E}^*\bullet\mathbf{E}+i\mu\mathbf{H}^*\bullet\mathbf{H}+i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H})-i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \end{cases}$$ (S2.2)

Introduce average sum and average difference.

$$\begin{cases} I\equiv\tfrac{1}{2}\left(\varepsilon\mathbf{E}^*\bullet\mathbf{E}+\mu\mathbf{H}^*\bullet\mathbf{H}\right) \\ J\equiv\tfrac{1}{2}\left(\varepsilon\mathbf{E}^*\bullet\mathbf{E}-\mu\mathbf{H}^*\bullet\mathbf{H}\right) \end{cases} \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}) \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}) \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ \tfrac{1}{2}\left[\mathbf{H}^*\bullet(\nabla\times\mathbf{E})-\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\right]=iI+\tfrac{1}{2}i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H})+i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ \tfrac{1}{2}\left[\mathbf{H}^*\bullet(\nabla\times\mathbf{E})+\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\right]=-iJ+\tfrac{1}{2}i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H})-i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ -iI+\tfrac{1}{2}\left[\mathbf{H}^*\bullet(\nabla\times\mathbf{E})-\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\right]=\tfrac{1}{2}i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H})+i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ iJ+\tfrac{1}{2}\left[\mathbf{H}^*\bullet(\nabla\times\mathbf{E})+\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\right]=\tfrac{1}{2}i\mu\beta\mathbf{H}^*\bullet(\nabla\times\mathbf{H})-i\varepsilon\beta\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ I+i\tfrac{1}{2}\left[\mathbf{H}^*\bullet(\nabla\times\mathbf{E})-\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\right]=-\tfrac{1}{2}\beta\left[\mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H})+\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E})\right] \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ J-i\tfrac{1}{2}\left[\mathbf{H}^*\bullet(\nabla\times\mathbf{E})+\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\right]=\tfrac{1}{2}\beta\left[\mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H})-\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E})\right] \end{cases}$$ (S2.3)

Meanwhile, we make use of the set of dot and cross products. Taking difference and sum, we can prepare for the Poynting vectors.

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ I+i\tfrac{1}{2}\begin{bmatrix}\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)+\nabla\bullet(\mathbf{E}^*\times\mathbf{H}) \\ +(\nabla\times\mathbf{H}^*)\bullet\mathbf{E}-(\nabla\times\mathbf{E}^*)\bullet\mathbf{H}\end{bmatrix}=-\tfrac{1}{2}\beta\begin{bmatrix}\mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H}) \\ +\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E})\end{bmatrix} \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ J-i\tfrac{1}{2}\begin{bmatrix}\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)-\nabla\bullet(\mathbf{E}^*\times\mathbf{H}) \\ +(\nabla\times\mathbf{H}^*)\bullet\mathbf{E}+(\nabla\times\mathbf{E}^*)\bullet\mathbf{H}\end{bmatrix}=\tfrac{1}{2}\beta\begin{bmatrix}\mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H}) \\ -\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E})\end{bmatrix} \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ I+i\tfrac{1}{2}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)+\nabla\bullet(\mathbf{E}^*\times\mathbf{H})\right] \\ \quad +i\tfrac{1}{2}\left[(\nabla\times\mathbf{H}^*)\bullet\mathbf{E}-(\nabla\times\mathbf{E}^*)\bullet\mathbf{H}\right]=-\tfrac{1}{2}\beta\left[\mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H})+\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E})\right] \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ J-i\tfrac{1}{2}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)-\nabla\bullet(\mathbf{E}^*\times\mathbf{H})\right] \\ \quad -i\tfrac{1}{2}\left[(\nabla\times\mathbf{H}^*)\bullet\mathbf{E}+(\nabla\times\mathbf{E}^*)\bullet\mathbf{H}\right]=\tfrac{1}{2}\beta\left[\mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H})-\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E})\right] \end{cases}$$ (S2.4)

Consider the following real and imaginary parts.

$$\begin{cases} \tfrac{1}{2}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)+\nabla\bullet(\mathbf{E}^*\times\mathbf{H})\right]=\mathrm{Re}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)\right] \\ -i\tfrac{1}{2}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)-\nabla\bullet(\mathbf{E}^*\times\mathbf{H})\right]=\mathrm{Im}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*)\right] \end{cases}$$ (S2.5)

Therefore, the difference and sum become the following.

$$\begin{cases} \mathbf{H}^*\cdot(\mathbf{FL})-\mathbf{E}^*\cdot(\mathbf{AL}): \ I+i\operatorname{Re}\left[\nabla\cdot\left(\mathbf{E}\times\mathbf{H}^*\right)\right] \\ \quad +i\tfrac{1}{2}\left[\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{E}-\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{H}\right]=-\tfrac{1}{2}\beta\left[\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)+\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)\right] \\ \mathbf{H}^*\cdot(\mathbf{FL})+\mathbf{E}^*\cdot(\mathbf{AL}): \ J+\operatorname{Im}\left[\nabla\cdot\left(\mathbf{E}\times\mathbf{H}^*\right)\right] \\ \quad -i\tfrac{1}{2}\left[\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{E}+\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{H}\right]=\tfrac{1}{2}\beta\left[\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)-\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)\right] \end{cases} \tag{S2.6}$$

From here on, we assume a medium under consideration to be lossless so that both of $\{\varepsilon,\mu\}$ are real for simplicity. Furthermore, we take β is real as well so that $\{\varepsilon,\mu,\beta\}\in\mathbb{R}$. Our system is henceforth Hermitian. Separating into real and imaginary parts.

$\{\varepsilon,\mu,\beta\}\in\mathbb{R} \ \Rightarrow$

$\mathbf{H}^*\cdot(\mathbf{FL})-\mathbf{E}^*\cdot(\mathbf{AL}):$

$$\begin{cases} I=\tfrac{1}{2}\operatorname{Im}\left[\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{E}-\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{H}\right]-\tfrac{1}{2}\beta\operatorname{Re}\left[\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)+\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)\right] \\ \nabla\cdot\mathbf{P}=-\tfrac{1}{2}\operatorname{Re}\left[\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{E}-\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{H}\right]-\tfrac{1}{2}\beta\operatorname{Im}\left[\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)+\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)\right] \end{cases} \tag{S2.7}$$

$\mathbf{H}^*\cdot(\mathbf{FL})+\mathbf{E}^*\cdot(\mathbf{AL}):$

$$\begin{cases} J+\nabla\cdot\mathbf{R} \\ \quad =-\tfrac{1}{2}\operatorname{Im}\left[\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{E}+\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{H}\right]+\tfrac{1}{2}\beta\operatorname{Re}\left[\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)-\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)\right] \\ \tfrac{1}{2}\operatorname{Re}\left[\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{E}+\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{H}\right]=-\tfrac{1}{2}\beta\operatorname{Im}\left[\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)-\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)\right] \end{cases}$$

Here, we are prompted to introduce several intermediaries for notational simplicity.

$$\begin{cases} \mathbf{A}_\pm\equiv\left[\mathbf{E}^*\cdot\left(\nabla\times\mathbf{H}\right)\mp\mathbf{H}^*\cdot\left(\nabla\times\mathbf{E}\right)\right]^* \\ \mathbf{B}_\pm\equiv\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)\mp\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right) \end{cases} \Rightarrow$$

$$\begin{cases} \mathbf{H}^*\cdot(\mathbf{FL})-\mathbf{E}^*\cdot(\mathbf{AL}): \begin{cases} I=\tfrac{1}{2}\operatorname{Im}\left(\mathbf{A}_+\right)-\tfrac{1}{2}\beta\operatorname{Re}\left(\mathbf{B}_-\right) \\ \nabla\cdot\mathbf{P}=-\tfrac{1}{2}\operatorname{Re}\left(\mathbf{A}_+\right)-\tfrac{1}{2}\beta\operatorname{Im}\left(\mathbf{B}_-\right) \end{cases} \\ \mathbf{H}^*\cdot(\mathbf{FL})+\mathbf{E}^*\cdot(\mathbf{AL}): \begin{cases} J+\nabla\cdot\mathbf{R}=-\tfrac{1}{2}\operatorname{Im}\left(\mathbf{A}_-\right)+\tfrac{1}{2}\beta\operatorname{Re}\left(\mathbf{B}_+\right) \\ \operatorname{Re}\left(\mathbf{A}_-\right)+\beta\operatorname{Im}\left(\mathbf{B}_+\right)=0 \end{cases} \end{cases} \tag{S2.8}$$

For an achiral medium with $\beta=0$ for a curl-β-based constitutive relations, we have a physically meaningful reduced set as follows.

$$\beta=0 \ \Rightarrow \ \begin{cases}\nabla\times\mathbf{E}=i\mu\mathbf{H} \\ \nabla\times\mathbf{H}=-i\varepsilon\mathbf{E}\end{cases}, \begin{cases}\nabla\cdot\mathbf{E}=0 \\ \nabla\cdot\mathbf{H}=0\end{cases} \Rightarrow$$

$$\mathbf{B}_+\equiv\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)-\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)=\mu\mathbf{H}^*\cdot\left(-i\varepsilon\mathbf{E}\right)-\varepsilon\mathbf{E}^*\cdot\left(i\mu\mathbf{H}\right)$$
$$=i\varepsilon\mu\left(\mathbf{H}^*\cdot\mathbf{E}+\mathbf{E}^*\cdot\mathbf{H}\right)=2i\varepsilon\mu\operatorname{Re}\left(\mathbf{E}\cdot\mathbf{H}^*\right)\equiv2i\varepsilon\mu\mathscr{K}$$

$$\mathbf{B}_-\equiv\mu\mathbf{H}^*\cdot\left(\nabla\times\mathbf{H}\right)+\varepsilon\mathbf{E}^*\cdot\left(\nabla\times\mathbf{E}\right)=\mu\mathbf{H}^*\cdot\left(-i\varepsilon\mathbf{E}\right)+\varepsilon\mathbf{E}^*\cdot\left(i\mu\mathbf{H}\right)$$
$$=i\varepsilon\mu\left(\mathbf{E}^*\cdot\mathbf{H}-\mathbf{H}^*\cdot\mathbf{E}\right)=i\varepsilon\mu2i\operatorname{Im}\left(\mathbf{E}^*\cdot\mathbf{H}\right)=-2\varepsilon\mu\operatorname{Im}\left(\mathbf{E}^*\cdot\mathbf{H}\right)=2\varepsilon\mu\operatorname{Im}\left(\mathbf{E}\cdot\mathbf{H}^*\right)\equiv2\varepsilon\mu\mathscr{C}$$

$$\begin{cases}\mathbf{B}_+=2i\varepsilon\mu\mathscr{K} \\ \mathbf{B}_-=2\varepsilon\mu\mathscr{C}\end{cases} \Rightarrow \ \{\mathbf{B}_+,\mathbf{B}_-\}=2\varepsilon\mu\{i\mathscr{K},\mathscr{C}\} \ \Rightarrow \ \mathbf{B}_-+\mathbf{B}_+=2\varepsilon\mu\left(\mathscr{C}+i\mathscr{K}\right) \tag{S2.9}$$

