

Article

Study of a Viscous Λ WDM Model: Near-Equilibrium Condition, Entropy Production, and Cosmological Constraints

Norman Cruz ^{1,2,*} , Esteban González ^{3,*}  and Jose Jovel ^{1,*} 

¹ Departamento de Física, Universidad de Santiago de Chile, Avenida Ecuador 3493, Santiago 9170124, Chile

² Center for Interdisciplinary Research in Astrophysics and Space Exploration (CIRAS), Universidad de Santiago de Chile, Avenida Libertador Bernardo O'Higgins 3363, Estación Central, Santiago 9170124, Chile

³ Dirección de Investigación y Postgrado, Universidad de Aconcagua, Pedro de Villagra 2265, Vitacura, Santiago 7630367, Chile

* Correspondence: norman.cruz@usach.cl (N.C.); esteban.gonzalez@uac.cl (E.G.); jose.jovel@usach.cl (J.J.)

Abstract: Extensions to a Λ DM model have been explored in order to face current tensions that occur within its framework, which encompasses broadening the nature of the dark matter (DM) component to include warmness and a non-perfect fluid description. In this paper, we investigated the late-time cosmological evolution of an exact solution recently found in the literature, which describes a viscous warm Λ DM model (Λ WDM) with a DM component that obeys a polytropic equation of state (EoS), which experiences dissipative effects with a bulk viscosity proportional to its energy density, with proportionality constant ξ_0 . This solution has the particularity of having a very similar behavior to the Λ CDM model for small values of ξ_0 , evolving also to a de Sitter type expansion in the very far future. We explore firstly the thermodynamic consistencies of this solution in the framework of Eckart's theory of non-perfect fluids, focusing on the fulfillment of the two following conditions: (i) the near-equilibrium condition and (ii) the positiveness of the entropy production. We explore the range of parameters of the model that allow to fulfill these two conditions at the same time, finding that a viscous WDM component is compatible with both ones, being in this sense, a viable model from the thermodynamic point of view. Furthermore, we constrained the free parameters of the model with the observational data coming from supernovae Ia (SNe Ia) and the observational Hubble parameter data (OHD), using these thermodynamics analyses to define the best priors for the cosmological parameters related to the warmness and the dissipation of the DM, showing that this viscous Λ WDM model can describe the combined SNe Ia+OHD data in the same way as the Λ CDM model. The cosmological constraint at 3σ CL gives us an upper limit on the bulk viscous constant of order $\xi_0 \sim 10^6$ Pa·s, which is in agreement with some previous investigations. Our results support that the inclusion of a dissipative WDM, as an extension of the standard cosmological model, leads to a both thermodynamically consistent and properly fitted cosmological evolution.

Keywords: viscosity; dark matter; cosmological constant; cosmological model



Citation: Cruz, N.; González, E.; Jovel, J. Study of a Viscous Λ WDM Model: Near-Equilibrium Condition, Entropy Production, and Cosmological Constraints. *Symmetry* **2022**, *14*, 1866. <https://doi.org/10.3390/sym14091866>

Academic Editor: Maxim Y. Khlopov

Received: 30 July 2022

Accepted: 31 August 2022

Published: 7 September 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

In the current cosmology, the observational evidence suggests that almost the total energy density of the universe is compound by the dark sector, roughly classified into 30% of dark matter (DM) and 70% of dark energy (DE) [1–4]. The DM component is responsible for the structure formation in the universe, while DE is responsible for the recent accelerating expansion of the universe [5]. The most simple model that includes these components, which fits very well the cosmological data, is the Λ CDM model [1,2], where DE is modeled by a positive cosmological constant (CC) Λ , and DM is described as a pressureless fluid known as cold DM (CDM). However, this model is not absent of problems from the theoretical and observational point of view. For example, the CC problem, where the value of the CC differs from theoretical field estimations in 60–120 orders of

magnitude than the observed value [6–8]. Furthermore, measurements of the Hubble parameter at the current time, H_0 , present a discrepancy of 4.4σ between the measurements obtained from Planck CMB and the local measurements obtained by A. G. Riess et al. [9]. Other tensions are the measurements of $\sigma_8 - \Omega_m$ (where σ_8 is the r.m.s. fluctuations of perturbations at $8 h^{-1}\text{Mpc}$ scale) coming from large-scale structure (LSS) observations and the extrapolated from Planck CMB (dependent on the ΛCDM model) [10,11], and the results from the experiment to detect the global EoR Signature (EDGES), which detect an excess of radiation in the reionization epoch that is not predicted by the ΛCDM model, specifically at $z \approx 17$ [12].

In order to try to overcome some of these problems, dissipative effects can be considered as a more realistic way of treating cosmic fluids. By the symmetry of a homogeneous and isotropic universe, the dissipative process is usually characterized by a bulk viscosity. In this sense, some authors have considered the non-inclusion of the CC in order to alleviate the CC problem, explaining the late-time acceleration behavior of the universe through a dissipative viscous fluid [13–24], as a natural choice since the effect of the bulk viscosity is to produce a negative pressure that leads to an acceleration in the universe expansion [25–27]. Furthermore, in [28,29], the authors discuss the H_0 tension problem in the context of dissipative fluids as a good chance to construct new cosmological models with non-ideal fluids. Even more, the $\sigma_8 - \Omega_m$ tension can be alleviated if one assumes a small amount of viscosity in the DM component [10], as well as the explanation of the excess of radiation predicted by the EDGES experiment [30].

On the other hand, bulk viscosity seems to be significant in cosmic evolution. For example, many observational properties of disk galaxies can be represented by a dissipative DM component [31,32]. For neutralino CDM, the bulk viscous pressure is present in the CDM fluid through the energy transferred from the CDM fluid to the radiation fluid [33]. Some authors propose that bulk viscosity can produce different cooling rates of the components of the cosmic fluid [34–36], or may be the result of non-conserving particle interactions [37]. Even more, from Landau and Lifshitz [38], the bulk viscosity can be interpreted from the macroscopic point of view as the existence of slow processes to restore the equilibrium state. At the perturbative level, viscous fluid dynamics provide also a simple and accurate framework with the purpose of extending the description into the nonlinear regime [39]. In addition, the viscous effect could be the result of the interaction between DM and DE fluids [40]. In this sense, diffusion in cosmology plays an important role, since a diffusive exchange of energy between dark energy and dark matter could occur [41,42]. Following this line, the bulk viscosity ζ depends, particularly, on the temperature and pressure of the dissipative fluid [43]. Therefore, a natural election for the bulk viscosity of the dissipative fluid is to consider a dependency proportional to the power of their energy density $\zeta = \zeta_0 \rho^m$, where $\zeta_0 > 0$ is a bulk viscous constant; this is an election that has been widely investigated in the literature [44–49]. Since the nature of DM is unknown up to date, and a dissipative effect can not be discarded [50], it is of physical interest to explore how a bulk viscous DM behaves in the ΛCDM model.

Another important possible extension of the standard cosmological model has also been investigated in recent decades. It is well known that a CDM is capable to explain the observed structure very well above $\sim 1\text{ Mpc}$, while it has issues explaining small-scale structure observations [51,52], such as the missing satellite problem [53,54]—this refers to the discrepancy of approximately 10 times more dwarf galaxies between the values obtained by the numerical simulations based on the ΛCDM model and the observed ones in the cluster of galaxies. In this sense, a warm DM (WDM) can potentially be a good candidate to explain small-scale structure observations that currently represent a challenge for a CDM. Many studies of the number of satellites in the Milky Way or small halos with dwarf galaxies appear to be in better agreement with the observations for a WDM than they are for a CDM [52,55,56,56]. One of the well-motivated WDM hypotheses implies an extension of the standard model of particle physics by three sterile (right-handed, gauge singlet) neutrinos [57–59], produced via mixing with active neutrinos in the early universe [57,60–64]. On the other hand, from the

perturbation point of view, the non-linear effects make the power spectrum of the WDM look very much similar to a CDM, and LSS such as filaments, sheets, and a large void suggest that a WDM reproduces the observed ones well[65]; the WDM is an interesting alternative from the point of view of cosmology and particle physics.

All the discrepancies mentioned above imply extensions of the Λ CDM model, such as dissipative effects in a WDM component, which leads to taking into account a relativistic thermodynamic theory of non-perfect fluids out of the equilibrium. Eckart was the first to develop such a theory [66], with a similar model proposed by Landau and Lifshitz [38]. However, it was later shown that Eckart's theory was a non-causal theory [67,68]. A causal theory was proposed by Israel and Stewart (IS) [69,70], which was reduced to Eckart's theory when the relaxation time for the bulk viscous effects were negligible [71]. Since the IS theory presents a much greater mathematical difficulty than Eckart's theory, this last one is considered as a first approximation in order to study viscous cosmologies [44,46,72–75]. Following this line, the authors in [27] study a universe filled with two fluids in the framework of Eckart's theory: (i) a perfect fluid as DE mimicking the dynamics of the CC, and (ii) a bulk viscous DM; thus, finding a good agreement at 3σ CL with the observational background data coming from the observational Hubble parameter (OHD), type Ia supernovae (SNe Ia), and strong lensing systems (SLS). It is important to mention that, in these theories, the bulk viscous pressure Π has to be lower than the equilibrium pressure p of the dissipative fluid, i.e.,

$$l = \left| \frac{\Pi}{p} \right| \ll 1, \quad (1)$$

which it is known as *the near-equilibrium condition*, and represents the assumption that the fluid is close to thermodynamic equilibrium.

According to Maartens, in the context of dissipate inflation [71], the condition to have an accelerated expansion due only to the negativeness of the viscous pressure Π in Eckart's and IS theories enters into direct contradiction with the near-equilibrium condition given by Equation (1). In this sense, as it has been proposed in [27,47], if a positive CC is considered in these theories, then the near-equilibrium condition could be preserved in some regimes. In addition, it was shown by J. Hua and H. Hu [74] that a dissipative DM in Eckart's theory with CC has a significantly better fit with the cosmological data than the Λ CDM model, which indicates that this model is competitive to fit the combined SNe Ia + CMB + BAO + OHD data. Nevertheless, the inclusion of the CC implies abandoning the idea of unified DM models with dissipation, whose advantage is to avoid the CC problem, but that leads to reinforcing the proposal of extending the standard model, keeping a DE component modeled by a CC. Another important point of the near-equilibrium condition given by Equation (1) is that we need a non-zero equilibrium pressure for the dissipative fluid, discarding the possibility of a CDM component. In this sense, a relativistic approach of dissipative fluids is consistent with a WDM component [51,56,57,65,76–78] in order to satisfy the near-equilibrium condition.

It is important to mention that, for cosmologies with perfect fluids, there is no entropy production because these fluids are in equilibrium and their thermodynamics are reversible. However, for cosmologies with non-perfect fluids, where irreversible process exists, there is a positive entropy production during the cosmic evolution [22,79–82]. The near-equilibrium condition and entropy production has been previously discussed in the literature. The near-equilibrium condition was studied, for example, in [80] for the IS theory with gravitational constant G and Λ that vary over time; while in [83], it was studied in Eckart's and IS theories for the case of a dissipative Boltzmann gas and without the inclusion of a CC. The entropy production was studied in [79] in Eckart's and IS theories for a dissipative DE; while in [22], the authors studied the entropy production in the full IS theory with a matter content represented by one dissipative fluid component, and the kinematics and thermodynamics properties of the solutions are discussed (the entropy production in cosmological viscous fluids has been more widely studied, and more references can be found in [84–89]).

