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Symmetry Analysis of a Model of Option Pricing and Hedging

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Abstract: The Guéant and Pu model of option pricing and hedging, which takes into account transaction costs, and the impact of operations on the market is studied by group analysis methods. The infinite-dimensional continuous group of equivalence transforms of the model is found. It is applied to get the group classification of the model under consideration. In addition to the general case, the classification contains three specifications of a free element in the equation, which correspond to models with groups of symmetries of a special kind. Optimal systems of subalgebras for some concrete models from the obtained classification are derived and used for the calculation of according invariant submodels.

Keywords: Guéant and Pu model; option pricing; group analysis of differential equations; symmetry analysis; symmetry group; Lie algebra; equivalence transformation; group classification; optimal system of subalgebras; invariant submodel

MSC: 35B06; 35Q91; 70G65

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1. Introduction

The classical Black–Scholes model [1,2] of option pricing dynamics is based on the perfect market hypothesis. Under this hypothesis, there are no execution costs and market participants use only the prevailing market prices and cannot influence the prices by their operations. The Black–Scholes model gives useful results when the underlying asset is liquid and the transaction amount is not too large for the market. However, the perfect market hypothesis contradicts to the market practice in many aspects, this fact makes the classical model too limited in application.

Last decades many researchers actively studied improvements of the classical Black–Scholes model, which would take into account the market illiquidity and the impact of transactions on prices. One may see works of Magill and Constantinides [3], Kyle [4], Leland [5], Cvitanić and Karatzas [6], Barles and Soner [7], Grossman [8], Platen and Schweizer [9], Sircar and Papanicolaou [10], Schönbucher and Wilmott [11], Bank and Baum [12], Çetin, Jarrow and Protter ([13] Section 4), Çetin and Rogers ([14] Section 6), Rogers and Singh [15].

New models proposed in these works have been investigated by many researchers both numerically and analytically. The work of Ibragimov and Gazizov [16] contains the first analytical investigation of the Black–Scholes equation by the group analysis methods [17,18].

Note that differential equations play a major role in solving problems of modeling processes and phenomena of the surrounding world and their importance does not decrease over the years (see, e. g., recent papers [19–21]). The group analysis (or symmetry analysis) of differential equations is one of the few theories that provide methods for finding exact solutions to nonlinear differential equations and systems of equations of wide classes. Since the middle of the 20th century, a huge number of results have been obtained on the group structure and exact solutions of many equations and systems of equations encountered in describing the dynamics of various physical processes, especially in gas dynamics, elasticity theory, etc. (see [17,18,22,23] and the bibliographies therein).

Let us determine that, following the classical works [17,18], we call a symmetry group of a differential equation a one-parameter group of transformations of independent and dependent variables. After such transformations, the differential equation under study does not change its form in the new variables. Each such group uniquely corresponds to a first-order differential operator, and these operators form a Lie algebra of the differential equation under consideration. The commutator is the Lie multiplication in such algebra. Therefore, when talking about the symmetry groups of a differential equation or about its group structure, at the same time we are talking about the Lie algebra of operators of the mentioned kind for this equation. Strict definitions of the listed objects are very cumbersome, they can be found by the reader in monographs [17,18] or in any other monograph on group analysis of differential equations.

In the last decade, in the works of Bordag [24,25], of Dyshaev and Fedorov [26–32] group properties of various nonlinear Black–Scholes type models were studied, and their invariant solutions and submodels were calculated. In the papers of Dyshaev and Fedorov, group classifications for various classes of nonlinear Black–Scholes type models were obtained.

Guéant and Pu in [33,34] carried out an analysis of options pricing taking into account transaction costs and the impact of operations on the market under the next assumptions:

(1) the risk-free rate r, the absolute risk aversion parameter γ and the volatility σ are constant;
(2) the process of market trading volume V_t is deterministic, non-negative, and bounded;
(3) there exists a maximum degree of participation ρ_m, i.e., processes ν are such that $|\nu_t| \leq \rho_m V_t$ almost everywhere;
(4) the number of shares in the hedged portfolio is $q_t = q_0 + \int_0^t \nu_s ds$;
(5) the price process is modeled by the stochastic differential equation $dS_t = \mu dt + \sigma dW_t$, where μ is the expected return of the underlying asset;
(6) to model execution costs, a continuous, non-negative, even, strictly convex function $L : \mathbb{R} \to \mathbb{R}_+$ is used, which is increasing on $\mathbb{R}_+$, $L(0) = 0$, and coercive, i.e., $\lim_{\rho\to+\infty} L(\rho)/\rho = +\infty$;
(7) the dynamics of the account X is described by the equation $dX_t = rX_t dt - \nu_t S_t dt - V_t L(\nu_t/V_t)dt$.

As result, Guéant and Pu derived a differential equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{1}{2}\sigma^2\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + V_t H(\theta_q), \tag{1}$$

where $H(p) = \sup\limits_{|\rho|\leq\rho_m} [p\rho - L(\rho)]$. It is a model of the dynamics of the indifference price $\theta(t, S, q)$ for a call option. This price depends on time t, the price of the underlying asset S, and the number of shares in the hedged portfolio q.

In [35], the group classification of the Guéant—Pu model (1) with a constant market trading volume V_t is obtained, and for all specifications of the free element H from the classification optimal systems of subalgebras of the Lie algebra is found, invariant solutions and submodels for subalgebras from the optimal systems are derived.

In the present paper, the Guéant—Pu model

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{1}{2}\sigma^2\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F(t, \theta_q) \tag{2}$$

is investigated. Here a free element F depends on t and θ_q, i.e., the market trading volume V_t may depend on t, in contrast to the model, which is considered in [35]. In the Section 2, the continuous group of equivalence transformations of Equation (2) is calculated. It correponds to an infinite-dimensional Lie algebra of the equation with three-dimensional finite part and with two basis operators, which coefficients are defined by two arbitrary functions of t and by their derivatives. In the Section 3 the search of the symmetry groups for general Equation (2) started. The equivalence transformations are used in the Section 4 for the search of the specifications of the free element F, such that $F_{\theta_q\theta_q} \not\equiv 0$, which corresponds to equations of form (2) with different Lie algebras. The obtained theorem on group classification is formulated in the Section 5. In the Section 6, optimal systems of subalgebras are found for the Lie algebra of model (2) with a general function $F(t,\theta_q)$ and with a specification $F = e^{rt}\Phi(\theta_q)$, which were obtained in the group classification. For every subalgebra from the optimal system, the invariant submodel of the Guéant—Pu model is calculated, if it exists.

2. Continuous Groups of Equivalence Transformations

Consider the Gueant—Pu Equation (2), where $\theta = \theta(t,S,q)$, $F(t,\theta_q)$ is a free element. Assume that $r\gamma\sigma\mu \neq 0$, $T > 0$. For the search of continouos equivalence transformations groups of Equation (2) we will consider the function F and all its derivatives as additional variables. Generators of such groups have a form

$$Y = \tau\partial_t + \xi\partial_S + \alpha\partial_q + \eta\partial_\theta + \zeta\partial_F,$$

where τ, ξ, α, η depend on t, S, q, θ, and ζ depends on t, S, q, θ, F, θ_t, θ_S, θ_q. Hereafter $\partial_\beta := \frac{\partial}{\partial\beta}$ is the partial derivative with respect to a variable β. Equation (2) with a new variable F we will consider in the system with additional equations

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{1}{2}\sigma^2\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F, \tag{3}$$

$$F_S = 0, \quad F_q = 0, \quad F_\theta = 0, \quad F_{\theta_t} = 0, \quad F_{\theta_S} = 0, \tag{4}$$

which show the dependence of F on t and θ_q only. System (3), (4) is considered as a manifold $\mathcal{M}$ in the expanded space of the corresponding variables. Let us act by the prolongated operator [17,18]

$$\underset{2}{Y} = Y + \eta^t\partial_{\theta_t} + \eta^S\partial_{\theta_S} + \eta^q\partial_{\theta_q} + \eta^{SS}\partial_{\theta_{SS}} + \zeta^t\partial_{F_t} + \zeta^S\partial_{F_S} + \zeta^q\partial_{F_q} + \zeta^\theta\partial_{F_\theta} + \\ + \zeta^{\theta_t}\partial_{F_{\theta_t}} + \zeta^{\theta_S}\partial_{F_{\theta_S}} + \zeta^{\theta_q}\partial_{F_{\theta_q}}$$

on the both sides of Equation (3). In order to use the geometric invariance criterion [17], we restrict the result on the manifold $\mathcal{M}$ and obtain the equation

$$\begin{aligned} &\eta^t - r\eta - (\mu - rS)\alpha + rq\xi + \mu\eta^S + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q)\left(\eta^S - \alpha - \frac{r}{2}(\theta_S - q)\tau\right) - \\ &-\zeta + \frac{1}{2}\sigma^2\eta^{SS}|_{\mathcal{M}} = \eta^t - r\eta + rS\alpha + rq\xi + (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\eta^S - \alpha) - \\ &-\frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau - \zeta + \frac{1}{2}\sigma^2\eta^{SS}|_{\mathcal{M}} = 0. \end{aligned} \tag{5}$$