We make sure that the achiral limit leads to a correct simple set.

$$\beta = 0 \;\Rightarrow\; \begin{cases} \nabla \times \mathbf{E} = i\mu\mathbf{H} \\ \nabla \times \mathbf{H} = -i\varepsilon\mathbf{E} \end{cases}, \quad \begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \;\Rightarrow\; \Rightarrow\; \begin{cases} \mathbf{H}^* \cdot (\mathbf{FL}) - \mathbf{E}^* \cdot (\mathbf{AL}): \\ \mathbf{H}^* \cdot (\mathbf{FL}) + \mathbf{E}^* \cdot (\mathbf{AL}): \end{cases}$$

$$\begin{cases} \mathbf{A}_+ \equiv \left(\nabla \times \mathbf{H}^*\right)\cdot\mathbf{E} - \left(\nabla \times \mathbf{E}^*\right)\cdot\mathbf{H} = \left(i\varepsilon\mathbf{E}^*\right)\cdot\mathbf{E} - \left(-i\mu\mathbf{H}^*\right)\cdot\mathbf{H} = 2iI \\ \mathbf{A}_- \equiv \left(\nabla \times \mathbf{H}^*\right)\cdot\mathbf{E} + \left(\nabla \times \mathbf{E}^*\right)\cdot\mathbf{H} = \left(i\varepsilon\mathbf{E}^*\right)\cdot\mathbf{E} + \left(-i\mu\mathbf{H}^*\right)\cdot\mathbf{H} = 2iJ \end{cases} \;\Rightarrow\; . \qquad \text{(s2.10)}$$

$$\begin{cases} \mathbf{H}^* \cdot (\mathbf{FL}) - \mathbf{E}^* \cdot (\mathbf{AL}): & \begin{cases} I = I \\ \nabla \cdot \mathbf{P} = 0 \end{cases} \\ \mathbf{H}^* \cdot (\mathbf{FL}) + \mathbf{E}^* \cdot (\mathbf{AL}): & \begin{cases} J + \nabla \cdot \mathbf{R} = -J \\ 0 = 0 \end{cases} \end{cases}$$

Let us verify the reduction to achiral media by utilizing the Maxwell equations transformed into the following dimensionless forms.

$$\beta = \kappa = 0 \;\Rightarrow\; \begin{cases} \nabla \times \mathbf{E} = i\mu\mathbf{H} \\ \nabla \times \mathbf{H} = -i\varepsilon\mathbf{E} \end{cases}, \quad \begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \;\Rightarrow$$

$$\mathbf{B}_\pm \equiv \mu\mathbf{H}^* \cdot (\nabla \times \mathbf{H}) \mp \varepsilon\mathbf{E}^* \cdot (\nabla \times \mathbf{E}) \;\Rightarrow$$

$$\mathbf{B}_+ + \mathbf{B}_- = 2\mu\mathbf{H}^* \cdot (\nabla \times \mathbf{H}) = 2\mu\mathbf{H}^* \cdot (-i\varepsilon\mathbf{E}) = -i2\varepsilon\mu\mathbf{E} \cdot \mathbf{H}^*$$

$$\begin{cases} \mathbf{B}_+ \equiv \mu\mathbf{H}^* \cdot (\nabla \times \mathbf{H}) - \varepsilon\mathbf{E}^* \cdot (\nabla \times \mathbf{E}) = \mu\mathbf{H}^* \cdot (-i\varepsilon\mathbf{E}) - \varepsilon\mathbf{E}^* \cdot (i\mu\mathbf{H}) \\ \quad = -i\varepsilon\mu\left(\mathbf{E} \cdot \mathbf{H}^* + \mathbf{E}^* \cdot \mathbf{H}\right) = -i2\varepsilon\mu\,\mathrm{Re}\left(\mathbf{E} \cdot \mathbf{H}^*\right) \equiv -i2\varepsilon\mu\mathcal{K} \\ \mathbf{B}_- \equiv \mu\mathbf{H}^* \cdot (\nabla \times \mathbf{H}) + \varepsilon\mathbf{E}^* \cdot (\nabla \times \mathbf{E}) = \mu\mathbf{H}^* \cdot (-i\varepsilon\mathbf{E}) + \varepsilon\mathbf{E}^* \cdot (i\mu\mathbf{H}) \\ \quad = -i\varepsilon\mu\left(\mathbf{E} \cdot \mathbf{H}^* - \mathbf{E}^* \cdot \mathbf{H}\right) = 2\varepsilon\mu\,\mathrm{Im}\left(\mathbf{E} \cdot \mathbf{H}^*\right) \equiv 2\varepsilon\mu\mathcal{C} \end{cases} \qquad \text{(s2.11)}$$

We thus find that $\{\mathbf{B}_+, \mathbf{B}_-\}$ are a pure imaginary parameter and a pure real parameter, respectively.

Section S3. Energy conservation laws for field-based constitutive relations

For a chiral media with the field-κ-based constitutive relations, let us examine the Maxwell equations.

$$\begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases}, \quad \begin{cases} \mathbf{D} = \varepsilon\mathbf{E} + i\kappa\mathbf{H} \\ \mathbf{B} = \mu\mathbf{H} - i\kappa\mathbf{E} \end{cases} \;\Rightarrow\; \begin{cases} \nabla \times \mathbf{E} = i\mathbf{B} \\ \nabla \times \mathbf{H} = -i\mathbf{D} \end{cases} \;\Rightarrow\; \begin{cases} \nabla \times \mathbf{E} = i\mu\mathbf{H} + \kappa\mathbf{E} \\ \nabla \times \mathbf{H} = -i\varepsilon\mathbf{E} + \kappa\mathbf{H} \end{cases}$$

$$K = \begin{pmatrix} \kappa & i\mu \\ -i\varepsilon & \kappa \end{pmatrix} \;\Rightarrow\; K^{-1} = \frac{1}{\kappa^2 - \varepsilon\mu} \begin{pmatrix} \kappa & -i\mu \\ i\varepsilon & \kappa \end{pmatrix} \;\Rightarrow\; . \qquad \text{(S3.1)}$$

$$\nabla \times \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = K \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} \;\Rightarrow\; \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = K^{-1}\nabla \times \begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = \frac{1}{\kappa^2 - \varepsilon\mu} \begin{Bmatrix} \kappa(\nabla \times \mathbf{E}) - i\mu(\nabla \times \mathbf{H}) \\ i\varepsilon(\nabla \times \mathbf{E}) + \kappa(\nabla \times \mathbf{H}) \end{Bmatrix}$$

This portion will be a little simpler than that for the curl-based constitutive relations.
Let us rework the sum and difference for the energy conservation laws.

$$\begin{cases} \nabla\times\mathbf{E} = i\mu\mathbf{H} + \kappa\mathbf{E} \\ \nabla\times\mathbf{H} = -i\varepsilon\mathbf{E} + \kappa\mathbf{H} \end{cases}, \quad \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \end{cases}$$

$$\begin{cases} (\mathbf{FL}): \ \nabla\times\mathbf{E} = i\mu\mathbf{H} + \kappa\mathbf{E} \\ (\mathbf{AL}): \ \nabla\times\mathbf{H} = -i\varepsilon\mathbf{E} + \kappa\mathbf{H} \end{cases} \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}): \ (\nabla\times\mathbf{E})\bullet\mathbf{H}^* = i\mu\mathbf{H}^*\bullet\mathbf{H} + \kappa\mathbf{E}\bullet\mathbf{H}^* \\ \mathbf{E}^*\bullet(\mathbf{AL}): \ (\nabla\times\mathbf{H})\bullet\mathbf{E}^* = -i\varepsilon\mathbf{E}^*\bullet\mathbf{E} + \kappa\mathbf{E}^*\bullet\mathbf{H} \end{cases} \quad . \tag{S3.2}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \ (\nabla\times\mathbf{E})\bullet\mathbf{H}^* - (\nabla\times\mathbf{H})\bullet\mathbf{E}^* = i\varepsilon\mathbf{E}^*\bullet\mathbf{E} + i\mu\mathbf{H}^*\bullet\mathbf{H} + \kappa(\mathbf{E}\bullet\mathbf{H}^* - \mathbf{E}^*\bullet\mathbf{H}) \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \ (\nabla\times\mathbf{E})\bullet\mathbf{H}^* + (\nabla\times\mathbf{H})\bullet\mathbf{E}^* = -i\varepsilon\mathbf{E}^*\bullet\mathbf{E} + i\mu\mathbf{H}^*\bullet\mathbf{H} + \kappa(\mathbf{E}\bullet\mathbf{H}^* + \mathbf{E}^*\bullet\mathbf{H}) \end{cases}$$

We introduce average sum and average difference.

$$\begin{cases} I \equiv \frac{1}{2}(\varepsilon\mathbf{E}^*\bullet\mathbf{E} + \mu\mathbf{H}^*\bullet\mathbf{H}) \\ J \equiv \frac{1}{2}(\varepsilon\mathbf{E}^*\bullet\mathbf{E} - \mu\mathbf{H}^*\bullet\mathbf{H}) \end{cases} \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \ \frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* - (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = iI + i\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \ \frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* + (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = -iJ + \kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases}$$
$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \ -iI + \frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* - (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = i\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \ +iJ + \frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* + (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = \kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases} \quad . \tag{S3.3}$$
$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \ I + i\frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* - (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = -\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \ J - i\frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* + (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = -i\kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases}$$

The formations with the field-based constitutive relations look simpler than those with the curl-based constitutive relations.