The aim of this paper is to explore the thermodynamic consistency in the description of a dissipative WDM component, as there is not enough investigated up to date, and if the constraints from the present cosmological data on the model that we propose are compatible with the consistency criteria found. Our model is described by an analytical solution obtained in [90] for a flat Friedman–Lemaître–Robertson–Walker (FLRW) maximally symmetric universe, dominated by a dissipative DM modeled by the barotropic EoS $p = (\gamma - 1)\rho$, where ρ is the energy density of the dissipative DM and γ is known as a barotropic index, and DE is modeled by the CC, in the framework of the Eckart’s theory. This solution was obtained using the expression $\xi = \xi_0 \rho^m$ for the bulk viscosity with the particular choice of $m = 1$, and it was studied in the context of the late and early-times singularities. Although this solution was found for a very particular election of m , it has the important characteristics that, for a positive CC, behaves very similarly to the Λ CDM model for all the cosmic time when $\xi_0 \rightarrow 0$, without singularity towards the past in asymptotic behavior known as “soft-Big Bang” [91,92], and with an asymptotic de Sitter expansion towards the future. This last behavior is of interest because the solution tends to the de Sitter expansion regardless of the value of ξ_0 and γ , as long as $\xi_0 < \gamma/3H_0$, which is a feature not found for other elections of m . Therefore, we focus our study on the late-time behavior of this solution, assuming that $\gamma \neq 1$ but close to 1, which represents a dissipative Λ WDM model with the same asymptotic late-time behavior as the Λ CDM model. In particular, we study the near-equilibrium condition and the positiveness of entropy production of this solution to find the constraints that these criteria impose on the model’s free parameters. We focus on the possibility to have a range of them satisfying all of these conditions, and we compare with the best fit values obtained from the cosmological constraint with the SNe Ia+OHD data. This study leads to some important clues about the physical behavior of the analytical solution, which represents a particular extension of the standard cosmological model. In this model, we have two more free parameters than the Λ CDM model, namely, γ and ξ_0 , but the gain lies in a more complete description of the nature of the DM component, suggested by a previous investigation made in the context to alleviate tensions in the standard cosmological model. Despite the fact that, in this work, we are not facing none of the mentioned tensions, our primary intention is to explore if the two extensions made to the standard model present also a consistent relativistic fluid description and not only a well-suitable fit with the cosmological data.

The outline of this paper is as follows: In Section 2, we summarize a solution that was found in [90], which represents the model of our study. In Section 3, we present general results about the near-equilibrium condition and the entropy production of the viscous fluid. In Section 4, we study the solution at late times, where in Section 4.1, we study the fulfillment of the near-equilibrium condition, while in Section 4.2, we study the entropy production of the dissipative fluid present in the model. In Section 5, we constraint the free parameters of our model with the SNe Ia and OHD data. In Section 6, we discuss these results, comparing them with the Λ CDM model, and we study the completeness of both, the near-equilibrium condition, and entropy production for the actual data. In addition, we find an upper limit for the present value of the bulk viscous constant. Finally, in Section 7, we present some conclusions and final discussions. The $8\pi G = c = 1$ units will be used in this work.

2. Exact Analytical Solution in Eckart’s Theory with CC

In this section, we briefly resume a de Sitter-like solution and an analytical solution found in [90] for a flat FLRW universe composed of a dissipative DM ruled by the barotropic EoS $p = (\gamma - 1)\rho$, with a bulk viscosity of the form $\xi = \xi_0 \rho^m$, and a DE given by the CC; the field equations, in the framework of Eckart’s theory, are given by [66,90]

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} + \frac{\Lambda}{3}, \quad (2)$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6}(\rho + 3P_{eff}) + \frac{\Lambda}{3}, \quad (3)$$

$$\dot{\rho} + 3H(\rho + P_{eff}) = 0, \quad (4)$$

where

$$P_{eff} = p + \Pi, \quad (5)$$

$$\Pi = -3H\zeta. \quad (6)$$

From these equations, it is possible to obtain a single evolution equation for the Hubble parameter $H = \dot{a}/a$, where a is the scale factor and “dot” accounts for the derivative with respect to the cosmic time t , which is given by

$$2\dot{H} + 3\gamma H^2 - 3\zeta_0 H(3H^2 - \Lambda)^m - \Lambda\gamma = 0. \quad (7)$$

We address the reader to see all the technical details in [90], where, from Equation (7), de Sitter-like and exact solutions for the cases of $m = 0$ and $m = 1$, with positive and negative CC, have been studied in the context of late- and early-time singularities. The results were compared with the Λ CDM model and, for this purpose, the differential Equation (7) is solved for $\zeta_0 = 0$ with the initial conditions $H(t = 0) = H_0$ and $a(t = 0) = 1$, which leads to

$$H(t) = \frac{H_0 \sqrt{\Omega_{\Lambda_0}} \left((\sqrt{\Omega_{\Lambda_0}} + 1) e^{3\gamma H_0 t \sqrt{\Omega_{\Lambda_0}}} - \sqrt{\Omega_{\Lambda_0}} + 1 \right)}{(\sqrt{\Omega_{\Lambda_0}} + 1) e^{3\gamma H_0 t \sqrt{\Omega_{\Lambda_0}}} + \sqrt{\Omega_{\Lambda_0}} - 1}, \quad (8)$$

$$a(t) = \left(\cosh\left(\frac{3\gamma \sqrt{\Omega_{\Lambda_0}} H_0 t}{2}\right) + \frac{\sinh\left(\frac{3\gamma \sqrt{\Omega_{\Lambda_0}} H_0 t}{2}\right)}{\sqrt{\Omega_{\Lambda_0}}} \right)^{\frac{2}{3\gamma}}, \quad (9)$$

where $\Omega_{\Lambda_0} = \Lambda/(3H_0^2)$. Note that Equation (8) tends asymptotically at very late times ($t \rightarrow \infty$) to the de Sitter solution $H_{dS} = H_0 \sqrt{\Omega_{\Lambda_0}}$.

We are particularly interested in the case of $m = 1$ of Equation (7) with a positive CC, where the following de Sitter-like solution ($\dot{H} = 0$) is found

$$E_{dS} = \sqrt{\Omega_{\Lambda_0}}, \quad (10)$$

with $E = H(t)/H_0$. It is important to note that Equation (10) is the usual de Sitter solution, written in its dimensionless form, and naturally appears in this dissipative scenario. On the other hand, the exact analytical solution ($\dot{H} \neq 0$) found takes the following expression in terms of the dimensionless parameters:

$$\tau = \frac{\Omega_{\zeta_0} \sqrt{\Omega_{\Lambda_0}} \log\left(\frac{(1-\Omega_{\Lambda_0})(\gamma-E\Omega_{\zeta_0})^2}{(E^2-\Omega_{\Lambda_0})(\gamma-\Omega_{\zeta_0})^2}\right)}{3\sqrt{\Omega_{\Lambda_0}}(\gamma^2-\Omega_{\zeta_0}^2\Omega_{\Lambda_0})} + \frac{\gamma \log\left(\frac{(\sqrt{\Omega_{\Lambda_0}}-1)(\sqrt{\Omega_{\Lambda_0}}+E)}{(\sqrt{\Omega_{\Lambda_0}}+1)(\sqrt{\Omega_{\Lambda_0}}-E)}\right)}{3\sqrt{\Omega_{\Lambda_0}}(\gamma^2-\Omega_{\zeta_0}^2\Omega_{\Lambda_0})}, \quad (11)$$

obtained with the border condition $H(t = 0) = H_0$, and where $\Omega_{\zeta_0} = 3\zeta_0 H_0$ and $\tau = tH_0$ are the dimensionless bulk viscous constant and cosmic time, respectively. The above solution is an implicit relation of $E(\tau)$. According to [90], Equation (11) presents a future singularity in a finite time known as Big-Rip [46,93–95] when $\Omega_{\zeta_0} > \gamma$. In this singularity, we have an infinite a , ρ and p , and therefore, the Ricci scalar diverges. Furthermore, it is

discussed that one interesting behavior of this solution can be obtained if we considered the opposite condition, i.e.,

$$\Omega_{\xi_0} < \gamma, \quad (12)$$

which leads to a universe with a behavior very similar to the Λ CDM model, which coincides as $\Omega_{\xi_0} \rightarrow 0$, as can be seen in Figure 1, where we have numerically found the behavior of E as a function of τ from Equation (11), taking into account the condition (12) with $\gamma = 1.002$, $\Omega_{\xi_0} = 0.001$, and $\Omega_{\Lambda_0} = 0.69$. For a further comparison, we also plotted the Λ CDM model.

Note that the solution (11) tends asymptotically for $\tau \rightarrow \infty$ to the usual de Sitter solution (10), which can be seen in Figure 1. Therefore, for the condition given by Equation (12) and for $\gamma \neq 1$ but close to 1, solution (11) represents a viscous Λ WDM model with a late-time behavior very similar to the Λ CDM model and with the same asymptotic de Sitter expansion.

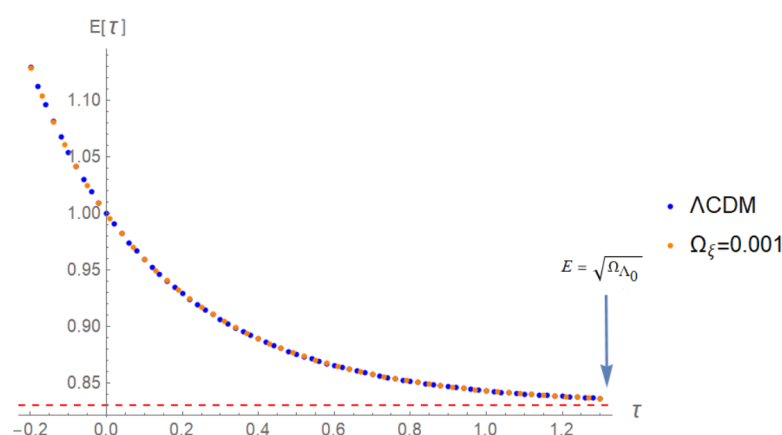


Figure 1. Numerical behavior of $E(\tau)$ obtained from Equation (11) at late times, for $\Omega_{\Lambda_0} = 0.69$, $\Omega_{\xi_0} = 0.001$ and $\gamma = 1.002$. For comparison, we also plotted the Λ CDM model obtained from Equation (8).

3. Near-Equilibrium Condition and Entropy Production

In the following section, we found the main expressions in terms of γ , and the dimensionless quantities E , Ω_{ξ_0} and Ω_{Λ_0} that arise from the near-equilibrium condition and the entropy production.

3.1. Near-Equilibrium Condition

As it was previously discussed, in Eckart's theory, it is necessary to fulfill the near-equilibrium condition (1). Following Maartens [71], and according to Equations (3) and (5), we can write

$$\frac{\ddot{a}}{a} = -\frac{1}{6}[\rho + 3(p + \Pi)] + \frac{\Lambda}{3}. \quad (13)$$

From the above expression, the condition to have an accelerated expansion driven only by the negativeness of the bulk viscous pressure Π , imposing $\ddot{a} > 0$ and taking $\Lambda = 0$, is

$$-\Pi > p + \frac{\rho}{3}. \quad (14)$$

This last result implies that the viscous stress is greater than the equilibrium pressure p of the fluid, i.e., the near-equilibrium condition is not fulfilled because in order to obtain an accelerated expansion, the fluid has to be far from equilibrium. This situation could be changed if a positive CC is included [27,47]. In this case, the condition $\ddot{a} > 0$ on Equation (13) leads to

$$-\Pi > \frac{-2\Lambda}{3} + p + \frac{\rho}{3}, \quad (15)$$

i.e., the near-equilibrium condition could be fulfilled in some regime, because from Equation (15), the viscous stress is not necessarily greater than the equilibrium pressure p . The near-equilibrium condition given by Equation (1) can be rewritten in terms of the dimensionless parameters, using Equation (6), and the EoS of the DM component, obtaining

$$l = \left| \frac{E(\tau)\Omega_{\xi_0}}{\gamma - 1} \right| \ll 1. \quad (16)$$

From the above equation, it is clear to see that a CDM component with $\gamma = 1$ is not compatible with the near-equilibrium condition, and only for some kind of WDM with $\gamma > 1$, this condition can be fulfilled. On the other hand, note that in the above expression, the solution given by $E(\tau)$ drives the behavior of l as a function of the cosmic time τ . In this sense, and since $E(\tau)$ is a decreasing function of time, the constraints on Ω_{ξ_0} are more restrictive as we look forward.

