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Meir–Keeler Type Contraction in Orthogonal *M*-Metric Spaces

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Abstract: In this article, we prove fixed point results for a Meir–Keeler type contraction due to orthogonal *M*-metric spaces. The results of the paper improve and extend some recent developments in fixed point theory. The extension is assured by the concluding remarks and followed by the main theorem. Finally, an application of the main theorem is established in proving theorems for some integral equations and integral-type contractive conditions.

Keywords: fixed point; orthogonal set; orthogonal metric spaces; orthogonal *M*-metric space; metric spaces

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1. Introduction

Although fixed point theory has many applications, it was primarily used to show the establishment and, in certain circumstances, uniqueness of a particular class of points obeying a given criterion. It depicts the solution of an equation, which can be of several types: integral equations, differential equations, matrixial equations, and so on. These elements are referred to as fixed points since they must be associated with an operator. The fundamental space in which a fixed point issue is given must have an abstract metric context, that is a mapping that specifies the distance between two random points. Initially, only metric spaces were studied since they fulfilled the important qualities that ensure the presence and uniqueness of fixed points: non-negativity, the identity of indiscernible, symmetry, and triangle inequality. It is worth mentioning here two recent references wherein fixed point theory has been used to study symmetry/gemological shape of fractals [1,2].

The Banach contraction principle was one of the most well-known and significant mathematical findings of the previous century. However, over time, various scholars have made a wide range of minor adjustments that have broadened and extended the concept of Banach in several ways. Studies are commonly conducted in the context of extended real metric spaces, semi-metrics, quasi-metrics, pseudo-metrics, fuzzy metric spaces, probabilistic metric spaces, partial metric spaces, *G*-metric spaces, and *M*-metric spaces nowadays.

In 1969, Meir and Keeler [3], discovered an intriguing result that is an extension of the well-known Banach Contraction Principle [4]. The strong Meir–Keeler function and the weaker Meir–Keeler function were introduced by Chen and Chang [5], based on observations made by Meir–Keeler [3]. These contraction has been extended by the many authors for which reader may refer [6–13]. Das and Gupta [14] proved some fixed theorems by using the rational expression. Further, Gupta and Saxena [15] and Samet et al. [16] generalized the results due to Das and Gupta [14]. In 2013, Samet et al. [16] proved a fixed point theorem of Meir- Keeler type that extends the result of Das and Gupta [14]. In 2015, Najeh et al. [17], established a fixed point theorem of the Meir–Keeler type, which extended the result of Gupta and Saxena [15]. The results of Najeh et al. [17] and Samet et al. [16] are further improved by Koti et al. [18].

Matthews [19] defined partial metric space for the first time in 1994. Eventually, many mathematicians worked on establishing partial metric spaces and fixed point theorems. In a recent study, Haghi et al. [20] noted that one should "be cautious when working with partial metric fixed point results" and prove certain fixed point theorems, which show that the analogous results in ordinary metric spaces may be used to get fixed point generalization to partial metric spaces. Asadi et al. [21] proposed the concept of M-metric space in 2014, which expanded the partial metric space and certain fixed point theorems proven there. Patle et al. [22] are the first to investigate the Pampaiu Hausdorff distance as a result of M-metric. Recently, Asim et al. [23] extended M-metric by introducing the M_ν-metric. In the literature on fixed point theory, there are numerous new advances on M-metric space.

Recently, Gordji et al. [24] expanded the literature on metric space by introducing the concept of orthogonality and establishing the fixed point result. There are several uses for this novel idea of an orthogonal set, as well as numerous forms of orthogonality. Eshaghi Gordji and Habibi [25,26] proved the fixed point in generalized orthogonal metric space and related results in orthogonal metric spaces. Furthermore, for more information, we refer the reader to [27–35]. Very recently, Uddin et al. [36] introduced the notion of orthogonal M-metric space (briefly $M_\perp$ metric spaces). In this paper, we establish new fixed point theorems for the Meir–Keeler type contractions in the context of $M_\perp$-metric space. Finally, an application of these results in proving fixed point theorems of integral type contraction conditions is also given.

2. Preliminaries

Definition 1 ([24]). *Consider a binary relation $\perp$ defined on a non-empty set E. If binary relation $\perp$ fulfils the undermentioned criteria:*

$$\exists\vartheta_0[(\text{for all } \varsigma \in E, \varsigma \perp \vartheta_0) \text{ or } (\text{for all } \varsigma \in E, \vartheta_0 \perp \varsigma)],$$

then pair, $(E, \perp)$ known as an orthogonal set and element ϑ_0 is called an orthogonal element. This O-set or orthogonal set is denoted by $(E, \perp)$.

Definition 2 ([24]). *Let $(E, \perp)$ be an orthogonal set (O-set). Any two elements $\vartheta, \varsigma \in E$ such that $\vartheta \perp \varsigma$, then $\vartheta, \varsigma \in E$ are said to be orthogonally related.*

An orthogonal set is illustrated in the following non-trivial examples.

Example 1. *Let $E = 2\mathbb{Z}$ and set a binary relation $\perp$ on $2\mathbb{Z}$ as $m \perp n$ if $m.n = 0$. Then $(2\mathbb{Z}, \perp)$ is an orthogonal set with 0 as an orthogonal element.*

Example 2. *Let E be set of all matrices of order n over $\mathbb{R}$ that is $E = M_n(\mathbb{R})$, a binary relation $\perp$ on $M_n(\mathbb{R})$ defined as $A \perp B$ if $AB = BA$. For a scalar matrix $S \in M_n(\mathbb{R})$, we have $SA = AS$ for all $A \in M_n(\mathbb{R})$.*

An O-set may have a unique, more than one or infinite orthogonal element.

Consider a binary relation $\perp$ on a non-empty set E with usual metric d defined on set E then, triplet $(E, \perp, d)$ is called O-metric space(briefly) or orthogonal metric space. Some basic characteristics of an O-set and O-metric space are given below.

