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Stability of Boundary Value Discrete Fractional Hybrid Equation of Second Type with Application to Heat Transfer with Fins

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Abstract: The development in the qualitative theory of fractional differential equations is accompanied by discrete analog which has been studied intensively in recent past. Suitable fixed point theorem is to be selected to study the boundary value discrete fractional equations due to the properties exhibited by fractional difference operators. This article aims at investigating the stability results in the sense of Hyers and Ulam with application of Mittag–Leffler function hybrid fractional order difference equation of second type. The symmetric structure of the operators defined in this article is vital in establishing the existence results by using Krasnoselkii's fixed point theorem. Banach contraction mapping principle and Krasnoselkii's fixed point theorem are employed to establish the uniqueness and existence results for solution of fractional order discrete equation. A problem on heat transfer with fins is provided as an application to considered hybrid type fractional order difference equation and the stability results are demonstrated with simulations.

Keywords: fractional order; discrete; Mittag–Leffler function; boundary value problems; Hyers Ulam stability

MSC: 26A33; 34B15; 39A30

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1. Introduction

In 1940, Ulam was the first to stimulate the concept of stability for the functional equations [1] and had posed several unsolved problems. In 1941, a problem posed by Ulam on stability of homomorphisms was solved by Hyers [2]. The stability concept named after the two mathematicians has inspired a large number of mathematicians and was recently employed for the analysis of stability of differential and difference equations. Hyers–Ulam stability results for differential equations was initiated by Obloza in [3,4] and the advancement on the results was carried out by Alsina and Ger [5] in 1998. Qarawani in [6] considered the Emden–Fowler and generalized differential equations of order 2 and performed the stability analysis in the sense of Hyers–Ulam. The work by Alqifiary and Jung in [7,8] investigated the Hyers–Ulam stability of 2nd order differential equations and discussed the generalized Hyers–Ulam stability for nth order differential equation using the Laplace transform method. The analysis of the qualitative properties of the dynamical systems are vital in understanding the physical behaviour of the real world phenomenon.

The arbitrary order calculus was known to the research community since 1695. The lack of interpretation of the arbitrary order equations in modelling real world problems had caused stagnation in growth of its theory. Development of digital computers during 20th century has led to considerable contribution to the theory of fractional calculus. Main reason for adapting the fractional order calculus in constructing models is due to its capability to bring memory factors into effect and also increases the accuracy of the models of real world. The fractional order of the system in addition to the hysteresis is also very crucial in bringing out the physical significance of the phenomena under consideration. The choice of the fractional order greatly depends on the behavioural aspect of the system under consideration. Fractional calculus has recently attracted many researchers in modelling real life applications [9–15]. Closed form solutions of the fractional order equations was studied in [16]. Shakeel et al. in [17] presented the results on exact solutions for Burger equation of fractional order using novel expansion method. Recently, Hyers–Ulam stability analysis was extended to differential equations of fractional order. In [18–20], Wang and Zhou investigated the Ulam stability of the fractional order equations. Ulam–Hyers Mittag–Leffler stability of fractional equations using fixed point approach was carried out by N. Egbhali [21] in 2016. Most recently the Ulam–Hyers Mittag–Leffler stability of non-linear fractional equation was studied in [22].

The theory of discrete fractional calculus was developed in recent decades. The discrete counterpart has not gained expected theoretical support as the continuous case. However, recent literature has proved that discrete time fractional calculus has gained equal importance from the mathematicians and engineers. Discrete fractional calculus is considered to have their origin in the works of [23,24]. Atici and Eloe in [25,26] and Holm in [27] have contributed some significant results on the discrete fractional sum and differences. The basic theory on discrete nabla fractional calculus was illustrated in [28,29]. Tumour-immune system was constructed using discrete fractional order difference equations and its chaotic behaviour are presented in [30]. Josephson junction discrete fractional model and its stability properties was discussed in [31]. Application of discrete fractional equation to pantograph equations can be found in [32]. Some discrete time fractional order problems with boundary conditions were investigated by authors in [33–37].

Non-linear differential equations with quadratic perturbations also known as hybrid type differential equations are of great interest to the mathematicians and engineers due to their ability to describe different dynamic models as special cases. The two types of fractional order perturbation equations are discussed in [38] and some recent contributions on hybrid type fractional order equations include [39–49]. The first work on Hyers–Ulam stability of fractional difference equation with boundary condition was carried out by Fulai Chen and Yong Zhou in 2013 [50]. Influenced by the above works with boundary conditions, we consider the non-linear discrete fractional equation of the form

$$\begin{aligned} \Delta_*^{\zeta}[v(\varpi) - \Theta(\varpi, v(\varpi))] = \mathcal{M}(\varpi + \zeta - 1, v(\zeta + \varpi - 1)), \varpi \in [0, \wp]_{\mathbb{N}_0}, 1 < \zeta < 2 \\ v(\zeta - 1) = g(v), \Delta[v(\zeta + \wp) - \Theta(\zeta + \wp, v(\zeta + \wp))] - \psi(v) = A. \end{aligned} \tag{1}$$

where Δ_*^{ζ} is the fractional Caputo difference operator, $A \in \mathbb{R}$, $\mathbb{N}_j = \{j, j+1, j+2, \dots\}$, $\Theta, \mathcal{M} : C\Big([\zeta - 1, \zeta + \wp]_{\mathbb{N}_{\zeta-1}}, \mathbb{R}\Big) \to \mathbb{R}$ is continuous in v, $g, \psi : C\Big([\zeta - 1, \zeta + \wp]_{\mathbb{N}_{\zeta-1}}\Big) \to \mathbb{R}$ is Lipschitz continuous in v with positive constant $K, \lambda \in (0,1)$. This article employs Caputo type fractional difference operator due to its practical interpretation of the real world problems.

Boundary values problems with conditions defined at the extremes has attracted engineers and scientists due to their ability to give analytical understanding and prediction of the phenomena over a period of time. The applications include finding the electrical potential of any given region, disciplines of physics such as elongated rods, thermostat in sensors, heat transfer through surfaces, and so on. The generation of heat in most real life scenarios are due to infrared, nuclear, chemical, or electrical activities. Temperature and flow of heat are the two most important factors in heat conduction problems which can be understood with greater level of accuracy by construction of models with boundary value problems. The boundary conditions that define the problems on heat conduction may be linear or non-linear. The non-linearity in the boundary conditions are widely due to the influence of power of temperature entering the boundary condition. Increasing interest towards study of discrete fractional boundary value problems and exploring real life applications this article provides an application to (1) in the form of heat transfer equation with fins. The main objective of the article is:

- Investigation of Hyers–Ulam–Mittag–Leffler stability for hybrid fractional order difference equation of second type;
- Application to heat transfer with fins.

The paper is structured as follows. Section 2 presents some necessary mathematical identities that are required throughout the paper. Existence and uniqueness of the solution of (1) are established in Section 3. Hyers–Ulam–Mittag–Leffler stability of the non-linear discrete fractional equation is introduced in Sections 4 and 5 demonstrates the application to heat transfer with fins and Section 6 concludes the paper.

2. Mathematical Background

This section provides necessary mathematical concepts that are used throughout this work.

Definition 1 ([51])**.** *Let $\zeta > 0$. The $\zeta - th$ fractional sum of v is defined by*

$$\Delta^{-\zeta} v(\varpi) = \frac{1}{\Gamma(\zeta)} \sum_{s=a}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} v(s), \tag{2}$$

where $\varpi^{(\zeta)}$ denotes the falling factorial.

Definition 2 ([51])**.** *Let $\beta > 0$ and $\kappa - 1 < \beta < \kappa$, where $\kappa = \lceil \beta \rceil \in \mathbb{N}$. Set $\zeta = \kappa - \beta$. The $\beta - th$ fractional difference is given by*

$$\begin{aligned} \Delta_*^{\beta} v(\varpi) &= \Delta^{-\zeta}(\Delta^{\kappa} v(\varpi)) \\ &= \frac{1}{\Gamma(\zeta)} \sum_{s=j}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} (\Delta^{\kappa} v)(s), \end{aligned} \tag{3}$$

where Δ^{κ} is the forward difference operator of order k.