Meanwhile, we make use of the set of dot and cross products. Taking difference and sum, we can prepare for the Poynting vectors. Applying vector identities,

$$\begin{cases} \mathbf{A}\times(\nabla\times\mathbf{B}) = \mathbf{A}\bullet(\nabla)\mathbf{B} - (\mathbf{A}\bullet\nabla)\mathbf{B} \\ \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\bullet\mathbf{A}) - \nabla^2\mathbf{A} \\ \nabla\bullet(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\bullet\mathbf{B} - (\nabla\times\mathbf{B})\bullet\mathbf{A} \end{cases} \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \end{cases}$$

$$\begin{cases} dif: \ (\nabla\times\mathbf{E})\bullet\mathbf{H}^* + (\nabla\times\mathbf{H})\bullet\mathbf{E}^* = \nabla\bullet(\mathbf{E}\times\mathbf{H}^*) - \nabla\bullet(\mathbf{E}^*\times\mathbf{H}) + (\nabla\times\mathbf{H}^*)\bullet\mathbf{E} + (\nabla\times\mathbf{E}^*)\bullet\mathbf{H} \\ sum: \ (\nabla\times\mathbf{E})\bullet\mathbf{H}^* - (\nabla\times\mathbf{H})\bullet\mathbf{E}^* = \nabla\bullet(\mathbf{E}\times\mathbf{H}^*) + \nabla\bullet(\mathbf{E}^*\times\mathbf{H}) + (\nabla\times\mathbf{H}^*)\bullet\mathbf{E} - (\nabla\times\mathbf{E}^*)\bullet\mathbf{H} \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \ I + i\frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* - (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = -\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \ J - i\frac{1}{2}\left[(\nabla\times\mathbf{E})\bullet\mathbf{H}^* + (\nabla\times\mathbf{H})\bullet\mathbf{E}^*\right] = -i\kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \ I + i\frac{1}{2}\begin{bmatrix} \nabla\bullet(\mathbf{E}\times\mathbf{H}^*) + \nabla\bullet(\mathbf{E}^*\times\mathbf{H}) \\ +(\nabla\times\mathbf{H}^*)\bullet\mathbf{E} - (\nabla\times\mathbf{E}^*)\bullet\mathbf{H} \end{bmatrix} = -\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\[2em] \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \ J - i\frac{1}{2}\begin{bmatrix} \nabla\bullet(\mathbf{E}\times\mathbf{H}^*) - \nabla\bullet(\mathbf{E}^*\times\mathbf{H}) \\ +(\nabla\times\mathbf{H}^*)\bullet\mathbf{E} + (\nabla\times\mathbf{E}^*)\bullet\mathbf{H} \end{bmatrix} = -i\kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL}) - \mathbf{E}^*\bullet(\mathbf{AL}): \\ I + i\frac{1}{2}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*) + \nabla\bullet(\mathbf{E}^*\times\mathbf{H})\right] + i\frac{1}{2}\left[(\nabla\times\mathbf{H}^*)\bullet\mathbf{E} - (\nabla\times\mathbf{E}^*)\bullet\mathbf{H}\right] = -\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \mathbf{H}^*\bullet(\mathbf{FL}) + \mathbf{E}^*\bullet(\mathbf{AL}): \\ J - i\frac{1}{2}\left[\nabla\bullet(\mathbf{E}\times\mathbf{H}^*) - \nabla\bullet(\mathbf{E}^*\times\mathbf{H})\right] - i\frac{1}{2}\left[(\nabla\times\mathbf{H}^*)\bullet\mathbf{E} + (\nabla\times\mathbf{E}^*)\bullet\mathbf{H}\right] = -i\kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases} \quad . \tag{S3.4}$$

Let us separate the above pair respectively into the following real and imaginary parts, while introducing the following pairs of auxiliary vectors.

$$\begin{cases} \mathbf{P} \equiv \mathrm{Re}(\mathbf{E}\times\mathbf{H}^*) \\ \mathbf{R} \equiv \mathrm{Im}(\mathbf{E}\times\mathbf{H}^*) \end{cases}, \begin{cases} \mathbf{A}_\pm \equiv \left[\mathbf{E}^*\bullet(\nabla\times\mathbf{H})\mp\mathbf{H}^*\bullet(\nabla\times\mathbf{E})\right]^* \\ \mathbf{B}_\pm \equiv \mu\mathbf{H}^*\bullet(\nabla\times\mathbf{H})\mp\varepsilon\mathbf{E}^*\bullet(\nabla\times\mathbf{E}) \end{cases} \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \end{cases}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \ I+i\nabla\bullet\mathbf{P}+i\tfrac{1}{2}\mathbf{A}_+ = -\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \ J+\nabla\bullet\mathbf{R}-i\tfrac{1}{2}\mathbf{A}_- = -i\kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases} \tag{S3.5}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \begin{cases} I-\tfrac{1}{2}\mathrm{Im}(\mathbf{A}_+) = -\kappa\,\mathrm{Im}(\mathbf{E}\bullet\mathbf{H}^*) \\ \nabla\bullet\mathbf{P}+\tfrac{1}{2}\mathrm{Re}(\mathbf{A}_+) = 0 \end{cases} \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \begin{cases} J+\nabla\bullet\mathbf{R}+\tfrac{1}{2}\mathrm{Im}(\mathbf{A}_-) = 0 \\ \tfrac{1}{2}\mathrm{Re}(\mathbf{A}_-) = \kappa\,\mathrm{Re}(\mathbf{E}\bullet\mathbf{H}^*) \end{cases} \end{cases}$$

Let us verify the reduction to achiral media. Likewise, the Maxwell equations get transformed into the following dimensionless forms.

$$\kappa=0 \ \Rightarrow \ \begin{cases} \nabla\times\mathbf{E}=i\mu\mathbf{H} \\ \nabla\times\mathbf{H}=-i\varepsilon\mathbf{E} \end{cases}, \begin{cases} \nabla\bullet\mathbf{E}=0 \\ \nabla\bullet\mathbf{H}=0 \end{cases} \Rightarrow \ \Rightarrow \begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \end{cases}$$

$$\begin{cases} \mathbf{A}_+ \equiv (\nabla\times\mathbf{H}^*)\bullet\mathbf{E}-(\nabla\times\mathbf{E}^*)\bullet\mathbf{H}=(i\varepsilon\mathbf{E}^*)\bullet\mathbf{E}-(-i\mu\mathbf{H}^*)\bullet\mathbf{H}=2iI \\ \mathbf{A}_- \equiv (\nabla\times\mathbf{H}^*)\bullet\mathbf{E}+(\nabla\times\mathbf{E}^*)\bullet\mathbf{H}=(i\varepsilon\mathbf{E}^*)\bullet\mathbf{E}+(-i\mu\mathbf{H}^*)\bullet\mathbf{H}=2iJ \end{cases} \Rightarrow \tag{S3.6}$$

$$\begin{cases} \mathbf{H}^*\bullet(\mathbf{FL})-\mathbf{E}^*\bullet(\mathbf{AL}): \begin{cases} I=I \\ \nabla\bullet\mathbf{P}=0 \end{cases} \\ \mathbf{H}^*\bullet(\mathbf{FL})+\mathbf{E}^*\bullet(\mathbf{AL}): \begin{cases} J+\nabla\bullet\mathbf{R}+J=0 \\ 0=0 \end{cases} \end{cases} \Rightarrow \begin{cases} \nabla\bullet\mathbf{P}=0 \\ \nabla\bullet\mathbf{R}+2J=0 \end{cases}$$

Section S4. Decompositions of the Poynting vectors

We define the vectors $\mathbf{T}_\pm$, while making use of the vector equality valid for the gradient ∇J of the reactive energy J defined in the main text.

$$\begin{cases} \mathbf{T}_+ \equiv \mathbf{H}^*\bullet(\nabla)\mathbf{E}-\mathbf{E}^*\bullet(\nabla)\mathbf{H}+(\mathbf{E}^*\bullet\nabla)\mathbf{H}-(\mathbf{H}^*\bullet\nabla)\mathbf{E} \\ \mathbf{T}_- \equiv \mathbf{E}^*\bullet(\nabla)\mathbf{H}+\mathbf{H}^*\bullet(\nabla)\mathbf{E}-(\mathbf{E}^*\bullet\nabla)\mathbf{H}-(\mathbf{H}^*\bullet\nabla)\mathbf{E} \end{cases}$$
$$\mathrm{Re}\left[\varepsilon\mathbf{E}^*\bullet(\nabla)\mathbf{E}-\mu\mathbf{H}^*\bullet(\nabla)\mathbf{H}\right] \tag{S4.1}$$
$$= \varepsilon\nabla\left(\tfrac{1}{2}|\mathbf{E}|^2\right)-\mu\nabla\left(\tfrac{1}{2}|\mathbf{H}|^2\right) \equiv \nabla\left[\tfrac{1}{2}\left(\varepsilon|\mathbf{E}|^2-\mu|\mathbf{H}|^2\right)\right] \equiv \nabla J$$

The matrix relation $\{\mathbf{E},\mathbf{H}\}=K^{-1}\nabla\times\{\mathbf{E},\mathbf{H}\}$ with Eq. (s08) is combined with the vector identity $\mathbf{M}\times(\nabla\times\mathbf{N})=\mathbf{M}\bullet(\nabla)\mathbf{N}-(\mathbf{M}\bullet\nabla)\mathbf{N}$ for any $\{\mathbf{M},\mathbf{N}\}$ in obtaining the following decomposition rules for the curl-β-based constitutive relations with $\varepsilon,\mu,\beta\in\mathbb{R}$.

For a chiral media with the curl-β-based constitutive relations, let us examine the complex Poynting vector. There are both self-terms $\left\{ \mathbf{E} \times \left(\nabla \times \mathbf{E}^* \right), \left(\nabla \times \mathbf{H} \right) \times \mathbf{H}^* \right\}$ and cross-terms $\left\{ \mathbf{E} \times \left(\nabla \times \mathbf{H}^* \right), \left(\nabla \times \mathbf{E} \right) \times \mathbf{H}^* \right\}$ for the complex Poynting vector when expanded as follows.

$$\left\{ \begin{array}{l} \mathbf{P} + i\mathbf{R} \equiv \mathbf{E} \times \mathbf{H}^* = \mathbf{E} \times \left[-i\frac{1}{\mu}\left(\nabla \times \mathbf{E} \right) - \beta \left(\nabla \times \mathbf{H} \right) \right]^* = i\frac{1}{\mu}\mathbf{E} \times \left(\nabla \times \mathbf{E}^* \right) - \beta \mathbf{E} \times \left(\nabla \times \mathbf{H}^* \right) \\ \mathbf{P} + i\mathbf{R} \equiv \mathbf{E} \times \mathbf{H}^* = \left[-\beta \left(\nabla \times \mathbf{E} \right) + i\frac{1}{\varepsilon}\left(\nabla \times \mathbf{H} \right) \right] \times \mathbf{H}^* = -\beta \left(\nabla \times \mathbf{E} \right) \times \mathbf{H}^* + i\frac{1}{\varepsilon}\left(\nabla \times \mathbf{H} \right) \times \mathbf{H}^* \end{array} \right. \tag{S4.2}$$

Both cross-terms carry the chirality parameter $\beta \neq 0$ as should do. There are two ways that are suitable for the electric-magnetic democracy.