The coefficients of the prolongated operator $\underset{2}{Y}$ are calculated using the total derivatives operators

$$D_t = \frac{\partial}{\partial t} + \theta_t\frac{\partial}{\partial\theta} + \ldots, \quad D_S = \frac{\partial}{\partial S} + \theta_S\frac{\partial}{\partial\theta} + \theta_{SS}\frac{\partial}{\partial\theta_S} + \ldots, \quad D_q = \frac{\partial}{\partial q} + \theta_q\frac{\partial}{\partial\theta} + \ldots,$$

$$\tilde{D}_t = \frac{\partial}{\partial t} + F_t\frac{\partial}{\partial F} + \ldots, \quad \tilde{D}_S = \frac{\partial}{\partial S} + F_S\frac{\partial}{\partial F} + \ldots, \quad \tilde{D}_q = \frac{\partial}{\partial q} + F_q\frac{\partial}{\partial F} + \ldots,$$

$$\tilde{D}_\theta = \frac{\partial}{\partial\theta} + F_\theta\frac{\partial}{\partial F} + \ldots, \quad \tilde{D}_{\theta_t} = \frac{\partial}{\partial\theta_t} + F_{\theta_t}\frac{\partial}{\partial F} + \ldots, \quad \tilde{D}_{\theta_S} = \frac{\partial}{\partial\theta_S} + F_{\theta_S}\frac{\partial}{\partial F} + \ldots$$

and the prolongation formulas (see detail in [17,18])

$$\begin{gathered}
\eta^t = D_t\eta - \theta_t D_t\tau - \theta_S D_t\xi - \theta_q D_t\alpha, \quad \eta^S = D_S\eta - \theta_t D_S\tau - \theta_S D_S\xi - \theta_q D_S\alpha, \\
\eta^q = D_q\eta - \theta_t D_q\tau - \theta_S D_q\xi - \theta_q D_q\alpha, \quad \eta^{SS} = D_S\eta^S - \theta_{St} D_S\tau - \theta_{SS} D_S\xi - \theta_{Sq} D_S\alpha, \\
\zeta^S = \tilde{D}_S\zeta - F_t\tilde{D}_S\tau - F_S\tilde{D}_S\xi - F_q\tilde{D}_S\alpha - F_\theta\tilde{D}_S\eta - F_{\theta_t}\tilde{D}_S\eta^t - F_{\theta_S}\tilde{D}_S\eta^S - F_{\theta_q}\tilde{D}_S\eta^q, \\
\zeta^q = \tilde{D}_q\zeta - F_t\tilde{D}_q\tau - F_S\tilde{D}_q\xi - F_q\tilde{D}_q\alpha - F_\theta\tilde{D}_q\eta - F_{\theta_t}\tilde{D}_q\eta^t - F_{\theta_S}\tilde{D}_q\eta^S - F_{\theta_q}\tilde{D}_q\eta^q, \\
\zeta^\theta = \tilde{D}_\theta\zeta - F_t\tilde{D}_\theta\tau - F_S\tilde{D}_\theta\xi - F_q\tilde{D}_\theta\alpha - F_\theta\tilde{D}_\theta\eta - F_{\theta_t}\tilde{D}_\theta\eta^t - F_{\theta_S}\tilde{D}_\theta\eta^S - F_{\theta_q}\tilde{D}_\theta\eta^q, \\
\zeta^{\theta_t} = \tilde{D}_{\theta_t}\zeta - F_t\tilde{D}_{\theta_t}\tau - F_S\tilde{D}_{\theta_t}\xi - F_q\tilde{D}_{\theta_t}\alpha - F_\theta\tilde{D}_{\theta_t}\eta - F_{\theta_t}\tilde{D}_{\theta_t}\eta^t - F_{\theta_S}\tilde{D}_{\theta_t}\eta^S - F_{\theta_q}\tilde{D}_{\theta_t}\eta^q, \\
\zeta^{\theta_S} = \tilde{D}_{\theta_S}\zeta - F_t\tilde{D}_{\theta_S}\tau - F_S\tilde{D}_{\theta_S}\xi - F_q\tilde{D}_{\theta_S}\alpha - F_\theta\tilde{D}_{\theta_S}\eta - F_{\theta_t}\tilde{D}_{\theta_S}\eta^t - F_{\theta_S}\tilde{D}_{\theta_S}\eta^S - F_{\theta_q}\tilde{D}_{\theta_S}\eta^q.
\end{gathered}$$

The result of the action of $\underset{2}{Y}$ on Equation (4) after restricting on the manifold $\mathcal{M}$ gives

$$\begin{gathered}
\zeta^S|_{\mathcal{M}} = \zeta_S - F_t\tau_S - F_{\theta_q}\eta^q_S|_{\mathcal{M}} = \zeta_S - F_t\tau_S - F_{\theta_q}(\eta_{Sq} + \theta_q\eta_{S\theta} - \theta_t(\tau_{Sq} + \theta_q\tau_{S\theta}) - \\
-\theta_S(\xi_{Sq} + \theta_q\xi_{S\theta}) - \theta_q(\alpha_{Sq} + \theta_q\alpha_{S\theta}))|_{\mathcal{M}} = 0, \\
\zeta^q|_{\mathcal{M}} = \zeta_q - F_t\tau_q - F_{\theta_q}\eta^q_q|_{\mathcal{M}} = \zeta_q - F_t\tau_q - F_{\theta_q}(\eta_{qq} + \theta_q\eta_{q\theta} - \theta_t(\tau_{qq} + \theta_q\tau_{q\theta}) - \\
-\theta_S(\xi_{qq} + \theta_q\xi_{q\theta}) - \theta_q(\alpha_{qq} + \theta_q\alpha_{q\theta}))|_{\mathcal{M}} = 0, \\
\zeta^\theta|_{\mathcal{M}} = \zeta_\theta - F_t\tau_\theta - F_{\theta_q}\eta^q_\theta|_{\mathcal{M}} = \zeta_\theta - F_t\tau_\theta - F_{\theta_q}(\eta_{q\theta} + \theta_q\eta_{\theta\theta} - \theta_t(\tau_{q\theta} + \theta_q\tau_{\theta\theta}) - \\
-\theta_S(\xi_{q\theta} + \theta_q\xi_{\theta\theta}) - \theta_q(\alpha_{q\theta} + \theta_q\alpha_{\theta\theta}))|_{\mathcal{M}} = 0 \\
\zeta^{\theta_t}|_{\mathcal{M}} = \zeta_{\theta_t} - F_{\theta_q}\eta^q_{\theta_t}|_{\mathcal{M}} = \zeta_{\theta_t} + F_{\theta_q}(\tau_q + \theta_q\tau_\theta)|_{\mathcal{M}} = 0, \\
\zeta^{\theta_S}|_{\mathcal{M}} = \zeta_{\theta_S} - F_{\theta_q}\eta^q_{\theta_S}|_{\mathcal{M}} = \zeta_{\theta_S} + F_{\theta_q}(\xi_q + \theta_q\xi_\theta)|_{\mathcal{M}} = 0.
\end{gathered} \tag{6}$$

The transition on the manifold $\mathcal{M}$ means the substitution for θ_t the right-hand side of (3) and vanishing of variables F_S, F_q, F_θ, F_{θ_t}, F_{θ_S}. It does not change the form of the last two equations in (6). Therefore, the separation of variables F_{θ_q} and θ_q gives $\zeta_{\theta_t} = 0$, $\zeta_{\theta_S} = 0$, $\tau_q = 0$, $\tau_\theta = 0$, $\xi_q = 0$, $\xi_\theta = 0$.

We substitute the prolongation formulas into Equation (5) and after the transition on $\mathcal{M}$ equate the result to zero:

$$\begin{gathered}
\eta_t - \theta_S\xi_t - \theta_q\alpha_t - r\eta + rS\alpha + rq\xi - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau - \zeta + \\
+(\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\eta_S + \theta_S\eta_\theta - \theta_S\xi_S - \theta_q(\alpha_S + \theta_S\alpha_\theta) - \alpha) + \\
+\frac{1}{2}\sigma^2(\eta_{SS} + 2\theta_S\eta_{S\theta} + \theta_S^2\eta_{\theta\theta} - 2\theta_{St}\tau_S + \theta_{SS}(\eta_\theta - \theta_q\alpha_\theta - 2\xi_S) - \\
-2\theta_{Sq}(\alpha_S + \theta_S\alpha_\theta) - \theta_S\xi_{SS} - \theta_q(\alpha_{SS} + 2\theta_S\alpha_{S\theta} + \theta_S^2\alpha_{\theta\theta})) + \\
+\left(r\theta + (\mu - rS)q - \mu\theta_S - \frac{1}{2}\sigma^2\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F\right)\times \\
\times\left(\eta_\theta - \tau_t - \theta_q\alpha_\theta - (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))\tau_S - \frac{\sigma^2}{2}\tau_{SS}\right) = 0.
\end{gathered} \tag{7}$$

Since all the functions in (7) do not depend on θ_{St}, θ_{Sq}, equate to zero the coefficients in Equation (7) at θ_{St}, θ_{Sq} and obtain the equalities $\tau_S = 0$, $\alpha_S + \theta_S\alpha_\theta = 0$; since α and its derivatives does not depend on θ_S, we have $\alpha_S = \alpha_\theta = 0$. Thus,

$$\tau_S = 0,\ \tau_q = 0,\ \tau_\theta = 0,\ \xi_q = 0,\ \xi_\theta = 0,\ \alpha_S = 0,\ \alpha_\theta = 0,\ \zeta_{\theta_t} = 0,\ \zeta_{\theta_S} = 0. \tag{8}$$

The first 3 equations in (6) now have the form

$$\zeta^S|_{\mathcal{M}} = \zeta_S - F_{\theta_q}(\eta_{Sq} + \theta_q\eta_{S\theta}) = 0, \quad \zeta^q|_{\mathcal{M}} = \zeta_q - F_{\theta_q}(\eta_{qq} + \theta_q\eta_{q\theta} - \theta_q\alpha_{qq}) = 0,$$
$$\zeta^\theta|_{\mathcal{M}} = \zeta_\theta - F_{\theta_q}(\eta_{q\theta} + \theta_q\eta_{\theta\theta}) = 0.$$

The separation of the variables F_{θ_q} and θ_q here gives

$$\begin{gathered} \eta_{S\theta} = 0, \quad \eta_{Sq} = 0, \quad \alpha_{qq} = \eta_{q\theta} = 0, \quad \eta_{qq} = 0, \quad \eta_{\theta\theta} = 0, \\ \zeta_S = 0, \quad \zeta_q = 0, \quad \zeta_\theta = 0, \quad \zeta_{\theta_t} = 0, \quad \zeta_{\theta_S} = 0. \end{gathered} \tag{9}$$