Definition 3 ([52])**.** *Let $\varpi \in \mathbb{N}_0, \omega \in (-1,1)$ and $\zeta \in \mathbb{R}^+$. The discrete one parameter Mittag–Leffler function is*

$$F_{\zeta}(\omega, \varpi) = \sum_{\kappa=0}^{\infty} (\omega)^{\kappa} \frac{(\varpi + \kappa(\zeta - 1))^{(\zeta\kappa)}}{\Gamma(\zeta\kappa + 1)}. \tag{4}$$

Lemma 1 ([53,54])**.** *Assume that $\beta > 0$ and $\mathcal{M}$ is defined on $\mathbb{N}_j$ then*

$$\begin{aligned} \Delta_*^{-\beta} \Delta_*^{\beta} \mathcal{M}(\varpi) &= \mathcal{M}(\varpi) - \sum_{\kappa=0}^{n-1} \frac{(\varpi - j)^{\kappa}}{\kappa!} \Delta^{\kappa}[\mathcal{M}(j)] \\ &= \mathcal{M}(\varpi) + c_0 + c_1 t + \ldots + c_{n-1} t^{(n-1)}, \end{aligned} \tag{5}$$

where $n_1 \geq \beta$ and $c_i \in \mathbb{R}, i = [1, n_1 - 1] \cap \mathbb{N}_1$.

Theorem 1 ([55])**.** *(Banach Contraction Mapping Principle) A contraction mapping on a complete metric space has exactly one fixed point.*

Theorem 2 ([56])**.** *Let the subset $\mathcal{H}$ of the Banach space B be non-empty, convex, bounded and closed. Let $\mathcal{P}_1 : B \to B$ and $\mathcal{P}_2 : \mathcal{H} \to B$ be two operators, such that*

(i) *The operator $\mathcal{P}_1$ is a contraction;*
(ii) *The operator $\mathcal{P}_2$ is completely continuous;*
(iii) $x = \mathcal{P}_1 x + \mathcal{P}_2 y$ *for all* $x \in \mathcal{H} \Rightarrow x \in \mathcal{H}$.

Then, there exists a solution for the equation $\mathcal{P}_1 x + \mathcal{P}_2 x = x$.

3. Existence and Uniqueness Results

Let the set $v = \{v(\varpi)\}_{\varpi=\zeta-1}^{\wp+\zeta}$ be denoted by B with norm

$$\|v\| = \Big\{max|v(\varpi)|, \varpi \in [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}}\Big\}.$$

Then, B is a Banach Space. Before establishing the results on existence of unique solution for the consider boundary value problem (1), we shall make the following assumptions. Let

(J_1) There exist non-zero real constants K, λ, such that

$$\begin{aligned} |g(v(\varpi)) - g(v_1(\varpi))| &\le K|v(\varpi) - v_1(\varpi)|, \\ |\psi(v(\varpi)) - \psi(v_1(\varpi))| &\le \lambda|v(\varpi) - v_1(\varpi)|. \end{aligned} \tag{6}$$

(J_2) There exist non zero real constants $\mathbb{L}, \mathbb{L}_1$, such that

$$\begin{aligned} |\mathcal{M}(\varpi, v(\varpi)) - \mathcal{M}(\varpi, v_1(\varpi))| &\le \mathbb{L}|v(\varpi) - v_1(\varpi)|, \\ |\Theta(\varpi, v(\varpi)) - \Theta(\varpi, v_1(\varpi))| &\le \mathbb{L}_1|v(\varpi) - v_1(\varpi)|. \end{aligned} \tag{7}$$

for all $v, v_1 \in B$.

Lemma 2. *A function* $v(\varpi) : \varpi \in [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}} \to \mathbb{R}$ *is a solution of the boundary value problem* (1) *iff* $v(\varpi)$ *is a solution of*

$$\begin{aligned} v(\varpi) =& \Theta(\varpi, v(\varpi)) - \Theta(\zeta-1, g(v)) + \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s-1+\zeta) + g(v) \\ &+ \frac{(\zeta-1-\varpi)}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp+\zeta-s-1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta) + (\zeta-1-\varpi)(A+\psi(v)), \end{aligned} \tag{8}$$

where $\varpi \in [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}}$.

Proof. Suppose that $v(\varpi)$ is a solution of (1). Using Lemma (1) with constants $c_0, c_1 \in \mathbb{R}$, we have

$$v(\varpi) = \Delta^{-\zeta}\mathcal{M}(\varpi-1+\zeta) - c_0 - c_1\varpi$$

$$v(\varpi) = \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi-s-1)^{(\zeta-1)} \mathcal{M}(s-1+\zeta) - c_0 - c_1\varpi, \tag{9}$$

for $\varpi \in [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}}$. Additionally, we have,

$\Delta[v(\varpi) - \Theta(\varpi, v(\varpi))] = \frac{1}{\Gamma(\zeta-1)} \sum_{s=0}^{\varpi-\zeta+1} (\varpi-s-1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta) - c_1,$
$\varpi \in [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}}$.

Using the boundary conditions, the constants are evaluated

$c_0 = \Theta(\zeta-1, g(v)) - \left[g(v) + \frac{\zeta-1-\varpi}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp+\zeta-s-1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta) + A + \psi(v)\right],$

$c_1 = \frac{1}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp+\zeta-s-1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta) - A - \psi(v).$

On the substitution of c_0 and c_1 in (9), we obtain (8).

Conversely, if $v(\varpi)$ is the solution of (8), it is clear that solution obtained from (8) satisfies (1). The proof is complete.

Define the operator $P : B \to B$ by

$$\begin{aligned}Pv(\varpi) =&\Theta(\varpi, v(\varpi)) - \Theta(\zeta - 1, g(v)) + \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v(s - 1 + \zeta)) \\ &+ g(v) + \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v(s - 1 + \zeta)) \\ &+ (\zeta - 1 - \varpi)(A + \psi(v)),\end{aligned} \tag{10}$$

for $\varpi \in [\zeta - 1, \wp + \zeta]_{\mathbb{N}_{\zeta-1}}$. It is obvious that $v(\varpi)$ is a solution of (1) if it is a fixed point of (10). □

Theorem 3. *If*

$$\begin{aligned}\rho =&\mathbb{L}_1(1 + K) + K + (\wp + 1)\lambda + \mathbb{L}\left[\frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\wp} (\wp + \zeta - s - 1)^{(\zeta-1)}\right. \\ &\left.+ \frac{(\wp + 1)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)}\right] < 1,\end{aligned} \tag{11}$$

then the boundary value problem (1) *has an unique solution in* B.

Proof. For each $\varpi \in [\zeta - 1, \wp + \zeta]_{\mathbb{N}_{\zeta-1}}$ and $v, v_1 \in B$,

$$\begin{aligned}|Pv(\varpi) - Pv_1(\varpi)| \leq& \mathbb{L}_1|v - v_1| + K\mathbb{L}_1|v - v_1| + \mathbb{L}\frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} [|v - v_1|] \\ &+ \mathbb{L}|(\zeta - 1 - \varpi)| \frac{1}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} [|v - v_1|] \\ &+ K|v - v_1| + |(\zeta - 1 - \varpi)|\lambda|v - v_1| \\ \leq& \left[\mathbb{L}_1 + K\mathbb{L}_1 + \mathbb{L}\frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} + K + |(\zeta - 1 - \varpi)|\lambda\right. \\ &\left.+ \mathbb{L}|(\zeta - 1 - \varpi)| \frac{1}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)}\right] \|v - v_1\| \\ \leq& \left[\mathbb{L}_1 + K\mathbb{L}_1 + \mathbb{L}\frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} + K\right. \\ &\left.+ \mathbb{L}(\wp + 1) \frac{1}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} + (\wp + 1)\lambda\right] \|v - v_1\| \\ \|Pv(\varpi) - Pv_1(\varpi)\| \leq& \rho\|v - v_1\|.\end{aligned}$$

Thus, P is a contraction mapping on B with $\rho < 1$. From Theorem 1, it is clear that P has a unique fixed point. The proof is complete. □

Theorem 4. *Assume that* J_1, J_2 *hold and there exists non-zero real constants* ξ_1, ξ_2, ξ_3, *such that*

$$|\mathcal{M}(\varpi, v(\varpi))| < \xi_1, \; |g(v(\varpi))| < \xi_2, \; |\psi(v(\varpi))| < \xi_3. \tag{12}$$

If

$$\mathbb{L}_1(1 + K) + K + (\wp + 1)M < 1, \tag{13}$$

then a solution for the problem (1) with boundary condition exists in B.