$$\mathbf{A} \times \left(\nabla \times \mathbf{B} \right) = \mathbf{A} \bullet \left(\nabla \right) \mathbf{B} - \left(\mathbf{A} \bullet \nabla \right) \mathbf{B} \quad \Rightarrow$$

$$\mathbf{P} + i\mathbf{R} \equiv \mathbf{E} \times \mathbf{H}^* = i\frac{1}{\mu}\mathbf{E} \times \left(\nabla \times \mathbf{E}^* \right) - \beta \mathbf{E} \times \left(\nabla \times \mathbf{H}^* \right)$$

$$= i\frac{1}{\mu}\left[\mathbf{E} \bullet \left(\nabla \right) \mathbf{E}^* - \left(\mathbf{E} \bullet \nabla \right) \mathbf{E}^* \right] - \beta \left[\mathbf{E} \bullet \left(\nabla \right) \mathbf{H}^* - \left(\mathbf{E} \bullet \nabla \right) \mathbf{H}^* \right] \tag{S4.3}$$

$$\mathbf{P} + i\mathbf{R} \equiv \mathbf{E} \times \mathbf{H}^* = -\beta \left(\nabla \times \mathbf{E} \right) \times \mathbf{H}^* + i\frac{1}{\varepsilon}\left(\nabla \times \mathbf{H} \right) \times \mathbf{H}^* = \beta \mathbf{H}^* \times \left(\nabla \times \mathbf{E} \right) - i\frac{1}{\varepsilon}\mathbf{H}^* \times \left(\nabla \times \mathbf{H} \right)$$

$$= -i\frac{1}{\varepsilon}\left[\mathbf{H}^* \bullet \left(\nabla \right) \mathbf{H} - \left(\mathbf{H}^* \bullet \nabla \right) \mathbf{H} \right] + \beta \left[\mathbf{H}^* \bullet \left(\nabla \right) \mathbf{E} - \left(\mathbf{H}^* \bullet \nabla \right) \mathbf{E} \right]$$

Resultantly, the decomposition of the Poynting vector gets complicated. Notwithstanding, we still discover the orbital-like and spin-like parts. The EM and reactive Poynting vectors are separated as follows in two ways (electric-based versus magnetic-based), when taking the real and imaginary parts of the above pair of formulas.

$$\left\{ \begin{array}{l} \mathbf{P} \equiv \mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^* \right) = -\mathrm{Im}\left\{ \frac{1}{\mu}\left[\mathbf{E} \bullet \left(\nabla \right) \mathbf{E}^* - \left(\mathbf{E} \bullet \nabla \right) \mathbf{E}^* \right] \right\} - \mathrm{Re}\left\{ \beta \left[\mathbf{E} \bullet \left(\nabla \right) \mathbf{H}^* - \left(\mathbf{E} \bullet \nabla \right) \mathbf{H}^* \right] \right\} \\ \quad = \mathrm{Im}\left\{ \frac{1}{\mu}\left[\mathbf{E}^* \bullet \left(\nabla \right) \mathbf{E} - \left(\mathbf{E}^* \bullet \nabla \right) \mathbf{E} \right] \right\} - \mathrm{Re}\left\{ \beta \left[\mathbf{E}^* \bullet \left(\nabla \right) \mathbf{H} - \left(\mathbf{E}^* \bullet \nabla \right) \mathbf{H} \right] \right\} \\ \mathbf{R} \equiv \mathrm{Im}\left(\mathbf{E} \times \mathbf{H}^* \right) = \mathrm{Re}\left\{ \frac{1}{\mu}\left[\mathbf{E} \bullet \left(\nabla \right) \mathbf{E}^* - \left(\mathbf{E} \bullet \nabla \right) \mathbf{E}^* \right] \right\} - \mathrm{Im}\left\{ \beta \left[\mathbf{E} \bullet \left(\nabla \right) \mathbf{H}^* - \left(\mathbf{E} \bullet \nabla \right) \mathbf{H}^* \right] \right\} \\ \quad = \mathrm{Re}\left\{ \frac{1}{\mu}\left[\mathbf{E}^* \bullet \left(\nabla \right) \mathbf{E} - \left(\mathbf{E}^* \bullet \nabla \right) \mathbf{E} \right] \right\} + \mathrm{Im}\left\{ \beta \left[\mathbf{E}^* \bullet \left(\nabla \right) \mathbf{H} - \left(\mathbf{E}^* \bullet \nabla \right) \mathbf{H} \right] \right\} \end{array} \right. \tag{S4.4}$$

Meanwhile,

$$\left\{ \begin{array}{l} \mathbf{P} \equiv \mathrm{Re}\left(\mathbf{E} \times \mathbf{H}^* \right) = \mathrm{Im}\left\{ \frac{1}{\varepsilon}\left[\mathbf{H}^* \bullet \left(\nabla \right) \mathbf{H} - \left(\mathbf{H}^* \bullet \nabla \right) \mathbf{H} \right] \right\} + \mathrm{Re}\left\{ \beta \left[\mathbf{H}^* \bullet \left(\nabla \right) \mathbf{E} - \left(\mathbf{H}^* \bullet \nabla \right) \mathbf{E} \right] \right\} \\ \mathbf{R} \equiv \mathrm{Im}\left(\mathbf{E} \times \mathbf{H}^* \right) = -\mathrm{Re}\left\{ \frac{1}{\varepsilon}\left[\mathbf{H}^* \bullet \left(\nabla \right) \mathbf{H} - \left(\mathbf{H}^* \bullet \nabla \right) \mathbf{H} \right] \right\} + \mathrm{Im}\left\{ \beta \left[\mathbf{H}^* \bullet \left(\nabla \right) \mathbf{E} - \left(\mathbf{H}^* \bullet \nabla \right) \mathbf{E} \right] \right\} \end{array} \right. \tag{S4.5}$$

Therefore, we encounter additional terms due to optical chirality.
Taking the averages of the two respective pairs, we get the followings.

$$\begin{cases} \mathbf{P} = \frac{1}{2}\mathrm{Im}\begin{Bmatrix} \frac{1}{\mu}\left[\mathbf{E}^*\boldsymbol{\cdot}(\nabla)\mathbf{E}-(\mathbf{E}^*\boldsymbol{\cdot}\nabla)\mathbf{E}\right] \\ +\frac{1}{\varepsilon}\left[\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{H}-(\mathbf{H}^*\boldsymbol{\cdot}\nabla)\mathbf{H}\right] \end{Bmatrix}+\frac{1}{2}\mathrm{Re}\begin{Bmatrix} \beta\left[\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{E}-(\mathbf{H}^*\boldsymbol{\cdot}\nabla)\mathbf{E}\right] \\ -\beta\left[\mathbf{E}^*\boldsymbol{\cdot}(\nabla)\mathbf{H}-(\mathbf{E}^*\boldsymbol{\cdot}\nabla)\mathbf{H}\right] \end{Bmatrix} \\[4mm] \mathbf{R} = \frac{1}{2}\mathrm{Re}\begin{Bmatrix} \frac{1}{\mu}\left[\mathbf{E}^*\boldsymbol{\cdot}(\nabla)\mathbf{E}-(\mathbf{E}^*\boldsymbol{\cdot}\nabla)\mathbf{E}\right] \\ -\frac{1}{\varepsilon}\left[\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{H}-(\mathbf{H}^*\boldsymbol{\cdot}\nabla)\mathbf{H}\right] \end{Bmatrix}+\frac{1}{2}\mathrm{Im}\begin{Bmatrix} \beta\left[\mathbf{E}^*\boldsymbol{\cdot}(\nabla)\mathbf{H}-(\mathbf{E}^*\boldsymbol{\cdot}\nabla)\mathbf{H}\right] \\ +\beta\left[\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{E}-(\mathbf{H}^*\boldsymbol{\cdot}\nabla)\mathbf{E}\right] \end{Bmatrix} \end{cases} \tag{S4.6}$$

In terms of the short-hand notations $\left\{\mathbf{O}_\beta,\mathbf{S}_\beta,J,\mathbf{T}_\pm\right\}$, we obtain the following concise set.

$$C_\beta^{curl}:\begin{cases} \mathbf{O}_\beta \equiv \frac{1}{2}\mathrm{Im}\left[\mu^{-1}\mathbf{E}^*\boldsymbol{\cdot}(\nabla)\mathbf{E}+\varepsilon^{-1}\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{H}\right] \\ \mathbf{S}_\beta \equiv -\frac{1}{2}\mathrm{Im}\left[\mu^{-1}\left(\mathbf{E}^*\boldsymbol{\cdot}\nabla\right)\mathbf{E}+\varepsilon^{-1}\left(\mathbf{H}^*\boldsymbol{\cdot}\nabla\right)\mathbf{H}\right] \\ \mathbf{P}=\mathbf{O}_\beta+\mathbf{S}_\beta+\frac{1}{2}\beta\,\mathrm{Re}\left(\mathbf{T}_+\right) \\ \mathbf{R}=\frac{1}{2}\left(\varepsilon\mu\right)^{-1}\nabla J+\frac{1}{2}\mathrm{Re}\left[\varepsilon^{-1}\left(\mathbf{H}^*\boldsymbol{\cdot}\nabla\right)\mathbf{H}-\mu^{-1}\left(\mathbf{E}^*\boldsymbol{\cdot}\nabla\right)\mathbf{E}\right]+\frac{1}{2}\beta\,\mathrm{Im}\left(\mathbf{T}_-\right) \end{cases} \tag{S4.7}$$

Meanwhile, for the field-κ-based constitutive relations with $\varepsilon,\mu,\kappa\in\mathbb{R}$, we obtain the following decomposition rules.