Definition 3 ([24]). *Consider a binary relation $\perp$ defined on a non-empty set E then, sequence $\{\vartheta_n\}$ is called an orthogonal sequence (briefly O-sequence) if*

$$(\text{for all } n \in \mathbb{N}, \vartheta_n \perp \vartheta_{n+1}) \text{ or } (\text{for all } n \in \mathbb{N}, \vartheta_{n+1} \perp \vartheta_n).$$

Definition 4 ([24]). *Assume that triplate $(E, \perp, d)$ be an O-metric space. If every Cauchy O-sequence is converges in E, then set E is called O-complete.*

Remark 1 ([24]). *Every complete metric space is O-complete and the converse is not true.*

Definition 5 ([24]). *Consider a binary relation $\perp$ on a non-empty set E with usual metric d defined on set E and assume that $(E, \perp, d)$ be an O-metric space and f be a self-map on E. If for each O-sequence $\{\vartheta_n\}_{n\in\mathbb{N}} \to \vartheta$ implies $f(\vartheta_n) \to f(\vartheta)$ as $n \to \infty$, then self-map f is called $\perp$- continuous at ϑ. In addition, f is said to be $\perp$- continuous on E if f is $\perp$- continuous in each $\vartheta \in E$.*

Remark 2. *The authors of [24] find, O-continuity in conventional metric spaces is weaker than classical continuity.*

Definition 6 ([24]). *Consider a binary relation $\perp$ on a non-empty set E and let pair $(E, \perp)$ be an O-set. A self-map $f : E \to E$ is called $\perp$-preserving if $f(\vartheta) \perp f(\varsigma)$ whenever $\vartheta \perp \varsigma$ and weakly $\perp$-preserving if $f(\vartheta) \perp f(\varsigma)$ or $f(\varsigma) \perp f(\vartheta)$ whenever $\vartheta \perp \varsigma$.*

3. Main Results

For the discussion that follows, the following notation will be helpful:

(i) $\Gamma_{\vartheta\varsigma} = \min\{\Gamma(\vartheta, \vartheta), \Gamma(\varsigma, \varsigma)\}$,
(ii) $\Gamma'_{\vartheta\varsigma} = \max\{\Gamma(\vartheta, \vartheta), \Gamma(\varsigma, \varsigma)\}$.

Definition 7 ([36]). *Consider a binary relation $\perp$ defined on a non-empty set E and function $\Gamma : E \times E \to \mathbb{R}^+$ is called $M_\perp$ - if the undermentioned criteria are fulfilled:*

(i) $\Gamma(\vartheta, \vartheta) = \Gamma(\varsigma, \varsigma) = \Gamma(\vartheta, \varsigma) \Leftrightarrow \vartheta = \varsigma$;
(ii) $\Gamma_{\vartheta\varsigma} \leq \Gamma(\vartheta, \varsigma)$, *for all* $\vartheta, \varsigma \in E$;
(iii) $\Gamma(\vartheta, \varsigma) = \Gamma(\varsigma, \vartheta)$, *for all* $\vartheta, \varsigma \in E$;
(iv) $\exists\, s(\geq 1) \in \mathbb{R}$ *such that for all* $\vartheta, \varsigma \in E$ *with* $\vartheta \perp \varsigma \perp \kappa$, *we have*

$$(\Gamma(\vartheta, \varsigma) - \Gamma_{\vartheta\varsigma}) \leq s[(\Gamma(\vartheta, z) - \Gamma_{\vartheta\kappa}) + (\Gamma(\kappa, \varsigma) - \Gamma_{z\varsigma})] - \Gamma(\kappa, \kappa),$$

then (E, Γ) is referred as an orthogonal M-metric space or $M_\perp$-metric space.

Definition 8 ([36]). *Suppose that (E, Γ) be a $M_\perp$-metric space and $\{\vartheta_n\}$ be a sequence in the set E, then*

(a) *a sequence*

$$\{\vartheta_n\} \to \vartheta \Leftrightarrow \lim_{n\to\infty}(\Gamma(\vartheta_n, \vartheta) - \Gamma_{\vartheta_n\vartheta}) = 0; \tag{1}$$

(b) *sequence $\{\vartheta_n\}$ is called $M_\perp$-Cauchy sequence if*

$$\lim_{n,m\to\infty}(\Gamma(\vartheta_n, \vartheta_m) - \Gamma_{\vartheta_n\vartheta_m}) \text{ and } \lim_{n,m\to\infty}(\Gamma'_{\vartheta_n\vartheta_m} - \Gamma_{\vartheta_n\vartheta_m}) \tag{2}$$

both exist and are finite;

(c) *if for every $M_\perp$-Cauchy sequence $\{\vartheta_n\} \to \vartheta$ such that*

$$\lim_{n\to\infty}\Gamma(\vartheta_n, \vartheta) - \Gamma_{\vartheta_n\vartheta} = 0 \text{ and } \lim_{n\to\infty}\Gamma'_{\vartheta_n\vartheta} - \Gamma_{\vartheta_n\vartheta} = 0,$$

then, (E, Γ) is referred as complete $M_\perp$-metric space.

Definition 9 ([36]). *Consider a binary relation $\perp$ defined on a non-empty set E and assume that pair (E, Γ) be an $M_\perp$-metric space. If for each O-sequence $\{\vartheta_n\}_{n\in\mathbb{N}} \to \vartheta$ implies $f(\vartheta_n) \to f(\vartheta)$ as $n \to \infty$ then a self-map f is $\perp$-continuous at $\vartheta \in E$. In addition, f is said to be $\perp$-continuous on E if f is $\perp$-continuous at each $\vartheta \in E$.*

Theorem 1. *Let (E, Γ) be an complete orthogonal M-metric space with coefficient $s \geq 1$ and $f : E \to E$ be $\perp$-preserving, $\perp$-continuous mappings and satisfying the following condition; for all $\epsilon > 0, \exists \delta > 0$ and for all $\vartheta, \varsigma \in E$ with $\vartheta \perp \varsigma$ such that*

$$\epsilon \leq \Gamma(\vartheta, \varsigma) < \epsilon + \delta \Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon. \tag{3}$$

Then T has a unique fixed point.