Proof. Let $\mathcal{H} = \{v \in B : \|v\| \leq \mathcal{S}\}$, where $\mathcal{S} \in \mathbb{R}_+$, such that

$$\mathcal{S} \geq \frac{2\Theta_0 + \xi_2 + \xi_3 + A + \mathbb{W}}{1 - \mathbb{L}_1(1+K)}, \tag{14}$$

where $\Theta_0 = \max_{\varpi \in [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}}} |\Theta(\varpi, 0)|$ and

$$\mathbb{W} = \frac{\xi_1}{\Gamma(\zeta)} \sum_{s=0}^{\wp} (\zeta + \wp - s - 1)^{(\zeta-1)} + (\wp+1)\left[\frac{\xi_1}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\zeta + \wp - s - 1)^{(\zeta-2)}\right].$$

It is clear that the subset $\mathcal{H} \in B$ is convex, bounded and closed. Define the operators $\mathbb{P}_1 : B \to B$ and $\mathbb{P}_2 : \mathcal{H} \to B$ as

$$\begin{aligned}
\mathbb{P}_1 v =& \Theta(\varpi, v(\varpi)) - \Theta(\zeta-1, g(v)) + g(v) + (\zeta - 1 - \varpi)(A + \psi(v)) \\
\mathbb{P}_2 v =& \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v(s-1+\zeta)) \\
&+ \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v(s-1+\zeta))
\end{aligned}$$

We shall proceed to show that the operators satisfy the conditions of Theorem 2.

Step 1: The operator $\mathbb{P}_1$ is a contraction. From the assumptions J_1 and J_2, we have

$$\begin{aligned}
|\mathbb{P}_1 v - \mathbb{P}_1 v_1| \leq& |\Theta(\varpi, v(\varpi)) - \Theta(\varpi, v_1(\varpi))| + |\Theta(\zeta-1, g(v)) - \Theta(\zeta-1, g(v_1))| \\
&+ |g(v) - g(v_1)| + |\zeta - 1 + \varpi||\psi(v) - \psi(v_1)| \\
\|\mathbb{P}_1 v - \mathbb{P}_1 v_1\| \leq& (\mathbb{L}_1(1+K) + K + (\wp+1)\lambda)\|v - v_1\|
\end{aligned}$$

It is evident from the condition (13) that the operator $\mathbb{P}_1$ is a contraction.

Step 2: We aim at proving $\mathbb{P}_2$ is completely continuous on $\mathcal{H}$.
Since the continuity of $\mathbb{P}_2$ is straightforward implication of continuity of $\mathcal{M}$, we proceed to prove the uniform boundedness of $\mathbb{P}_2$.

$$\begin{aligned}
|\mathbb{P}_2 v| =& \left| \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s-1+\zeta, v(s-1+\zeta)) \right. \\
&\left. + \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta, v(s-1+\zeta)) \right| \\
\leq& \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} |\mathcal{M}(s-1+\zeta, v(s-1+\zeta))| \\
&+ \frac{|(\zeta - 1 - \varpi)|}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} |\mathcal{M}(s-1+\zeta, v(s-1+\zeta))| \\
\leq& \frac{\xi_1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} + \frac{\xi_1(\wp+1)}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \\
\leq& \mathbb{W}
\end{aligned}$$

Thus, the uniform boundedness on $\mathcal{H}$ of $\mathbb{P}_2$ is confirmed.
We now prove the equicontinuity of $\mathbb{P}_2$.
Let for any $\varepsilon > 0$, there exist $\varpi_1, \varpi_2 \in [\zeta - 1, \wp + \zeta]_{\mathbb{N}_{\zeta-1}}$ with $\varpi_1 < \varpi_2$, such that

$$\begin{aligned}
&\left|\frac{\xi_1}{\Gamma(\zeta)}\left[\sum_{s=0}^{\varpi_2-\zeta}(\varpi_2-s-1)^{(\zeta-1)}-\sum_{s=0}^{\varpi_1-\zeta}(\varpi_1-s-1)^{(\zeta-1)}\right]\right| \\
&\quad+|(\zeta-1-\varpi_2)-(\zeta-1-\varpi_1)|\left[\frac{\xi_1}{\Gamma(\zeta-1)}\sum_{s=0}^{\wp+1}(\zeta+\wp-s-1)^{(\zeta-2)}\right]<\epsilon.
\end{aligned} \tag{15}$$

In this case,

$$\begin{aligned}
|\mathbb{P}_2v(\varpi_2)-\mathbb{P}_2v(\varpi_1)| &= \left|\frac{1}{\Gamma(\zeta)}\sum_{s=0}^{\varpi_2-\zeta}(\varpi_2-s-1)^{(\zeta-1)}\mathcal{M}(s-1+\zeta,v(s-1+\zeta))\right. \\
&\quad+\frac{(\zeta-1-\varpi_2)}{\Gamma(\zeta-1)}\sum_{s=0}^{\wp+1}(\wp+\zeta-s-1)^{(\zeta-2)}\mathcal{M}(s-1+\zeta,v(s-1+\zeta)) \\
&\quad-\frac{1}{\Gamma(\zeta)}\sum_{s=0}^{\varpi_1-\zeta}(\varpi_1-s-1)^{(\zeta-1)}\mathcal{M}(s-1+\zeta,v(s-1+\zeta)) \\
&\quad\left.-\frac{(\zeta-1-\varpi_1)}{\Gamma(\zeta-1)}\sum_{s=0}^{\wp+1}(\wp+\zeta-s-1)^{(\zeta-2)}\mathcal{M}(s-1+\zeta,v(s-1+\zeta))\right| \\
&\leq\frac{1}{\Gamma(\zeta)}\left|\left[\sum_{s=0}^{\varpi_2-\zeta}(\varpi_2-s-1)^{(\zeta-1)}-\sum_{s=0}^{\varpi_1-\zeta}(\varpi_1-s-1)^{(\zeta-1)}\right]\right. \\
&\quad\left.\mathcal{M}(s-1+\zeta,v(s-1+\zeta))\right|+\left|[(\zeta-1-\varpi_2)-(\zeta-1-\varpi_1)]\right. \\
&\quad\left.\frac{1}{\Gamma(\zeta-1)}\sum_{s=0}^{\wp+1}(\wp+\zeta-s-1)^{(\zeta-2)}\mathcal{M}(s-1+\zeta,v(s-1+\zeta))\right| \\
&\leq\left|\frac{\xi_1}{\Gamma(\zeta)}\left[\sum_{s=0}^{\varpi_2-\zeta}(\varpi_2-s-1)^{(\zeta-1)}-\sum_{s=0}^{\varpi_1-\zeta}(\varpi_1-s-1)^{(\zeta-1)}\right]\right| \\
&\quad+|(\zeta-1-\varpi_2)-(\zeta-1-\varpi_1)|\left[\frac{\xi_1}{\Gamma(\zeta-1)}\sum_{s=0}^{\wp+1}(\zeta+\wp-s-1)^{(\zeta-2)}\right] \\
|\mathbb{P}_2v(\varpi_2)-\mathbb{P}_2v(\varpi_1)| &\leq\epsilon.
\end{aligned}$$

Therefore, the operator $\mathbb{P}_2$ is equi-continuous and Arzela–Ascoli's theorem guarantees the completely continuity of $\mathbb{P}_2$.