For a chiral media with the field-κ-based constitutive relations, let us examine the complex Poynting vector. There are both self-terms $\left\{\mathbf{E}\times(\nabla\times\mathbf{E}^*),(\nabla\times\mathbf{H})\times\mathbf{H}^*\right\}$ and cross-terms $\left\{\mathbf{E}\times(\nabla\times\mathbf{H}^*),(\nabla\times\mathbf{E})\times\mathbf{H}^*\right\}$ for the complex Poynting vector when expanded as follows.

$$\begin{cases} \mathbf{P}+i\mathbf{R}\equiv\mathbf{E}\times\mathbf{H}^*=\mathbf{E}\times\left[\dfrac{i\varepsilon(\nabla\times\mathbf{E})+\kappa(\nabla\times\mathbf{H})}{\kappa^2-\varepsilon\mu}\right]^*=\dfrac{-i\varepsilon\mathbf{E}\times(\nabla\times\mathbf{E}^*)+\kappa\mathbf{E}\times(\nabla\times\mathbf{H}^*)}{\kappa^2-\varepsilon\mu} \\[4mm] \mathbf{P}+i\mathbf{R}\equiv\mathbf{E}\times\mathbf{H}^*=\left[\dfrac{\kappa(\nabla\times\mathbf{E})-i\mu(\nabla\times\mathbf{H})}{\kappa^2-\varepsilon\mu}\right]\times\mathbf{H}^*=\dfrac{\kappa(\nabla\times\mathbf{E})\times\mathbf{H}^*-i\mu(\nabla\times\mathbf{H})\times\mathbf{H}^*}{\kappa^2-\varepsilon\mu} \end{cases} \tag{S4.8}$$

Both cross-terms carry the chirality parameter $\beta\neq 0$ as should do. There are two ways that are suitable for the electric-magnetic democracy.

$$\textcolor{red}{\mathbf{A}\times(\nabla\times\mathbf{B})=\mathbf{A}\boldsymbol{\cdot}(\nabla)\mathbf{B}-(\mathbf{A}\boldsymbol{\cdot}\nabla)\mathbf{B}\ \Rightarrow}$$

$$\mathbf{P}+i\mathbf{R}\equiv\mathbf{E}\times\mathbf{H}^*=\dfrac{-i\varepsilon\mathbf{E}\times(\nabla\times\mathbf{E}^*)+\kappa\mathbf{E}\times(\nabla\times\mathbf{H}^*)}{\kappa^2-\varepsilon\mu}$$
$$=\dfrac{-i\varepsilon}{\kappa^2-\varepsilon\mu}\left[\mathbf{E}\boldsymbol{\cdot}(\nabla)\mathbf{E}^*-(\mathbf{E}\boldsymbol{\cdot}\nabla)\mathbf{E}^*\right]+\dfrac{\kappa}{\kappa^2-\varepsilon\mu}\left[\mathbf{E}\boldsymbol{\cdot}(\nabla)\mathbf{H}^*-(\mathbf{E}\boldsymbol{\cdot}\nabla)\mathbf{H}^*\right]$$
$$\mathbf{P}+i\mathbf{R}\equiv\mathbf{E}\times\mathbf{H}^*=\dfrac{\kappa(\nabla\times\mathbf{E})\times\mathbf{H}^*-i\mu(\nabla\times\mathbf{H})\times\mathbf{H}^*}{\kappa^2-\varepsilon\mu}=-\dfrac{\kappa\mathbf{H}^*\times(\nabla\times\mathbf{E})-i\mu\mathbf{H}^*\times(\nabla\times\mathbf{H})}{\kappa^2-\varepsilon\mu} \tag{S4.9}$$
$$=i\dfrac{\mu}{\kappa^2-\varepsilon\mu}\left[\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{H}-(\mathbf{H}^*\boldsymbol{\cdot}\nabla)\mathbf{H}\right]-\dfrac{\kappa}{\kappa^2-\varepsilon\mu}\left[\mathbf{H}^*\boldsymbol{\cdot}(\nabla)\mathbf{E}-(\mathbf{H}^*\boldsymbol{\cdot}\nabla)\mathbf{E}\right]$$

Resultantly, the decomposition of the Poynting vector gets complicated. Notwithstanding, we still discover the orbital-like and spin-like parts. The EM and reactive Poynting vectors are separated as follows in two ways (electric-based versus magnetic-based), when taking the real and imaginary parts of the above pair of formulas.

$$\begin{cases} \mathbf{P} = \dfrac{\varepsilon}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left[\mathbf{E}\bullet(\nabla)\mathbf{E}^* - (\mathbf{E}\bullet\nabla)\mathbf{E}^*\right] + \dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left[\mathbf{E}\bullet(\nabla)\mathbf{H}^* - (\mathbf{E}\bullet\nabla)\mathbf{H}^*\right] \\[2mm] \quad = -\dfrac{\varepsilon}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left[\mathbf{E}^*\bullet(\nabla)\mathbf{E} - (\mathbf{E}^*\bullet\nabla)\mathbf{E}\right] + \dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left[\mathbf{E}^*\bullet(\nabla)\mathbf{H} - (\mathbf{E}^*\bullet\nabla)\mathbf{H}\right] \\[2mm] \mathbf{R} = -\dfrac{\varepsilon}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left[\mathbf{E}\bullet(\nabla)\mathbf{E}^* - (\mathbf{E}\bullet\nabla)\mathbf{E}^*\right] + \dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left[\mathbf{E}\bullet(\nabla)\mathbf{H}^* - (\mathbf{E}\bullet\nabla)\mathbf{H}^*\right] \\[2mm] \quad = -\dfrac{\varepsilon}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left[\mathbf{E}^*\bullet(\nabla)\mathbf{E} - (\mathbf{E}^*\bullet\nabla)\mathbf{E}\right] - \dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left[\mathbf{E}^*\bullet(\nabla)\mathbf{H} - (\mathbf{E}^*\bullet\nabla)\mathbf{H}\right] \end{cases} \tag{S4.10}$$

Meanwhile,

$$\begin{cases} \mathbf{P} \equiv \mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^*\right) = -\dfrac{\mu}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left[\mathbf{H}^*\bullet(\nabla)\mathbf{H} - (\mathbf{H}^*\bullet\nabla)\mathbf{H}\right] - \dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left[\mathbf{H}^*\bullet(\nabla)\mathbf{E} - (\mathbf{H}^*\bullet\nabla)\mathbf{E}\right] \\[2mm] \mathbf{R} \equiv \mathrm{Im}\left(\mathbf{E}\times\mathbf{H}^*\right) = \dfrac{\mu}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left[\mathbf{H}^*\bullet(\nabla)\mathbf{H} - (\mathbf{H}^*\bullet\nabla)\mathbf{H}\right] - \dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left[\mathbf{H}^*\bullet(\nabla)\mathbf{E} - (\mathbf{H}^*\bullet\nabla)\mathbf{E}\right] \end{cases} \tag{S4.11}$$

Therefore, we encounter additional terms due to optical chirality. Taking the averages of the two respective pairs, we get the followings.

$$\begin{cases} \mathbf{P} = -\tfrac{1}{2}\dfrac{1}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left\{ \begin{array}{l} \varepsilon\left[\mathbf{E}^*\bullet(\nabla)\mathbf{E} - (\mathbf{E}^*\bullet\nabla)\mathbf{E}\right] \\ +\mu\left[\mathbf{H}^*\bullet(\nabla)\mathbf{H} - (\mathbf{H}^*\bullet\nabla)\mathbf{H}\right] \end{array} \right\} + \tfrac{1}{2}\dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left\{ \begin{array}{l} \mathbf{E}^*\bullet(\nabla)\mathbf{H} - (\mathbf{E}^*\bullet\nabla)\mathbf{H} \\ -\mathbf{H}^*\bullet(\nabla)\mathbf{E} + (\mathbf{H}^*\bullet\nabla)\mathbf{E} \end{array} \right\} \\[4mm] \mathbf{R} = \tfrac{1}{2}\dfrac{1}{\kappa^2 - \varepsilon\mu}\,\mathrm{Re}\left\{ \begin{array}{l} -\varepsilon\left[\mathbf{E}^*\bullet(\nabla)\mathbf{E} - (\mathbf{E}^*\bullet\nabla)\mathbf{E}\right] \\ +\mu\left[\mathbf{H}^*\bullet(\nabla)\mathbf{H} - (\mathbf{H}^*\bullet\nabla)\mathbf{H}\right] \end{array} \right\} - \tfrac{1}{2}\dfrac{\kappa}{\kappa^2 - \varepsilon\mu}\,\mathrm{Im}\left\{ \begin{array}{l} \mathbf{E}^*\bullet(\nabla)\mathbf{H} - (\mathbf{E}^*\bullet\nabla)\mathbf{H} \\ +\mathbf{H}^*\bullet(\nabla)\mathbf{E} - (\mathbf{H}^*\bullet\nabla)\mathbf{E} \end{array} \right\} \end{cases} \tag{S4.12}$$

In terms of the short-hand notations $\left\{\mathbf{O}_\beta, \mathbf{S}_\beta, J, \mathbf{T}_\pm\right\}$, we obtain the following concise set.

$$C_\kappa^{field} : \begin{cases} \mathbf{O}_\kappa \equiv \tfrac{1}{2}\,\mathrm{Im}\left[\dfrac{\varepsilon\mathbf{E}^*\bullet(\nabla)\mathbf{E} + \mu\mathbf{H}^*\bullet(\nabla)\mathbf{H}}{\varepsilon\mu - \kappa^2}\right] \\[4mm] \mathbf{S}_\kappa \equiv -\tfrac{1}{2}\,\mathrm{Im}\left[\dfrac{\varepsilon(\mathbf{E}^*\bullet\nabla)\mathbf{E} + \mu(\mathbf{H}^*\bullet\nabla)\mathbf{H}}{\varepsilon\mu - \kappa^2}\right] \\[4mm] \mathbf{P} = \mathbf{O}_\kappa + \mathbf{S}_\kappa + \tfrac{1}{2}\dfrac{\kappa}{\varepsilon\mu - \kappa^2}\,\mathrm{Re}\left(\mathbf{T}_+\right) \\[4mm] \mathbf{R} = \tfrac{1}{2}\dfrac{\nabla J}{\varepsilon\mu - \kappa^2} - \tfrac{1}{2}\,\mathrm{Re}\left[\dfrac{\varepsilon(\mathbf{E}^*\bullet\nabla)\mathbf{E} - \mu(\mathbf{H}^*\bullet\nabla)\mathbf{H}}{\varepsilon\mu - \kappa^2}\right] + \tfrac{1}{2}\dfrac{\kappa}{\varepsilon\mu - \kappa^2}\,\mathrm{Im}\left(\mathbf{T}_+\right) \end{cases} \tag{S4.13}$$

Here, $\mathbf{M}^*\bullet(\nabla)\mathbf{M} = M_j^*\left(\partial M_j/\partial x_i\right)\hat{\mathbf{e}}_i$ is an orbital-like parameter, while $\left(\mathbf{M}^*\bullet\nabla\right)\mathbf{M} = M_j^*\left(\partial M_i/\partial x_j\right)\hat{\mathbf{e}}_i$ a spin-like parameter. Besides, summations are implied for repeated indices with $\left\{x_i, \hat{\mathbf{e}}_i\right\}$ being a coordinate and its unit vector.

The above pair of decomposition rules express both EM and reactive Poynting vectors in their sub-parts in the presence of medium chirality. Although both of $\{\mathbf{P}, \mathbf{R}\}$ are odd in β, we have sought circumstances under which they happen to be even in β.