Proof. The orthogonality of a non-empty set implies that $\exists\ \vartheta_0 \in E$, satisfying

$$(\text{for all } \varsigma \in E, \vartheta_0 \perp \varsigma) \text{ or } (\text{for all } \varsigma \in E, \varsigma \perp \vartheta_0).$$

It follows that $\vartheta_0 \perp f\vartheta_0$ or $f\vartheta_0 \perp \vartheta_0$. Let $\vartheta_1 = f\vartheta_0, \vartheta_2 = f\vartheta_1, \ldots, \vartheta_{n+1} = f\vartheta_n$ for $n \in \mathbb{N}$. By (3) and $\perp$-preservence of f, we have

$$\begin{aligned} d_n &= \Gamma(\vartheta_n, \vartheta_{n+1}) \\ &= \Gamma(f\vartheta_{n-1}, f\vartheta_n) \\ &\leq \Gamma(\vartheta_{n-1}, \vartheta_n) \\ &= d_{n-1}, \end{aligned}$$

so, the sequence $\{d_n\}$ is bounded below and decreasing, thus $d_n \to d$ for some $d \in \mathbb{R}^+$. On contrary, suppose that $d > 0$, therefore $\Gamma(\vartheta_n, \vartheta_{n+1}) \geq d$. On the other hand for $d > 0$, there exists $\delta(d) > 0$ such that

$$d \leq \Gamma(\vartheta_n, \vartheta_{n+1}) \leq d + \delta(d) \Rightarrow \Gamma(f\vartheta_n, f\vartheta_{n-1}) = \Gamma(\vartheta_n, \vartheta_{n-1}) < d, \tag{4}$$

which is contraction, so $d = 0$ that is

$$\lim_{n\to\infty} \Gamma(\vartheta_n, \vartheta_{n+1}) = 0 \tag{5}$$

then,

$$\lim_{n\to\infty} \min\{\Gamma(\vartheta_n, \vartheta_n), \Gamma(\vartheta_{n-1}, \vartheta_{n-1})\} = \lim_{n\to\infty} \Gamma_{\vartheta_n, \vartheta_{n-1}} \leq \lim_{n\to\infty} \Gamma(\vartheta_n, \vartheta_{n-1}) = 0,$$

and

$$\lim_{n\to\infty} \Gamma_{\vartheta_m, \vartheta_n} = 0, \lim_{n\to\infty} \Gamma'_{\vartheta_m, \vartheta_n} = 0. \tag{6}$$

Since, $\lim_{n\to\infty} \Gamma(\vartheta_n, \vartheta_n) = 0$. Now, we want to show that

$$\lim_{m,n\to\infty} \Gamma(\vartheta_m, \vartheta_n) = 0.$$

Let, it is not true. So for some $\epsilon > 0$, we have

$$\lim_{m,n\to\infty} \sup \Gamma(\vartheta_m, \vartheta_n) > 2\epsilon.$$

In addition, by hypothesis, there exists $\delta > 0$, such that

$$\epsilon \leq \Gamma(\vartheta, \varsigma) < \epsilon + \delta \Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon.$$

The aforementioned inequality is valid when δ is substituted with $\delta' = \min\{\delta, \epsilon\}$, by (5)

$$\exists \mathbb{N} > 0, \text{ for all } n > \mathbb{N} \Rightarrow \Gamma(\vartheta_n, \vartheta_{n+1}) < \frac{\delta'}{3s}$$

and for $m, n > \mathbb{N}$, $\Gamma(\vartheta_m, \vartheta_n) > \frac{2\epsilon}{s}$. This implies, since

$$\Gamma(\vartheta_n, \vartheta_{n+1}) < \frac{\epsilon}{\delta} \text{ and } \epsilon + \delta' < 2\epsilon < \Gamma(\vartheta_m, \vartheta_n),$$

such that $\exists\, i, m < i < n$ with

$$\epsilon + \frac{2\delta'}{3} < \Gamma(\vartheta_m, \vartheta_i) - \Gamma_{\vartheta_m, \vartheta_i} < \epsilon + \delta', \tag{7}$$

even so, for all m and i

$$\begin{aligned}\Gamma(\vartheta_m, \vartheta_i) - \Gamma_{\vartheta_m, \vartheta_{m+1}} &\le s\Big[\Gamma(\vartheta_m, \vartheta_{m+1}) + \Gamma(\vartheta_{m+1}, \vartheta_{i+1}) + \Gamma(\vartheta_{i+1}, \vartheta_i)\Big] \\ &\le s\Big[\frac{\delta'}{3s} + \frac{\epsilon}{s} + \frac{\delta'}{3s}\Big] \\ &= \frac{2\delta'}{3} + \epsilon,\end{aligned}$$

which is in contradiction to (7). So by (6) and $\lim_{m,n\to\infty} \Gamma(\vartheta_m, \vartheta_n) = 0$, we see that the O-sequence $\{\vartheta_n\}$ is a Cauchy O-sequence in E. Since E is O-complete, then there exists $\vartheta^* \in E$ such that

$$\lim_{n\to\infty}(\Gamma(\vartheta_n, \vartheta^*) - \Gamma_{\vartheta_n, \vartheta^*}) = 0.$$

Further, we demonstrate that f has a fixed point in E. Beacuse $\Gamma(\vartheta_n, \vartheta_n) \to 0$, then $\min\{\Gamma(\vartheta_n, \vartheta_n), \Gamma(\vartheta^*, \vartheta^*)\} \to 0$ implies that $\Gamma(\vartheta_n, \vartheta^*) \to 0$ thus by the hypothesis $\Gamma(f\vartheta_n, f\vartheta^*)$ $\le \Gamma(\vartheta_n, \vartheta^*) \to 0$ and also $\min\{\Gamma(f\vartheta_n, f\vartheta_n), \Gamma(f\vartheta^*, f\vartheta^*)\} \le \Gamma(f\vartheta_n, f\vartheta^*) \to 0$, this implies that $f\vartheta_n \to f\vartheta^*$.

$$\begin{aligned}\Gamma(\vartheta^*, f\vartheta^*) &\le \lim_{n\to\infty} \sup \Gamma(\vartheta^*, \vartheta_n) + \lim_{n\to\infty} \sup \Gamma(\vartheta_n, f\vartheta^*) \\ &= \lim_{n\to\infty} \sup \Gamma(\vartheta_n, f\vartheta^*) \\ &= \lim_{n\to\infty} \sup \Gamma(f\vartheta_{n-1}, f\vartheta^*) \\ &< \lim_{n\to\infty} \sup \Gamma(\vartheta_{n-1}, \vartheta^*),\end{aligned}$$

this implies that $\Gamma(\vartheta^*, f\vartheta^*) = 0$. Since $\min\{\Gamma(\vartheta^*, \vartheta^*), \Gamma(f\vartheta^*, f\vartheta^*)\} \le \Gamma(\vartheta^*, f\vartheta^*) = 0$, and also $\Gamma(f\vartheta^*, f\vartheta^*) \le \Gamma(\vartheta^*, \vartheta^*)$, then $\Gamma(f\vartheta^*, f\vartheta^*) = 0$ this implies that $\Gamma_{\vartheta^*, f\vartheta^*} = 0$ thus $\Gamma(\vartheta^*, f\vartheta^*) = \Gamma_{\vartheta^*, f\vartheta^*} = \Gamma(f\vartheta^*, f\vartheta^*) = \Gamma(\vartheta^*, \vartheta^*)$; this implies that $f\vartheta^* = \vartheta^*$.