Step 3: We aim to prove that $v=\mathbb{P}_1v+\mathbb{P}_2v_1,$ for all $v_1\in\mathcal{H}\Rightarrow v\in\mathcal{H}$.
Let $v\in B, v_1\in\mathcal{H}$, such that $v=\mathbb{P}_1v+\mathbb{P}_2v_1$.

$$\begin{aligned}
|v(\varpi)| &\leq|\mathbb{P}_1v|+|\mathbb{P}_2v_1| \\
&\leq|\Theta(\varpi,v(\varpi))-\Theta(\zeta-1,g(v))+g(v)+A+\psi(v)|+\mathbb{P}_2v_1 \\
&\leq|\Theta(\varpi,v(\varpi))-\Theta(\varpi,0)+\Theta(\varpi,0)|+|g(v)|+|\psi(v)|+A \\
&\quad+|\Theta(\zeta-1,g(v))-\Theta(\varpi,0)+\Theta(\varpi,0)|+\mathbb{W} \\
&\leq[\mathbb{L}_1(1+K)]|v|+2\Theta_0+A+\mathbb{W}+\xi_2+\xi_3 \\
\|v(\varpi)\| &\leq\frac{2\Theta_0+A+W+\xi_2+\xi_3}{1-(\mathbb{L}_1(1+K))} \\
&\leq\mathcal{S}.
\end{aligned}$$

It is evident that $v(\varpi)\in\mathcal{H}$ and ensures the existence of at least one solution for the problem (1).

This completes the proof of the theorem. □

4. Hyers–Ulam Stability

In this section, we introduce the concept of Hyers–Ulam–Mittag–Leffler stability in the following definitions. Consider the Equation (1) and the following inequalities:

$$\left|\Delta_*^{\zeta}[v_1(\varpi)-\Theta(\varpi,v_1(\varpi))]-\mathcal{M}(\varpi-1+\zeta,v_1(\varpi-1+\zeta))\right|\leq\epsilon,\varpi\in[0,\wp]_{\mathbb{N}_0}. \tag{16}$$

$$\left|\Delta_*^{\zeta}[v_1(\varpi)-\Theta(\varpi,v_1(\varpi))]-\mathcal{M}(\varpi+\zeta-1,v_1(\varpi+\zeta-1))\right|\leq\epsilon\phi(\zeta-1+\varpi),\varpi\in[0,\wp]_{\mathbb{N}_0}. \tag{17}$$

$$\left|\Delta_*^{\zeta}[v_1(\varpi)-\Theta(\varpi,v_1(\varpi))]-\mathcal{M}(\varpi+\zeta-1,v_1(\varpi+\zeta-1))\right|\leq\epsilon F_{\zeta}(\lambda,\varpi),\varpi\in[0,\wp]_{\mathbb{N}_0}. \tag{18}$$

Definition 4. *Equation (1) is Hyers–Ulam stable, if a positive real number $\gamma>0$ exists for every $\epsilon>0$ and solution $v_1\in B$ of (16) then there is a solution $v\in B$ of (1) with*

$$|v_1(\varpi)-v(\varpi)|\leq\gamma\epsilon,\ \varpi\in[\zeta-1,\wp+\zeta]_{\mathbb{N}_{\zeta-1}}.$$

Definition 5. *Equation (1) is Hyers–Ulam–Rassias stable with respect to ϕ if a positive real number $\gamma>0$ exists for every $\epsilon>0$ and solution $v_1\in B$ of (17) then there is a solution $v\in B$ of (1) with*

$$|v_1(\varpi)-v(\varpi)|\leq\gamma\phi(\varpi)\epsilon,\ \varpi\in[\zeta-1,\wp+\zeta]_{\mathbb{N}_{\zeta-1}}.$$

Definition 6. *Equation (1) is Hyers–Ulam–Mittag–Leffler stable with Mittag–Leffler function $F_{\zeta}(\lambda,\varpi)$ if a positive real number $\gamma_1>0$ exists for every $\epsilon>0$ and solution $v_1\in B$ of (18) then there is a solution $v\in B$ of (1) with*

$$|v_1(\varpi)-v(\varpi)|\leq\gamma\epsilon F_{\zeta}(\lambda,\varpi),\ \varpi\in[\zeta-1,\wp+\zeta]_{\mathbb{N}_{\zeta-1}}.$$

Before proceeding to prove the stability results let us consider the following remarks which are natural consequences of the above paragraphs. We now will define functions h_1,h_2 and h_3 for proving the inequalities that are required to ensure the stability in the sense of Hyers and Ulam with Mittag–Leffler function.

Remark 1. *A function $v_1\in B$ is a solution of (16) if, and only if, a function*

$$h_1:[\zeta-1,\wp+\zeta]_{\mathbb{N}_{\zeta-1}}\to\mathbb{R}$$

exists such that:

(i) $|h_1(\varpi-1+\zeta)|\leq\epsilon,\varpi\in[0,\wp]_{\mathbb{N}_0}$;

(ii) $\Delta_*^{\zeta}[v_1(\varpi)-\Theta(\varpi,v_1(\varpi))]=h_1(\varpi-1+\zeta)+\mathcal{M}(\varpi-1+\zeta,v_1(\varpi-1+\zeta)),\varpi\in[0,\wp]_{\mathbb{N}_0}$.

Remark 2. *A function $v_1\in B$ is a solution of (17) if, and only if, a function*

$$h_2:[\zeta-1,\wp+\zeta]_{\mathbb{N}_{\zeta-1}}\to\mathbb{R}$$

exists such that:

(i) $|h_2(\varpi-1+\zeta)|\leq\epsilon\phi(\varpi-1+\zeta),\varpi\in[0,\wp]_{\mathbb{N}_0}$;

(ii) $\Delta_*^{\zeta}[v_1(\varpi)-\Theta(\varpi,v_1(\varpi))]=h_2(\varpi-1+\zeta)+\mathcal{M}(\varpi-1+\zeta,v_1(\varpi-1+\zeta)),\varpi\in[0,\wp]_{\mathbb{N}_0}$.

Remark 3. *A function $v_1\in B$ is a solution of (18) if and only if a function*

$$h_3:[\zeta-1,\wp+\zeta]_{\mathbb{N}_{\zeta-1}}\to\mathbb{R}$$

exists such that:

(i) $|h_3(\varpi-1+\zeta)|\leq\epsilon F_{\zeta}(\mu,\varpi),\varpi\in[0,\wp]_{\mathbb{N}_0}$;

(ii) $\Delta_{*}^{\zeta}[v_1(\varpi) - \Theta(\varpi, v_1(\varpi))] = h_3(\varpi - 1 + \zeta) + \mathcal{M}(\varpi - 1 + \zeta, v_1(\varpi - 1 + \zeta)), \varpi \in [0, \wp]_{\mathbb{N}_0}$.