Because $J_{LR}^\alpha = 0$ for an obliquely colliding pair of plane waves, the conservation law involving the reactive Poynting vector is simplified respectively into $\nabla\bullet\mathbf{R} + \tfrac{1}{2}\,\mathrm{Im}\left(\mathbf{A}_+\right) = \tfrac{1}{2}\beta\,\mathrm{Re}\left(\mathbf{B}_+\right)$ and $\nabla\bullet\mathbf{R} + \tfrac{1}{2}\,\mathrm{Im}\left(\mathbf{A}_+\right) = 0$, while $\mathbf{R}$ is simplified into the following.

$$\begin{cases} \mathbf{R} = \frac{1}{2}\mathrm{Re}\left[\varepsilon^{-1}\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H} - \mu^{-1}\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E}\right] + \frac{1}{2}\beta\,\mathrm{Im}\left(\mathbf{T}_-\right) \\ \mathbf{R} = \frac{1}{2}\mathrm{Re}\left[\dfrac{\mu\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H} - \varepsilon\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E}}{\varepsilon\mu - \kappa^2}\right] + \frac{1}{2}\dfrac{\kappa}{\varepsilon\mu - \kappa^2}\,\mathrm{Im}\left(\mathbf{T}_+\right) \end{cases} \qquad (S4.14)$$

The generic relation $\mathrm{Re}\left[\varepsilon\mathbf{E}^*\cdot(\nabla)\mathbf{E} - \mu\mathbf{H}^*\cdot(\nabla)\mathbf{H}\right] \equiv \nabla J$ has already been discussed. Therefore, both formulas are reduced for an achiral medium with $\beta = \kappa = 0$ to $\nabla J = 2\varepsilon\mu\mathbf{R}$. In other words, the spatial gradient of the reactive energy density is proportional to the reactive Poynting vector.

All relations handled in this section take forms based on the electric-magnetic democracy.

Section S5. Spin angular momentum (AM) density

Consider a spin angular momentum (AM) density $\mathbf{M} \equiv \mathrm{Im}\left(\frac{1}{2}\varepsilon\mathbf{E}^*\times\mathbf{E} + \frac{1}{2}\mu\mathbf{H}^*\times\mathbf{H}\right)$. Firstly, let us derive the chirality conservation law involving $\nabla\cdot\mathbf{M}$ for a chiral media with the field-κ-based constitutive relations.

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A} \;\Rightarrow\; \begin{cases} \nabla\times\mathbf{E} = \dfrac{\varepsilon\mu\beta}{1-\varepsilon\mu\beta^2}\mathbf{E} + \dfrac{i\mu}{1-\varepsilon\mu\beta^2}\mathbf{H} \\ \nabla\times\mathbf{H} = \dfrac{-i\varepsilon}{1-\varepsilon\mu\beta^2}\mathbf{E} + \dfrac{\varepsilon\mu\beta}{1-\varepsilon\mu\beta^2}\mathbf{H} \end{cases} \Rightarrow$$

$$\mathbf{M} \equiv \mathrm{Im}\left(\tfrac{1}{2}\varepsilon\mathbf{E}^*\times\mathbf{E} + \tfrac{1}{2}\mu\mathbf{H}^*\times\mathbf{H}\right) \;\Rightarrow\; \nabla\cdot\mathbf{M} \equiv \mathrm{Im}\left[\tfrac{1}{2}\varepsilon\nabla\cdot\left(\mathbf{E}^*\times\mathbf{E}\right) + \tfrac{1}{2}\mu\nabla\cdot\left(\mathbf{H}^*\times\mathbf{H}\right)\right]$$

$$2\nabla\cdot\mathbf{M} \equiv \mathrm{Im}\left[\varepsilon\left(\nabla\times\mathbf{E}^*\right)\cdot\mathbf{E} - \varepsilon\left(\nabla\times\mathbf{E}\right)\cdot\mathbf{E}^* + \mu\left(\nabla\times\mathbf{H}^*\right)\cdot\mathbf{H} - \mu\left(\nabla\times\mathbf{H}\right)\cdot\mathbf{H}^*\right]$$

$$= \mathrm{Im}\left[\begin{array}{l} \varepsilon\left(\dfrac{\varepsilon\mu\beta}{1-\varepsilon\mu\beta^2}\mathbf{E}^* + \dfrac{-i\mu}{1-\varepsilon\mu\beta^2}\mathbf{H}^*\right)\cdot\mathbf{E} - \varepsilon\left(\dfrac{\varepsilon\mu\beta}{1-\varepsilon\mu\beta^2}\mathbf{E} + \dfrac{i\mu}{1-\varepsilon\mu\beta^2}\mathbf{H}\right)\cdot\mathbf{E}^* \\ + \mu\left(\dfrac{i\varepsilon}{1-\varepsilon\mu\beta^2}\mathbf{E}^* + \dfrac{\varepsilon\mu\beta}{1-\varepsilon\mu\beta^2}\mathbf{H}^*\right)\cdot\mathbf{H} - \mu\left(\dfrac{-i\varepsilon}{1-\varepsilon\mu\beta^2}\mathbf{E} + \dfrac{\varepsilon\mu\beta}{1-\varepsilon\mu\beta^2}\mathbf{H}\right)\cdot\mathbf{H}^* \end{array}\right]$$

$$= \mathrm{Im}\left[\varepsilon\dfrac{-i\mu}{1-\varepsilon\mu\beta^2}\mathbf{H}^*\cdot\mathbf{E} - \varepsilon\dfrac{i\mu}{1-\varepsilon\mu\beta^2}\mathbf{H}\cdot\mathbf{E}^* + \mu\dfrac{i\varepsilon}{1-\varepsilon\mu\beta^2}\mathbf{E}^*\cdot\mathbf{H} - \mu\dfrac{-i\varepsilon}{1-\varepsilon\mu\beta^2}\mathbf{E}\cdot\mathbf{H}^*\right] \Rightarrow$$

$$= \dfrac{1}{1-\varepsilon\mu\beta^2}\mathrm{Im}\left[-i\varepsilon\mu\mathbf{H}^*\cdot\mathbf{E} - i\varepsilon\mu\mathbf{H}\cdot\mathbf{E}^* + i\varepsilon\mu\mathbf{E}^*\cdot\mathbf{H} + i\varepsilon\mu\mathbf{E}\cdot\mathbf{H}^*\right]$$

$$= \dfrac{-\varepsilon\mu}{1-\varepsilon\mu\beta^2}\mathrm{Re}\left(\mathbf{H}^*\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{E}^* + \mathbf{E}^*\cdot\mathbf{H} + \mathbf{E}\cdot\mathbf{H}^*\right) = \dfrac{-4\varepsilon\mu}{1-\varepsilon\mu\beta^2}\mathrm{Re}\left(\mathbf{E}\cdot\mathbf{H}^*\right), \quad \begin{cases} \mathscr{C} \equiv \mathrm{Im}\left(\mathbf{E}\cdot\mathbf{H}^*\right) \\ \mathscr{K} \equiv \mathrm{Re}\left(\mathbf{E}\cdot\mathbf{H}^*\right) \end{cases}$$

$$\nabla\cdot\mathbf{M} + \dfrac{2\varepsilon\mu}{1-\varepsilon\mu\beta^2}\mathrm{Re}\left(\mathbf{E}\cdot\mathbf{H}^*\right) = 0 \;\Rightarrow\; \nabla\cdot\mathbf{M} + \dfrac{2\varepsilon\mu}{1-\varepsilon\mu\beta^2}\mathscr{K} = 0 \qquad\qquad (S5.1)$$

As a byproduct, $\nabla\cdot\mathbf{M}$ is symmetric in β.

Secondly, let us then derive the chirality conservation law involving $\nabla\cdot\mathbf{M}$ for a chiral media with the field-κ-based constitutive relations.

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=(\nabla\times\mathbf{A})\cdot\mathbf{B}-(\nabla\times\mathbf{B})\cdot\mathbf{A} \quad\Rightarrow\quad \begin{cases}\nabla\times\mathbf{E}=i\mu\mathbf{H}+\kappa\mathbf{E}\\ \nabla\times\mathbf{H}=-i\varepsilon\mathbf{E}+\kappa\mathbf{H}\end{cases}\Rightarrow$$

$$\mathbf{M}\equiv\mathrm{Im}\left(\tfrac{1}{2}\varepsilon\mathbf{E}^{*}\times\mathbf{E}+\tfrac{1}{2}\mu\mathbf{H}^{*}\times\mathbf{H}\right)\quad\Rightarrow\quad \nabla\cdot\mathbf{M}\equiv\mathrm{Im}\left[\tfrac{1}{2}\varepsilon\nabla\cdot\left(\mathbf{E}^{*}\times\mathbf{E}\right)+\tfrac{1}{2}\mu\nabla\cdot\left(\mathbf{H}^{*}\times\mathbf{H}\right)\right]$$

$$2\nabla\cdot\mathbf{M}\equiv\mathrm{Im}\left[\varepsilon\left(\nabla\times\mathbf{E}^{*}\right)\cdot\mathbf{E}-\varepsilon\left(\nabla\times\mathbf{E}\right)\cdot\mathbf{E}^{*}+\mu\left(\nabla\times\mathbf{H}^{*}\right)\cdot\mathbf{H}-\mu\left(\nabla\times\mathbf{H}\right)\cdot\mathbf{H}^{*}\right]$$

$$=\mathrm{Im}\left[\begin{array}{l}\varepsilon\left(-i\mu\mathbf{H}^{*}+\kappa\mathbf{E}^{*}\right)\cdot\mathbf{E}-\varepsilon\left(i\mu\mathbf{H}+\kappa\mathbf{E}\right)\cdot\mathbf{E}^{*}\\ +\mu\left(i\varepsilon\mathbf{E}^{*}+\kappa\mathbf{H}^{*}\right)\cdot\mathbf{H}-\mu\left(-i\varepsilon\mathbf{E}+\kappa\mathbf{H}\right)\cdot\mathbf{H}^{*}\end{array}\right] \tag{S5.2}$$

$$=\mathrm{Im}\left(\begin{array}{l}\kappa\varepsilon|\mathbf{E}|^{2}-i\varepsilon\mu\mathbf{H}^{*}\cdot\mathbf{E}-\kappa\varepsilon|\mathbf{E}|^{2}-i\varepsilon\mu\mathbf{H}\cdot\mathbf{E}^{*}\\ +\kappa\mu|\mathbf{H}|^{2}+i\varepsilon\mu\mathbf{E}^{*}\cdot\mathbf{H}-\kappa\mu|\mathbf{H}|^{2}+i\varepsilon\mu\mathbf{E}\cdot\mathbf{H}^{*}\end{array}\right)\Rightarrow$$

$$=\mathrm{Im}\left[-i\varepsilon\mu\mathbf{H}^{*}\cdot\mathbf{E}-i\varepsilon\mu\mathbf{H}\cdot\mathbf{E}^{*}+i\varepsilon\mu\mathbf{E}^{*}\cdot\mathbf{H}+i\varepsilon\mu\mathbf{E}\cdot\mathbf{H}^{*}\right]$$

$$=\varepsilon\mu\,\mathrm{Re}\left(-\mathbf{H}^{*}\cdot\mathbf{E}-\mathbf{H}\cdot\mathbf{E}^{*}+\mathbf{E}^{*}\cdot\mathbf{H}+\mathbf{E}\cdot\mathbf{H}^{*}\right)=0\quad\Rightarrow\quad \nabla\cdot\mathbf{M}=0$$