3.2. Entropy Production

The first law of thermodynamics is given by

$$TdS = dU + pdV, \quad (17)$$

where T , S , U and V are the temperature, entropy, the total thermal energy and the three-dimensional volume of the cosmic fluid, respectively. The total thermal energy of the fluid and the physical three-dimensional volume of the universe are given, respectively, by $U = \rho V$ and $V = V_0 a^3$ (where V_0 is the volume at the present time). For a DM, the energy density is given by $\rho = n(mc^2 + 3k_B T)$, and for the particular case of CDM, the energy density is associated with the rest mass of DM given by $\rho = nmc^2$, in full units. With these, we get from Equation (17), the Gibbs Equation [82]

$$dS = - \left(\frac{\rho + p}{Tn^2} \right) dn + \frac{d\rho}{Tn}, \quad (18)$$

where $n = N/V$ is the number of particle density. The following integrability condition must hold on the thermodynamical variables ρ and n

$$\left[\frac{\partial}{\partial \rho} \left(\frac{\partial S}{\partial n} \right) \right]_{\rho} = \left[\frac{\partial}{\partial n} \left(\frac{\partial S}{\partial \rho} \right) \right]_{n}, \quad (19)$$

then, we considered the thermodynamic assumption in which the temperature is a function of the number of particles density and the energy density, i.e., $T(n, \rho)$. With this, the above integrability condition becomes in [79,82],

$$n \frac{\partial T}{\partial n} + (\rho + p) \frac{\partial T}{\partial \rho} = T \frac{\partial p}{\partial \rho}. \quad (20)$$

We study the case of a perfect fluid and a viscous fluid separately in order to compare our result with the model without viscosity.

For a perfect fluid, the particle 4-current is taken to be $n^\alpha_{;\alpha} = 0$, where “;” accounts for the covariant derivative, which together with the conservation equation, leads to the following expressions for the particle density and the energy density, respectively:

$$\dot{n} + 3Hn = \frac{\dot{N}}{N} = 0, \quad (21)$$

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (22)$$

Assuming that the energy density depends on the temperature and the volume, i.e., $\rho(T, V)$ [79], we have the following relation:

$$\frac{d\rho}{da} = \frac{\partial\rho}{\partial T} \frac{dT}{da} + \frac{3V}{a} \frac{\partial\rho}{\partial V}. \quad (23)$$

Using Equations (22) and (23), and the EoS, it can be shown (as in [79]) that the barotropic temperature from (20) is given by

$$\frac{T}{T_0} = \frac{\rho}{\rho_0} a^3 = \left(\frac{\rho}{\rho_0} \right)^{\frac{\gamma-1}{\gamma}}. \quad (24)$$

This last expression was previously found by Martens in [82], using the method of characteristics. Additional to this, from Equation (18), together with Equations (21) and (22), and the EoS, we have $dS = 0$, or consequently, $dS/dt = 0$, which implies that there is no entropy production in the cosmic expansion, i.e., the fluid is adiabatic.

For a viscous fluid, an average 4-velocity is chosen in a way that there is no particle flux [43,66]; thus, in this frame, the particle 4-current is taken again as $n_{;\alpha}^{\alpha} = 0$ and Equation (21) is still valid. On the other hand, from Eckart's theory, we have the following conservation equation, according to Equations (4) and (5):

$$\dot{\rho} + 3H(\rho + p + \Pi) = 0, \quad (25)$$

which, together with Equation (21) and EoS (18), give us the following expression for the entropy production [79,81]

$$\frac{dS}{d\tau} = \frac{3E^2\Omega_{\xi_0}\rho}{nT}, \quad (26)$$

written in a dimensionless form. Therefore, the entropy production in the viscous expanding universe is, in principle, always positive, and we recovered the behavior of a perfect fluid when $\Omega_{\xi_0} = 0$. As we will see later, this positiveness requires some constraints under the free parameters of the solution, specifically in the expression for the temperature of the dissipative fluid.

4. Study of the Exact Solution

In this section, we study, at the same time, the exact solution (11) under the condition (12) in terms of fulfillment of the near-equilibrium condition and the positiveness of the entropy production. To that end, we focus our analysis in two defined late-time epochs of validity for the solution: (i) the actual time $\tau = 0$ for which $E = 1$, and (ii) the very late times $\tau \rightarrow \infty$ for which $E = \sqrt{\Omega_{\Lambda_0}}$. We will extend the analysis for $\tau > 0$ and for $\tau < 0$. It is important to mention that the asymptotic behavior of the solution (11), given by the de Sitter solution (10) when the condition (12) holds, leads to a universe dominated only by the CC, in which the dissipative WDM is diluted due to the universe expansion, as can be seen by evaluating Equation (10) in the Friedmann Equation (2)

$$H_{dS}^2 = \frac{\rho}{3} + \frac{\Lambda}{3}, \quad (27)$$

which leads to $\rho = 0$. Therefore, in this asymptotic behavior, we do not have a dissipative fluid to study the near-equilibrium condition and the entropy production. Nevertheless, we can study these two conditions as an asymptotic regime of the solution while $\rho \rightarrow 0$.

4.1. Near-Equilibrium Condition of the Exact Solution

Note that the near-equilibrium condition (16), considering that $1 < \gamma \leq 2$, can be rewritten as

$$E(\tau) \ll \frac{\gamma - 1}{\Omega_{\xi_0}} = \frac{\gamma}{\Omega_{\xi_0}} - \frac{1}{\Omega_{\xi_0}}, \quad (28)$$

an expression that tells us that as long as $E = H/H_0$ is much smaller than $(\gamma - 1)/\Omega_{\xi_0}$, the solution must be near to the thermodynamic equilibrium. This opens the possibility that the solution is able to fulfill this condition, considering that $E(t)$, given by Equation (11), decreases asymptotically to the de Sitter solution (10) as $\tau \rightarrow \infty$ under the condition (12), as shown in Figure 1. Furthermore, at the current time ($\tau = 0$), the condition (28) leads to

$$\Omega_{\xi_0} \ll \gamma - 1, \quad (29)$$

and for the very late times ($\tau \rightarrow \infty$), leads to

$$\sqrt{\Omega_{\Lambda_0}} \ll \frac{\gamma - 1}{\Omega_{\xi_0}} = \frac{\gamma}{\Omega_{\xi_0}} - \frac{1}{\Omega_{\xi_0}}. \quad (30)$$

The fulfillment of the condition (29) implies the fulfillment of the condition (30), because $0 < \Omega_{\Lambda_0} \leq 1$, and from the condition (29), we get $1 \ll (\gamma - 1)/\Omega_{\xi_0}$. Furthermore, note that the fulfillment of the condition (29) implies the fulfillment of the condition (12). In summary, the fulfillment of the condition (29) implies the fulfillment of the near-equilibrium condition from $0 \leq \tau < \infty$ and the condition (12), for which the solutions (11) behave as the de Sitter solution at very late times. It is important to note that the fulfillment of the near-equilibrium condition depends only on the values of γ and Ω_{ξ_0} and not on the values of Ω_{Λ_0} , with the characteristic that for a value of γ closer to 1 (CDM), a smaller value of Ω_{ξ_0} must be considered. Even more, for $1 < \gamma \leq 2$, we can see that $\Omega_{\xi_0} < 1$. On the other hand, the condition (29) is independent of the behavior of the solution because the election $E(\tau = 0) = 1$ is arbitrary, but this does not imply that the condition (30) is always fulfilled, because this condition depends on the behavior of the solution. In this sense, note that if we do not satisfy the condition (12) in Equation (16), then E diverges and the solution will be far from near equilibrium in a finite time in the Big-Rip scenario.

For $\tau < 0$, it is still possible to fulfill the near-equilibrium condition (28), as we mentioned above, while E is much smaller than $(\gamma - 1)/\Omega_{\xi_0}$. In Figure 2, we depict the near-equilibrium condition (16) as a function of Ω_{ξ_0} and E for the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The red zone represents the values for which we are far from the near equilibrium and the green line represents the near-equilibrium condition when $\Omega_{\xi_0} = 0.001$. Note that for these last values, we are far from the equilibrium when $\Omega_{\xi_0} \geq 0.002$ ($l \geq 1$) for all E , and for $\Omega_{\xi_0} = 0.001$, we are far from the near equilibrium when $E \geq 2$, i.e., we are far from the near-equilibrium condition for a time given by $\tau = -0.6001$, which is roughly equivalent to 8.63389 Gyrs backward in time (0.630211 times the life of the Λ CDM universe), according to Equation (11).

From Figure 2, we can see that when E grows, we can make Ω_{ξ_0} more closer to zero in order to fulfill the near-equilibrium condition. For the solution (11), under the condition (12), this means that for $\tau < \infty$, for which E grows, we can stay in the near equilibrium while Ω_{ξ_0} be small enough to satisfy the condition (28). This behavior is due to the election of the dissipation of the form $\xi = \xi_0 \rho$, because the dissipative pressure $\Pi = -3H\xi$ behaves as $\rho^{3/2}$ and the equilibrium pressure behaves as ρ , and therefore, considering that in this case ρ grows to the past, then the dissipative pressure grows more quickly than the equilibrium pressure and ξ_0 acts as a modulator of this growth.

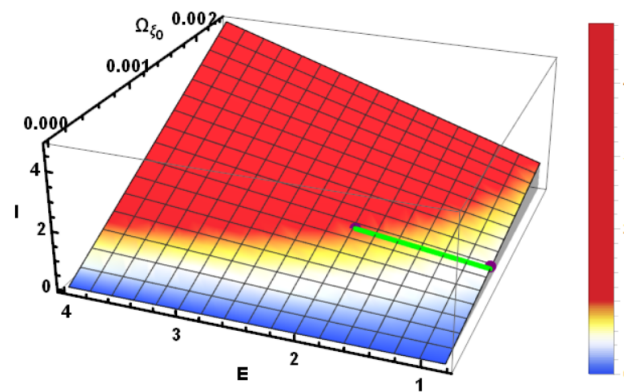


Figure 2. Behavior of l , given by Equation (16), for $0 \leq \Omega_{\xi_0} \leq 0.02$ and $\sqrt{\Omega_{\Lambda_0}} \leq E \leq 4$. We also consider the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The green line represents the near-equilibrium condition when $\Omega_{\xi_0} = 0.001$ and the red zone represents the values for which we are far from the near-equilibrium condition ($l > 1$).

4.2. Entropy Production of the Exact Solution

In order to obtain the entropy production of the dissipative fluid from Equation (26), we need to find their temperature from Equation (20). Rewriting the conservation Equation (25) in the form

$$\frac{d\rho}{da} = -\frac{3\rho}{a}(\gamma - 3H\xi_0), \quad (31)$$

we can rewrite Equation (23) as

$$\rho(\gamma - 3H\xi_0) = -\frac{a}{3} \frac{\partial \rho}{\partial T} \frac{dT}{da} - V \frac{\partial \rho}{\partial V}. \quad (32)$$

Then, from Equation (20), and using Equation (5), we have

$$n \frac{\partial T}{\partial n} + \rho(\gamma - 3H\xi_0) \frac{\partial T}{\partial \rho} = T \left[(\gamma - 1) - 3H\xi_0 - 3\xi_0 \rho \frac{\partial H}{\partial \rho} \right], \quad (33)$$

an expression that, with together Equation (32), leads to

$$\frac{dT}{T} = -3 \frac{da}{a} \left[(\gamma - 3H\xi_0) - 1 - 3\xi_0 \rho \frac{\partial H}{\partial \rho} \right]. \quad (34)$$

Thus, using Equations (2) and (31), we get, from Equation (34), the following expression:

$$\frac{dT}{T} = \frac{d\rho}{\rho} \left[1 - \frac{\left(\frac{2}{3} \sqrt{3(\rho + \Lambda)} + \xi_0 \rho \right)}{\frac{2}{3} \sqrt{3(\rho + \Lambda)} (\gamma - \sqrt{3(\rho + \Lambda)} \xi_0)} \right], \quad (35)$$

which has the following solution in our dimensionless notation:

$$\begin{aligned} \ln \left(\frac{T}{T_0} \right) &= \ln \left(\frac{\rho}{\rho_0} \right) \\ &+ \frac{2\Omega_{\xi_0} \sqrt{\Omega_{\Lambda}} \left[\operatorname{arctanh} \left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E} \right) - \operatorname{arctanh} \left(\sqrt{\Omega_{\Lambda_0}} \right) \right]}{(\gamma^2 - \Omega_{\Lambda_0} \Omega_{\xi_0}^2)} \\ &- \frac{\gamma \ln \left(\frac{\rho}{\rho_0} \right) + [\gamma(2 + \gamma) - \Omega_{\Lambda_0} \Omega_{\xi_0}^2] \ln \left(\frac{\gamma - E \Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}^2} \right)}{(\gamma^2 - \Omega_{\Lambda_0} \Omega_{\xi_0}^2)}. \end{aligned} \quad (36)$$

On the other hand, by integrating Equation (31) with the help of Equation (2), we obtain in our dimensionless notation the expression

$$\ln a^3 = \frac{2\Omega_{\xi_0}\sqrt{\Omega_{\Lambda}}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right) - \operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2 - \Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)} + \frac{-\gamma\ln\left(\frac{\rho}{\rho_0}\right) + 2\gamma\ln\left(\frac{\gamma - E\Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}}\right)}{\left(\gamma^2 - \Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}, \quad (37)$$

from which we can express Equation (36) in terms of the scale factor as follows:

$$\ln\left(\frac{T}{T_0}\right) = \ln\left(\frac{\rho}{\rho_0}\right) + \ln\left(\frac{\gamma - E\Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}}\right) + \ln a^3. \quad (38)$$

Hence, the temperature of the dissipative fluid as a function of the scale factor is given by

$$T = T_0\left(\frac{\rho}{\rho_0}\right)\left(\frac{\gamma - E\Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}}\right)a^3, \quad (39)$$

which reduced to the expression for the temperature of a perfect fluid, given by Equation (24), when $\Omega_{\xi_0} = 0$.