Theorem 5. *Assume that J_1 and J_2 hold. Let $v_1 \in B$ be a solution of (18) and $v \in B$ be a solution of boundary value problem* (1)*. Then, (1) is Hyers–Ulam stable provided*

$$[\mathbb{L}_1(1+K) + K + (\wp+1)\lambda]\Gamma(\zeta+1)\Gamma(\wp+1) + \mathbb{L}\Gamma(\zeta+\wp+1)(\zeta+1) < \Gamma(\zeta+1)\Gamma(\wp+1). \tag{19}$$

Proof. Inequality (16) and Remark (1) implies

$$\begin{aligned}
&\Bigg| v_1(\varpi) - \Theta(\varpi, v_1(\varpi)) + \Theta(\zeta-1, g(v_1)) - g(v_1) - (\zeta - 1 - \varpi)[A + \psi(v_1)] \\
&\quad - \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v_1(s - 1 + \zeta)) \\
&\quad - \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v_1(s - 1 + \zeta)) \Bigg| \\
&\quad \leq \epsilon \frac{\Gamma(\wp + \zeta + 1)}{\zeta\Gamma(\zeta)\Gamma(\wp+1)}.
\end{aligned} \tag{20}$$

From (8) and (20), we obtain

$$\begin{aligned}
|v_1(\varpi) - v(\varpi)| \leq &\Bigg| v_1(\varpi) - \Theta(\varpi, v(\varpi)) + \Theta(\zeta-1, g(v)) - g(v) - (\zeta - 1 - \varpi)[A + \psi(v)] \\
&- \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v(s - 1 + \zeta)) \\
&- \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v(s - 1 + \zeta)) \Bigg| \\
\leq &\Bigg| v_1(\varpi) - \Theta(\varpi, v(\varpi)) + \Theta(\zeta-1, g(v)) - g(v) - (\zeta - 1 - \varpi)[A + \psi(v)] \\
&- \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v(s - 1 + \zeta)) \\
&- \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v(s - 1 + \zeta)) \\
&- \Theta(\varpi, v_1(\varpi)) + \Theta(\zeta-1, g(v_1)) - g(v_1) - (\zeta - 1 - \varpi)[A + \psi(v_1)] \\
&- \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v_1(s - 1 + \zeta)) \\
&- \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v_1(s - 1 + \zeta)) \\
&+ \Theta(\varpi, v_1(\varpi)) - \Theta(\zeta-1, g(v_1)) + g(v_1) + (\zeta - 1 - \varpi)[A + \psi(v_1)] \\
&+ \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s - 1 + \zeta, v_1(s - 1 + \zeta)) \\
&+ \frac{(\zeta - 1 - \varpi)}{\Gamma(\zeta - 1)} \sum_{s=0}^{\wp+1} (\wp + \zeta - s - 1)^{(\zeta-2)} \mathcal{M}(s - 1 + \zeta, v_1(s - 1 + \zeta)) \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\frac{L}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} + \frac{L|(\zeta - 1 - \varpi)|}{\Gamma(\zeta-1)} \right. \\
&\quad \left. \sum_{s=0}^{\wp} (\wp + \zeta - s - 2)^{(\zeta-2)} \right] |v_1 - v| + \epsilon \frac{\Gamma(\wp+\zeta+1)}{\Gamma(\zeta+1)\Gamma(\wp+1)} \\
&\leq \left[\mathbb{L}_1(1+K) + K + (1+\wp)\lambda + \frac{\mathbb{L}\Gamma(\zeta+\wp+1)}{\Gamma(\zeta+1)\Gamma(\wp+1)}(1+\zeta) \right] |v_1 - v| \\
&\quad + \epsilon \frac{\Gamma(\wp+\zeta+1)}{\zeta\Gamma(\zeta)\Gamma(\wp+1)} \\
\|v_1 - v\| &\leq \gamma\epsilon.
\end{aligned}
$$

Therefore, if (19) holds, then (1) is stable in the sense of Hyers–Ulam with constant $\gamma = \frac{\Gamma(\wp+\zeta+1)}{\Gamma(\zeta+1)\Gamma(\wp+1) - [(\mathbb{L}_1(1+K)+K+(\wp+1)\lambda)\Gamma(\zeta+1)\Gamma(\wp+1) + \mathbb{L}\Gamma(\zeta+\wp+1)(\zeta+1)]}$. □

Theorem 6. *Assume that J_1 and J_2 and the following condition holds. Consider an increasing function $\phi : [\zeta-1, \wp+\zeta]_{\mathbb{N}_{\zeta-1}} \to \mathbb{R}^+$ and a constant $\delta > 0$ such that:*

$$
\frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \phi(s-1+\zeta) \leq \delta\epsilon\phi(s-1+\zeta), \varpi \in [0, \wp]_{\mathbb{N}_0}. \tag{21}
$$

Let $v_1 \in B$ be a solution of (17) and $v \in B$ be a solution of (1). Then, (1) is Ulam–Hyers–Rassias stable provided (19) holds.

Proof. Theorem (6) follows from the proof of Theorem (5). □

Theorem 7. *Assume that J_1 and J_2 holds. Let $v_1 \in B$ be a solution of (18) and $v \in B$ be a solution of (1) Then, (1) is Hyers–Ulam–Mittag–Leffler stable provided (19) holds.*

Proof. Inequality (18) and Remark (3) implies:

$$
\begin{aligned}
&\Bigg| v_1(\varpi) - \Theta(\varpi, v_1(\varpi)) + \Theta(\zeta-1, g(v_1)) - g(v_1) - (\zeta - 1 - \varpi)[A + \psi(v_1)] \\
&\quad - \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s-1+\zeta, v_1(s-1+\zeta)) \\
&\quad - \frac{(\zeta-1-\varpi)}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp+\zeta-s-1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta, v_1(s-1+\zeta)) \Bigg| \\
&\quad \leq \frac{\epsilon}{\mu} F_\zeta(\mu, \varpi).
\end{aligned} \tag{22}
$$

From (8) and (22), we obtain:

$$
\begin{aligned}
|v_1(\varpi) - v(\varpi)| \leq &\Bigg| v_1(\varpi) - \Theta(\varpi, v(\varpi)) + \Theta(\zeta-1, g(v)) - g(v) - (\zeta - 1 - \varpi)[A + \psi(v)] \\
&- \frac{1}{\Gamma(\zeta)} \sum_{s=0}^{\varpi-\zeta} (\varpi - s - 1)^{(\zeta-1)} \mathcal{M}(s-1+\zeta, v(s-1+\zeta)) \\
&- \frac{(\zeta-1-\varpi)}{\Gamma(\zeta-1)} \sum_{s=0}^{\wp+1} (\wp+\zeta-s-1)^{(\zeta-2)} \mathcal{M}(s-1+\zeta, v(s-1+\zeta)) \Bigg| \\
\leq &|v_1(\varpi) - Pv_1(\varpi)| + |Pv_1(\varpi) - Pv(\varpi)| \\
\|v_1 - v\| \leq &\gamma_1 \epsilon F_\zeta(\mu, \varpi).
\end{aligned}
$$

Therefore, if (19) holds then (1) is Hyers–Ulam–Mittag–Leffler stable with constant $\gamma_1 = \frac{\Gamma(\zeta+1)\Gamma(\wp+1)}{\mu[\Gamma(\zeta+1)\Gamma(\wp+1)-[(\mathbb{L}_1(1+K)+K+(\wp+1)\lambda)\Gamma(\zeta+1)\Gamma(\wp+1)+\mathbb{L}\Gamma(\zeta+\wp+1)(\zeta+1)]]}$. □

5. Applications

The rate of heat flow can be increased by enhancing one of the following three factors: surface area, difference in temperature, and convective heat transfer coefficient. The direction of transfer of heat from the region of the high temperature to the region of the lower temperature. Temperature difference between the object and its surroundings does have an impact on the heat transfer. Since there are certain limitations in varying the temperature (depends on the process) and heat transfer coefficient (cannot be increased beyond certain values), the only possible and most economical way of enhancing heat transfer is by introducing fins to the system. These metallic surfaces of different shapes with its length greater than its diameter or thickness have adiabatic or cooled tip [57]. The contact at the base of the surface may be perfect or imperfect. The general differential equation representing fins with constant thermal conductivity (λ) is

$$\frac{d^2 v(x)}{dx^2} + \frac{1}{A(x)}\frac{dA(x)}{dx}\frac{du(x)}{dx} - \frac{\hbar P(x)}{\lambda A(x)} = 0, 0 \le x \le \mathcal{L}, \tag{23}$$

where $\mathcal{L}$ is the length of the fin, heat transfer coefficient is denoted by $\hbar$, $P(x)$, and $A(x)$ are the perimeter of the fin and area of cross section. The boundary conditions for the heat transfer through a fin can be defined in three different ways. The first two ways depend on the nature of contact (complete or incomplete) that the fin has with the surface and the third on being tip of fin that are insulated. For a fin with an insulated tip, the boundary condition at $x = L$ is $\frac{du}{dx} = 0$. The real life applications of these fins varies from radiators in cars, natural cooling of bike engines, heat sinks in CPUs, power plants, and hydrogen fuel cells. The fins type application in nature includes ears of Fennec foxes and Jack rabbits used for release of heat that is generated by the blood flow in their body.