By substitution into Equation (7) equalities (8) and (9) we get

$$\begin{aligned} \eta_t - \theta_S\xi_t - \theta_q\alpha_t - r\eta + rS\alpha + rq\xi + (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\eta_S + \theta_S\eta_\theta - \theta_S\xi_S - \alpha) - \\ -\frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau - \zeta + \frac{1}{2}\sigma^2(\eta_{SS} + \theta_{SS}(\eta_\theta - 2\xi_S) - \theta_S\xi_{SS}) + \\ +(\eta_\theta - \tau_t)\left(r\theta + (\mu - rS)q - \mu\theta_S - \frac{1}{2}\sigma^2\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F\right) = 0. \end{aligned}$$

We separate this equation by the variables θ_{SS}, θ_S, since ζ does not depend on them in view of (9), and after a reduction we obtain

$$\theta_{SS} : 2\xi_S = \tau_t, \quad \theta_S^2 : \eta_\theta - 2\xi_S - r\tau + \tau_t = 0, \tag{10}$$

$$\theta_S : \gamma\sigma^2 e^{r(T-t)}(rq\tau + q\xi_S - \alpha + \eta_S - q\tau_t) - \mu\xi_S + \mu\tau_t - \xi_t - \frac{\sigma^2}{2}\xi_{SS} = 0, \tag{11}$$

$$\begin{aligned} 1 : \eta_t - \theta_q\alpha_t - r\eta + rS\alpha + rq\xi + (\mu - \gamma\sigma^2 e^{r(T-t)}q)(\eta_S - \alpha) - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}q^2\tau - \\ -\zeta + \frac{\sigma^2}{2}\eta_{SS} + (\eta_\theta - \tau_t)\left(r\theta + (\mu - rS)q - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}q^2 + F\right) = 0. \end{aligned} \tag{12}$$

From $2\xi_S = \tau_t$ in (10) in view of $\tau_S = 0$ due to (8) we get $\xi_{SS} = 0$. Substitution $2\xi_S = \tau_t$ from the first equation to the second one in (10) yields $\eta_\theta = r\tau$. Now we differentiate (11) by q and using (8), (9) we get $r\tau + \xi_S - \alpha_q - \tau_t = 0$. Substitute $\xi_S = \tau_t/2$ from (10), then $\alpha_q = r\tau - \tau_t/2$. Next, by differentiating (11) by S and using (8) and (9), we obtain $\gamma\sigma^2 e^{r(T-t)}\eta_{SS} - \xi_{tS} = 0$, or $\gamma\sigma^2 e^{r(T-t)}\eta_{SS} = \tau_{tt}/2$. Therefore, $\eta_{SSS} = 0$. Thus,

$$\xi_{SS} = 0, \quad \alpha_q = r\tau - \frac{\tau_t}{2}, \quad \eta_\theta = r\tau, \quad \eta_{SS} = \frac{e^{r(t-T)}}{2\gamma\sigma^2}\tau_{tt}, \quad \eta_{SSS} = 0. \tag{13}$$

The differentiation of Equation (12) by S twice with the substitution of the vanishing functions from (8), (9) and (13) gives $\eta_{tSS} - r\eta_{SS} = 0$. The substitution of η_{SS} from (13) here leads to $\tau_{ttt} = 0$. By the differentiation of Equation (12) by θ_q we obtain $\alpha_t + \zeta_{\theta_q} = 0$. The differentiation by q gives $\alpha_{tq} = 0$, i.e., due to (13) $r\tau_t - \tau_{tt}/2 = 0$. Since $r \neq 0$, this differential equation implies the equality $\tau_{tt} = 0$, hence $\tau_t = 0$. Then due to (10), (13)

$$\tau_t = 0, \quad \alpha_q = r\tau, \quad \xi_S = 0, \quad \eta_{SS} = 0. \tag{14}$$

From (8), (9), (14) it follows that τ is a constant, $\xi = \xi(t)$, $\alpha = r\tau q + A(t)$. Substituting these equalities into (11) we get $\gamma\sigma^2 e^{r(T-t)}(\eta_S - A(t)) - \xi_t = 0$. Therefore, due to (9), (13), (14) $\eta = r\tau\theta + B(t)q + (A(t) + e^{r(t-T)}\xi'(t)/(\gamma\sigma^2))S + C(t)$. So,

$$\xi = \xi(t), \quad \alpha = r\tau q + A(t), \quad \eta = r\tau\theta + B(t)q + \left(A(t) + \frac{e^{r(t-T)}\xi'(t)}{\gamma\sigma^2}\right)S + C(t). \tag{15}$$

Note that arbitrary functions used to represent the solution of the considered system of equations here and further we denote with capital letters A, B, C, etc. Substituting equalities (15) into (12) and shortening we obtain

$$\begin{gathered}(B'-rB+r\xi-\xi')q+C'-rC+\frac{\mu}{\gamma\sigma^2}e^{r(t-T)}\xi'+\\+\left(A'+\frac{e^{r(t-T)}}{\gamma\sigma^2}\xi''\right)S-A'\theta_q-\zeta+r\tau F=0.\end{gathered}\tag{16}$$

The differentiation by q of Equation (16) implies that $B'-rB+r\xi-\xi'=0$, hence $B(t)=\xi(t)+De^{rt}$. Next, differentiate by S Equation (16) and obtain

$$B=\xi+De^{rt},\quad A=-\frac{e^{-rT}}{\gamma\sigma^2}\int_{t_0}^{t}e^{rs}\xi''(s)ds+c.\tag{17}$$

We substitute (17) into (16) and (15) and get

$$\begin{gathered}\xi=\xi(t),\quad \alpha=r\tau q-\frac{e^{-rT}}{\gamma\sigma^2}\int_{t_0}^{t}e^{rs}\xi''(s)ds+c,\\ \eta=r\tau\theta+(\xi(t)+De^{rt})q+\frac{e^{-rT}}{\gamma\sigma^2}\left(e^{rt}\xi'(t)-\int_{t_0}^{t}e^{rs}\xi''(s)ds\right)S+cS+C(t),\\ \zeta=C'(t)-rC(t)+\frac{\mu}{\gamma\sigma^2}e^{r(t-T)}\xi'(t)+\frac{e^{r(t-T)}}{\gamma\sigma^2}\xi''(t)\theta_q+r\tau F.\end{gathered}$$

We formulate this result in the form of a theorem. To do this, we fix alternately one of the integration constants or an arbitrary function, equating the others to zero, and obtain the corresponding basic operator of the resulting Lie algebra. By Y_k we denote here these basis operators. If coefficients of an operator are defined by a function and its derivatives, we use this function denotation as the lower index for such operator.

Theorem 1. *The Lie algebra of continuous equivalence transformations for Equation* (2) *is generated by the operators*

$$\begin{gathered}Y_1:=e^{rt}q\partial_\theta,\quad Y_2:=\partial_q+S\partial_\theta,\quad Y_3:=\partial_t+rq\partial_q+r\theta\partial_\theta+rF\partial_F,\\ Y_\phi:=\phi(t)\partial_\theta+(\phi'(t)-r\phi(t))\partial_F,\quad Y_\psi:=\psi(t)\partial_S-\frac{e^{-rT}}{\gamma\sigma^2}\int_{t_0}^{t}e^{rs}\psi''(s)ds\partial_q+\\ +\left(\psi(t)q+\frac{e^{-rT}}{\gamma\sigma^2}\left(e^{rt}\psi'(t)-\int_{t_0}^{t}e^{rs}\psi''(s)ds\right)S\right)\partial_\theta+\\ +\left(\frac{\mu}{\gamma\sigma^2}e^{r(t-T)}\psi'(t)+\frac{e^{r(t-T)}}{\gamma\sigma^2}\psi''(t)\theta_q\right)\partial_F.\end{gathered}$$

Solving the Lie equations for the obtained Lie algebras and taking the projections on the variables t, θ_q, F we get

$$\begin{gathered}Y_1:\bar{\theta}_q=\theta_q+a_1e^{rt};\quad Y_3:\bar{t}=t+a_3,\quad \bar{F}=e^{ra_1}F;\\ Y_\phi:\bar{F}=F-r\phi(t)+\phi'(t);\quad Y_\psi:\bar{\theta}_q=\theta_q+\psi(t),\\ \bar{F}=F+\frac{\mu}{\gamma\sigma^2}e^{r(t-T)}\psi'(t)+\frac{e^{r(t-T)}}{2\gamma\sigma^2}\psi(t)\psi''(t)+\frac{e^{r(t-T)}}{\gamma\sigma^2}\psi''(t)\theta_q.\end{gathered}\tag{18}$$

Remark 1. *A Lie algebra is called principal [17] for Equation* (2)*, if it is admissible for* (2) *with any specification of F. From* (18) *it follows that the principal Lie algebra of Equation* (2) *is generated by Y_2 and by Y_ϕ at $\phi(t)=e^{rt}$. Indeed, for such ϕ the group of transformations, which is generated by Y_ϕ, does not change t, θ_q and F.*

Remark 2. *We see that the Lie algebra of continuous equivalence transformations for Equation (2) is infinite-dimensional, since its operators depend on arbitrary functions ϕ and ψ. Note that such equation with a function F depending on θ_q only has a 5-dimensional Lie algebra of continuous equivalence transformations (see [35]). It generated by Y_2, Y_3, Y_ϕ for $\phi(t) \equiv 1$, Y_ϕ for $\phi(t) = e^{rt}$ and Y_ψ at $\psi(t) \equiv 1$.*

3. Calculation of the Symmetry Groups in General Case

Our purpose is to obtain the so-called group classification [17] for equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F(t, \theta_q). \tag{19}$$

For this aim, firstly we will search generators of the symmetry groups for the equation under general assumptions.