Note that it is possible to obtain an expression for the temperature of the dissipative fluid that does not depend explicitly on ρ , by combining Equations (37) and (39), which leads to

$$T = T_0 a^{\frac{3}{\gamma}(\gamma - \gamma^2 + \Omega_{\xi_0}^2 \Omega_{\Lambda_0})} \left(\frac{\gamma - E\Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}}\right)^3 \times \left[\left(\frac{E + \sqrt{\Omega_{\Lambda_0}}}{E - \sqrt{\Omega_{\Lambda_0}}}\right)\left(\frac{1 - \sqrt{\Omega_{\Lambda_0}}}{1 + \sqrt{\Omega_{\Lambda_0}}}\right)\right]^{\left(\frac{\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}}{\gamma}\right)}, \quad (40)$$

and from which we can see that the temperature is always positive in two cases: (i) when $\gamma - E\Omega_{\xi_0} > 0$ and $\gamma - \Omega_{\xi_0} > 0$, or (ii) when $\gamma - E\Omega_{\xi_0} < 0$ and $\gamma - \Omega_{\xi_0} < 0$. Therefore, if we fulfill the near-equilibrium condition (28), then we obtain a positive expression for the temperature, since the case (i), and from the same condition given by Equation (28), we can also fulfill the condition (12) from which the solution (11) tends asymptotically to the future at the usual de Sitter solution. Note that case (ii) implies that the fluid is far from the near equilibrium and the solution (11) has a Big-Rip singularity. On the other hand, considering that the solution (11) is a decreasing function with time when the condition (12) holds, then the cubic term in Equation (40) is also a decreasing function; thus, considering that $E(\tau \rightarrow \infty) \rightarrow \sqrt{\Omega_{\Lambda_0}}$, a decreasing temperature with time requires that

$$\frac{a^{\frac{3}{\gamma}(\gamma - \gamma^2 + \Omega_{\xi_0}^2 \Omega_{\Lambda_0})}}{(E - \sqrt{\Omega_{\Lambda_0}})^{\left(\frac{\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}}{\gamma}\right)}} \rightarrow 0, \quad (41)$$

which is only possible, considering that $a(\tau \rightarrow \infty) \rightarrow \infty$, when the exponent of the power law for the scale factor is negative, i.e., if

$$\Omega_{\Lambda_0} < \frac{\gamma}{\Omega_{\xi_0}} \frac{(\gamma - 1)}{\Omega_{\xi_0}}, \quad (42)$$

which is always true because $0 < \Omega_{\Lambda_0} \leq 1$ and the fulfillment of the condition (12) implies that $1 < \gamma/\Omega_{\xi_0}$, as well as the fulfillment of the condition (29) implies that $1 \ll (\gamma - 1)/\Omega_{\xi_0}$. It is important to note that for a CDM ($\gamma = 1$), the exponent of the power law for the scale factor is always positive, and the temperature is an increasing function with time, which represents a contradictory behavior for an expanding universe.

In Figure 3, we present the numerical behavior of the temperature T of the dissipative fluid, given by Equation (39), as a function of the scale factor a . We rewrite E as a function of the energy density ρ from Equation (2), and we use the expression for ρ given by Equation (37). For the free parameters, we use the values of $T_0 = 1$, $\Omega_{\xi_0} = 0.001$, $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. We also present the behavior of the temperature when $\gamma = 1$. It is important to mention that the difference between the initial value of the temperature for the WDM case and their final value is 0.0102895 for a scale factor that is 3.6 times more bigger than the actual size of the universe, this is a consequence of being close to the near-equilibrium condition, which makes that temperature decrease very slowly to zero.

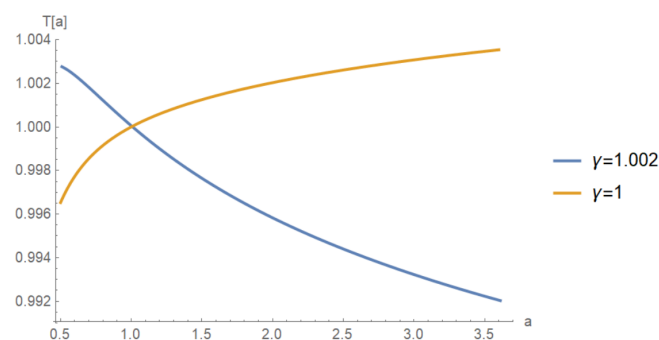


Figure 3. Numerical behavior of T , given by Equation (39), for $0.5 \leq a \leq 3.5$. We also consider the fixed values of $T_0 = 1$, $\Omega_{\xi_0} = 0.001$, and $\Omega_{\Lambda_0} = 0.69$.

With the temperature of the dissipative fluid given by Equation (39), we can calculate the entropy from Equations (21) and (26), which in our dimensionless notation, takes the following form:

$$\frac{dS}{d\tau} = \frac{3E^2\Omega_{\xi_0}\rho}{nT} = \frac{3E^2\rho_0\Omega_{\xi_0}(\gamma - \Omega_{\xi_0})}{n_0T_0(\gamma - E\Omega_{\xi_0})}. \quad (43)$$

The positiveness of the entropy production depends on the same both cases obtained before the positiveness of the temperature of the dissipative fluid. Therefore, the fulfillment of the near-equilibrium condition implies the positiveness of the entropy production. Note that for the asymptotic de Sitter solution, the entropy production goes to a constant but, in this case, $\rho \rightarrow 0$ (which implies a null Ω_{ξ_0}), and then we have a null entropy production. On the other hand, the entropy production goes to infinity, similar to the temperature of the dissipative fluid, when we do not satisfy the condition (12) in a finite time in the Big-Rip singularity.

In Figure 4, we show the numerical behavior of the entropy production, given by Equation (43), as a function of time τ for $\Omega_{\xi_0} = 0.001$, $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. We also show the behavior of the entropy production when $\Omega_{\xi_0} = 2.4$. For this last one, the near-equilibrium condition is not satisfied, since this value enters into contradiction with Equation (12), and the entropy production diverges in a finite time given by [90]

$$\tau_s = \frac{2\Omega_{\xi_0} \log \left[\left(\frac{1 - \sqrt{\Omega_{\Lambda_0}}}{1 + \sqrt{\Omega_{\Lambda_0}}} \right)^{\frac{\gamma}{2\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}}} (1 - \Omega_{\Lambda_0})^{\frac{1}{2}} \left(\frac{-\Omega_{\xi_0}}{\gamma - \Omega_{\xi_0}} \right) \right]}{3(\gamma^2 - \Omega_{\xi_0}^2 \Omega_{\Lambda_0})}, \quad (44)$$

which, according to the values given to the free parameters, this is equal to $\tau_s = 0.346571$, which is roughly equivalent to 4.98626 Gyrs from the present time.

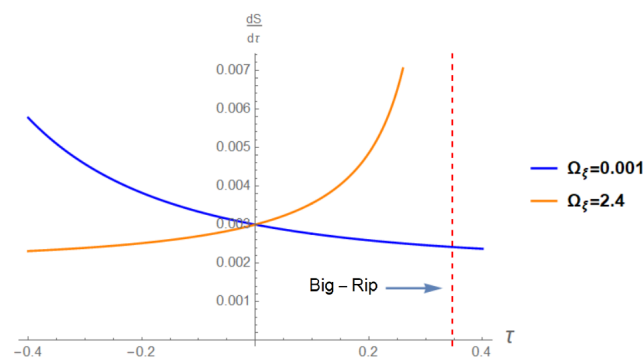


Figure 4. Numerical behavior of $dS/d\tau$, given by Equation (43), for $-0.4 \leq \tau \leq 0.4$. We also consider the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The red dashed line represents the time τ_s , given by Equation (44), in which the Big-Rip singularity occurs.

In summary, the condition given by Equation (29) together with the condition $\Omega_{\xi_0} < 1$, for the present time, describe a viscous WDM model which is compatible with the near-equilibrium condition and presents a proper physical behavior of the temperature (a decreasing function with the scale factor), and entropy production (without future Big-Rip singularity). Therefore, all of the previous thermodynamics analysis will help us to define the best prior definition for our cosmological parameters γ and Ω_{ξ_0} , in order to constraint with the cosmological data. Accordingly, our model has two more free parameters than the standard Λ CDM model, which appears from the possibility of a more general model of the DM component, that takes into account a warm nature with a non-perfect fluid description.

5. Cosmological Constraints

In this section, we shall constraint the free parameters of the viscous Λ WDM model with the SNe Ia data coming from the Pantheon sample [96], which consists of 1048 data points in the redshift range $0.01 \leq z \leq 2.3$; and the OHD compiled by Magaña et al. [97], which consists of 51 data points in the redshift range $0.07 \leq z \leq 2.36$. To do so, we compute the best-fit parameters and their respective confidence regions with the affine-invariant Markov Chain Monte Carlo (MCMC) method [98], implemented in the pure-Python code *emcee* [99], by setting 30 chains or “walkers”. As a convergence test, we compute the autocorrelation time of the chains τ_{corr} every 50 steps, provided by the *emcee* module. If the current step is larger than $50\tau_{corr}$, and if the values of τ_{corr} changed by less than 1%, then we will consider that the chains are converged and the code is stopped. The first $5\tau_{corr}$ steps are discarded as “burn-in” steps. This convergence test is complemented with the mean acceptance fraction, which should be between 0.2 and 0.5 [99], and can be modified by the stretch move provided by the *emcee* module.

Since we are implementing a Bayesian statistical analysis, we need to construct the Gaussian likelihood

$$\mathcal{L} = \mathcal{N} \exp \left(-\frac{\chi_I^2}{2} \right). \quad (45)$$

Here, $\mathcal{N}$ is a normalization constant, which does not influence in the MCMC analysis, and χ_I^2 is the merit function, where I stands for each dataset considered, namely, SNe Ia, OHD and their joint analysis in which $\chi_{joint}^2 = \chi_{SNe}^2 + \chi_{OHD}^2$.

The merit function for the OHD data is constructed as

$$\chi_{OHD}^2 = \sum_{i=1}^{51} \left[\frac{H_i - H_{th}(z_i, \theta)}{\sigma_{H,i}} \right]^2, \quad (46)$$

where H_i is the observational Hubble parameter at redshift z_i with an associated error $\sigma_{H,i}$, all of them were provided by the OHD sample, H_{th} is the theoretical Hubble parameter at the same redshift, and θ encompasses the free parameters of the model under study. It is important to mention that in our MCMC analysis, we consider the value of the Hubble parameter at the current time, H_0 , as a free parameter, which is written as $H_0 = 100 \frac{\text{km/s}}{\text{Mpc}} h$, with h dimensionless and for which we consider the Gaussian prior $G(0.7403, 0.0142)$, according to the value of H_0 obtained by A. G. Riess et al. [9].