Let ς^* be another point in E such that $f\varsigma^* = \varsigma^*$. Then by choice of ϑ_0 in the begining of the proof, we have

$$(\vartheta_0 \perp \vartheta^* \text{ and } \vartheta_0 \perp \varsigma^*) \text{ or } (\vartheta^* \perp \vartheta_0 \text{ and } \varsigma^* \perp \vartheta_0).$$

Since f is $\perp$-preserving, we have

$$(f^n\vartheta_0 \perp f^n\vartheta^* \text{ and } f^n\vartheta_0 \perp f^n\varsigma^*) \text{ or } (f^n\vartheta^* \perp f^n\vartheta_0 \text{ and } f^n\vartheta^* \perp f^n\vartheta_0), \text{ for all } n \in \mathbb{N}.\text{Thus}$$

$$\Gamma(\vartheta^*, \varsigma^*) = \Gamma(f\vartheta^*, f\varsigma^*) < \Gamma(\vartheta^*, \varsigma^*),$$

which implies that $\Gamma(\vartheta^*, \varsigma^*) = 0$, so $\vartheta^* = \varsigma^*$. □

Example 3. *Let $E = [0, 2]$ and a function $\Gamma : E \times E \to \mathbb{R}^+$ is defined as $\Gamma(\vartheta, \varsigma) = |\vartheta - \varsigma|$, for all $\vartheta, \varsigma \in E$ and binary relation $\perp$ on E is given by $\vartheta \perp \varsigma$ if $\vartheta\varsigma \le 2\vartheta$ this implies that pair (E, Γ) is an O-complete M-metric space. Now, a self-map f defined on set E is given as*

$$f(\vartheta) = \begin{cases} \frac{\vartheta^2}{4}, \vartheta \in [0, 1] \\ 0, \vartheta \in (1, 2]. \end{cases}$$

Then the undermentioned conditions are met:

(i) *If* $\vartheta = 0$ *and* $\varsigma \in [0,1]$, *then* $f\vartheta = 0$ *and* $f\varsigma = \frac{\varsigma^2}{4}$;
(ii) *If* $\vartheta = 0$ *and* $\varsigma \in (1,2]$, *then* $f\vartheta = f\varsigma = 0$;
(iii) *If* $\varsigma \in [0,1]$ *and* $[0,1]$, *then* $f\vartheta = \frac{\vartheta^2}{4}$ *and* $f\varsigma = \frac{\varsigma^2}{4}$;
(iv) *If* $\varsigma \in [0,1]$ *and* $\vartheta \in (1,2]$, *then* $f\varsigma = \frac{\varsigma^2}{4}$ *and* $f\vartheta = 0$.

These cases implies that $f\vartheta f\varsigma \leq 2f\vartheta$, *hence* f *is* $\perp$ *preserving. In addition,* $|f\vartheta - f\varsigma| = \left|\frac{\vartheta^2}{4} - \frac{\varsigma^2}{4}\right| \leq |\vartheta - \varsigma|$. *For given* $\epsilon > 0$ *and* $\delta > 0$ *with* $\epsilon \leq |\vartheta - \varsigma| < \epsilon + \delta$, *we get* $|f\vartheta - f\varsigma| < \epsilon$, *for all* $\vartheta, \varsigma \in E$. *Therefore all the condition of Theorem 1 are satisfied. Hence,* f *has* 0 *as unique fixed point in* E.

Remark 3. *If we assume* f *as in Example 3, then it is easy consequence to verify that* f *satisfies all the conditions of Theorem 1 and has fixed point* 0. *However,* f *is not continuous, so hypothesis of theorem of [3] not satisfied. This ensure that main Theorem 1 has extension over thereom of [3].*

Theorem 2. *Consider a binary relation* $\perp$ *defined on a non-empty set* E *and assume that* (E, Γ) *be a complete* $M_\perp$*-metric space with* $s \in [1, \infty)$ *and self map* f *is* $\perp$*-continuous and* $\perp$*-preserving on* E*, satisfying the following condition, for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\epsilon \leq \phi(\max\{\frac{(1+\Gamma(\vartheta, f\vartheta))(\Gamma(\varsigma, f\varsigma))}{(1+\Gamma(\vartheta,\varsigma))}, \frac{(\Gamma(\vartheta, f\vartheta))(\Gamma(\varsigma, f\varsigma))}{\Gamma(\vartheta,\varsigma)}, \Gamma(\vartheta,\varsigma))\} < \epsilon + \delta(\epsilon)$$

$$\Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon \tag{8}$$

for all $\vartheta, \varsigma \in E$ *or* $\varsigma \neq f\varsigma$ *with* $\vartheta \perp \varsigma$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuous monotonic increasing function such that* $\phi(t) < kt$, *for all* $t > 0$, $k \in (0,1)$ *and* $\phi(0) = 0$. *Then* f *has unique fixed point* $\vartheta^* \in E$. *Moreover, for all* $\vartheta \in E$ *the sequence* $\{f^n(\vartheta)\}$ *converges* $\vartheta^* \in E$.

Proof. The orthogonality of a non-empty set implies that $\exists\, \vartheta_0 \in E$, satisfying

$$(\text{for all } \varsigma \in E, \vartheta_0 \perp \varsigma) \text{ or } (\text{for all } \varsigma \in E, \varsigma \perp \vartheta_0).$$

It follows that $\vartheta_0 \perp f\vartheta_0$ or $f\vartheta_0 = \vartheta_0$. Let $\vartheta_1 = f\vartheta_0$, $\vartheta_2 = f\vartheta_1$, $\vartheta_3 = f\vartheta_2, \ldots, \vartheta_{n+1} = f\vartheta_n$, for all $n \in \mathbb{N}$.