When the Gauss laws $\nabla\cdot\mathbf{A}=\nabla\cdot\mathbf{B}=0$ are taken into consideration, the vector identity $\nabla\times(\mathbf{A}\times\mathbf{B})=(\nabla\cdot\mathbf{B})\mathbf{A}-(\nabla\cdot\mathbf{A})\mathbf{B}-(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$ is reduced to $\nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$. Resultantly,

$$\mathbf{M}\equiv\mathrm{Im}\left(\tfrac{1}{2}\varepsilon\mathbf{E}^{*}\times\mathbf{E}+\tfrac{1}{2}\mu\mathbf{H}^{*}\times\mathbf{H}\right)\Rightarrow$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\nabla\cdot\mathbf{B})\mathbf{A}-(\nabla\cdot\mathbf{A})\mathbf{B}-(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$\begin{cases}\nabla\cdot\mathbf{A}=0\\ \nabla\cdot\mathbf{B}=0\end{cases}\Rightarrow\quad \nabla\times(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}\quad\Rightarrow \tag{S5.3}$$

$$\tfrac{1}{2}\nabla\times\mathbf{M}=\tfrac{1}{4}\nabla\times\mathrm{Im}\left(\varepsilon\mathbf{E}^{*}\times\mathbf{E}+\mu\mathbf{H}^{*}\times\mathbf{H}\right)$$

$$=\tfrac{1}{4}\left\{\varepsilon\,\mathrm{Im}\left[\left(\mathbf{E}\cdot\nabla\right)\mathbf{E}^{*}-\left(\mathbf{E}^{*}\cdot\nabla\right)\mathbf{E}\right]+\mu\,\mathrm{Im}\left[\left(\mathbf{H}\cdot\nabla\right)\mathbf{H}^{*}-\left(\mathbf{H}^{*}\cdot\nabla\right)\mathbf{H}\right]\right\}$$

$$=-\tfrac{1}{2}\varepsilon\,\mathrm{Im}\left[\left(\mathbf{E}^{*}\cdot\nabla\right)\mathbf{E}\right]-\tfrac{1}{2}\mu\,\mathrm{Im}\left[\left(\mathbf{H}^{*}\cdot\nabla\right)\mathbf{H}\right]=-\tfrac{1}{2}\mathrm{Im}\left[\varepsilon\left(\mathbf{E}^{*}\cdot\nabla\right)\mathbf{E}+\left(\mathbf{H}^{*}\cdot\nabla\right)\mathbf{H}\right]$$

Consequently, both spin-like parts get related to the half-curl of the spin AM density, but with distinct divisors.

$$\begin{cases}C_{\beta}^{curl}:\quad \mathbf{S}_{\beta}\equiv-\tfrac{1}{2}\mathrm{Im}\left[\mu^{-1}\left(\mathbf{E}^{*}\cdot\nabla\right)\mathbf{E}+\varepsilon^{-1}\left(\mathbf{H}^{*}\cdot\nabla\right)\mathbf{H}\right]=\tfrac{1}{2}\left(\varepsilon\mu\right)^{-1}\nabla\times\mathbf{M}\\[2mm] C_{\kappa}^{field}:\quad \mathbf{S}_{\kappa}\equiv-\tfrac{1}{2}\mathrm{Im}\left[\dfrac{\varepsilon\left(\mathbf{E}^{*}\cdot\nabla\right)\mathbf{E}+\mu\left(\mathbf{H}^{*}\cdot\nabla\right)\mathbf{H}}{\varepsilon\mu-\kappa^{2}}\right]=\tfrac{1}{2}\left(\varepsilon\mu-\kappa^{2}\right)^{-1}\nabla\times\mathbf{M}\end{cases} \tag{S5.4}$$

The spin AM density $\mathbf{M}$ serves as a helicity flux for the EM helicity.

With the assumption of $\varepsilon,\mu\in\mathbb{R}$, let us prepare both $\mathrm{Im}\left(\mathbf{E}^{*}\times\mathbf{E}\right)$ and $\mathrm{Im}\left(\mathbf{H}^{*}\times\mathbf{H}\right)$ for $\mathbf{M}\equiv\mathrm{Im}\left(\tfrac{1}{2}\varepsilon\mathbf{E}^{*}\times\mathbf{E}+\tfrac{1}{2}\mu\mathbf{H}^{*}\times\mathbf{H}\right)$. Let us start with the following cross product between the electric fields.

$$\begin{cases} \mathbf{E}_{LR}^{\alpha} = \mathbf{Q}_{L} - iZ\mathbf{Q}_{R}^{\alpha} \\ \mathbf{H}_{LR}^{\alpha} = -iZ^{-1}\mathbf{Q}_{L} + \mathbf{Q}_{R}^{\alpha} \end{cases} \Rightarrow$$

$$\left(\mathbf{E}_{LR}^{\alpha}\right)^{*} \times \mathbf{E}_{LR}^{\alpha} = \left(\mathbf{Q}_{L} - iZ\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \left(\mathbf{Q}_{L} - iZ\mathbf{Q}_{R}^{\alpha}\right) = \left[\left(\mathbf{Q}_{L}\right)^{*} + iZ\left(\mathbf{Q}_{R}^{\alpha}\right)^{*}\right] \times \left(\mathbf{Q}_{L} - iZ\mathbf{Q}_{R}^{\alpha}\right)$$

$$= \left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L} + Z^{2}\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha} - iZ\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha} + iZ\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{L} \qquad (S5.5)$$

$$= \left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L} + Z^{2}\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha} - 2iZ\,\mathrm{Re}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right] \Rightarrow$$

$$\mathrm{Im}\left[\left(\mathbf{E}_{LR}^{\alpha}\right)^{*} \times \mathbf{E}_{LR}^{\alpha}\right] = \mathrm{Im}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L}\right] + Z^{2}\,\mathrm{Im}\left[\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right] - 2Z\,\mathrm{Re}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right]$$

Likewise, we form the following cross product between the magnetic fields.

$$\left(\mathbf{H}_{LR}^{\alpha}\right)^{*} \times \mathbf{H}_{LR}^{\alpha} = \left(-iZ^{-1}\mathbf{Q}_{L} + \mathbf{Q}_{R}^{\alpha}\right)^{*} \times \left(-iZ^{-1}\mathbf{Q}_{L} + \mathbf{Q}_{R}^{\alpha}\right)$$

$$= \left[iZ^{-1}\left(\mathbf{Q}_{L}\right)^{*} + \left(\mathbf{Q}_{R}^{\alpha}\right)^{*}\right] \times \left(-iZ^{-1}\mathbf{Q}_{L} + \mathbf{Q}_{R}^{\alpha}\right)$$

$$= Z^{-2}\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L} + \left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha} + iZ^{-1}\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha} - iZ^{-1}\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{L} \qquad (S5.6)$$

$$= Z^{-2}\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L} + \left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha} + 2iZ^{-1}\,\mathrm{Re}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right] \Rightarrow$$

$$\mathrm{Im}\left[\left(\mathbf{H}_{LR}^{\alpha}\right)^{*} \times \mathbf{H}_{LR}^{\alpha}\right] = Z^{-2}\,\mathrm{Im}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L}\right] + \mathrm{Im}\left[\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right] + 2Z\,\mathrm{Re}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right]$$

Resultantly, the average spin AM is found below in a sort of both symmetric and anti-symmetric forms.

$$\mathbf{M}_{LR}^{\alpha} \equiv \mathrm{Im}\left[\tfrac{1}{2}\varepsilon\left(\mathbf{E}_{LR}^{\alpha}\right)^{*} \times \mathbf{E}_{LR}^{\alpha} + \tfrac{1}{2}\mu\left(\mathbf{H}_{LR}^{\alpha}\right)^{*} \times \mathbf{H}_{LR}^{\alpha}\right]$$

$$= \tfrac{1}{2}\left(\varepsilon + Z^{-2}\mu\right)\mathrm{Im}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L}\right] + \tfrac{1}{2}\left(\varepsilon Z^{2} + \mu\right)\mathrm{Im}\left[\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right]$$

$$+ Z\left(\mu - \varepsilon\right)\mathrm{Re}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right] \qquad (S5.7)$$

$$= \varepsilon\,\mathrm{Im}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{L}\right] + \mu\,\mathrm{Im}\left[\left(\mathbf{Q}_{R}^{\alpha}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right] + Z\left(\mu - \varepsilon\right)\mathrm{Re}\left[\left(\mathbf{Q}_{L}\right)^{*} \times \mathbf{Q}_{R}^{\alpha}\right]$$

Consider the vector identity together with a pair of the pertinent Gauss laws.

$$\nabla \times \left(\mathbf{A} \times \mathbf{B}\right) = \left(\nabla \cdot \mathbf{B}\right)\mathbf{A} - \left(\nabla \cdot \mathbf{A}\right)\mathbf{B} - \left(\mathbf{A} \cdot \nabla\right)\mathbf{B} + \left(\mathbf{B} \cdot \nabla\right)\mathbf{A} \Rightarrow \begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \Rightarrow$$

$$\begin{cases} \nabla \times \left(\mathbf{E}^{*} \times \mathbf{E}\right) = \left(\nabla \cdot \mathbf{E}\right)\mathbf{E}^{*} - \left(\nabla \cdot \mathbf{E}^{*}\right)\mathbf{E} - \left(\mathbf{E}^{*} \cdot \nabla\right)\mathbf{E} + \left(\mathbf{E} \cdot \nabla\right)\mathbf{E}^{*} = -\left(\mathbf{E}^{*} \cdot \nabla\right)\mathbf{E} + \left(\mathbf{E} \cdot \nabla\right)\mathbf{E}^{*} \\ \nabla \times \left(\mathbf{H}^{*} \times \mathbf{H}\right) = \left(\nabla \cdot \mathbf{H}\right)\mathbf{H}^{*} - \left(\nabla \cdot \mathbf{H}^{*}\right)\mathbf{E} - \left(\mathbf{H}^{*} \cdot \nabla\right)\mathbf{H} + \left(\mathbf{H} \cdot \nabla\right)\mathbf{H}^{*} = -\left(\mathbf{H}^{*} \cdot \nabla\right)\mathbf{H} + \left(\mathbf{H} \cdot \nabla\right)\mathbf{H}^{*} \end{cases} \qquad (S5.8)$$