On the other hand, the merit function for the SNe Ia data is constructed as

$$\chi_{\text{SNe}}^2 = \sum_{i=1}^{1048} \left[\frac{\mu_i - \mu_{th}(z_i, \theta)}{\sigma_{\mu,i}} \right]^2, \quad (47)$$

where μ_i is the observational distance modulus of each SNe Ia at redshift z_i with an associated error $\sigma_{\mu,i}$, μ_{th} is the theoretical distance modulus for each SNe Ia at the same redshift, and θ encompasses the free parameters of the model under study. Following this line, the theoretical distance modulus can be obtained, for a flat FLRW space-time, from the expression

$$\mu_{th}(z_i, \theta) = 5 \log_{10} \left[\frac{d_L(z_i, \theta)}{\text{Mpc}} \right] + \bar{\mu}, \quad (48)$$

where $\bar{\mu} = 5[\log_{10}(c) + 5]$, c is the speed of light given in units of km/s and d_L is the luminosity distance given by

$$d_L(z_i, \theta) = (1 + z_i) \int_0^{z_i} \frac{dz'}{H(z', \theta)}. \quad (49)$$

In the Pantheon sample, the distance estimator is obtained using a modified version of Tripp's formula [100], with two nuisance parameters calibrated to zero with the BEAMS whit Bias Correction (BBC) method [101]. Hence, the observational distance modulus for each SNe Ia is given by

$$\mu_i = m_{B,i} - \mathcal{M}, \quad (50)$$

where $m_{B,i}$ is the corrected apparent B-band magnitude of a fiducial SNe Ia at redshift z_i , all of them provided by the pantheon sample (Currently available online in GitHub repository <https://github.com/dscolnic/Pantheon> (accessed on 30 August 2022)). The corrected apparent B-band magnitude $m_{B,i}$ for each SNe Ia with their respective redshifts z_i and errors $\sigma_{m_{B,i}}$ are available in the document `lcpam_full_long.txt`, and $\mathcal{M}$ is a nuisance parameter which must be jointly estimated with the free parameters θ of the model under study. Therefore, we can rewrite the merit function (47) in matrix notation (denoted by bold symbols) as

$$\chi_{\text{SNe}}^2 = \mathbf{M}(z, \theta, \mathcal{M})^\dagger \mathbf{C}^{-1} \mathbf{M}(z, \theta, \mathcal{M}), \quad (51)$$

where $[\mathbf{M}(z, \theta, \mathcal{M})]_i = m_{B,i} - \mu_{th}(z_i, \theta) - \mathcal{M}$, and $\mathbf{C} = \mathbf{D}_{stat} + \mathbf{C}_{sys}$ are the total uncertainties covariance matrix, with $\mathbf{D}_{stat} = \text{diag}(\sigma_{m_{B,i}}^2)$ being the statistical uncertainties of m_B and $\mathbf{C}_{sys}$ being the systematic uncertainties in the BBC approach (Currently available online in GitHub repository <https://github.com/dscolnic/Pantheon> in the document `sys_full_long.txt` (accessed on 30 August 2022)).

Finally, to marginalize over the nuisance parameters $\bar{\mu}$ and $\mathcal{M}$, we define $\bar{\mathcal{M}} = \bar{\mu} + \mathcal{M}$, and the merit function (51) is expanded as [102]

$$\chi_{\text{SNe}}^2 = A(z, \theta) - 2B(z, \theta)\bar{\mathcal{M}} + C\bar{\mathcal{M}}^2, \quad (52)$$

where

$$A(z, \theta) = \mathbf{M}(z, \theta, \bar{\mathcal{M}} = 0)^\dagger \mathbf{C}^{-1} \mathbf{M}(z, \theta, \bar{\mathcal{M}} = 0), \quad (53)$$

$$B(z, \theta) = \mathbf{M}(z, \theta, \bar{\mathcal{M}} = 0)^\dagger \mathbf{C}^{-1} \mathbf{1}, \quad (54)$$

$$C = \mathbf{1}C^{-1}\mathbf{1}. \quad (55)$$

Therefore, by minimizing the expanded merit function (52) with respect to $\tilde{\mathcal{M}}$, it is obtained $\tilde{\mathcal{M}} = B(z, \theta)/C$, and the expanded merit function is reduced to

$$\chi_{\text{SNe}}^2 = A(z, \theta) - \frac{B(z, \theta)^2}{C}, \quad (56)$$

which depends only on the free parameters of the model under study.

It is important to mention that the expanded and minimized merit function (56) provides the same information as the merit function (51) since the best-fit parameters minimize the merit function, and their corresponding value can be used as an indicator of the goodness-of-fit independently of the dataset used: the smaller the value of $\chi_{\min}^2$ is, the better is the fit. In this sense, in principle the value of $\chi_{\min}^2$ obtained for the best-fit parameters can be reduced by adding free parameters to the model under study, resulting in overfitting. Hence, we compute the Bayesian criterion information (BIC) [103] to compare the goodness-of-fit statistically. This criterion adds a penalization in the value of $\chi_{\min}^2$ that depends on the total number of free parameters of the model, θ_N , according to the expression

$$\text{BIC} = \theta_N \ln(n) + \chi_{\min}^2, \quad (57)$$

where n is the total number of data points in the corresponding data sample. Thus, when two different models are compared, the one most favored by the observations statistically corresponds to the one with the smallest value of BIC. In general, a difference of 2–6 in BIC is evidence against the model with higher BIC, a difference of 6–10 is strong evidence, and a difference of >10 is very strong evidence.

Since in the merit function of the two datasets, the respective model is considered as the Hubble parameter as a function of the redshift (see Equations (46) and (49)), then we numerically integrate Equation (7) with $m = 1$, which can be rewritten, considering that $\dot{z} = -(1+z)H$, as

$$\frac{dH}{dz} = \frac{1}{2(1+z)} \left[3\gamma H - 3\tilde{\xi}_0 (3H^2 - \Lambda) - \frac{\Lambda\gamma}{H} \right], \quad (58)$$

using as the initial condition $H(z=0) = H_0 = 100 \frac{\text{km/s}}{\text{Mpc}} h$, and taking into account that $\tilde{\xi}_0 = \Omega_{\tilde{\xi}_0}/(3H_0)$ and $\Lambda = 3H_0^2(1 - \Omega_{m0})$; this last one derived from Equation (2), which leads to $\Omega_{m0} + \Omega_{\Lambda0} = 1$. Furthermore, for a further comparison, we also constraint the free parameters of the Λ CDM model, whose respective Hubble parameter as a function of the redshift is given by

$$H(z) = 100 \frac{\text{km/s}}{\text{Mpc}} h \sqrt{\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}}. \quad (59)$$

Therefore, the free parameters of the viscous Λ WDM model are $\theta = \{h, \Omega_{m0}, \Omega_{\tilde{\xi}_0}, \gamma\}$, and the free parameters of the Λ CDM model are $\theta = \{h, \Omega_{m0}\}$. For the free parameters Ω_{m0} , $\Omega_{\tilde{\xi}_0}$ and γ , we consider the following priors: $\Omega_{m0} \in F(0, 1)$, $\gamma \in G(1.00, 0.02)$ and $0 < \Omega_{\tilde{\xi}_0} < \gamma - 1$; where F stands for flat prior, and the prior for $\Omega_{\tilde{\xi}_0}$ is derived from the constraint given by Equation (29).

In Table 1, we present the total steps, the mean acceptance fraction (MAF) and the autocorrelation time τ_{corr} of each free parameter, obtained when the convergence test is fulfilled during our MCMC analysis for both the viscous Λ WDM and Λ CDM models. The values of the MAF are obtained for a value of the stretch move of $a = 7$ for the Λ CDM model, and a value of $a = 3$ for the viscous Λ WDM model.

Table 1. Final values of the total number of steps, mean acceptance fraction (MAF) and autocorrelation time τ_{corr} for each model-free parameters, obtained when the convergence test described in Section 5 is fulfilled for a MCMC analysis with 30 chains or “walkers”. The values of the MAF are obtained for a value of the stretch move of $a = 7$ for the Λ CDM model, and a value of $a = 3$ for the viscous Λ WDM model.

Data	Total Steps	MAF	τ_{corr}			
			h	Ω_{m0}	Ω_{ξ_0}	γ
Λ CDM Model						
SNe Ia	1050	0.370	16.5	17.5	...	...
OHD	1000	0.367	14.9	17.1	...	...
SNe Ia+OHD	800	0.364	15.8	15.4	...	...
Viscous Λ WDM Model						
SNe Ia	2700	0.385	45.8	44.3	49.6	51.9
OHD	2450	0.377	39.6	45.5	48.2	48.9
SNe Ia+OHD	2700	0.379	43.0	45.9	50.5	53.3

6. Results and Discussion

The best-fit values for the Λ CDM and viscous Λ WDM models, obtained for the SNe Ia data, OHD, and in their joint analysis, as well as their corresponding goodness-of-fit criteria, are presented in Table 2. The uncertainties presented correspond to 1σ (68.3%), 2σ (95.5%) and 3σ (99.7%) of confidence level (CL). In Figures 5 and 6, we depict the joint and marginalized credible regions of the free parameter space of the Λ CDM and viscous Λ WDM model, respectively. The admissible regions presented in the joint regions correspond to 1σ , 2σ and 3σ CL.

Table 2. Best-fit values and goodness-of-fit criteria for the viscous Λ WDM model with free parameters h , Ω_{m0} , Ω_{ξ_0} and γ ; in addition, for the Λ CDM model with free parameters h and Ω_{m0} , obtained in the MCMC analysis described in Section 5 for the SNe Ia data, OHD, and in their joint analysis. The uncertainties correspond to 1σ (68.3%), 2σ (95.5%) and 3σ (99.7%) of confidence level (CL), respectively. The best-fit values for the Λ CDM model are used for the sake of comparison with the viscous Λ WDM model.

Data	Best-Fit Values				Goodness of Fit	
	h	Ω_{m0}	$\Omega_{\xi_0} (\times 10^{-2})$	γ	χ^2_{min}	BIC
ΛCDM Model						
SNe Ia	0.740 ^{+0.014 +0.028 +0.043} −0.013 −0.028 −0.040	0.299 ^{+0.022 +0.045 +0.064} −0.022 −0.042 −0.058	...	...	1026.9	1040.8
OHD	0.720 ^{+0.009 +0.018 +0.026} −0.009 −0.017 −0.025	0.241 ^{+0.014 +0.027 +0.038} −0.014 −0.027 −0.037	...	...	28.6	36.5
SNe Ia+OHD	0.710 ^{+0.008 +0.016 +0.023} −0.008 −0.016 −0.022	0.259 ^{+0.012 +0.024 +0.034} −0.012 −0.022 −0.036	...	...	1058.3	1072.3
Viscous ΛWDM Model						
SNe Ia	0.741 ^{+0.014 +0.029 +0.044} −0.014 −0.028 −0.044	0.293 ^{+0.023 +0.048 +0.066} −0.022 −0.043 −0.064	0.980 ^{+1.232 +2.954 +4.318} −0.716 −0.943 −0.979	1.023 ^{+0.015 +0.031 +0.045} −0.011 −0.019 −0.021	1026.9	1054.7
OHD	0.721 ^{+0.009 +0.018 +0.026} −0.010 −0.020 −0.029	0.237 ^{+0.017 +0.037 +0.053} −0.017 −0.033 −0.045	1.026 ^{+1.250 +2.641 +3.763} −0.738 −0.988 −1.019	1.023 ^{+0.014 +0.030 +0.048} −0.012 −0.019 −0.022	28.5	44.2
SNe Ia+OHD	0.709 ^{+0.009 +0.017 +0.026} −0.009 −0.017 −0.027	0.261 ^{+0.015 +0.030 +0.047} −0.015 −0.030 −0.041	1.633 ^{+1.381 +2.871 +4.186} −1.059 −1.544 −1.627	1.026 ^{+0.015 +0.031 +0.045} −0.013 −0.021 −0.024	1056.9	1084.9

From the best-fit parameters presented in Table 2, it is possible to see that there are no remarkable differences between the best-fit values for h and Ω_{m0} obtained for the Λ CDM model and the viscous Λ WDM model. This is an expected behavior due to the similarity between the two models at late times, as well as in the past, considering the best-fit values obtained for Ω_{ξ_0} . From the point of view of the goodness-of-fit criteria, we can conclude that the two models are able to describe the SNe Ia, OHD and SNe Ia+OHD data, with very similar values of χ^2_{min} , again due to the similarity of the behavior of the two models, especially at late times.