From contraction mapping (8), we observe that

$$\Gamma(f\vartheta, f\varsigma) < \mathbb{J}(\vartheta, \varsigma), \tag{9}$$

where

$$\mathbb{J}(\vartheta,\varsigma) = \phi(\max\{\frac{(1+\Gamma(\vartheta, f\vartheta))(\Gamma(\varsigma, f\varsigma))}{(1+\Gamma(\vartheta,\varsigma))}, \frac{(\Gamma(\vartheta, f\vartheta))(\Gamma(\varsigma, f\varsigma))}{\Gamma(\vartheta,\varsigma)}, \Gamma(\vartheta,\varsigma)\}).$$

Since f is $\perp$-preserving and $\{\vartheta_n\}$ is O-sequence, we get

$$\begin{aligned}
\mathbb{J}(\vartheta_n, \vartheta_{n+1}) = \phi(\max\{&\frac{(1+\Gamma(\vartheta_n, f\vartheta_n))(\Gamma(\vartheta_{n+1}, f\vartheta_{n+1}))}{(1+\Gamma(\vartheta_n, \vartheta_{n+1}))}, \frac{(\Gamma(\vartheta_n, f\vartheta_n))(\Gamma(\vartheta_{n+1}, f\vartheta_{n+1})}{\Gamma(\vartheta_n, \vartheta_{n+1})}, \\
&\Gamma(\vartheta_{n-1}, \vartheta_n))\} \\
= \phi(\max\{&\frac{(1+\Gamma(\vartheta_n, \vartheta_{n+1}))(\Gamma(\vartheta_{n+1}, \vartheta_{n+2}))}{(1+\Gamma(\vartheta_n, \vartheta_{n+1}))}, \frac{(\Gamma(\vartheta_n, \vartheta_{n+1}))(\Gamma(\vartheta_{n+1}, \vartheta_{n+2}))}{\Gamma(\vartheta_n, \vartheta_{n+1})}, \\
&\Gamma(\vartheta_n, \vartheta_{n+1}))\} \\
= \phi(&\max\{\Gamma(\vartheta_{n+1}, \vartheta_{n+2}), \Gamma(\vartheta_{n+1}, \vartheta_{n+2}), \Gamma(\vartheta_n, \vartheta_{n+1})\}) \\
= \phi(&\max\{\Gamma(\vartheta_{n+1}, \vartheta_{n+2}), \Gamma(\vartheta_n, \vartheta_{n+1})\}).
\end{aligned}$$

Let
$$\max\{\Gamma(\vartheta_{n+1},\vartheta_{n+2}),\Gamma(\vartheta_n,\vartheta_{n+1})\}=\Gamma(\vartheta_{n+1},\vartheta_{n+2}),$$
then
$$\mathbb{J}(\vartheta_n,\vartheta_{n+1})=\phi(\Gamma(\vartheta_{n+1},\vartheta_{n+2}))$$
(9) implies that
$$\begin{aligned}\Gamma(\vartheta_{n+1},\vartheta_{n+2})&=\Gamma(f\vartheta_n,f\vartheta_{n+1})\\&<\mathbb{J}(\vartheta_n,\vartheta_{n+1})\\&=\phi(\Gamma(\vartheta_{n+1},\vartheta_{n+2})\\&<k\Gamma(\vartheta_{n+1},\vartheta_{n+2})\end{aligned}$$
which is in contradiction. Thus,
$$\mathbb{J}(\vartheta_n,\vartheta_{n+1})=\phi(\Gamma(\vartheta n,\vartheta_{n+1}))$$
this implies that
$$\Gamma(f\vartheta_{n-1},f\vartheta_n)<k\Gamma(\vartheta_{n-1},\vartheta_n)<\ldots<k^n\Gamma(\vartheta_0,\vartheta_1),\text{for all } n\in\mathbb{N}. \tag{10}$$
For any $j>n$, where $j,n\in\mathbb{Z}_+$, we have
$$\begin{aligned}\Gamma(\vartheta_n,\vartheta_j)&=\Gamma(f\vartheta_{n-1},f\vartheta_{j-1})\\&\leq k\Gamma(\vartheta_{n-1},\vartheta_{j-1})\\&\ldots\\&\leq k^n\Gamma(\vartheta_0,\vartheta_{j-n})\end{aligned}$$
Hence,
$$\begin{aligned}\Gamma(\vartheta_n,\vartheta_j)-\Gamma_{\vartheta_n,\vartheta_j}&\leq k^n(s\Gamma(\vartheta_0,\vartheta_1)+s\Gamma(\vartheta_1,\vartheta_{j-n})\\&\leq k^n(s\Gamma(\vartheta_0,\vartheta_1)+s^2\Gamma(\vartheta_1,\vartheta_2)+s^2\Gamma(\vartheta_2,\vartheta_{j-n}))\\&\leq k^n(s\Gamma(\vartheta_0,\vartheta_1)+s^2\Gamma(\vartheta_1,\vartheta_2)+\ldots+s^{j-n}\Gamma(\vartheta_{j-n-1},\vartheta_{j-n}))\\&\leq k^n(s\Gamma(\vartheta_0,\vartheta_1)+k^ns^2k\Gamma(\vartheta_0,\vartheta_1)+\ldots+k^ns^{j-n}k^{j-1}\Gamma(\vartheta_0,\vartheta_1))\\&\leq sk^n(1+sk+(sk)^2+\ldots)\Gamma(\vartheta_0,\vartheta_1)\\&=\frac{sk^n}{1-sk}\Gamma(\vartheta_0,\vartheta_1).\end{aligned}$$
Since $k\in[0,1)$ and $k>0$, it follows from the above inequality that
$$\Gamma(\vartheta_n,\vartheta_j)-\Gamma_{\vartheta_n,\vartheta_j}\to 0 \text{ as } n,j\to\infty.$$
Similary,
$$\Gamma'_{(\vartheta_n,\vartheta_j)}-\Gamma_{\vartheta_n,\vartheta_j}\to 0 \text{ as } n,j\to\infty$$
and so $\Gamma_{\vartheta_n,\vartheta^*}\to 0$ as $n\to\infty$. Hence, we have $\Gamma(\vartheta_n,\vartheta^*)\to 0$ as $n\to\infty$. Then $\Gamma(\vartheta^*,\vartheta^*)=0=\Gamma_{\vartheta_n,\vartheta^*}$. Thus, ϑ_n is a O-cauchy sequnce in E. Since E is O-complete M-metric, there exists $\vartheta^*\in E$ such that
$$\Gamma(\vartheta_n,\vartheta^*)-\Gamma_{\vartheta_n,\vartheta^*}\to 0 \text{ as } n\to\infty.$$