Example 1. *Here in this example, we consider an adiabatic type fin used for heat transfer with constant perimeter and cross-sectional area. Then, the discrete fractional order form of (23) is*

$$\Delta^{\frac{3}{2}} v(\varpi) - \frac{(\hbar P)}{\vartheta A_c} v(\varpi + 0.5) = 0,\ \varpi \in [0, 10]_{\mathbb{N}_0}, \tag{24}$$

with boundary conditions $v(0.5) = 0.01$ and $\Delta v(11.5) = 0$.
Let the heat transfer coefficient $\hbar = 1W/m^2K$, thermal conductivity $\vartheta = 1.750kW/mK$, perimeter of fin $P = 2.2m$ and area of cross section of fin $A_c = 0.3850m^2$. Then, $\mathcal{M}(\varpi, v) = \frac{(\hbar P)}{\vartheta A_c} v(\varpi + 0.5)$ for every $\varpi \in [0.5, 11.5]$.
Hence, J_1 and J_2 are satisfied with $\mathbb{L} = 0.0033, K = 0.01$, and $\zeta = \frac{3}{2}$. Then, from Theorem (3), we observe $\rho = 0.2545 < 1$

Thus, (24) has a unique solution. In order to prove Hyers–Ulam stability of (24), we shall check the inequality (16). Let $v(\varpi) = \frac{\varpi^{(2)}}{15}$ and $\epsilon = 0.5$.

$$\begin{aligned}\left|\Delta^{\frac{3}{2}} v(\varpi) - \frac{(\hbar P)}{\vartheta A_c} v(\varpi + 0.5)\right| &= \left|\Delta^{-0.5}\Delta^2 \frac{\varpi^{(2)}}{15} - \frac{(\hbar P)}{\vartheta A_c}\frac{(\varpi+0.5)^{(2)}}{15}\right| \\ &= |0.5158 - 0.0287| \\ &= 0.4907 < \epsilon.\end{aligned}$$

Thus, the inequality (16) is satisfied. Similarly, from the inequality (18), we obtain

$$\begin{aligned}\left|\Delta^{\frac{3}{2}}v(\varpi) - \frac{(\hbar P)}{\vartheta A_c}v(\varpi+0.5)\right| &= 0.4907 \\ &< \epsilon F_{1.5}(0.1,1) \\ &= 0.55.\end{aligned}$$

Therefore, Theorems (5) and (7) imply Hyers–Ulam stablility and Hyers–Ulam–Mittag–Leffler Stability of (24).

The effect that thermal conductivity of the surface has on the stability of the system is analyzed with different values of ϑ. The values that are obtained at different order of the system are tabulated in Table 1. The values are plotted in Figure 1 and the stability conditions for the considered parameter values are less than 1. Hence, it coincides with our theory. A 3-dimensional plot for continuously varying thermal conductivity ϑ, fractional order ζ and corresponding values of ρ is presented in Figure 2. It can be observed from Figures 1 and 2 that the increase in thermal conductivity results in stability of the system. The values tabulated in Table 1 also reflects the corresponding impact of fractional order and thermal conductivity on the stability of the system. From the 3-dimensional plot it is visible that for thermal conductivity of 1.5 *kW*/*mK and fractional order $v = 1.99$ the stability condition attains greater value near to 1, proving that at $v = 1.99$ and for lesser values of ϑ the system becomes unstable.*

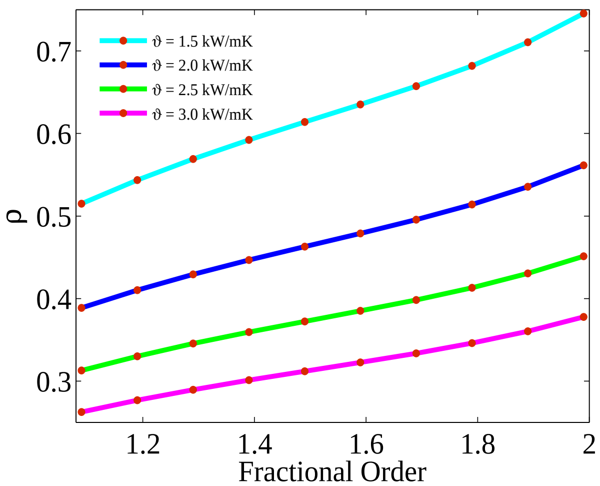

Figure 1. Results illustrating impact of fractional order (ζ) on stability Condition (ρ).

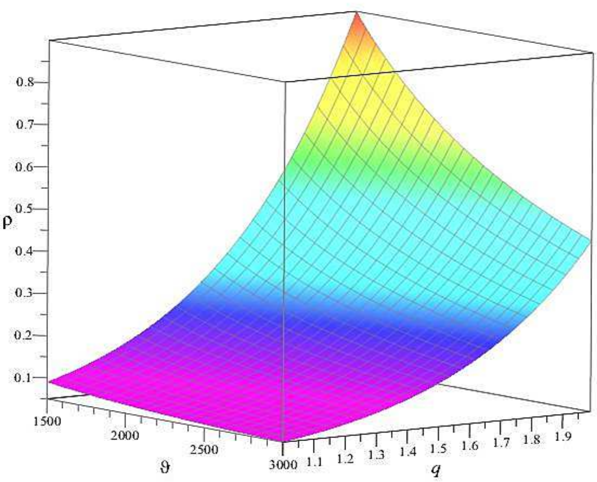

Figure 2. The 3-dimensional plot of results illustrating impact of fractional order (ζ) and thermal conductivity (ϑ) on (ρ).

Table 1. Impact of fractional order ζ on the stability condition ρ.

v	$\vartheta = 1.5$ kW/mK	$\vartheta = 2.0$ kW/mK	$\vartheta = 2.5$ kW/mK	$\vartheta = 3$ kW/mK
		ρ		
1.09	0.5148	0.3886	0.3129	0.2624
1.19	0.5435	0.4107	0.3301	0.2768
1.29	0.5690	0.4293	0.3454	0.2895
1.39	0.5922	0.4466	0.3593	0.3011
1.49	0.6138	0.4629	0.3723	0.3119
1.59	0.6351	0.4788	0.3851	0.3225
1.69	0.6573	0.4955	0.3984	0.3336
1.79	0.6819	0.5139	0.4131	0.3460
1.89	0.7106	0.5355	0.4304	0.3603
1.99	0.7453	0.5615	0.4512	0.3777

Example 2. *Consider the non-linear fractional difference equation*

$$\Delta^{\frac{9}{5}} v(\varpi) - \frac{(\varpi+0.8)^{-(0.95)}}{25} \sin(v(\varpi+0.8)) = 0, \ \varpi \in [0,12]_{\mathbb{N}_0}, \tag{25}$$

with boundary conditions $v(0.8) = \frac{v}{10}$ *and* $\Delta v(13.8) = 0.02$.
Take $\mathcal{M}(\varpi, v) = \frac{\Gamma(\varpi+1.8)}{25\Gamma(\varpi+2.75)} \sin(v(\varpi+0.8))$ *for every* $\varpi \in [0.8, 13.8]$.
Hence, J_1 *and* J_2 *are satisfied with* $\mathbb{L} = \frac{\Gamma(\varpi+1.8)}{25\Gamma(\varpi+2.75)}$*,* $K = 0.1$ *and* $\zeta = \frac{9}{5}$*. Then, from Theorem (3), we obtain* $\rho = 0.7592 < 1$*. Thus, (25) has a unique solution. With* $v(\varpi) = \frac{\varpi^{(2)}}{10}$ *and* $\epsilon = 0.4$*, the inequality (16) implies*

$$\begin{aligned} &\left| \Delta^{\frac{9}{5}} v(\varpi) - \frac{(\varpi+0.8)^{-(0.95)}}{25} \sin(v(\varpi+0.8)) \right| \\ &\quad = \left| \Delta^{-0.2}\Delta^2 \frac{\varpi^{(2)}}{10} - \frac{(\varpi+0.8)^{-(0.95)}}{25} \sin\left(\frac{(\varpi+0.8)^{(2)}}{10} \right) \right| \\ &\quad = 0.3736 < \epsilon. \end{aligned}$$

The inequality (16) holds, similarly with the inequality (18), we obtain

$$\begin{aligned} \left| \Delta^{\frac{9}{5}} v(\varpi) - \frac{(\varpi+0.8)^{-(0.95)}}{25} \sin(v(\varpi+0.8)) \right| &= 0.3736 \\ &< \epsilon F_{1.8}(0.1,1) \\ &= 0.44. \end{aligned}$$

From Theorems (5) and (7), we establish Hyers–Ulam stablility and Hyers–Ulam–Mittag–Leffler stability of (25).