On Equation (19) we act by the second prolongation $\underset{2}{X} = X + \eta^q\partial_{\theta_q} + \eta^S\partial_{\theta_S} + \eta^t\partial_{\theta_t} + \eta^{SS}\partial_{\theta_{SS}}$ for a generator $X = \tau\partial_t + \xi\partial_S + \alpha\partial_q + \eta\partial_\theta$ of a continuous group of transformations, where functions τ, ξ, α, η depend on t, S, q, θ. So,

$$\begin{gathered}\eta^t - r\eta + rS\alpha + rq\xi + (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\eta^S - \alpha) - \\ -\frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau + \frac{\sigma^2}{2}\eta^{SS} - F_t\tau - F_{\theta_q}\eta^q|_{\mathcal{M}} = 0.\end{gathered} \tag{20}$$

After the substitution into (20) of the prolongation formulas and the restriction on the manifold $\mathcal{M}$, using the Equation (19) for θ_t, we obtain

$$\begin{gathered}\left(r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F\right)\times \\ \times\Big(\eta_\theta - \tau_t - \theta_S\xi_\theta - \theta_q\alpha_\theta - (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\tau_S + \theta_S\tau_\theta) + \\ + F_{\theta_q}(\tau_q + \theta_q\tau_\theta) - \frac{\sigma^2}{2}(\theta_{SS}\tau_\theta + \theta_S\tau_{SS} + 2\theta_S\tau_{S\theta} + \theta_S^2\tau_{\theta\theta})\Big) - \\ -\left(r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F\right)^2\tau_\theta + \\ +\eta_t - \theta_S\xi_t - \theta_q\alpha_t - r\eta + rS\alpha + rq\xi + \\ (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\eta_S + \theta_S\eta_\theta - \theta_S(\xi_S + \theta_S\xi_\theta) - \theta_q(\alpha_S + \theta_S\alpha_\theta) - \alpha) - \\ -\frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau - F_t\tau - F_{\theta_q}(\eta_q + \theta_q\eta_\theta - \theta_S(\xi_q + \theta_q\xi_\theta) - \theta_q(\alpha_q + \theta_q\alpha_\theta)) + \\ + \frac{\sigma^2}{2}(\eta_{SS} + 2\theta_S\eta_{S\theta} + \theta_S^2\eta_{\theta\theta} - 2\theta_{tS}(\tau_S + \theta_S\tau_\theta) + \theta_{SS}(\eta_\theta - \theta_q\alpha_\theta - 2\xi_S - 3\theta_S\xi_\theta) - \\ -2\theta_{Sq}(\alpha_S + \theta_S\alpha_\theta) - \theta_S(\xi_{SS} + 2\theta_S\xi_{S\theta} + \theta_S^2\xi_{\theta\theta}) - \theta_q(\alpha_{SS} + 2\theta_S\alpha_{S\theta} + \theta_S^2\alpha_{\theta\theta})) = 0.\end{gathered} \tag{21}$$

The differentiation of this equation by the variables θ_{Sq} and θ_{tS} leads to the equations $\tau_S = 0$, $\tau_\theta = 0$, $\alpha_S = 0$, $\alpha_\theta = 0$.

Equating the coefficient at θ_{SS} in (21) to zero, obtain $\xi_\theta = 0$, $\tau_t - 2\xi_S - F_{\theta_q}\tau_q = 0$, and using the equality $\tau_S = 0$ we get $\xi_{SS} = 0$. Therefore,

$$\tau_S = 0, \quad \tau_\theta = 0, \quad \alpha_S = 0, \quad \alpha_\theta = 0, \quad \xi_\theta = 0, \quad \xi_{SS} = 0, \quad \tau_t - 2\xi_S - F_{\theta_q}\tau_q = 0. \tag{22}$$

Applying these equalities in (21) we obtain the equality

$$\Big(r\theta + (\mu - rS)q - \mu\theta_S - \frac{1}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2 + F)(\eta_\theta - \tau_t + F_{\theta_q}\tau_q\Big) + \eta_t - \\ -\theta_S\xi_t - \theta_q\alpha_t - r\eta + (\mu + \gamma\sigma^2 e^{r(T-t)}(\theta_S - q))(\eta_S + \theta_S\eta_\theta - \theta_S\xi_S - \alpha) + \\ +rS\alpha + rq\xi - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}(\theta_S - q)^2\tau - F_t\tau - F_{\theta_q}(\eta_q + \theta_q\eta_\theta - \theta_S\xi_q - \theta_q\alpha_q) + \\ +\frac{1}{2}\sigma^2(\eta_{SS} + 2\theta_S\eta_{S\theta} + \theta_S^2\eta_{\theta\theta}) = 0.$$

We separate this equation by the variable θ_S taking into account the last equation from (22) and get the equations

$$\theta_S^2 : \gamma e^{r(T-t)}(\eta_\theta - r\tau) + \eta_{\theta\theta} = 0, \tag{23}$$

$$\theta_S : \gamma\sigma^2 e^{r(T-t)}(-q\xi_S + \eta_S - \alpha + rq\tau) + F_{\theta_q}\xi_q + \sigma^2\eta_{S\theta} - \xi_t + 2\mu\xi_S = 0, \tag{24}$$

$$1 : \Big(r\theta + (\mu - rS)q - \frac{\gamma\sigma^2}{2}e^{r(T-t)}q^2 + F\Big)(\eta_\theta - \tau_t + F_{\theta_q}\tau_q) + \eta_t - \theta_q\alpha_t - r\eta + \\ +rS\alpha + rq\xi + (\mu - \gamma\sigma^2 e^{r(T-t)}q)(\eta_S - \alpha) - \frac{r}{2}\gamma\sigma^2 e^{r(T-t)}q^2\tau - F_t\tau - \\ -F_{\theta_q}(\eta_q + \theta_q\eta_\theta - \theta_q\alpha_q) + \frac{\sigma^2}{2}\eta_{SS} = 0. \tag{25}$$

From (22) it follows that $\xi = A(t,q)S + B(t,q)$, Equation (23) implies that $\eta_\theta = r\tau + C_0(t,S,q)e^{-\gamma e^{r(T-t)}\theta}$. Therefore,

$$\xi = A(t,q)S + B(t,q), \quad \eta = r\theta\tau + C(t,S,q)e^{-\gamma e^{r(T-t)}\theta} + D(t,S,q). \tag{26}$$

Substitute these equalities into (24), (25) and get

$$\gamma\sigma^2 e^{r(T-t)}(-qA + D_S - \alpha + rq\tau) - A_tS - B_t + F_{\theta_q}(A_qS + B_q) + 2\mu A = 0, \tag{27}$$

$$\Big(r\theta + (\mu - rS)q - \frac{\gamma\sigma^2}{2}e^{r(T-t)}q^2 + F\Big)(r\tau - \gamma e^{r(T-t)}Ce^{-\gamma e^{r(T-t)}\theta} - 2A) + \\ +r\theta\tau_t + C_te^{-\gamma e^{r(T-t)}\theta} + r\gamma e^{r(T-t)}\theta Ce^{-\gamma e^{r(T-t)}\theta} + D_t - r^2\theta\tau - rCe^{-\gamma e^{r(T-t)}\theta} - rD - \\ -\theta_q\alpha_t + rS\alpha + rq(AS + B) + (\mu - \gamma\sigma^2 e^{r(T-t)}q)(C_Se^{-\gamma e^{r(T-t)}\theta} + D_S - \alpha) - \\ -\frac{r}{2}\gamma\sigma^2 e^{r(T-t)}q^2\tau - F_t\tau + \frac{\sigma^2}{2}(C_{SS}e^{-\gamma e^{r(T-t)}\theta} + D_{SS}) - \\ -F_{\theta_q}(r\theta\tau_q + C_qe^{-\gamma e^{r(T-t)}\theta} + D_q + \theta_q(r\tau - \gamma e^{r(T-t)}Ce^{-\gamma e^{r(T-t)}\theta}) - \theta_q\alpha_q) = 0.$$

In the last equation the variable θ is present explicitly, after the reduction of similar terms the equation has a form $a + be^{q\theta} = 0$, $q \neq 0$. Hence $a = b = 0$ and we have the equations

$$a = \Big((\mu - rS)q - \frac{\gamma\sigma^2}{2}e^{r(T-t)}q^2 + F\Big)(r\tau - 2A) + D_t - rD - \\ -\theta_q\alpha_t + rS\alpha + rq(AS + B) + (\mu - \gamma\sigma^2 e^{r(T-t)}q)(D_S - \alpha) - \\ -\frac{r}{2}\gamma\sigma^2 e^{r(T-t)}q^2\tau - F_t\tau + \frac{\sigma^2}{2}D_{SS} - F_{\theta_q}(D_q + r\theta_q\tau - \theta_q\alpha_q) = 0, \tag{28}$$

$$b = -\Big((\mu - rS)q - \frac{\gamma\sigma^2}{2}e^{r(T-t)}q^2 + F\Big)\gamma e^{r(T-t)}C + C_t - rC + \\ +(\mu - \gamma\sigma^2 e^{r(T-t)}q)C_S + \frac{\sigma^2}{2}C_{SS} - F_{\theta_q}(C_q - \gamma e^{r(T-t)}C\theta_q) = 0. \tag{29}$$

4. Calculation of the Group Classification in the Case $F_{\theta_q\theta_q} \neq 0$

Let us continue the calculations using the assumption $F_{\theta_q\theta_q} \neq 0$. Differentiating the last equation in (22), (27) and (29) by θ_q, we obtain that $\tau_q = 0$, $A_q = 0$, $B_q = 0$, $C = 0$. Taking into account form (26) of ξ, we get $A = \tau_t/2$. Hence

$$\tau_q = 0, \quad A_q = 0, \quad A = \frac{\tau_t}{2}, \quad B_q = 0, \quad C = 0. \tag{30}$$

Differentiate (27) by S and due to (30) obtain the equality $\gamma\sigma^2 e^{r(T-t)} D_{SS} = \tau_{tt}/2$, hence $D_{SSS} = 0$ and $D_{SSq} = 0$. Therefore, the differentiation of Equation (28) twice by S gives $-rD_{SS} + D_{tSS} = 0$ and substituting the expression for D_{SS} we get $\tau_{ttt} = 0$.