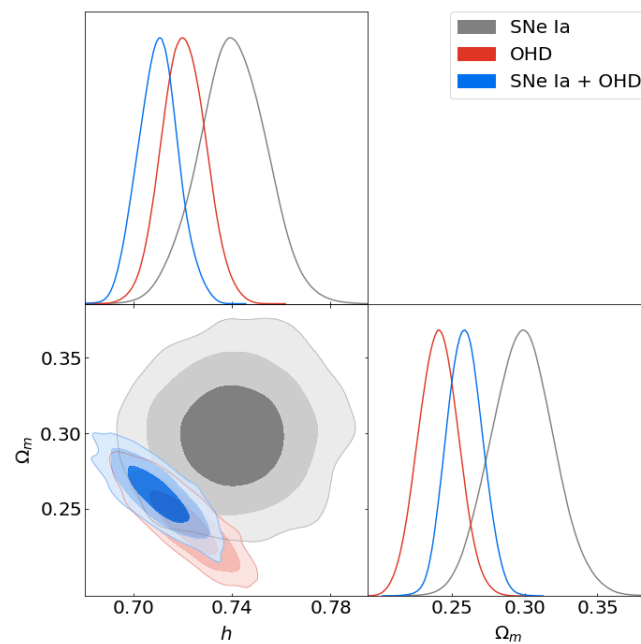


Figure 5. Joint and marginalized regions of the free parameters h and Ω_{m0} for the Λ CDM model, obtained in the MCMC analysis described in Section 5. The admissible regions presented in the joint regions correspond to 1σ (68.3%), 2σ (95.5%) and 3σ (99.7%) of confidence level (CL), respectively. The best-fit values for each model-free parameter are shown in Table 2.

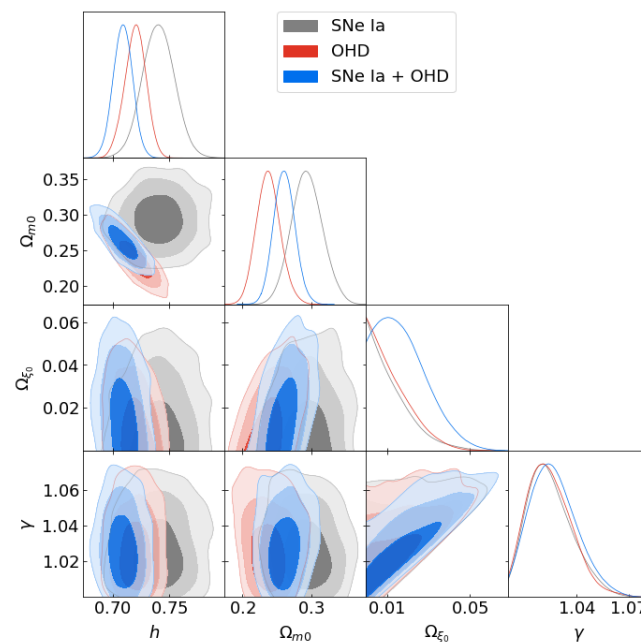


Figure 6. Joint and marginalized regions of the free parameters h , Ω_{m0} , Ω_{x0} and γ for the viscous Λ WDM model, obtained in the MCMC analysis described in Section 5. The admissible regions presented in the joint regions correspond to 1σ (68.3%), 2σ (95.5%) and 3σ (99.7%) of confidence level (CL), respectively. The best-fit values for each model-free parameter are shown in Table 2.

A remarkable result is that the viscous Λ WDM model exhibits a slightly lower value of χ^2_{min} for the SNe Ia+OHD data than the Λ CDM model, despite the fact that the Λ WDM model has a greater value of BIC than the Λ CDM model. This translates into a better fit for the Λ WDM model because the two extra free parameters of the viscous Λ WDM model are a consequence of a more general description of the DM component, which takes into account a warm nature and a non-perfect fluid description, as suggested by previous investigations

that use these alternatives to face tensions of the standard model, and they are not added by hand in order to force a better fit. Furthermore, the Λ CDM model assumes beforehand a CDM (one less free parameter) and all their matter components are described as perfect fluids (another less free parameter) which leads to a good fit of the combined SNe Ia+OHD data with less free parameters, but the price to pay are the problems mentioned above that the Λ CDM model experiences today. Therefore, we have an alternative model which goes beyond the standard Λ CDM model by considering a dissipative WDM, which has the capability to describe the SNe Ia and OHD data in the same way as the Λ CDM model, together with a more general description of DM nature.

On the other hand, the best fit values contrasted with the combined SNe Ia+OHD data, for γ and Ω_{ξ_0} are given by $1.02555^{+0.04453}_{-0.02424}$ and $0.01633^{+0.04185}_{-0.01627}$, respectively, at 3σ CL. Note that both deviations around the mean value satisfy the near-equilibrium condition Equation (29), which means that these values are compatible with our previous theoretical thermodynamics conclusion ($\Omega_{\xi_0} < 1$ and $\gamma \neq 1$). For the small values of Ω_{ξ_0} and γ given by the data, we are far from the near-equilibrium condition at $H = 21.61855H_0$, according to Equation (16), which means that we are far from the near equilibrium at a redshift of $z = 11.63228$, according to Equation (58); by then, we can ensure the near-equilibrium condition for the actual data measurement at $z \sim 2.3$.

Furthermore, it is important to mention that with the data, we can obtain the actual size of dissipation for our Λ WDM model. According to our dimensionless expression $\Omega_{\xi_0} = 3\xi_0 H_0$, ξ_0 has dimensions of time, and the bulk viscosity is given by $\xi = \xi_0 \rho$, which writing in full dimensions ($c \neq 1$) is given by $\xi = \xi_0 \varepsilon$, being ε the energy density of matter (that has the same dimension of pressure according to the EoS). Then, ξ has the dimensions $\text{Pa} \times \text{s}$ which correspond to the dimensions of viscosity. Following the cosmological constraint, we obtain $\Omega_{\xi_0} < 0.05818$ (a generalization of our theoretical constraint $\Omega_{\xi_0} < 1$). By then, using the definition of Ω_{ξ_0} , we have the following constraint on bulk viscosity

$$\xi < \varepsilon \times \frac{0.05818}{3H_0}. \quad (60)$$

If we introduce the critical density, given by $\varepsilon_0 = \frac{3c^2 H_0^2}{8\pi G}$, and since our best fit values for the Hubble parameter and matter density are $H_0 = 70.9238 \frac{\text{km/s}}{\text{Mpc}} = (4.3507 \times 10^{17} \text{s})^{-1}$ and $\Omega_m = 0.26073$, respectively, then the restriction over the upper limit of bulk viscosity is given by

$$\xi < 1.87082 \times 10^6 \text{ Pa} \cdot \text{s}. \quad (61)$$

For this particular case of values, if we consider the upper error of the best fit values at 3σ CL, the value of ξ remains of the order of $\xi \lesssim 10^6 \text{ Pa} \cdot \text{s}$. Note that from the previous thermodynamic analysis ($\Omega_{\xi_0} < 1$), and using the value of H_0 and Ω_m given by [1,9], we will find that

$$\xi < 3.628 \times 10^7 \text{ Pa} \cdot \text{s}, \quad (62)$$

which is a similar value with respect to the previous value found by B. D. Normann and I. Brevik in [48], in the context of a model with a viscous DM component with bulk viscosity of the form $\xi = \xi_0 \rho^{\frac{1}{2}}$, and DE component given by the CC. They also suggest (see also [49]) that the value $\xi \sim 10^6 \text{ Pa} \cdot \text{s}$ for the present viscosity is reasonable. Furthermore, other investigations suggest the same orders of magnitude [48,49,89,104], but since these models are not identical, some discrepancies are expected with our results.

7. Conclusions

We have discussed throughout this work the near-equilibrium condition, entropy production and cosmological constraint of a cosmological model filled with a dissipative WDM, where the bulk viscosity is proportional to the energy density, and a positive CC, which is described by an exact solution previously found in [90]. Assuming the condition given by Equation (12), this solution behaves very similar to the standard cosmological

model for small values of Ω_{ξ_0} , as we can see in Figure 1, avoiding the appearance of a future singularity in a finite time (Big-Rip).

We have shown that the presence of the CC—together with a small viscosity from the expression (16), and considering the condition given by Equation (12)—leads to a near equilibrium regime for the WDM component.

The WDM component has a temperature that decreases very slowly with the cosmic expansion as a result of being close to the near-equilibrium condition, contrary to the non-physical behavior found for the dust case ($\gamma = 1$), where the temperature increased with the scale factor. Furthermore, we have shown that the second law of thermodynamics is fulfilled as long as the conditions (12) and (28) are satisfied. On the contrary, the entropy production would diverge in a finite time if a Big-Rip singularity occurs.

To fulfill the two criteria discussed in this work, we need to have $\Omega_{\xi_0} < 1$ and a WDM component with an EoS satisfying the constraint given in Equation (29).

It is important to mention that, in our WDM model, we need to have $\Omega_{\xi_0} \ll \gamma - 1$. For small values of Ω_{ξ_0} (in particularly $\Omega_{\xi_0} = 0.001$), the model enters into an agreement with some previous results found, for example in [105], where cosmological bounds on the EoS for the DM were found, and the inclusion of the CC is considered. The bounds for a constant barotropic index are $-1.50 \times 10^{-6} < \omega < 1.13 \times 10^{-6}$ ($\omega = \gamma - 1$), if there is no entropy production, and $-8.78 \times 10^{-3} < \omega < 1.86 \times 10^{-3}$, if the adiabatic speed of sound vanishes, both at 3σ of confidence level. Another example can be found in [106] where, using WMAP+BAO+HO observations, the EoS at the present time is given by $\omega = 0.00067^{+0.00011}_{-0.00067}$.

Highlighting again, we have to mention that the asymptotic behavior of the exact solution at the infinite future, given by (10), corresponds to the usual de Sitter solution, which indicates that this solution describes a dissipative WDM that could reproduce the same asymptotic behavior of the standard model. As long as the exact solution tends to this value, the near-equilibrium condition (28) can be satisfied. Of course, the de Sitter solution has a constant temperature, according to Equation (20); since in this case we have a null density and null pressure of the fluid, the entropy production is zero, according to Equation (26).

We have shown in this work that, previously to any constraining from the cosmological data, the study of thermodynamics consistencies required by the Eckart's approach, such as the near-equilibrium condition and entropy production, leads to important constraints on the cosmological parameters, such as the given one by Equation (1), which implies the necessity of WDM, in agreement with some previous results found in [105–107]. On the other hand, the constraint (12) tells us that Big-Rip singularities are avoided at late times if the near-equilibrium condition is preserved, even though the exact solution explored behaves very similarly to the standard model, and opens the possibility of a more realistic fluid description of the DM containing dissipation processes within the Eckart's framework, giving us physically important clues about the EoS of this component (γ) and the size of dissipation involved (Ω_{ξ_0}). In this sense, this thermodynamic analysis had the aim of finding a model that satisfies the near-equilibrium condition, together with a proper behavior of the temperature and entropy production, helping us to find the best prior to the cosmological constraints discussed in Section 5.

The results of the cosmological constraint are shown in Table 2, as well as their corresponding goodness-of-fit criteria. A greater value of BIC for the Λ WDM model, in comparison to the value obtained for the Λ CDM model, is a reflection of considering a more general theory, with two more free parameters (γ and Ω_{ξ_0}) than the standard Λ CDM model (γ fixed to 1 and a perfect DM component), obtaining also a slightly better fit for the combined SNe Ia+OHD data for the Λ WDM model, being in this sense a more realistic model that describes in the same way (slightly better) the SNe Ia and OHD data similar to the standard cosmological model does.

With the best fit values for Ω_{ξ_0} and γ at 3σ CL, obtained for the combined SNe Ia+OHD data, we can conclude that the near-equilibrium condition is fully satisfied, even for a redshift

of $z = 11$, according to Equation (58); then, a proper physical behavior of the temperature (decreasing function with the scale factor), and entropy production (without future Big-Rip singularity), is obtained. Furthermore, our data analysis suggests that the actual value of bulk viscosity has an upper value of the order of 10^6 Pa·s, in agreement with some previous investigations. It is also important to mention that the model, from the point of view of the thermodynamic analysis (and even from the cosmological data), does not rule out values smaller than this, since the space of values obtained from the theoretical ground at the present time is given by $0 < \Omega_{\xi_0} < 1$, and allows, in principle, to get smaller values that can be more acceptable from the viscous hydrodynamic point of view.

Finally, the main contribution of the present study is to show that the standard model with the two extensions, i.e., a dissipative WDM component—in order to face the recent cosmological tension found—fulfills the criteria of a consistent relativistic fluid description under some constraints for the free parameters of the model, and also can describe the combined SNe Ia+OHD data in the same way as the Λ CDM. Therefore, our exact solution describes a physically viable model for a dissipative WDM component, which is supported by many investigations that have extended the possibilities for the DM nature to face the tensions of the standard model.