Of course, we are left only with the spin-like parts under the condition $\varepsilon, \mu \in \mathbb{R}$.

$$\mathbf{M} \equiv \mathrm{Im}\left(\tfrac{1}{2}\varepsilon\mathbf{E}^* \times \mathbf{E} + \tfrac{1}{2}\mu\mathbf{H}^* \times \mathbf{H}\right) \;\Rightarrow\; \nabla \times \mathbf{M} \equiv \mathrm{Im}\left[\tfrac{1}{2}\varepsilon\nabla \times \left(\mathbf{E}^* \times \mathbf{E}\right) + \tfrac{1}{2}\mu\nabla \times \left(\mathbf{H}^* \times \mathbf{H}\right)\right]$$

$$\varepsilon, \mu \in \mathbb{R} \;\Rightarrow$$

$$2\nabla \times \mathbf{M} \equiv \mathrm{Im}\left[\varepsilon\left(\mathbf{E}\cdot\nabla\right)\mathbf{E}^* - \varepsilon\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E} + \mu\left(\mathbf{H}\cdot\nabla\right)\mathbf{H}^* - \mu\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H}\right]$$

$$= -2\,\mathrm{Im}\left[\varepsilon\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E} + \mu\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H}\right] \;\Rightarrow$$

$$\begin{cases} \mathbf{S}_\beta \equiv -\tfrac{1}{2}\mathrm{Im}\left[\mu^{-1}\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E} + \varepsilon^{-1}\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H}\right] \\[2ex] \mathbf{S}_\kappa \equiv -\tfrac{1}{2}\mathrm{Im}\left[\dfrac{\varepsilon\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E} + \mu\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H}}{\varepsilon\mu - \kappa^2}\right] \end{cases} \Rightarrow \qquad\qquad (\text{S5.9})$$

$$\tfrac{1}{2}\nabla \times \mathbf{M} = -\tfrac{1}{2}\mathrm{Im}\left[\varepsilon\left(\mathbf{E}^*\cdot\nabla\right)\mathbf{E} + \mu\left(\mathbf{H}^*\cdot\nabla\right)\mathbf{H}\right] = \begin{cases} \left(\varepsilon\mu\right)\mathbf{S}_\kappa \\[1ex] \left(\varepsilon\mu - \kappa^2\right)\mathbf{S}_\kappa \end{cases}$$

There is only a slight difference between the pair of constitutive relations.

Section S6. Evaluations of dot- and cross-products for oblique collisions

Let us evaluate various self- and cross-products between $\left\{\mathbf{Q}_L, \mathbf{Q}_R^\alpha\right\}$. There are several easier parameters, that consist of the waves of same types.

$$\begin{cases} \left(\mathbf{Q}_L\right)^* \cdot \mathbf{Q}_L \equiv \left|Q_L\right|^2, & \left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_L = \mathrm{i}\left|Q_L\right|^2 \hat{\boldsymbol{\xi}}_- \\[1ex] \left(\mathbf{Q}_R^\alpha\right)^* \cdot \mathbf{Q}_R^\alpha \equiv \left|Q_R^\alpha\right|^2, & \left(\mathbf{Q}_R^\alpha\right)^* \times \mathbf{Q}_R^\alpha = -\mathrm{i}\left|Q_R^\alpha\right|^2 \hat{\boldsymbol{\eta}}_- \end{cases} \qquad\qquad (\text{S6.1})$$

In comparison, the following set of parameters handles both of $\left\{\mathbf{Q}_L, \mathbf{Q}_R^\alpha\right\}$ so that care should be exercised as follows.

$$\left(\mathbf{Q}_L\right)^* \cdot \mathbf{Q}_R^\alpha = \left(Q_L\right)^* Q_R^\alpha \left[\tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{z}} - \mathrm{i}\hat{\boldsymbol{\xi}}_+\right)\exp\left(\mathrm{i}k_L\xi_-\right)\right]^* \cdot \left[\tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{z}} + \mathrm{i}\hat{\boldsymbol{\eta}}_+\right)\exp\left(\mathrm{i}k_R\eta_-\right)\right]$$

$$= \left(Q_L\right)^* Q_R^\alpha \tfrac{1}{2}\exp\left[-\mathrm{i}\left(k_L\xi_- - k_R\eta_-\right)\right]\left(\hat{\mathbf{z}} + \mathrm{i}\hat{\boldsymbol{\xi}}_+\right)\cdot\left(\hat{\mathbf{z}} + \mathrm{i}\hat{\boldsymbol{\eta}}_+\right)$$

$$= \left(Q_L\right)^* Q_R^\alpha \tfrac{1}{2}\left(1 - \hat{\boldsymbol{\eta}}_+ \cdot \hat{\boldsymbol{\xi}}_+\right)\exp\left[-\mathrm{i}\left(k_L\xi_- - k_R\eta_-\right)\right] \qquad\qquad (\text{S6.2})$$

$$\left(\mathbf{Q}_L\right)^* \cdot \mathbf{Q}_R^\alpha = \left|\left(Q_L\right)^* Q_R^\alpha\right| \tfrac{1}{2}\left(1 - \cos\alpha\right)\exp\left[-\mathrm{i}\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\right] \;\Rightarrow$$

$$\begin{cases} \mathrm{Re}\left[\left(\mathbf{Q}_L\right)^* \cdot \mathbf{Q}_R^\alpha\right] = \left|\left(Q_L\right)^* Q_R^\alpha\right| \tfrac{1}{2}\left(1 - \cos\alpha\right)\cos\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right) \\[1ex] \mathrm{Im}\left[\left(\mathbf{Q}_L\right)^* \cdot \mathbf{Q}_R^\alpha\right] = -\left|\left(Q_L\right)^* Q_R^\alpha\right| \tfrac{1}{2}\left(1 - \cos\alpha\right)\sin\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right) \end{cases}$$

Here, we made use of the definitions for $\left\{\Gamma_{LR}^\alpha, \delta_{LR}^\alpha\right\}$. Likewise, consider the cross product.

$$\left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_R^\alpha = \left(Q_L\right)^* Q_R^\alpha \left[\tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{z}}+i\hat{\boldsymbol{\eta}}_+\right)\exp\left(ik_R\eta_-\right)\right] \times \left[\tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{z}}-i\hat{\boldsymbol{\xi}}_+\right)\exp\left(ik_L\xi_-\right)\right]^*$$

$$= \left(Q_L\right)^* Q_R^\alpha \left[\tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{z}}-i\hat{\boldsymbol{\xi}}_+\right)\exp\left(ik_L\xi_-\right)\right]^* \times \left[\tfrac{1}{\sqrt{2}}\left(\hat{\mathbf{z}}+i\hat{\boldsymbol{\eta}}_+\right)\exp\left(ik_R\eta_-\right)\right]$$

$$= \left(Q_L\right)^* Q_R^\alpha \tfrac{1}{2}\exp\left[-i\left(k_L\xi_- - k_R\eta_-\right)\right]\left(\hat{\mathbf{z}}+i\hat{\boldsymbol{\xi}}_+\right)\times\left(\hat{\mathbf{z}}+i\hat{\boldsymbol{\eta}}_+\right)$$

$$= \left(Q_L\right)^* Q_R^\alpha \tfrac{1}{2}\exp\left[-i\left(k_L\xi_- - k_R\eta_-\right)\right]\left[i\hat{\mathbf{z}}\times\left(\hat{\boldsymbol{\eta}}_+ - \hat{\boldsymbol{\xi}}_+\right)-\hat{\boldsymbol{\xi}}_+\times\hat{\boldsymbol{\eta}}_+\right]$$

$$= \left(Q_L\right)^* Q_R^\alpha \tfrac{1}{2}\exp\left[-i\left(k_L\xi_- - k_R\eta_-\right)\right]\left[i\left(\hat{\boldsymbol{\xi}}_+ - \hat{\boldsymbol{\eta}}_+\right)\times\hat{\mathbf{z}}-\hat{\boldsymbol{\xi}}_+\times\hat{\boldsymbol{\eta}}_+\right] \qquad (S6.3)$$

$$= \left(Q_L\right)^* Q_R^\alpha \tfrac{1}{2}\exp\left[-i\left(k_L\xi_- - k_R\eta_-\right)\right]\left[i\left(\hat{\boldsymbol{\xi}}_- - \hat{\boldsymbol{\eta}}_-\right)+\sin\alpha\hat{\mathbf{z}}\right] \Rightarrow$$

$$\left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_R^\alpha = \left|\left(Q_L\right)^* Q_R^\alpha\right|\exp\left[-i\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\right]\tfrac{1}{2}\left[i\left(\hat{\boldsymbol{\xi}}_- - \hat{\boldsymbol{\eta}}_-\right)+\sin\alpha\hat{\mathbf{z}}\right]$$

$$\mathbf{Q}_R^\alpha \times \left(\mathbf{Q}_L\right)^* = \left(Q_L\right)^* Q_R^\alpha$$

$$\mathrm{Im}\left[\mathbf{Q}_R^\alpha \times \left(\mathbf{Q}_L\right)^*\right] = -\tfrac{1}{2}\left|\left(Q_L\right)^* Q_R^\alpha\right|\cos\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\left(\hat{\boldsymbol{\xi}}_- - \hat{\boldsymbol{\eta}}_-\right)$$

Taking real and imaginary parts,

$$\begin{cases}\mathrm{Re}\left[\left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_R^\alpha\right] = \left|\left(Q_L\right)^* Q_R^\alpha\right|\tfrac{1}{2}\left[\sin\alpha\cos\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\hat{\mathbf{z}}+\sin\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\left(\hat{\boldsymbol{\xi}}_- - \hat{\boldsymbol{\eta}}_-\right)\right] \\ \mathrm{Im}\left[\left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_R^\alpha\right] = \left|\left(Q_L\right)^* Q_R^\alpha\right|\tfrac{1}{2}\left[\cos\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\left(\hat{\boldsymbol{\xi}}_- - \hat{\boldsymbol{\eta}}_-\right)-\sin\alpha\sin\left(\Gamma_{LR}^\alpha + \delta_{LR}^\alpha\right)\hat{\mathbf{z}}\right]\end{cases} \qquad (S6.4)$$

It turns out that both parts $\left\{\mathrm{Re}\left[\left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_R^\alpha\right], \mathrm{Im}\left[\left(\mathbf{Q}_L\right)^* \times \mathbf{Q}_R^\alpha\right]\right\}$ contain not only the longitudinal vector $\hat{\boldsymbol{\xi}}_- - \hat{\boldsymbol{\eta}}_-$ but also the transverse vector $\hat{\mathbf{z}}$.

In addition, it is stressed that all formulas presented in this section hold true for both types of constitutive relations.