Next, differentiating Equation (27) by q and Equation (28) by θ_q and S we obtain

$$-\frac{\tau_t}{2} + D_{Sq} - \alpha_q + r\tau = 0, \quad D_{Sq} = 0, \quad \alpha_q = r\tau - \frac{\tau_t}{2}.$$

Differentiate (28) by S and q and get

$$-r(r\tau - \tau_t) + r\alpha_q + rA - \gamma\sigma^2 e^{r(T-t)} D_{SS} = r\tau_t - \tau_{tt}/2 = 0.$$

From this equation and the equality $\tau_{ttt} = 0$ it follows that $\tau_t = 0$.

Therefore, τ is a constant, $\alpha_q = r\tau$, $\alpha = rq\tau + E(t)$. Substitute it in Equation (27) and obtain $\gamma\sigma^2 e^{r(T-t)}(D_S - E) - B_t = 0$. Hence,

$$\xi = B(t), \quad \alpha = rq\tau + E(t), \quad D = G(t,q) + E(t)S + \frac{e^{r(t-T)}}{\gamma\sigma^2} B'(t)S. \tag{31}$$

Substituting (31) into (28) and reducing we get

$$r\tau F + G_t + E'S + \frac{e^{r(t-T)}}{\gamma\sigma^2} B''S - E'\theta_q - rG + rBq + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2} B' - B'q - \tau F_t - G_q F_{\theta_q} = 0. \tag{32}$$

Differentiate (32) by θ_q and q and get $G_{qq} = 0$. Then $G = H(t)q + J(t)$ and the separation of Equation (32) by q and S gives

$$\begin{gathered} G = H(t)q + J(t), \quad E' + \frac{e^{r(t-T)}}{\gamma\sigma^2} B'' = 0, \quad H' - rH + rB - B' = 0, \\ r\tau F + J' - E'\theta_q - rJ + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2} B' - \tau F_t - HF_{\theta_q} = 0. \end{gathered} \tag{33}$$

The third equation in (33) implies that $B = H + Ke^{rt}$. Substitute this equality into the second equation in (33), then

$$B(t) = H(t) + Ke^{rt}, \quad E(t) = -\int_{t_0}^{t} \frac{e^{r(s-T)}}{\gamma\sigma^2}(H''(s) + r^2 Ke^{rs})ds + L. \tag{34}$$

Now equalities (31) implies that

$$\begin{gathered} \xi = H(t) + Ke^{rt}, \quad \alpha = rq\tau - \int_{t_0}^{t} \frac{e^{r(s-T)}}{\gamma\sigma^2}(H''(s) + r^2 Ke^{rs})ds + L, \\ \eta = r\theta\tau + \left(-\int_{t_0}^{t} \frac{e^{r(s-T)}}{\gamma\sigma^2}(H''(s) + r^2 Ke^{rs})ds + L\right)S + \\ + \frac{e^{r(t-T)}}{\gamma\sigma^2}(H'(t) + rKe^{rt})S + H(t)q + J(t). \end{gathered} \tag{35}$$

Substituting (34) into the last equation in (33) we get

$$r\tau F - \tau F_t - HF_{\theta_q} + J' - rJ + \frac{e^{r(t-T)}}{\gamma\sigma^2}(H'' + r^2Ke^{rt})\theta_q + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(H' + rKe^{rt}) = 0. \quad (36)$$

This equation has the form $r\tau F - \tau F_t - H(t)F_{\theta_q} + u(t)\theta_q + v(t) = 0$. Consider possible situations.

4.1. The Case $\tau = 0$, $H \equiv 0$

If $\tau = 0$, $H \equiv 0$, then $K = 0$, $J' - rJ = 0$, $J = J_0e^{rt}$. Due to (35) we get the generators of symmetry groups $X_1 = e^{rt}\partial_\theta$, $X_2 = \partial_q + S\partial_\theta$ for arbitrary F, such that $F_{\theta_q\theta_q} \neq 0$.

4.2. The Case $\tau \neq 0$, $H \equiv 0$

If $\tau \neq 0$, $H \equiv 0$, then $F = \Phi_1(\theta_q)e^{rt} + b(t)\theta_q + c(t)$. Using the equivalence transformation of the group, which is generated by Y_ϕ (18) with ϕ, such that $\phi' - r\phi + c = 0$, we obtain $F = \Phi_1(\theta_q)e^{rt} + b(t)\theta_q$. Since $F_{\theta_q\theta_q} \not\equiv 0$, then $\Phi_1'' \not\equiv 0$. Substitute F in (36), then

$$r\tau b\theta_q - \tau b'\theta_q + J' - rJ + \frac{e^{r(t-T)}}{\gamma\sigma^2}r^2Ke^{rt}\theta_q + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}rKe^{rt} = 0.$$

Separating by the variable θ_q, obtain

$$b(t) = b_0e^{rt} + \frac{rKe^{r(2t-T)}}{\tau\gamma\sigma^2}, \quad J(t) = J_0e^{rt} - \frac{\mu Ke^{r(2t-T)}}{\gamma\sigma^2}.$$

Denote $\Phi(\theta) := \Phi_1(\theta_q) + b_0\theta_q$, then $\Phi'' = \Phi_1'' \not\equiv 0$. Thus,

$$\begin{gathered} F = \Phi(\theta_q)e^{rt} + 2rbe^{2rt}\theta_q, \quad \Phi'' \neq 0,\ b \in \mathbb{R}, \\ \tau = \tau_0, \quad \xi = 2\tau\gamma\sigma^2be^{r(t+T)}, \quad \alpha = rq\tau - r\tau be^{2rt} + L, \\ \eta = r\theta\tau + r\tau be^{2rt}S + LS + J_0e^{rt} - 2\mu\tau be^{2rt}. \end{gathered}$$

Therefore, we obtained the specialization and the symmetry group, which is generated by operators

$$\begin{gathered} X_1 = e^{rt}\partial_\theta, \quad X_2 = \partial_q + S\partial_\theta, \\ X_3 = \partial_t + 2\gamma\sigma^2b_1e^{r(t+T)}\partial_S + (rq - rb_1e^{2rt})\partial_q + (r\theta + rb_1e^{2rt}S - 2\mu b_1e^{2rt})\partial_\theta. \end{gathered}$$

4.3. The Case $\tau = 0$, $H \not\equiv 0$

If $\tau = 0$, $H \not\equiv 0$, then $u \not\equiv 0$, otherwise, $F_{\theta_q\theta_q} \equiv 0$. Therefore, $F = a(t)\theta_q^2 + b(t)\theta_q + c(t)$, $a \not\equiv 0$. We use the equivalence transformation of the group with the generator Y_ψ (18), where ψ is a solution of the equation

$$\psi''(t) + \gamma\sigma^2e^{r(T-t)}(2a(t)\psi(t) + b(t)) = 0,$$

and get $F = a(t)\theta_q^2 + c(t)$, then by a transformation with a generator Y_ϕ we obtain the equivalent function $F = a(t)\theta_q^2$. Then (36) implies the equation

$$J' + \frac{e^{r(t-T)}}{\gamma\sigma^2}(H'' + r^2Ke^{rt})\theta_q - rJ + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(H' + rKe^{rt}) - 2Ha\theta_q = 0.$$

Therefore,

$$H'' = 2\gamma\sigma^2e^{r(T-t)}a(t)H - r^2Ke^{rt}, \quad J' - rJ + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(H' + rKe^{rt}) = 0. \quad (37)$$

Solving the second equation in (37) we get

$$J = J_0 e^{rt} - \mu \frac{e^{r(t-T)}}{\gamma\sigma^2}(H + Ke^{rt}).$$

Then (35) has the form

$$\begin{gathered}\tau = 0, \quad \xi = H(t) + Ke^{rt}, \quad \alpha = -2\int_{t_0}^{t} a(s)H(s)ds + L,\\ \eta = \left(-2\int_{t_0}^{t} a(s)H(s)ds + L\right)S + \frac{e^{r(t-T)}}{\gamma\sigma^2}(H'(t) + rKe^{rt})S + \\ + H(t)q + J_0 e^{rt} - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(H(t) + Ke^{rt}).\end{gathered} \tag{38}$$

Let $\Psi(t)$ is a partial solution of the first equation in (37) for $K = 1$, then a general solution of the equation is $H(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + K\Psi(t)$, where φ_1 and φ_2 are two linearly independent solutions of the homogeneous equation $H'' = 2\gamma\sigma^2 e^{r(T-t)}a(t)H$. Therefore, (38) implies that

$$\begin{gathered}X_1 = e^{rt}\partial_\theta, \quad X_2 = \partial_q + S\partial_\theta,\\ X_3 = \varphi_1(t)\partial_S - 2\int_{t_0}^{t} a(s)\varphi_1(s)ds\partial_q + \\ + \left(\frac{e^{r(t-T)}\varphi_1'(t)}{\gamma\sigma^2}S - 2S\int_{t_0}^{t} a(s)\varphi_1(s)ds + \varphi_1(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_1(t)\right)\partial_\theta,\\ X_4 = \varphi_2(t)\partial_S - 2\int_{t_0}^{t} a(s)\varphi_2(s)ds\partial_q + \\ + \left(\frac{e^{r(t-T)}\varphi_2'(t)}{\gamma\sigma^2}S - 2S\int_{t_0}^{t} a(s)\varphi_2(s)ds + \varphi_2(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_2(t)\right)\partial_\theta,\\ X_5 = (\Psi(t) + e^{rt})\partial_S - 2\int_{t_0}^{t} a(s)\Psi(s)ds\partial_q + \\ + \left(\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi'(t) + re^{rt})S - 2S\int_{t_0}^{t} a(s)\Psi(s)ds + \Psi(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi(t) + e^{rt})\right)\partial_\theta.\end{gathered}$$