Now,

$$\begin{aligned}\Gamma(\vartheta^*, T\vartheta^*) &\leq \lim_{n\to\infty}\sup(\Gamma(\vartheta^*, \vartheta_n)) + \sup\lim_{n\to\infty}(\Gamma(\vartheta_n, T\vartheta^*)) \\ &= \lim_{n\to\infty}\sup(\Gamma(\vartheta_n, T\vartheta^*)) \\ &= \lim_{n\to\infty}\sup(\Gamma(f\vartheta_{n-1}, f\vartheta^*)) \\ &\leq \lim_{n\to\infty}\sup k(\Gamma(\vartheta_{n-1}, \vartheta^*)) = 0.\end{aligned}$$

and by (iv), we have

$$\begin{aligned}\Gamma(\vartheta^*, f\vartheta^*) &\leq \sup k[\Gamma(\vartheta_{n-1}, f\vartheta^*) - \Gamma_{\vartheta_{n-1}, f\vartheta^*} + \Gamma(f\vartheta^*, \vartheta^*) - \Gamma_{f\vartheta^*, \vartheta^*}] - \Gamma(f\vartheta^*, f\vartheta^*) \\ &\leq k\Gamma(f\vartheta^*, \vartheta^*).\end{aligned}$$

So $\Gamma(\vartheta^*, f\vartheta^*) = 0$, Thus $0 \leq \Gamma(f\vartheta^*, f\vartheta^*) \leq k\Gamma(\vartheta^*, \vartheta^*) = 0$,
thus

$$\Gamma(f\vartheta^*, f\vartheta^*) = \Gamma(\vartheta^*, f\vartheta^*) = \Gamma(\vartheta^*, \vartheta^*) = 0.$$

Therefore, $f\vartheta^* = \vartheta^*$. Let ς^* be another point in E such that $f\varsigma^* = \varsigma^*$. Then by choice of ϑ_0 in the begining of the proof, we have

$$(\vartheta_0 \perp \vartheta^* \text{ and } \vartheta_0 \perp \varsigma^*) \text{ or } (\vartheta^* \perp \vartheta_0 \text{ or } \varsigma^* \perp \vartheta_0).$$

Since f is $\perp$-preserving, we have

$$(f^n\vartheta_0 \perp f^n\vartheta^* \text{ and } f^n\vartheta_0 \perp f^n\varsigma^*) \text{ or } (f^n\vartheta^* \perp f^n\vartheta_0 \text{ and } f^n\varsigma^* \perp f^n\vartheta_0), \text{ for all } n \in \mathbb{N}. \text{ Thus}$$

$$\Gamma(\vartheta^*, \varsigma^*) = \Gamma(f\vartheta^*, f\varsigma^*) < k\Gamma(\vartheta^*, \varsigma^*),$$

which implies that $\Gamma(\vartheta^*, \varsigma^*) = 0$, so $\vartheta^* = \varsigma^*$. □

Remark 4. *Theorem 2 is $M_\perp$-metric version generalization of theorem 2.1 Koti et al. [18], theorem 2.1 of Samet et al. [16] and theorem 2.1 of Najeh et al. [17].*

Remark 5. *Theorem 2.1 of Samet et al. [16] generalised theorem 1 of Das and Gupta [14], theorem 2.1 of Najeh et al. [17] also generalised theorem 1 of Gupta and Saxena [15], The above main Theorem 2 generalised the main results of [14,15] due to $M_\perp$-metric space version.*

Remark 6. *Theorem 2 is generalised version of theorem 3.1 of Asadi et al. [21] and Banach contraction principle in the setting of $M_\perp$-metric space via rational expression.*

Corollary 1. *Let $E \neq \phi$ be a O-set with binary relation $\perp$ defined on E and mapping $\Gamma : E \times E \to \mathbb{R}^+$ such that pair (E, Γ) is O-complete M-metric space. If self map f is $\perp$-continuous and $\perp$-preserving on E, meets with following exiom:*
for all $\epsilon > 0$, $\exists\delta > 0$ satisfying

$$\begin{aligned}&\epsilon \leq \phi\Big(\frac{(1+\Gamma(\vartheta, f\vartheta))\Gamma(\varsigma, f\varsigma)}{1+\Gamma(\vartheta, \varsigma)} + \frac{\Gamma(\vartheta, f\vartheta)\Gamma(\varsigma, f\varsigma)}{\Gamma(\vartheta, \varsigma)} + \Gamma(\vartheta, \varsigma)\Big) < \epsilon + \delta(\epsilon) \\ &\Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon,\end{aligned}$$

for all $\vartheta, \varsigma \in E$, with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$ or $\varsigma \neq f\varsigma$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotonic increasing function such that $\phi(t) < kt$, for all $t > 0$, $k \in (0,1)$ and $\phi(0) = 0$. Then $\exists$ a unique point $\varsigma^* \in E$ such that $f\varsigma^* = \varsigma^*$. In addition, for all $\vartheta \in E$, the sequence $\{f^n(\vartheta)\} \to \varsigma^* \in E$.

Corollary 2. *Let $E \neq \phi$ be a O-set with binary relation $\perp$ defined on E and mapping $\Gamma : E \times E \to \mathbb{R}^+$ such that pair (E, Γ) is O-complete M-metric space. If self map f is $\perp$-continuous and $\perp$-*

preserving on E, meets with following exiom:
for all $\epsilon > 0$, $\exists\, \delta > 0$ satisfying

$$\begin{aligned} &\epsilon \leq \alpha(\frac{(1+\Gamma(\vartheta, f\vartheta))\Gamma(\varsigma, f\varsigma)}{1+\Gamma(\vartheta,\varsigma)} + \frac{\Gamma(\vartheta, f\vartheta)\Gamma(\varsigma, f\varsigma)}{\Gamma(\vartheta,\varsigma)} + \Gamma(\vartheta,\varsigma)) < \epsilon + \delta(\epsilon) \\ &\Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon, \end{aligned}$$

for all $\vartheta, \varsigma \in E$, with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$ or $\varsigma \neq f\varsigma$, where $\alpha \in (0, \frac{k}{3})$ and $k \in (0,1)$. Then $\exists$ a unique point $\varsigma^* \in E$ such that $f\varsigma^* = \varsigma^*$. In addition, for all $\vartheta \in E$, the sequence $\{f^n(\vartheta)\} \to \varsigma^* \in E$.