Author Contributions: Investigation, N.C., E.G. and J.J.; Methodology, N.C., E.G. and J.J.; Software, E.G.; Supervision, N.C.; Validation, N.C., E.G. and J.J.; Visualization, E.G.; Writing original draft preparation, J.J. All authors have read and agreed to the published version of the manuscript.

Funding: Norman Cruz is supported by Universidad de Santiago de Chile (USACH), through Proyecto DICYT N° 042131CM, Vicerrectoría de Investigación, Desarrollo e Innovación. Jose Jovel is supported by ANID-PFCHA/Doctorado Nacional/2018-21181327.

Acknowledgments: Esteban González acknowledges the support of Dirección de Investigación y Postgrado at Universidad de Aconcagua.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

DM	Dark matter
DE	Dark energy
CC	Cosmological constant
EoS	Equation of state

References

1. Planck Collaboration. Planck 2018 results. VI. Cosmological parameters. *arXiv* **2020**, arXiv:1807.06209.
2. Hinshaw, G.; Larson, D.; Komatsu, E.; Spergel, D.N.; Bennett, C.L.; Dunkley, J.; Nolte, M.R.; Halpern, M.; Hill, R.S.; Odegard, N.; et al. Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. *Astrophys. J. Suppl.* **2013**, *208*, 19. [[CrossRef](#)]
3. Alam, S.; Ata, M.; Bailey, S.; Beutler, F.; Bizyaev, D.; Blazek, J.A.; Bolton, A.S.; Brownstein, J.R.; Burden, A.; Chuang, C.-H.; et al. The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: Cosmological analysis of the DR12 galaxy sample. *Mon. Not. R. Astron. Soc.* **2017**, *470*, 2617–2652. [[CrossRef](#)]
4. Cao, S.; Ryan, J.; Ratra, B. Using Pantheon and DES supernova, baryon acoustic oscillation, and Hubble parameter data to constrain the Hubble constant, dark energy dynamics, and spatial curvature. *Mon. Not. R. Astron. Soc.* **2021**, *504*, 300–310. [[CrossRef](#)]
5. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; et al. Measurements of Ω and Λ from 42 High-Redshift Supernovae. *Astrophys. J.* **1999**, *517*, 565–586. [[CrossRef](#)]
6. Weinberg, S. The Cosmological Constant Problem. *Rev. Mod. Phys.* **1989**, *61*, 1–23. [[CrossRef](#)]
7. Carroll, S.M.; Press, W.H.; Turner, E.L. The cosmological constant. *Annu. Rev. Astron. Astrophys.* **1992**, *30*, 499–542. [[CrossRef](#)]
8. Zlatev, I.; Wang, L.; Steinhardt, P.J. Quintessence, cosmic coincidence, and the cosmological constant. *Phys. Rev. Lett.* **1999**, *82*, 896–899. [[CrossRef](#)]

9. Riess, A.G.; Casertano, S.; Yuan, W.; Macri, L.M.; Scolnic, D. Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond Λ CDM. *Astrophys. J.* **2019**, *876*, 85. [\[CrossRef\]](#)
10. Anand, S.; Chaubal, P.; Mazumdar, A.; Mohanty, S. Cosmic viscosity as a remedy for tension between PLANCK and LSS data. *J. Cosmol. Astropart. Phys.* **2017**, *11*, 5. [\[CrossRef\]](#)
11. Raveri, M. Are cosmological data sets consistent with each other within the Λ cold dark matter model? *Phys. Rev. D* **2016**, *93*, 043522. [\[CrossRef\]](#)
12. Bowman, J.D.; Rogers, A.E.E.; Monsalve, R.A.; Mozdzen, T.J.; Mahesh, N. An absorption profile centred at 78 megahertz in the sky-averaged spectrum. *Nature* **2018**, *555*, 67–70. [\[CrossRef\]](#) [\[PubMed\]](#)
13. Cruz, M.; Cruz, N.; Lepe, S. Accelerated and decelerated expansion in a causal dissipative cosmology. *Phys. Rev. D* **2017**, *96*, 124020. [\[CrossRef\]](#)
14. Fabris, J.C.; Goncalves, S.V.B.; de Sa Ribeiro, R. Bulk viscosity driving the acceleration of the Universe. *Gen. Relativ. Gravit.* **2006**, *38*, 495–506. [\[CrossRef\]](#)
15. Li, B.; Barrow, J.D. Does Bulk Viscosity Create a Viable Unified Dark Matter Model? *Phys. Rev. D* **2009**, *79*, 103521. [\[CrossRef\]](#)
16. Hipólito-Ricaldi, W.S.; Velten, H.E.S.; Zimdahl, W. Viscous dark fluid universe. *Phys. Rev. D* **2010**, *82*, 063507. [\[CrossRef\]](#)
17. Avelino, A.; Nucamendi, U. Can a matter-dominated model with constant bulk viscosity drive the accelerated expansion of the universe? *J. Cosmol. Astropart. Phys.* **2009**, *904*, 6. [\[CrossRef\]](#)
18. Avelino, A.; Nucamendi, U. Exploring a matter-dominated model with bulk viscosity to drive the accelerated expansion of the Universe. *J. Cosmol. Astropart. Phys.* **2010**, *1008*, 9. [\[CrossRef\]](#)
19. Sasidharan, A.; Mathew, T.K. Bulk viscous matter and recent acceleration of the Universe. *Eur. Phys. J. C* **2015**, *75*, 348. [\[CrossRef\]](#)
20. Sasidharan, A.; Mathew, T.K. Phase space analysis of bulk viscous matter dominated universe. *J. High Energy Phys.* **2016**, *1606*, 138. [\[CrossRef\]](#)
21. Mohan, N.D.J.; Sasidharan, A.; Mathew, T.K. Bulk viscous matter and recent acceleration of the universe based on causal viscous theory. *Eur. Phys. J. C* **2017**, *77*, 849. [\[CrossRef\]](#)
22. Cruz, N.; González, E.; Palma, G. Exact analytical solution for an Israel–Stewart cosmology. *Gen. Relativ. Gravit.* **2020**, *52*, 62. [\[CrossRef\]](#)
23. Cruz, N.; González, E.; Palma, G. Testing dissipative dark matter in causal thermodynamics. *Mod. Phys. Lett. A* **2021**, *36*, 2150032. [\[CrossRef\]](#)
24. Almada, A.H.; Aspeitia, M.A.G.; Rodríguez-Meza, M.A.; Motta, V. A hybrid model of viscous and Chaplygin gas to tackle the Universe acceleration. *arXiv* **2021**, arXiv:2103.16733.
25. Nojiri, S.; Odintsov, S.D. Inhomogeneous Equation of State of the Universe: Phantom Era, Future Singularity and Crossing the Phantom Barrier. *Phys. Rev. D* **2005**, *72*, 023003. [\[CrossRef\]](#)
26. Capozziello, S.; Cardone, V.F.; Elizalde, E.; Nojiri, S.; Odintsov, S.D. Observational constraints on dark energy with generalized equations of state. *Phys. Rev. D* **2006**, *73*, 043512. [\[CrossRef\]](#)
27. Herrera-Zamorano, L.; Hernández-Almada, A.; García-Aspeitia, M.A. Constraints and cosmography of CDM in presence of viscosity. *Eur. Phys. J. C* **2020**, *80*, 637. [\[CrossRef\]](#)
28. Elizalde, E.; Khurshudyan, M.; Odintsov, S.D.; Myrzakulov, R. An analysis of the H_0 tension problem in a universe with a viscous dark fluid. *Phys. Rev. D* **2020**, *102*, 123501. [\[CrossRef\]](#)
29. Normann, B.D.; Brevik, I.H. Could the Hubble tension be resolved by bulk viscosity? *arXiv* **2021**, arXiv:2107.13533.
30. Bhatt, J.R.; Mishra, A.K.; Nayak, A.C. Viscous dark matter and 21 cm cosmology. *Phys. Rev. D* **2019**, *100*, 063539. [\[CrossRef\]](#)
31. Foot, R.; Vagnozzi, S. Dissipative hidden sector dark matter. *Phys. Rev. D* **2015**, *91*, 023512. [\[CrossRef\]](#)
32. Foot, R.; Vagnozzi, S. Solving the small-scale structure puzzles with dissipative dark matter. *J. Cosmol. Astropart. Phys.* **2016**, *2016*, 13. [\[CrossRef\]](#)
33. Hofmann, S.; Schwarz, D.J.; Stoecker, H. Damping scales of neutralino cold dark matter. *Phys. Rev. D* **2001**, *64*, 083507. [\[CrossRef\]](#)
34. Schweizer, M.A. Transient and transport coefficients for radiative fluids. *Astrophys. J. Part 1* **1982**, *258*, 798–811. [\[CrossRef\]](#)
35. Udey, N.; Israel, W. General relativistic radiative transfer: The 14-moment approximation. *Mon. Not. R. Astron. Soc.* **1982**, *199*, 1137–1147. [\[CrossRef\]](#)
36. Zimdahl, W. ‘Understanding’ cosmological bulk viscosity. *Mon. Not. R. Astron. Soc.* **1996**, *280*, 1239. [\[CrossRef\]](#)
37. Zimdahl, W.; Pavón, D. Cosmology with adiabatic matter creation. *Int. J. Mod. Phys. D* **1994**, *3*, 327–330. [\[CrossRef\]](#)
38. Landau, L.D.; Lifshitz, E.M. *Fluid Mechanics*; Butterworth-Heinemann: Oxford, UK, 1959; Volume 6.
39. Blas, D.; Floerchinger, S.; Garny, M.; Tetradis, N.; Wiedemann, U.A. Large scale structure from viscous dark matter. *J. Cosmol. Astropart. Phys.* **2015**, *2015*, 49. [\[CrossRef\]](#)
40. Calogero, S.; Velten, H. Cosmology with matter diffusion. *J. Cosmol. Astropart. Phys.* **2013**, *11*, 25. [\[CrossRef\]](#)
41. Benisty, D.; Guendelman, E.I. Interacting Diffusive Unified Dark Energy and Dark Matter from Scalar Fields. *Eur. Phys. J. C* **2017**, *77*, 396. [\[CrossRef\]](#)
42. Benisty, D.; Guendelman, E.; Haba, Z. Unification of dark energy and dark matter from diffusive cosmology. *Phys. Rev. D* **2019**, *99*, 123521; Erratum in: *Phys. Rev. D* **2020**, *101*, 049901. doi: 10.1103/PhysRevD.99.123521. [\[CrossRef\]](#)
43. Weinberg, S.; Dicke, R.H. *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*; John Wiley and Sons, Inc.: Hoboken, NJ, USA, 1972.