4.4. The Case $\tau \neq 0$, $H \not\equiv 0$

For the case $\tau \neq 0$, $H \not\equiv 0$ make a replacement

$$F = e^{rt}\Phi(t, \theta_q) + \frac{1}{\tau}e^{rt}\int_{t_0}^{t} e^{-rs}u(s)ds\theta_q + \frac{1}{\tau}e^{rt}\int_{t_0}^{t} e^{-rs}v(s)ds$$

and obtain the equation $\tau\Phi_t + H(t)\Phi_{\theta_q} = g(t)$ for some g. Therefore, we have $F = e^{rt}\Phi(\theta_q - \int H(t)dt/\tau) + b_1(t)\theta_q + c_1(t)$. After using the equivalence transformation of the group for Y_ψ (18) with $\psi = \int H(t)dt/\tau$ we obtain $F = e^{rt}\Phi(\theta_q) + b(t)\theta_q + c(t)$, $\Phi'' \not\equiv 0$. Substitute the result in (36), then

$$\begin{gathered}r\tau b\theta_q + r\tau c - \tau b'\theta_q - \tau c' - e^{rt}H\Phi' - Hb + J' - rJ + \\ + \frac{e^{r(t-T)}}{\gamma\sigma^2}(H'' + r^2Ke^{rt})\theta_q + \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(H' + rKe^{rt}) = 0.\end{gathered}$$

Hence $\Phi(\theta_q) = a_0 + a_1\theta_q + a\theta_q^2$, by the equivalence transformation for X_ψ, where

$$\psi''(t) + \gamma\sigma^2 e^{r(T-t)}(2ae^{rt}\psi(t) + a_1 + b(t)) = 0,$$

then by an equivalence transformation for a group with a generator X_ϕ obtain $F = ae^{rt}\theta_q^2$ with a constant $a \neq 0$. So, we obtain a partial case to the previous one, but with a nonzero τ, which gives additional symmetry. Thus,

$$X_1 = e^{rt}\partial_\theta, \quad X_2 = \partial_q + S\partial_\theta, \quad X_3 = \partial_t + rq\partial_q + r\theta\partial_\theta,$$
$$X_4 = \varphi_1(t)\partial_S - 2a\int_{t_0}^t e^{rs}\varphi_1(s)ds\partial_q +$$
$$+\left(\frac{e^{r(t-T)}\varphi_1'(t)}{\gamma\sigma^2}S - 2aS\int_{t_0}^t e^{rs}\varphi_1(s)ds + \varphi_1(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_1(t)\right)\partial_\theta,$$
$$X_5 = \varphi_2(t)\partial_S - 2a\int_{t_0}^t e^{rs}\varphi_2(s)ds\partial_q +$$
$$+\left(\frac{e^{r(t-T)}\varphi_2'(t)}{\gamma\sigma^2}S - 2aS\int_{t_0}^t e^{rs}\varphi_2(s)ds + \varphi_2(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_2(t)\right)\partial_\theta,$$
$$X_6 = (\Psi(t) + e^{rt})\partial_S - 2a\int_{t_0}^t e^{rs}\Psi(s)ds\partial_q +$$
$$+\left(\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi'(t) + re^{rt})S - 2aS\int_{t_0}^t e^{rs}\Psi(s)ds + \Psi(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi(t) + e^{rt})\right)\partial_\theta.$$

Instead of the first equation in (37), we have the equation with constant coefficients

$$H'' - 2a\gamma\sigma^2 e^{rT}H + r^2Ke^{rt} = 0.$$

Therefore, we can calculate a solution of this equation analitically. If $a\gamma > 0$, $a \neq r^2/2\gamma\sigma^2 e^{rT}$, then

$$\varphi_1(t) = e^{\sqrt{2a\gamma\sigma^2 e^{rT}}t}, \quad \varphi_2(t) = e^{-\sqrt{2a\gamma\sigma^2 e^{rT}}t}, \quad \Psi(t) = \frac{r^2Ke^{rt}}{r^2 - 2a\gamma\sigma^2 e^{rT}}. \tag{39}$$

For $a\gamma > 0$, $a = r^2/2\gamma\sigma^2 e^{rT}$ we have

$$\varphi_1(t) = e^{\sqrt{2a\gamma\sigma^2 e^{rT}}t}, \quad \varphi_2(t) = e^{-\sqrt{2a\gamma\sigma^2 e^{rT}}t}, \quad \Psi(t) = -\frac{rKte^{rt}}{2}. \tag{40}$$

Finally, if $a\gamma < 0$, then

$$\varphi_1(t) = \sin\sqrt{-2a\gamma\sigma^2 e^{rT}}t, \quad \varphi_2(t) = \cos\sqrt{-2a\gamma\sigma^2 e^{rT}}t, \quad \Psi(t) = \frac{r^2Ke^{rt}}{r^2 - 2a\gamma\sigma^2 e^{rT}}. \tag{41}$$

The equality $a = r^2/2\gamma\sigma^2 e^{rT}$ in this case is not possible.

5. Theorem on Group Classification

Let us formulate the results of calculations in the previous section as the following theorem on group classification. We denote here by X_k obtained basis operators in the corresponding Lie algebras.

Theorem 2. *Let $r, \gamma, \sigma, \mu, T \in \mathbb{R}$.*

1. The Lie algebra for the equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{\gamma\sigma^2}{2}e^{r(T-t)}(\theta_S - q)^2 + F(t, \theta_q), \tag{42}$$

where F is not equivalent to $a(t)\theta_q^2$ or $e^{rt}\Phi(\theta_q) + b_0e^{rt}\theta_q + b_1e^{2rt}\theta_q$, $F_{\theta_q\theta_q} \not\equiv 0$, is generated by the operators

$$X_1 := e^{rt}\partial_\theta, \quad X_2 := \partial_q + S\partial_\theta. \tag{43}$$

2. The Lie algebra for the equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{\gamma\sigma^2}{2}e^{r(T-t)}(\theta_S - q)^2 + e^{rt}\Phi(\theta_q) + be^{2rt}\theta_q, \tag{44}$$

where $b \in \mathbb{R}$, Φ is a nonlinear function, which is not equivalent to $a\theta_q^2$, is generated by the operators

$$\begin{gathered} X_1 := e^{rt}\partial_\theta, \quad X_2 := \partial_q + S\partial_\theta, \\ X_3 := \partial_t + 2\gamma\sigma^2 be^{r(t+T)}\partial_S + (rq - rbe^{2rt})\partial_q + (r\theta + rbe^{2rt}S - 2\mu be^{2rt})\partial_\theta. \end{gathered} \tag{45}$$

3. The Lie algebra for the equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{\gamma\sigma^2}{2}e^{r(T-t)}(\theta_S - q)^2 + a(t)\theta_q^2,$$

where $a(t)$ is a nonzero function, which is not equivalent to a_0e^{rt}, is generated by the operators

$$\begin{gathered} X_1 := e^{rt}\partial_\theta, \quad X_2 := \partial_q + S\partial_\theta, \\ X_3 := \varphi_1(t)\partial_S - 2\int_{t_0}^t a(s)\varphi_1(s)ds\partial_q + \\ +\left(\frac{e^{r(t-T)}\varphi_1'(t)}{\gamma\sigma^2}S - 2S\int_{t_0}^t a(s)\varphi_1(s)ds + \varphi_1(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_1(t)\right)\partial_\theta, \\ X_4 := \varphi_2(t)\partial_S - 2\int_{t_0}^t a(s)\varphi_2(s)ds\partial_q + \\ +\left(\frac{e^{r(t-T)}\varphi_2'(t)}{\gamma\sigma^2}S - 2S\int_{t_0}^t a(s)\varphi_2(s)ds + \varphi_2(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_2(t)\right)\partial_\theta, \\ X_5 := (\Psi(t) + e^{rt})\partial_S - 2\int_{t_0}^t a(s)\Psi(s)ds\partial_q + \\ +\left(\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi'(t) + re^{rt})S - 2S\int_{t_0}^t a(s)\Psi(s)ds + \Psi(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi(t) + e^{rt})\right)\partial_\theta. \end{gathered}$$

Here φ_1, φ_2 are linearly independent solutions of the equation $H''(t) = 2\gamma\sigma^2e^{r(T-t)}a(t)H(t)$, Ψ is a partial solution of the equation $H''(t) = 2\gamma\sigma^2e^{r(T-t)}a(t)H(t) - r^2e^{rt}$.