Corollary 3. *Let $E \neq \phi$ be a O-set with binary relation $\perp$ defined on E and mapping $\Gamma : E \times E \to \mathbb{R}^+$ such that pair (E, Γ) is O-complete M-metric space. If self map f is $\perp$-continuous and $\perp$-preserving on E, meets with following exiom:*
for all $\epsilon > 0$, $\exists\delta > 0$ such that

$$\begin{aligned} &\epsilon \leq \phi(\max\{\frac{(1+\Gamma(\vartheta, f\varsigma))\Gamma(\varsigma, f\varsigma)}{1+\Gamma(\vartheta,\varsigma)}, \Gamma(\vartheta,\varsigma)\}) < \epsilon + \delta(\epsilon) \\ &\Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon, \end{aligned}$$

for all $\vartheta, \varsigma \in E$, with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$ or $\varsigma \neq f\varsigma$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotonic increasing function such that $\phi(t) < kt$, for all $t > 0$, $k \in (0,1)$ and $\phi(0) = 0$. Then $\exists$ a unique point $\varsigma^* \in E$ such that $f\varsigma^* = \varsigma^*$. In addition, for all $\vartheta \in E$, the sequence $\{f^n(\vartheta)\} \to \varsigma^* \in E$

Corollary 4. *Let $E \neq \phi$ be a O-set with binary relation $\perp$ defined on E and mapping $\Gamma : E \times E \to \mathbb{R}^+$ such that pair (E, Γ) is O-complete M-metric space. If self map f is $\perp$-continuous and $\perp$-preserving on E, meets with following exiom:*
for all $\epsilon > 0$, $\exists\delta > 0$ such that

$$\begin{aligned} &\epsilon \leq \phi(\frac{(1+\Gamma(\vartheta, f\varsigma))\Gamma(\varsigma, f\varsigma)}{1+\Gamma(\vartheta,\varsigma)} + \Gamma(\vartheta,\varsigma)) < \epsilon + \delta(\epsilon) \\ &\Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon, \end{aligned}$$

for all $\vartheta, \varsigma \in E$, with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$ or $\varsigma \neq f\varsigma$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous monotonic increasing function such that $\phi(t) < kt$, for all $t > 0$, $k \in (0,1)$ and $\phi(0) = 0$. Then $\exists$ a unique point $\varsigma^* \in E$ such that $f\varsigma^* = \varsigma^*$. In addition, for all $\vartheta \in E$, the sequence $\{f^n(\vartheta)\} \to \varsigma^* \in E$.

Corollary 5. *Let $E \neq \phi$ be a O-set with binary relation $\perp$ defined on E and mapping $\Gamma : E \times E \to \mathbb{R}^+$ such that pair (E, Γ) is O-complete M-metric space. If self map f is $\perp$-continuous and $\perp$-preserving on E, meets with following exiom:*
for all $\epsilon > 0$, $\exists\delta > 0$ such that

$$\begin{aligned} &\epsilon \leq \alpha(\frac{(1+\Gamma(\vartheta, f\varsigma))\Gamma(\varsigma, f\varsigma)}{1+\Gamma(\vartheta,\varsigma)} + \Gamma(\vartheta,\varsigma)) < \epsilon + \delta(\epsilon) \\ &\Rightarrow \Gamma(f\vartheta, f\varsigma) < \epsilon, \end{aligned}$$

for all $\vartheta, \varsigma \in E$ with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$ or $\varsigma \neq f\varsigma$, where $\alpha \in (0, \frac{k}{2})$ and $k \in (0,1)$. Then $\exists$ a unique point $\varsigma^* \in E$ such that $f\varsigma^* = \varsigma^*$. In addition, for all $\vartheta \in E$, the sequence $\{f^n(\vartheta)\} \to \varsigma^* \in E$.

4. Fixed Points for Integral Type Contractions

Theorem 3. *Let (E, Γ) be a O-complete M-metric space and let selfmap f is $\perp$-preserving on E. Consider that $\exists$ a map $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following:*

(i) $h(0) = 0, t > 0 \Rightarrow h(t) > 0,$
(ii) *h is non-decreasing and right continuous,*
(iii) *for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\epsilon \leq h(\mathbb{J}(\vartheta, \varsigma)) \leq \epsilon + \delta \Rightarrow h(\Gamma(f\vartheta, f\varsigma)) < \epsilon \tag{11}$$

for all $\vartheta, \varsigma \in E$, with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$, where

$$\mathbb{J}(\vartheta, \varsigma)) = \phi(\max\{\frac{(1+\Gamma(\vartheta, f\vartheta))(\Gamma(\varsigma, f\varsigma))}{(1+\Gamma(\vartheta, \varsigma))}, \frac{(\Gamma(\vartheta, f\vartheta))(\Gamma(\varsigma, f\varsigma))}{\Gamma(\vartheta, \varsigma))}, \Gamma(\vartheta, \varsigma))\}$$

and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is monotonic continuous increasing function such that $\phi(t) < kt, k \in (0,1), t > 0$ and $\phi(0) = 0$. Then Equation (8) is satisfied.

Proof. Fix $\epsilon > 0$, so $h(\epsilon) > 0$. Hence, by (11), there exist $\delta > 0$ such that

$$\text{for all } \vartheta, \varsigma \text{ with } \vartheta \perp \varsigma, \vartheta \neq \varsigma, h(\epsilon) \leq h(\mathbb{J}(\vartheta, \varsigma)) < h(\epsilon) + \delta$$

$$\Rightarrow h(\Gamma(f\vartheta, f\varsigma)) < h(\epsilon).$$

According to right continuity of h, there exist $\delta > 0$ such that

$$h(\epsilon + \delta_1) < h(\epsilon) + \delta,$$

fix $\vartheta, \varsigma \in E$ with $\vartheta \neq \varsigma$ such that

$$\epsilon \leq \mathbb{J}(\vartheta, \varsigma) < \epsilon + \delta.$$

Since h is a non-decreasing mapping, we have

$$h(\epsilon) \leq h(\mathbb{J}(\vartheta, \varsigma)) < h(\epsilon + \delta_1) < h(\epsilon) + \delta$$

so, we have

$$h(\Gamma(f\vartheta, f\varsigma)) < h(\epsilon)$$

which implies that

$$\Gamma(f\vartheta, f\varsigma) < \epsilon.$$

□

Corollary 6. *Let (E, Γ) be a orthogonal M-metric space and let self-map f is $\perp$-preserving on E and a locally integrable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with*

(i) $\int_0^t h(s)ds > 0$ *for* $t > 0$;

(ii) *and for each $\epsilon > 0$, $\exists\, \delta > 0$ satisfying*

$$\epsilon \leq \int_0^{\mathbb{J}(\vartheta, \varsigma)} h(s)ds < \epsilon + \delta$$

$$\Rightarrow \int_0^{\Gamma(f\vartheta, f\varsigma)} h(s)ds < \epsilon$$

for all $\vartheta, \varsigma \in E$ with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$. Then Equation (8) is satisfied.