The impact of varying fractional order ζ *on the stability of the problem* (25) *is represented in Figure 3 by plotting* ρ *defined in (11). The start and end points along the x-axis are based on the range of fractional order* $\zeta \in (1,2)$*. The corresponding values obtained are tabulated in Table 2. From Figure 3 it can be understood that the value of* ρ *for all the values* $\zeta \in (1,2)$ *is less than 1 coinciding with the theory. Figure 3 illustrates that the considered system (25) remains stable for* $\zeta \in (1,2)$*, but the value of* ρ *increases with increasing fractional order* ζ*. Thus, for choice of different parameter values the system (25) may be unstable when the order of the system is higher. The regression equation for the values tabulated in Table 2 is given by* $Y = 0.2052\ X + 0.5215$.

Table 2. Impact of fractional order ζ on ρ.

ζ	1.09000	1.1900	1.29000	1.3900	1.4900	1.5900	1.6900	1.7900	1.8900	1.9900
ρ	0.7416	0.7688	0.7913	0.8105	0.8275	0.8440	0.8616	0.8821	0.9077	0.9405

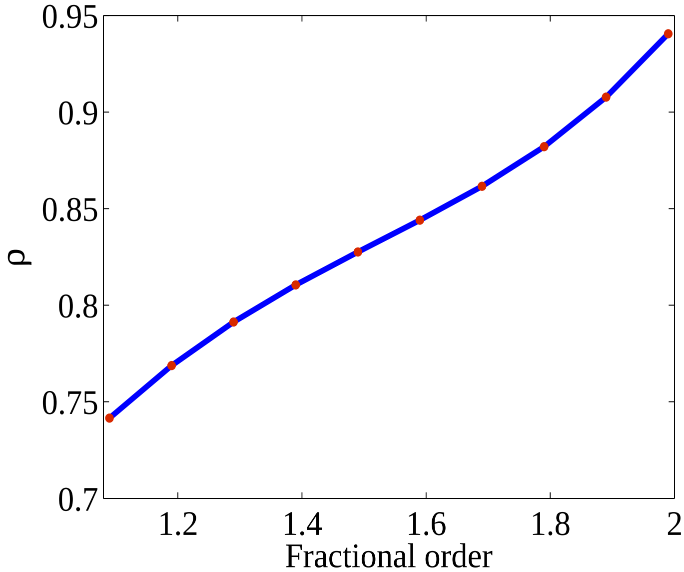

Figure 3. Impact of fractional order (ζ) on stability Condition (ρ).

6. Conclusions

The hybrid type boundary value problem with fractional order discrete time is considered and existence results along with the uniqueness of the solution are established. The stability in the sense of Hyers and Ulam using Mittag–Leffler function is illustrated for the boundary value problem. The application of heat transfer between surfaces using fins is taken into consideration and theoretical results obtained for general case are applied to check the suitability. The 2D and 3D simulations are provided to strengthen the results. The thermal conductivity (ϑ) and fractional order υ are taken as a parameters of study and is varied between 1.5 kW/mK and 3.0 kW/mK and $(1,2)$, respectively, to analyze their impact. It can be observed that the boundary problem of heat transfer with fins remains stable for all the values of $\vartheta \in (1.5, 3.0)$ and for fractional order $\zeta \in (1,2)$. The work can be further extended to explore the stability properties of hybrid type sum-difference equations with variable order.

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References

1. Ulam, S.M. *A Collection of Mathematical Problems*; Number 8; Interscience Publishers: Geneva, Switzerland, 1960.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [CrossRef] [PubMed]
3. Obłoza, M. Hyers stability of the linear differential equation. *Rocz. Nauk.-Dydakt. Pr. Math.* **1993**, *13*, 259–270.
4. Obłoza, M. Connections between Hyers and Lyapunov stability of the ordinary differential equations. *Rocz. Nauk.-Dydakt. Pr. Math.* **1997**, *14*, 141–146.
5. Alsina, C.; Ger, R. On some inequalities and stability results related to the exponential function. *J. Inequalities Appl.* **1998**, *2*, 373–380. [CrossRef]
6. Qarawani, M.N. Hyers-Ulam Stability of a Generalized Second-Order Nonlinear Differential Equation. *Appl. Math.* **2012**, *3*, 1857–1861. [CrossRef]
7. Alqifiary, Q.H.; Jung, S.M. On the Hyers-Ulam stability of differential equations of second order. *Abstr. Appl. Anal.* **2014**, *2014*, 483707. [CrossRef]
8. Alqifiary, Q.H.; Jung, S.M. Laplace transform and generalized Hyers-Ulam stability of linear differential equations. *Electron. J. Differ. Equ* **2014**, *2014*, 483707.
9. Podlubny, I. Fractional differential equations. *Math. Sci. Eng.* **1999**, *198*, 41–119.
10. Selvam, G.M.; Alzabut, J.; Dhakshinamoorthy, V.; Jonnalagadda, J.M.; Abodayeh, K. Existence and stability of nonlinear discrete fractional initial value problems with application to vibrating eardrum. *Math. Biosci. Eng.* **2021**, *18*, 3907–3921. [CrossRef]
11. Baleanu, D.; Mohammadi, H.; Rezapour, S. A fractional differential equation model for the COVID-19 transmission by using the Caputo–Fabrizio derivative. *Adv. Differ. Equ.* **2020**, *2020*, 299. [CrossRef]
12. Farman, M.; Akgül, A.; Baleanu, D.; Imtiaz, S.; Ahmad, A. Analysis of fractional order chaotic financial model with minimum interest rate impact. *Fractal Fract.* **2020**, *4*, 43. [CrossRef]
13. Khan, A.; Abdeljawad, T.; Gomez-Aguilar, J.; Khan, H. Dynamical study of fractional order mutualism parasitism food web module. *Chaos Solitons Fractals* **2020**, *134*, 109685. [CrossRef]
14. Selvam, A.G.M.; Baleanu, D.; Alzabut, J.; Vignesh, D.; Abbas, S. On Hyers–Ulam Mittag-Leffler stability of discrete fractional Duffing equation with application on inverted pendulum. *Adv. Differ. Equ.* **2020**, *2020*, 456. [CrossRef]
15. Tabouche, N.; Berhail, A.; Matar, M.; Alzabut, J.; Selvam, A.; Vignesh, D. Existence and stability analysis of solution for Mathieu fractional differential equations with applications on some physical phenomena. *Iran. J. Sci. Technol. Trans. A Sci.* **2021**, *45*, 973–982. [CrossRef]
16. Shakeel, M.; Shah, N.A.; Chung, J.D. Novel Analytical Technique to Find Closed Form Solutions of Time Fractional Partial Differential Equations. *Fractal Fract.* **2022**, *6*, 24. [CrossRef]
17. Shakeel, M.; Ul-Hassan, Q.M.; Ahmad, J.; Naqvi, T. Exact solutions of the time fractional BBM-Burger equation by (G'/G) novel-expansion method. *Adv. Math. Phys.* **2014**, *2014*, 181594. [CrossRef]
18. Wang, J.; Lv, L.; Zhou, Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Differ. Equ.* **2011**, *63*, 1–10. [CrossRef]
19. Wang, J.; Zhou, Y.; Fec, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl.* **2012**, *64*, 3389–3405. [CrossRef]
20. Wang, J.; Zhang, Y. Ulam–Hyers–Mittag-Leffler stability of fractional-order delay differential equations. *Optimization* **2014**, *63*, 1181–1190. [CrossRef]
21. Eghbali, N.; Kalvandi, V.; Rassias, J.M. A fixed point approach to the Mittag-Leffler-Hyers-Ulam stability of a fractional integral equation. *Open Math.* **2016**, *14*, 237–246. [CrossRef]
22. Niazi, A.U.K.; Wei, J.; Rehman, M.U.; Denghao, P. Ulam-Hyers-Mittag-Leffler stability for nonlinear fractional neutral differential equations. *Sb. Math.* **2018**, *209*, 1337. [CrossRef]
23. Agarwal, R.P. Certain fractional q-integrals and q-derivatives. In *Mathematical Proceedings of the Cambridge Philosophical Society*; Cambridge University Press: Cambridge, UK, 1969; Volume 66, pp. 365–370.
24. Diaz, J.; Osler, T. Differences of fractional order. *Math. Comput.* **1974**, *28*, 185–202. [CrossRef]
25. Atici, F.M.; Eloe, P.W. A transform method in discrete fractional calculus. *Int. J. Differ. Equ.* **2007**, *2*, 165–176.
26. Atici, F.; Eloe, P. Initial value problems in discrete fractional calculus. *Proc. Am. Math. Soc.* **2009**, *137*, 981–989. [CrossRef]
27. Holm, M. Sum and difference compositions in discrete fractional calculus. *Cubo* **2011**, *13*, 153–184. [CrossRef]
28. Anastassiou, G.A. Nabla discrete fractional calculus and nabla inequalities. *Math. Comput. Model.* **2010**, *51*, 562–571. [CrossRef]
29. Anastassiou, G.A. Foundations of nabla fractional calculus on time scales and inequalities. *Comput. Math. Appl.* **2010**, *59*, 3750–3762. [CrossRef]
30. Alzabut, J.; Selvam, A.; Dhakshinamoorthy, V.; Mohammadi, H.; Rezapour, S. On Chaos of Discrete Time Fractional Order Host-Immune-Tumor Cells Interaction Model. *J. Appl. Math. Comput.* **2022**, 1–26. [CrossRef]
31. Selvam, A.G.M.; Vignesh, D. Stability of discrete fractional Josephson junction. *Adv. Math. Sci. J.* **2021**, *10*, 137–144. [CrossRef]
32. Alzabut, J.; Selvam, A.G.M.; El-Nabulsi, R.A.; Dhakshinamoorthy, V.; Samei, M.E. Asymptotic stability of nonlinear discrete fractional pantograph equations with non-local initial conditions. *Symmetry* **2021**, *13*, 473. [CrossRef]
33. Atıcı, F.M.; Eloe, P.W. Two-point boundary value problems for finite fractional difference equations. *J. Differ. Equ. Appl.* **2011**, *17*, 445–456. [CrossRef]