4. The Lie algebra for the equation

$$\theta_t = r\theta + (\mu - rS)q - \mu\theta_S - \frac{\sigma^2}{2}\theta_{SS} - \frac{\gamma\sigma^2}{2}e^{r(T-t)}(\theta_S - q)^2 + ae^{rt}\theta_q^2,$$

where a is a nonzero constant, is generated by the operators

$$X_1 := e^{rt}\partial_\theta, \quad X_2 := \partial_q + S\partial_\theta, \quad X_3 := \partial_t + rq\partial_q + r\theta\partial_\theta,$$
$$X_4 := \varphi_1(t)\partial_S - 2a\int_{t_0}^t e^{rs}\varphi_1(s)ds\partial_q +$$
$$+\left(\frac{e^{r(t-T)}\varphi_1'(t)}{\gamma\sigma^2}S - 2aS\int_{t_0}^t e^{rs}\varphi_1(s)ds + \varphi_1(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_1(t)\right)\partial_\theta,$$
$$X_5 := \varphi_2(t)\partial_S - 2a\int_{t_0}^t e^{rs}\varphi_2(s)ds\partial_q +$$
$$+\left(\frac{e^{r(t-T)}\varphi_2'(t)}{\gamma\sigma^2}S - 2aS\int_{t_0}^t e^{rs}\varphi_2(s)ds + \varphi_2(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}\varphi_2(t)\right)\partial_\theta,$$
$$X_6 := (\Psi(t) + e^{rt})\partial_S - 2a\int_{t_0}^t e^{rs}\Psi(s)ds\partial_q +$$
$$+\left(\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi'(t) + re^{rt})S - 2aS\int_{t_0}^t e^{rs}\Psi(s)ds + \Psi(t)q - \mu\frac{e^{r(t-T)}}{\gamma\sigma^2}(\Psi(t) + e^{rt})\right)\partial_\theta,$$

where $\varphi_1, \varphi_2, \Psi$ are from (39), (40), *or* (41), *depending on the sign of $a\gamma$ and the value of a.*

Remark 3. *In the second part of this theorem at $b = 0$ and in the fourth one we have the market trading volume $V_t = ae^{rt}$ with a constant $a \neq 0$, as multiplier at a function of θ_q in an expression for F. If $\Phi \equiv 0$ in the second part, then the market trading volume is $V_t = be^{2rt}$. In the third part of the theorem $V_t = a(t)$.*

Remark 4. *A theorem on the group classification of Equation* (2) *with a free element F depending on θ_q only is obtained in [35]. It contains the specifications $F = e^{\nu\theta_q}$ and $F = \theta_q^2$, which correspond to additional symmetries of the equation.*

6. Application to the Search of Some Submodels

Using a symmetry group for a differential equation we can reduce the number of variables on which an unknown function depends by the dimension of the considered group. If the resulting equation can be integrated, we obtain an exact solution of the original equation, invariant with respect to the group of symmetries under consideration. If the resulting equation is not integrable, following L.V. Ovsyannikov [22], we will call such an equation an invariant submodel of the initial equation (initial model).

In order to find invariant solutions or submodels that are not translated into each other by transformations of variables, we must find the so-called optimal system of subalgebras of the Lie algebra of the equation under study. To do this, the internal automorphisms of this algebra are used, which can be found through nonzero structural constants of the algebra. Below we will do this for the two simplest Lie algebras of the symmetry groups generators obtained in Theorem 2 on the group classification. Invariant solutions or invariant submodels for different subalgebras of the optimal system are not equivalent, i.e., they cannot be obtained from each other by replacing variables.

6.1. Optimal System of Subalgebras and Submodels for the General Case

Lie algebra L_2 (43) is commutative, hence it has no continuous groups of internal automorphisms. Thus, its optimal system of one-dimensional subalgebras is $\Theta_1 = \{\langle X_2\rangle, \langle X_1 + cX_2\rangle, c \in \mathbb{R}\}$.

The subalgebra $\langle X_2 \rangle$ has the invariants $J_1 = t$, $J_2 = S$, $J_3 = \theta - qS$. Writing $J_3 = w(J_1, J_2)$ we obtain the form of the corresponding invariant solution $\theta = w(t, S) + Sq$. Substitute it into Equation (42) and obtain the submodel

$$w_t = rw - \mu w_S - \frac{\sigma^2}{2} w_{SS} - \frac{\gamma\sigma^2}{2} e^{r(T-t)} w_S^2 + F(t, S),$$

which is invariant for $\langle X_2 \rangle$. Analogously we get the invariants t, $\frac{e^{rt}}{c} + S$, $\theta - \frac{e^{rt}}{c} q - Sq$ for the subalgebra $\langle X_1 + cX_2 \rangle$, $c \neq 0$. The invariant submodel for it has the form

$$w_t = rw - \mu w_S - \frac{\sigma^2}{2} w_{SS} - \frac{\gamma\sigma^2}{2} e^{r(T-t)} w_S^2 + F\left(t, \frac{e^{rt}}{c} + S\right),$$

where $\theta = w(t, S) + \frac{e^{rt}}{c} q + Sq$. If $c = 0$, then the subalgebra $\langle X_1 \rangle$ has no invariant submodels, since its invariants t, S, q do not depend on θ.

6.2. Optimal System of Subalgebras and Submodels for the Specification $F = \Phi(\theta_q)e^{rt}$

Nonzero structural constants for a Lie algebra with a basis $\{X_1, X_2, \ldots, X_n\}$ are coefficients c_{ij}^k in the decomposition of a commutator $[X_i, X_j]$ [17] by the basis: $[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$. Generators of continuous groups of internal automorphisms can be calculated by the formula $E_i = \sum_{j,k=1}^{n} c_{ij}^k e_j \partial_{e_k}$, where e_j are coefficients in the decomposition of an element of the Lie algebra by its basis, which depend on group parameters.

Consider the Lie algebra L_3 with basis (45). For L_3 we have $c_{23}^1 = -c_{32}^1 = -2\gamma\sigma^2 be^{rT}$, $c_{23}^2 = -c_{32}^2 = r$. The integration of the Lie equations for the generators gives $E_2 : \bar{e}_1 = e_1 - 2\gamma\sigma^2 be^{rT} e_3 a_2$, $\bar{e}_2 = e_2 + re_3 a_2$; $E_3 : \bar{e}_1 = e_1 - \frac{2\gamma\sigma^2}{r} be^{rT} e_2 (1 - e^{-ra_3})$, $\bar{e}_2 = e^{-ra_3} e_2$. Also, we add a mirror automorphism $E_- : \bar{e}_1 = -\bar{e}_1$, which does not change the commutators of the basis operators of this Lie algebra L_3.

Let $b \neq 0$, for $e_3 \neq 0$ by the internal automorphism E_2 obtain $e_2 = 0$, then we have $(e_1, e_2, e_3) = (c, 0, 1)$ after scaling, i.e., we get $cX_1 + X_3$, $c \in \mathbb{R}$. If $e_3 = 0$, $e_2 \neq 0$, then by E_3 get $e_1 = 0$, $(e_1, e_2, e_3) = (0, 1, 0)$, if we take

$$a_3 = -\frac{1}{r} \ln\left(1 - \frac{re^{-rT} e_1}{2\gamma\sigma^2 be_2}\right)$$

in the case $\frac{re^{-rT} e_1}{2\gamma\sigma^2 be_2} < 1$. If $\frac{re^{-rT} e_1}{2\gamma\sigma^2 be_2} \geq 1$, we will use E_- to go to the previous case. For $e_2 = e_3 = 0$ we have $(e_1, e_2, e_3) = (1, 0, 0)$. Thus, $\Theta_1^1 = \{\langle X_1 \rangle, \langle X_2 \rangle, \langle cX_1 + X_3 \rangle, c \in \mathbb{R}\}$.

Let us search for a system of two-dimensional subalgebras for L_3 with $b \neq 0$. For the basis vector X_1 of the one-dimensional subalgebra $\langle X_1 \rangle$, consider the second basis vector in the form $\alpha X_2 + \beta X_3$, then the commutator has the form $[X_1, \alpha X_2 + \beta X_3] = 0$. We get subalgebras $\langle X_1, X_2 \rangle$ for $e_3 = 0$, $\langle X_1, X_3 \rangle$ for $e_3 \neq 0$, if we use E_2.

For the basis vector X_2, consider the second basis vector in the form $\alpha X_1 + \beta X_3$. Their commutator is $[X_2, \alpha X_1 + \beta X_3] = r\beta X_2 - 2\beta\gamma\sigma^2 be^{rT} X_1$. Therefore, a subalgebra is formed at $\beta = 0$, which is already found in the form $\langle X_1, X_2 \rangle$.

For $cX_1 + X_3$, consider the second basis vector in the form $\alpha X_1 + \beta X_2$. Then we have $[cX_1 + X_3, \alpha X_1 + \beta X_2] = 2\beta\gamma\sigma^2 be^{rT} X_1 - r\beta X_2$ and get the subalgebra $\langle cX_1 + X_3, 2\gamma\sigma^2 be^{rT} X_1 - rX_2 \rangle$. By E_3 and E_- reduce it to $\langle cX_1 + X_3, X_2 \rangle$.

Thus, we proved the optimal systems of one-dimensional and two-dimensional subalgebras of the Lie algebra under consideration.

Lemma 1. *Optimal systems of one-dimensional and two-dimensional subalgebras of Lie algebra L_3 (45) with $b \neq 0$ are*

$$\Theta_1^1 = \{\langle X_1\rangle, \langle X_2\rangle, \langle cX_1 + X_3\rangle, c \in \mathbb{R}\},\ \Theta_2^1 = \{\langle X_1, X_2\rangle, \langle X_1, X_3\rangle, \langle cX_1 + X_3, X_2\rangle, c \in \mathbb{R}\}.$$

Here a denotation Θ_k^j is used for the optimal system of k-dimensional subalgebras in the j-th case.

In the case $b = 0$, for $e_3 \neq 0$ we obtain the vector $(c, 0, 1)$, $c \in \mathbb{R}$, using E_2. If $e_3 = 0$, then using E_3, E_- we get $(1, 1, 0)$, $(1, 0, 0)$, $(0, 1, 0)$. So, $\Theta_1^2 = \{\langle X_1\rangle, \langle X_2\rangle, \langle X_1 + X_2\rangle, \langle cX_1 + X_3\rangle, c \in \mathbb{R}\}$. In this case, we have two-dimensional subalgebras $\langle X_1, X_2\rangle$, $\langle X_1, X_3\rangle$ also. Moreover, $[X_2, \alpha X_1 + \beta X_3] = r\beta X_2$, and we have the subalgebra $\langle X_2, \alpha X_1 + \beta X_3\rangle$ for any $\alpha, \beta \in \mathbb{R}$. If $\beta = 0$, it will be a partial case of $\langle X_1, X_2\rangle$, for $\beta \neq 0$ we obtain the subalgebra $\langle cX_1 + X_3, X_2\rangle$. Since $[cX_1 + X_3, \alpha X_1 + \beta X_2] = -r\beta X_2$, we obtain another subalgebra $\langle cX_1 + X_3, X_1\rangle$.