44. Murphy, G.L. Big-Bang Model Without Singularities. *Phys. Rev. D* **1973**, *8*, 4231–4233. [\[CrossRef\]](#)
45. Padmanabhan, T.; Chitre, S. Viscous universes. *Phys. Lett. A* **1987**, *120*, 433. [\[CrossRef\]](#)
46. Brevik, I.; Gorbunova, O. Dark energy and viscous cosmology. *Gen. Relativ. Gravit.* **2005**, *37*, 2039–2045. [\[CrossRef\]](#)
47. Cruz, N.; González, E.; Lepe, S.; Gómez, D.S.C. Analysing dissipative effects in the Λ CDM model. *J. Cosmol. Astropart. Phys.* **2018**, *12*, 17. [\[CrossRef\]](#)
48. Normann, B.D.; Brevik, I. Characteristic Properties of Two Different Viscous Cosmology Models for the Future Universe. *Mod. Phys. Lett. A* **2017**, *32*, 1750026. [\[CrossRef\]](#)
49. Normann, B.D.; Brevik, I. General Bulk-Viscous Solutions and Estimates of Bulk Viscosity in the Cosmic Fluid. *Entropy* **2016**, *18*, 215. [\[CrossRef\]](#)
50. Brevik, I.; Grøn, Ø.; de Haro, J.; Odintsov, S.D.; Saridakis, E.N. Viscous Cosmology for Early- and Late-Time Universe. *Int. J. Mod. Phys. D* **2017**, *26*, 1730024. [\[CrossRef\]](#)
51. Bode, P.; Ostriker, J.P. Halo formation in warm dark matter models. *Astrophys. J.* **2001**, *556*, 93–107. [\[CrossRef\]](#)
52. de Vega, H.J.; Sanchez, N.G. Warm dark matter in the galaxies: theoretical and observational progresses. Highlights and conclusions of the chalonge meudon workshop 2011. *arXiv* **2011**, arXiv:1109.3187.
53. Klypin, A.; Kravtsov, A.V.; Valenzuela, O.; Prada, F. Where are the missing Galactic satellites? *Astrophys. J.* **1999**, *522*, 82–92. [\[CrossRef\]](#)
54. Moore, B.; Ghigna, S.; Governato, F.; Lake, G.; Quinn, T.R.; Stadel, J.; Tozzi, P. Dark Matter Substructure within Galactic Halos. *Astrophys. J. Lett.* **1999**, *524*, L19–L22. [\[CrossRef\]](#)
55. Newton, O.; Leo, M.; Cautun, M.; Jenkins, A.; Frenk, C.S. Constraints on the properties of warm dark matter using the satellite galaxies of the Milky Way. *J. Cosmol. Astropart. Phys.* **2021**, *8*, 62. [\[CrossRef\]](#)
56. Viel, M.; Lesgourgues, J.; Haehnelt, M.G.; Matarrese, S.; Riotto, A. Constraining warm dark matter candidates including sterile neutrinos and light gravitinos with WMAP and the Lyman-alpha forest. *Phys. Rev. D* **2005**, *71*, 063534. [\[CrossRef\]](#)
57. Boyarsky, A.; Lesgourgues, J.; Ruchayskiy, O.; Viel, M. Lyman-alpha constraints on warm and on warm-plus-cold dark matter models. *J. Cosmol. Astropart. Phys.* **2009**, *5*, 12. [\[CrossRef\]](#)
58. Asaka, T.; Blanchet, S.; Shaposhnikov, M. The ν msm, dark matter and neutrino masses. *Phys. Lett.* **2005**, *B631*, 151–156. [\[CrossRef\]](#)
59. Asaka, T.; Shaposhnikov, M. The ν MSM, dark matter and baryon asymmetry of the universe [rapid communication]. *Phys. Lett. B* **2005**, *620*, 17–26. [\[CrossRef\]](#)
60. Dodelson, S.; Widrow, L.M. Sterile-neutrinos as dark matter. *Phys. Rev. Lett.* **1994**, *72*, 17–20. [\[CrossRef\]](#) [\[PubMed\]](#)
61. Shi, X.; Fuller, G.M. A new dark matter candidate: Non-thermal sterile neutrinos. *Phys. Rev. Lett.* **1999**, *82*, 2832–2835. [\[CrossRef\]](#)
62. Abazajian, K.; Fuller, G.M.; Patel, M. Sterile neutrino hot, warm, and cold dark matter. *Phys. Rev. D* **2001**, *64*, 023501. [\[CrossRef\]](#)
63. Asaka, T.; Laine, M.; Shaposhnikov, M. Lightest sterile neutrino abundance within the numsm. *J. High Energy Phys.* **2007**, *1*, 91. [\[CrossRef\]](#)
64. Laine, M.; Shaposhnikov, M. Sterile neutrino dark matter as a consequence of ν MSM-induced lepton asymmetry. *J. Cosmol. Astropart. Phys.* **2008**, *0806*, 031. [\[CrossRef\]](#)
65. Colombi, S.; Dodelson, S.; Widrow, L.M. Large scale structure tests of warm dark matter. *Astrophys. J.* **1996**, *458*, 1. [\[CrossRef\]](#)
66. Eckart, C. The Thermodynamics of Irreversible Processes. III. Relativistic Theory of the Simple Fluid. *Phys. Rev.* **1940**, *58*, 919. [\[CrossRef\]](#)
67. Israel, W. Nonstationary irreversible thermodynamics: A Causal relativistic theory. *Ann. Phys.* **1976**, *100*, 310–331. [\[CrossRef\]](#)
68. Muller, I. Zum Paradoxon der Wärmeleitungstheorie. *Z. Phys.* **1967**, *198*, 329–344. [\[CrossRef\]](#)
69. Israel, W.; Stewart, J.M. Transient relativistic thermodynamics and kinetic theory. *Ann. Phys.* **1979**, *118*, 341–372. [\[CrossRef\]](#)
70. Israel, W.; Stewart, J.M. On Transient Relativistic Thermodynamics and Kinetic Theory. II. *R. Soc. Lond. A* **1977**, *357*, 59–75.
71. Maartens, R. Dissipative cosmology. *Class. Quantum Gravity* **1995**, *12*, 1455–1465. [\[CrossRef\]](#)
72. Hernández-Almada, A. Cosmological test on viscous bulk models using Hubble Parameter measurements and type Ia Supernovae data. *Eur. Phys. J. C* **2019**, *79*, 751. [\[CrossRef\]](#)
73. Wilson, J.R.; Mathews, G.J.; Fuller, G.M. Bulk Viscosity, Decaying Dark Matter, and the Cosmic Acceleration. *Phys. Rev. D* **2007**, *75*, 043521. [\[CrossRef\]](#)
74. Hu, J.; Hu, H. Viscous universe with cosmological constant. *Eur. Phys. J. Plus* **2020**, *135*, 718. [\[CrossRef\]](#)
75. Cataldo, M.; Cruz, N.; Lepe, S. Viscous dark energy and phantom evolution. *Phys. Lett. B* **2005**, *619*, 5–10. [\[CrossRef\]](#)
76. Viel, M.; Becker, G.D.; Bolton, J.S.; Haehnelt, M.G. Warm dark matter as a solution to the small scale crisis: New constraints from high redshift Lyman- α forest data. *Phys. Rev. D* **2013**, *88*, 043502. [\[CrossRef\]](#)
77. Asaka, T.; Shaposhnikov, M.; Kusenko, A. Opening a new window for warm dark matter. *Phys. Lett. B* **2006**, *638*, 401–406. [\[CrossRef\]](#)
78. Avelino, A.; Cruz, N.; Nucamendi, U. Testing the EoS of dark matter with cosmological observations. *arXiv* **2012**, arXiv:1211.4633.
79. Tamayo, D. Thermodynamics of viscous dark energy. *arXiv* **2020**, arXiv:2006.14153.
80. Cornejo-Pérez, O.; Belinchón, J.A. Exact solutions of a flat full causal bulk viscous frw cosmological model through factorization. *Int. J. Mod. Phys. D* **2013**, *22*, 1350031. [\[CrossRef\]](#)
81. Cruz, M.; Cruz, N.; Lepe, S. Phantom solution in a non-linear Israel–Stewart theory. *Phys. Lett. B* **2017**, *769*, 159. [\[CrossRef\]](#)
82. Maartens, R. Causal Thermodynamics in Relativity. *arXiv* **1996**, arXiv:astro-ph/9609119.
83. Hiscock, W.A.; Salmons, J. Dissipative Boltzmann–Robertson–Walker cosmologies. *Phys. Rev. D* **1991**, *43*, 3249–3258. [\[CrossRef\]](#)

84. Cruz, M.; Lepe, S.; Odintsov, S.D. Thermodynamically allowed phantom cosmology with viscous fluid. *Phys. Rev. D* **2018**, *98*, 083515. [\[CrossRef\]](#)
85. Mohan, N.J.; Krishna, P.; Sasidharan, A.; Mathew, T.K. Dynamical system analysis and thermal evolution of the causal dissipative model. *arXiv* **2018**, arXiv:1807.04041.
86. Mathew, T.K.; Aswathy, M.B.; Manoj, M. Cosmology and thermodynamics of FLRW universe with bulk viscous stiff fluid. *Eur. Phys. J. C* **2014**, *74*, 3188. [\[CrossRef\]](#)
87. Brevik, I.; Timoshkin, A.V. Thermodynamic aspects of entropic cosmology with viscosity. *Int. J. Mod. Phys. D* **2021**, *30*, 2150008. [\[CrossRef\]](#)
88. Jerin Mohan, N.D.; Mathew, T.K. On the feasibility of truncated Israel-Stewart model in the context of late acceleration. *Class. Quantum Gravity* **2021**, *38*, 145016. [\[CrossRef\]](#)
89. Brevik, I. Viscosity-Induced Crossing of the Phantom Barrier. *Entropy* **2015**, *17*, 6318–6328. [\[CrossRef\]](#)
90. Cruz, N.; González, E.; Jovel, J. Singularities and Soft-Big Bang in a viscous Λ CDM model. *Phys. Rev. D* **2022**, *105*, 024047. [\[CrossRef\]](#)
91. Rebhan, E. ‘Soft bang’ instead of ‘big bang’: Model of an inflationary universe without singularities and with eternal physical past time. *Astron. Astrophys.* **2000**, *353*, 1–9.
92. Novello, M.; Elbaz, E. Soft big bang model induced by nonminimal coupling. *Nuovo C. B* **1994**, *109*, 741–746. [\[CrossRef\]](#)
93. McInnes, B. The dS/CFT Correspondence and the Big Smash. *J. High Energy Phys.* **2002**, *0208*, 029. [\[CrossRef\]](#)
94. Nojiri, S.; Odintsov, S.D.; Tsujikawa, S. Properties of singularities in (phantom) dark energy universe. *Phys. Rev. D* **2005**, *71*, 063004. [\[CrossRef\]](#)
95. Bouhmadi-López, M.; Kiefer, C.; Martín-Moruno, P. Phantom singularities and their quantum fate: General relativity and beyond—A CANTATA COST action topic. *Gen. Relativ. Gravit.* **2019**, *51*, 135. [\[CrossRef\]](#)
96. Scolnic, D.M.; Jones, D.O.; Rest, A.; Pan, Y.C.; Chornock, R.; Foley, R.J.; Huber, M.E.; Kessler, R.; Narayan, G.; Riess, A.G.; et al. The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample. *Astrophys. J.* **2018**, *859*, 101. [\[CrossRef\]](#)
97. Magaña, J.; Amante, M.H.; Garcia-Aspeitia, M.A.; Motta, V. The Cardassian expansion revisited: Constraints from updated Hubble parameter measurements and type Ia supernova data. *Mon. Not. R. Astron. Soc.* **2018**, *476*, 1036. [\[CrossRef\]](#)
98. Goodman, J.; Weare, J. Ensemble samplers with affine invariance. *Commun. Appl. Math. Comput. Sci.* **2010**, *5*, 65–80. [\[CrossRef\]](#)
99. Foreman-Mackey, D.; Conley, A.; Meierjürgen, W.; Hogg, D.W.; Lang, D.; Marshall, P.; Price-Whelan, A.; Sanders, J.; Zuntz, J. emcee: The MCMC Hammer. *Publ. Astron. Soc. Pac.* **2013**, *125*, 306. [\[CrossRef\]](#)
100. Tripp, R. A two-parameter luminosity correction for Type Ia supernovae. *Astron. Astrophys.* **1998**, *331*, 815.
101. Kessler, R.; Scolnic, D. Correcting Type Ia Supernova Distances for Selection Biases and Contamination in Photometrically Identified Samples. *Astrophys. J.* **2017**, *836*, 56. [\[CrossRef\]](#)
102. Lazkoz, R.; Nesseris, S.; Perivolaropoulos, L. Exploring cosmological expansion parametrizations with the gold SNIa data set. *J. Cosmol. Astropart. Phys.* **2005**, *2005*, 10. [\[CrossRef\]](#)
103. Schwarz, G. Estimating the Dimension of a Model. *Ann. Stat.* **1978**, *6*, 461. doi: 10.1214/aos/1176344136. [\[CrossRef\]](#)
104. Brevik, I. Temperature Variation in the Dark Cosmic Fluid in the Late Universe. *Mod. Phys. Lett. A* **2016**, *31*, 1650050. [\[CrossRef\]](#)
105. Mueller, C.M. Cosmological bounds on the equation of state of dark matter. *Phys. Rev. D* **2005**, *71*, 047302. [\[CrossRef\]](#)
106. Kumar, S.; Xu, L. Observational constraints on variable equation of state parameters of dark matter and dark energy after Planck. *Phys. Lett. B* **2014**, *737*, 244–247. [\[CrossRef\]](#)
107. Boyanovsky, D. Sterile Neutrinos as Dark Matter: Alternative Production Mechanisms in the Early Universe. *Universe* **2021**, *7*, 264. [\[CrossRef\]](#)