34. Sitthiwirattham, T. Existence and uniqueness of solutions of sequential nonlinear fractional difference equations with three-point fractional sum boundary conditions. *Math. Methods Appl. Sci.* **2015**, *38*, 2809–2815. [CrossRef]
35. Chasreechai, S.; Kiataramkul, C.; Sitthiwirattham, T. On nonlinear fractional sum-difference equations via fractional sum boundary conditions involving different orders. *Math. Probl. Eng.* **2015**, *2015*, 519072. [CrossRef]
36. Zhou, H.; Alzabut, J.; Yang, L. On fractional Langevin differential equations with anti-periodic boundary conditions. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3577–3590. [CrossRef]
37. Seemab, A.; Ur Rehman, M.; Alzabut, J.; Hamdi, A. On the existence of positive solutions for generalized fractional boundary value problems. *Bound. Value Probl.* **2019**, *2019*, 186. [CrossRef]
38. Herzallah, M.A.; Baleanu, D. On fractional order hybrid differential equations. *Abstr. Appl. Anal.* **2014**, *2014*, 389386. [CrossRef]
39. Alzabut, J.; Selvam, A.G.M.; Vignesh, D.; Gholami, Y. Solvability and stability of nonlinear hybrid Δ-difference equations of fractional-order. *Int. J. Nonlinear Sci. Numer. Simul.* **2021**. [CrossRef]
40. Sun, S.; Zhao, Y.; Han, Z.; Li, Y. The existence of solutions for boundary value problem of fractional hybrid differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **2012**, *17*, 4961–4967. [CrossRef]
41. Ahmad, B.; Ntouyas, S.K.; Alsaedi, A. Existence results for a system of coupled hybrid fractional differential equations. *Sci. World J.* **2014**, *2014*, 426438. [CrossRef]
42. Ahmad, B.; Ntouyas, S.K. An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions. *Abstr. Appl. Anal.* **2014**, *2014*, 705809. [CrossRef]
43. Dhage, B.C.; Ntouyas, S.K. Existence results for boundary value problems for fractional hybrid differential inclusions. *Topol. Methods Nonlinear Anal.* **2014**, *44*, 229–238. [CrossRef]
44. Sitho, S.; Ntouyas, S.K.; Tariboon, J. Existence results for hybrid fractional integro-differential equations. *Bound. Value Probl.* **2015**, *2015*, 1–13. [CrossRef]
45. Bashiri, T.; Vaezpour, S.M.; Park, C. Existence results for fractional hybrid differential systems in Banach algebras. *Adv. Differ. Equ.* **2016**, *2016*, 57. [CrossRef]
46. Abdelouaheb, A.; Ahcene, D. Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle. *Ural Math. J.* **2019**, *5*, 3–12.
47. Lachouri, A.; Ardjouni, A.; Djoudi, A. Existence and Ulam stability results for nonlinear hybrid implicit Caputo fractional differential equations. *Math. Moravica* **2020**, *24*, 109–122. [CrossRef]
48. Chasreechai, S.; Sitthiwirattham, T. Existence results of initial value problems for hybrid fractional sum-difference equations. *Discret. Dyn. Nat. Soc.* **2018**, *2018*, 5268528. [CrossRef]
49. Shammakh, W.; Selvam, A.G.M.; Dhakshinamoorthy, V.; Alzabut, J. A study of generalized hybrid discrete pantograph equation via Hilfer fractional operator. *Fractal Fract.* **2022**, *6*, 152. [CrossRef]
50. Chen, F.; Zhou, Y. Existence and Ulam stability of solutions for discrete fractional boundary value problem. *Discret. Dyn. Nat. Soc.* **2013**, *2013*, 459161. [CrossRef]
51. Goodrich, C.; Peterson, A.C. *Discrete Fractional Calculus*; Springer: Berlin/Heidelberg, Germany, 2015; Volume 10.
52. Wu, G.C.; Baleanu, D.; Zeng, S.D.; Luo, W.H. Mittag-Leffler function for discrete fractional modelling. *J. King Saud Univ.-Sci.* **2016**, *28*, 99–102. [CrossRef]
53. Abdeljawad, T. On Riemann and Caputo fractional differences. *Comput. Math. Appl.* **2011**, *62*, 1602–1611. [CrossRef]
54. Lv, W. Existence of solutions for discrete fractional boundary value problems with a p-Laplacian operator. *Adv. Differ. Equ.* **2012**, *2012*, 163. [CrossRef]
55. Banach, S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
56. Dhage, B. A nonlinear alternative with applications to nonlinear perturbed differential equations. *Nonlinear Stud.* **2006**, *13*, 343–354.
57. Kraus, A.; Aziz, A.; Welty, J. *Extended Surface Heat Transfer*; John Wiley & Sons, Inc.: New York, NY, USA, 2001.