Lemma 2. *The optimal system of one-dimensional and two-dimensional subalgebras of Lie algebra L_3 (45) with $b_1 = 0$ are $\Theta_1^2 = \{\langle X_1\rangle, \langle X_2\rangle, \langle X_1 + X_2\rangle, \langle cX_1 + X_3\rangle, c \in \mathbb{R}\}$ and $\Theta_2^2 = \{\langle X_1, X_2\rangle, \langle cX_1 + X_3, X_1\rangle, \langle cX_1 + X_3, X_2\rangle, c \in \mathbb{R}\}$.*

The subalgebras $\langle X_1\rangle$, $\langle X_1, X_2\rangle$, $\langle X_1, X_3\rangle$ do not have invariant submodels, since $\langle X_1\rangle$ does not have invariants depending on θ.

Consider the case $b = 0$, then the subalgebra $\langle X_1 + X_2\rangle$ has invariants t, S, $\theta - (e^{rt} + S)q$, therefore, an invariant solution has the form $\theta = w(t, S) + (e^{rt} + S)q$ and the invariant submodel is

$$w_t = rw - \mu w_S - \frac{\sigma^2}{2} w_{SS} - \frac{\gamma\sigma^2}{2} e^{r(T-t)} w_S^2 + F(t, e^{rt} + S).$$

The subalgebra $\langle cX_1 + X_3\rangle$ has invariants $x := qe^{-rt}$, S, $\theta e^{-rt} - ct$, hence we will look for an invariant solution in the form $\theta = cte^{rt} + e^{rt} w(qe^{-rt}, S)$, where w is a function of two variables. Substitute it into (44) and obtain the invariant for $\langle cX_1 + X_3\rangle$ submodel

$$\Phi(w_x) + rxw_x = \frac{\sigma^2}{2} w_{SS} + \mu w_S + \frac{\gamma\sigma^2}{2} e^{rT} (w_S - x)^2 + (rS - \mu)x + c.$$

The subalgebra $\langle cX_1 + X_3, X_1\rangle$ has no invariants depending on θ and, therefore, invariant submodels. Let us find the invariant submodel with respect to $\langle cX_1 + X_3, X_2\rangle$. Consider a function $G = G(x, S, y)$, where $x := qe^{-rt}$, $y := \theta e^{-rt} - ct$ are invariants for the subalgebra $\langle cX_1 + X_3\rangle$. Then $X_2 G = e^{-rt} G_x + Se^{-rt} G_y$ and invariants of the subalgebra $\langle cX_1 + X_3, X_2\rangle$ are S and $y - Sx = (\theta - Sq)e^{-rt} - ct$. Therefore, we will search an invariant solution for this subalgebra in the form $\theta = cte^{rt} + Sq + e^{rt} w(S)$. The invariant submodel will have the form

$$w''(S) + \frac{2\mu}{\sigma^2} w'(S) + \gamma e^{rT} w'(S)^2 - \frac{2}{\sigma^2}\Phi(S) + \frac{2c}{\sigma^2} = 0.$$

7. Conclusions

In this paper, we develop a theorem on group classification of the Guéant and Pu model of the option pricing taking into account transaction costs and the impact of operations on the market. For this aim, the Lie algebra of generators of continuous groups of equivalence transformations is calculated. For the general case, and for the case of the equation with the right-hand side $F = e^{rt}\Phi(\theta_q)$, optimal systems of subalgebras and corresponding invariant submodels are derived. The results of this work will be applied to the analogous research into the Guéant and Pu model with the specifications $F = e^{rt}\Phi(\theta_q) + be^{2rt}$, $F = a(t)\theta_q^2$, $F = ae^{rt}\theta_q^2$, which is presented in the obtained theorem on group classification. The knowledge of the group structure of the studied models obtained in this way will make it possible to calculate their exact solutions and conservation laws.

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References

1. Black, F. The pricing of Commodity Contracts. *J. Financ. Econ.* **1976**, *3*, 167–179. [CrossRef]
2. Black, F.; Scholes, M. The pricing of options and corporate liabilities. *J. Political Econ.* **1973**, *81*, 637–659. [CrossRef]
3. Magill, M.J.P.; Constantinides, G.M. Portfolio selection with transactions costs. *J. Econ. Theory* **1976**, *13*, 245–263. [CrossRef]
4. Kyle, A.S. Continuous auctions and insider trading. *Econometrica* **1985**, *53*, 1315–1335. [CrossRef]
5. Leland, H.E. Option pricing and replication with transactions costs. *J. Financ.* **1985**, *40*, 1283–1301. [CrossRef]
6. Cvitanić, J.; Karatzas, I. Hedging and portfolio optimization under transaction costs: A martingale approach. *Math. Financ.* **1996**, *6*, 133–165. [CrossRef]
7. Barles, G.; Soner, H.M. Option pricing with transaction costs and a nonlinear Black—Scholes equation. *Financ. Stochastics* **1998**, *2*, 369–397. [CrossRef]
8. Grossman, S. An analysis of the implications for stock and futures price volatility of program trading and dynamic hedging strategies. *J. Bus.* **1998**, *61*, 275–298. [CrossRef]
9. Platen, E.; Schweizer, M. On feedback effects from hedging derivatives. *Math. Financ.* **1998**, *8*, 67–84. [CrossRef]
10. Sircar, R.; Papanicolaou, G. Generalized Black—Scholes models accounting for increased market volatility from hedging strategies. *Appl. Math. Financ.* **1998**, *5*, 45–82. [CrossRef]
11. Schönbucher, P.; Wilmott, P. The feedback-effect of hedging in illiquid markets. *SIAM J. Appl. Math.* **2000**, *61*, 232–272.
12. Bank, P.; Baum, D. Hedging and portfolio optimization in financial markets with a large trader. *Math. Financ.* **2004**, *14*, 1–18. [CrossRef]
13. Çetin, U.; Jarrow, R.; Protter, P. Liquidity risk and arbitrage pricing theory. *Financ. Stoch.* **2004**, *8*, 311–341. [CrossRef]
14. Çetin, U.; Rogers, L.C. Modelling liquidity effects in discrete time. *Math. Financ.* **2007**, *17*, 15–29. [CrossRef]
15. Rogers, L.C.; Singh, L.S. The cost of illiquidity and its effects on hedging. *Math. Financ.* **2010**, *20*, 597–615. [CrossRef]
16. Gazizov, R.K.; Ibragimov, N.H. Lie symmetry analysis of differential equations in finance. *Nonlinear Dyn.* **1998**, *17*, 387–407. [CrossRef]
17. Ovsiannikov, L.V. *Group Analysis of Differential Equations*; Academic Press: New York, NY, USA, 1982.
18. Olver, P.J. *Applications of Lie Groups to Differential Equations*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1986.
19. Turab, A.; Sintunavarat, W. A unique solution of the iterative boundary value problem for a second-order differential equation approached by fixed point results. *Alex. Eng. J.* **2021**, *60*, 5797–5802. [CrossRef]
20. Li, W.-J.; Yu, Y.-H. Oscillatory behavior of third-order nonlinear differential equations with a sublinear neutral term. *Acta Math. Appl. Sin. Engl. Ser.* **2022**, *38*, 484–496. [CrossRef]
21. Santra, S.S.; Scapellato, A. Some conditions for the oscillation of second-order differential equations with several mixed delays. *J. Fixed Point Theory Appl.* **2022**, *24*, 18. [CrossRef]
22. Ovsyannikov, L.V. The "podmodeli" programm. Gas dynamics. *J. Appl. Math. Mech.* **1994**, *58*, 601–627. [CrossRef]
23. Annin, B.D.; Bytev, V.O.; Senashev, C.I. *Group Properties of the Equations of Elasticity and Plasticity*; Nauka: Novosibirsk, Russia, 1985. (In Russian)
24. Bordag, L.A. *Geometrical Properties of Differential Equations: Applications of the Lie Group Analysis in Financial Mathematics*; World Scientific Publishing Company: Singapore, 2015.
25. Bordag, L.A.; Mikaelyan, A. Models of self-financing hedging strategies in illiquid markets: Symmetry reductions and exact solution. *J. Lett. Math. Phys.* **2011**, *96*, 191–207. [CrossRef]
26. Dyshaev, M.M. Group analysis of a nonlinear generalization for Black—Scholes equation. *Chelyabinsk Phys. Math. J.* **2016**, *1*, 7–14.

27. Dyshaev, M.M.; Fedorov, V.E. Symmetry analysis and exact solutions for a nonlinear model of the financial markets theory. *Math. Notes NEFU* **2016**, *23*, 28–45.
28. Fedorov, V.E.; Dyshaev, M.M. Group classification for a general nonlinear model of option pricing. *Ural Math. J.* **2016**, *2*, 37–44. [CrossRef]
29. Dyshaev, M.M.; Fedorov, V.E. Symmetries and exact solutions of a nonlinear pricing options equation. *Ufa Math. J.* **2017**, *9*, 29–40. [CrossRef]
30. Fedorov, V.E.; Dyshaev, M.M. Invariant solutions for nonlinear models in illiquid markets. *Math. Methods Appl. Sci.* **2018**, *41*, 8963–8972. [CrossRef]
31. Dyshaev, M.M.; Fedorov, V.E. Comparing of some sensitivities (Greeks) for nonlinear models of option pricing with market illiquidity. *Math. Notes NEFU* **2019**, *26*, 94–108.
32. Fedorov, V.E.; Dyshaev, M.M. Group classification for a class of non-linear models of the RAPM type. *Commun. Nonlinear Sci. Numer. Simul.* **2021**, *92*, 105471. [CrossRef]
33. Guéant, O.; Pu, J. Option pricing and hedging with execution costs and market impact. *arXiv* **2015**, arXiv:1311.4342.
34. Guéant, O. *The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2016.
35. Yadrikhinskiy, K.V.; Fedorov, V.E.; Dyshaev, M.M. Group analysis of the Guéant and Pu model of option pricing and hedging. In *Nonlinear Physical Science*; Springer Nature: Singapore, 2021; pp. 173–204.