Corollary 7. *Let $\mathbb{D} = \{O | O : [0, \infty) \to [0, \infty)\}$ such that*

(i) *O is continuous and non-decreasing,*

(ii) *$O(0) = 0$ and $O(t) > 0$, for all $t > 0$.*

Consider (E, Γ) be a $M_\perp$-metric space and let self-map f is $\perp$-preserving on E. Assume for every $\epsilon > 0$, $\exists\, \delta(\epsilon)$ satisfying

$$\epsilon \leq O(\mathbb{J}(\vartheta, \varsigma)) < \epsilon + \delta(\delta)$$

$$\Rightarrow O(\Gamma(f\vartheta, f\varsigma)) < \epsilon$$

for all $\vartheta, \varsigma \in E$ with $\vartheta \perp \varsigma$ and $\vartheta \neq \varsigma$, where $O \in \mathbb{D}$. Then Equation (8) is satisfied.

Corollary 8. *Let (E,Γ) be a O-complete M-metric space and let self map f is $\perp$-preserving on E. Let a locally integrable function $h:[0,\infty)\to[0,\infty)$ satisfying*

$$\int_0^t h(s)ds>0,\forall t>0.$$

Consider that f meets the undermentioned condition for all $\vartheta,\varsigma\in E$ with $\vartheta\perp\varsigma$ and $\vartheta\neq\varsigma$

$$\int_0^{\Gamma(f\vartheta,f\varsigma)} h(t)dt\leq\mu\int_0^{\mathbb{J}(\vartheta,\varsigma)} h(t)dt,$$

where $\mu\in(0,1)$. Then f has a unique fixed point $\varsigma^*\in E$. Moreover, for any $\vartheta\in E$, the sequence $\{f^n\}$ converges to ς^*.

Proof. Let $\epsilon>0$, it is easy to observe that (8) is satisfied for $\delta(\epsilon)=\epsilon(\frac{1}{\mu}-1)$. Then (11) is holds and this completes the proof. □

5. Applications

Consider the following integral equation

$$u(t)=\int_a^b H(t,s,u(s))ds+g(t),\forall t\in[a,b] \tag{12}$$

where $t\in[a,b]$, $H:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ and $b>a\geq0$.

In this part of study, we will propose an existence theorem for a solution of Equation (12) that belongs to $E=(C[a,b],\mathbb{R})$, set of all continuous function defined on $I=[a,b]$ by using the obtained main result in (8). Consider

$$(fu)(t)=\int_a^b H(t,s,u(s))ds+g(t),u\in E,t\in[a,b].$$

The existence of solution of (12) is equivalent to the existence of a fixed point of f. It is well known that E endowed with M-metric defined by $\Gamma=\sup_{t\in I}|u(t)-v(t)|$, for all $u,v\in E$ with $u\perp v$ if $u(t)\leq v(t)$, for all $t\in[a,b]$ forms a O-complete M-metric space. Suppose that the following condition holds:

(i) $K:[a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ is continous and $g:[a,b]\to\mathbb{R}$;

(ii) $H(t,s,.)>0$ and $\int_a^b H(t,s,.):\mathbb{R}\to\mathbb{R}$ is increasing for all $t,s\in I$;

(iii) for all $u,v\in E,s,t\in I$ and $\alpha\in(0,1),$ we have

$$\Big|H(t,s,u(s))-H(t,s,v(s))\Big|^2\leq\frac{\alpha^2}{4(b-a)^2}\Big(\Big|u(s)-v(s)\Big|\Big)^2.$$

Theorem 4. *Assume that condition (i)–(iii) are satisfied. Then integral Equation (12) has unique solution.*

Proof. As $H(t,s,.)>0$ for all $t,s\in[a,b]$, we have

$$\begin{aligned}(fu)(t)&=\int_a^b H(t,s,u(s))ds+g(t),\\&\leq\int_a^b H(t,s,v(s))ds+g(t),\\&=(fv)(t).\end{aligned}$$

Hence, f is $\perp$-preserving. Let $\{\mu_n\}$ be an O-Cauchy sequence converging to $\mu \in E$. Then

$$\mu_0(t) \leq \mu_1(t) \leq \mu_2(t) \leq \mu_3(t) \leq \ldots\ldots.\mu_n(t) \leq \ldots\ldots \leq \mu(t), \text{for all } t \in [0,\lambda],$$

this implies that $\mu_n \perp \mu$, for all $t \in [0,\lambda]$. As f is $\perp$-preserving then, $f(\mu_n) \to f(\mu)$. Therefore, f is O-continous. Now for $u, v \in E$, we have

$$\begin{aligned}
\Big|(fu)(t) - (fv)(t)\Big|^2 &\leq \Big| \int_a^b \Big(H(t,s,u(s)) - H(t,s,v(s))\Big) ds \Big|^2 \\
&\leq \int_a^b 1^2 ds \int_a^b \Big|\Big(H(t,s,u(s)) - H(t,s,v(s))\Big)\Big|^2 ds \\
&\leq (b-a) \int_a^b \Big|\Big(H(t,s,u(s)) - H(t,s,v(s))\Big)\Big|^2 ds \\
&\leq (b-a) \frac{\alpha^2}{4(b-a)^2} \int_a^b \Big|u(s) - v(s)\Big|^2 ds \\
&\leq \Big[\frac{\alpha}{2}\Gamma(u,v)\Big]^2
\end{aligned}$$

$$\leq \Big[\frac{\alpha}{2} \max\Big\{\frac{(1+\Gamma(u,fu))(\Gamma(v,fv))}{(1+\Gamma(u,v))}, \frac{(\Gamma(u,fu))(\Gamma(v,fv))}{\Gamma(u,v)}, \Gamma(u,v))\Big\}\Big]^2,$$

and so

$$\Gamma(fu,fv) \leq \Big[\frac{\alpha}{2} \max\Big\{\frac{(1+\Gamma(u,fu))(\Gamma(v,fv))}{(1+\Gamma(u,v))}, \frac{(\Gamma(u,fu))(\Gamma(v,fv))}{\Gamma(u,v)}, \Gamma(u,v))\Big\}\Big].$$

Hence, by Equation (8) integral Equation (12) has unique solution. □

6. Conclusions

In this paper, we proved fixed point results for a Meir–Keeler type contraction due to orthogonal M-metric spaces. An application of the main theorem was established in proving theorems for some integral equations and integral-type contractive conditions. The fact has been substantially furnished with examples. Further, some problems can be studied for these types of contractions. For example, the fixed-circle problem can be studied using these new contractions on different generalized metric spaces. We hope that the results examined in this paper will contribute significantly and scientifically to the theoy of fixed point and will help researchers to further advance their research work in the field of fixed point theory.

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