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Some Generalizations of $(\nabla\nabla)^{\Delta}$–Gronwall–Bellman–Pachpatte Dynamic Inequalities on Time Scales with Application

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Abstract: As a new usage of Leibniz integral rule on time scales, we proved some new extensions of dynamic Gronwall–Pachpatte-type inequalities on time scales. Our results extend some existing results in the literature. Some integral and discrete inequalities are obtained as special cases of the main results. The inequalities proved here can be used in the analysis as handy tools to study the stability, boundedness, existence, uniqueness and oscillation behavior for some kinds of partial dynamic equations on time scales. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Gronwall's inequality; dynamic inequality; time scales; Leibniz integral rule on time scales

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1. Introduction

Gronwall-Bellman's inequality [1] in the integral form stated: Let π and f be continuous and nonnegative functions defined on $[a,b]$, and let π_0 be nonnegative constant. Then the inequality

$$\pi(t) \leq \pi_0 + \int_a^t f(s)\pi(s)ds, \quad \text{for all} \quad t \in [a,b], \tag{1}$$

implies that

$$\pi(t) \leq \pi_0 \exp\left(\int_a^t f(s)ds\right), \quad \text{for all} \quad t \in [a,b].$$

Baburao G. Pachpatte [2] proved the discrete version of (1). In particular, he proved that: If $\pi(n)$, $a(n)$, $\gamma(n)$ are nonnegative sequences defined for $n \in \mathbb{N}_0$ and $a(n)$ is non-decreasing for $n \in \mathbb{N}_0$, and if

$$\pi(n) \leq a(n) + \sum_{s=0}^{n-1} \gamma(n)\pi(n), n \in \mathbb{N}_0, \tag{2}$$

then

$$\pi(n) \leq a(n) \prod_{s=0}^{n-1} [1+\gamma(n)], n \in \mathbb{N}_0.$$

Bohner and Peterson [3] unify the integral form (2) and the discrete form (1) by introducing a dynamic inequality on a time scale $\mathbb{T}$ stated: If π, ζ are right dense continuous functions and $\gamma \geq 0$ is regressive and right dense continuous functions, then

$$\pi(t) \leq \zeta(t) + \int_{t_0}^{t} \pi(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all} \quad t \in \mathbb{T},$$

implies

$$\pi(t) \leq \zeta(t) + \int_{t_0}^{t} e_\gamma(t, \sigma(\eta))\zeta(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all} \quad t \in \mathbb{T},$$

For Gronwall-Bellman inequalities in two independent variables on time scales, Anderson [4] studied the following result.

$$\omega(\pi(t,s)) \leq a(t,s) + c(t,s) \int_{t_0}^{t} \int_{s}^{\infty} \omega'(\pi(\tau,\eta))[d(\tau,\eta)w(\pi(\tau,\eta)) + b(\tau,\eta)]\nabla\eta\Delta\tau, \tag{3}$$

where π, a, c, d be nonnegative continuous functions defined for $(t,s) \in \mathbb{T} \times \mathbb{T}$, and b be a nonnegative continuous function for $(t,s) \in [t_0, \infty)_\mathbb{T} \times [t_0, \infty)_\mathbb{T}$ and $\omega \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\omega' > 0$ for $\pi > 0$.

Time scales calculus with the objective to unify discrete and continuous analysis was introduced by S. Hilger [5]. For additional subtleties on time scales, we allude the peruser to the books by Bohner and Peterson [3,6].

Gronwall–Bellman-type inequalities, which have many applications in the qualitative and quantitative behavior, like stability, boundedness, existence, uniqueness and oscillation behavior, have been developed by many mathematicians and several refinements and extensions have been done to the previous results, we refer the reader to [7–16].

Theorem 1 ([13])**.** *Leibniz Integral Rule on Time Scales. In the following by $\Psi^\Delta(r_1, r_2)$ we mean the delta derivative of $\Psi(r_1, r_2)$ with respect to r_1. Similarly, $\Psi^\nabla(r_1, r_2)$ is understood. If Ψ , Ψ^Δ and Ψ^∇ are continuous, and $u, h : \mathbb{T} \to \mathbb{T}$ are delta differentiable functions, then the following formulas holds $\forall r_1 \in \mathbb{T}^\phi$.*

(i) $\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\Delta r_2\right]^\Delta = \int_{u(r_1)}^{h(r_1)} \Psi^\Delta(r_1, r_2)\Delta r_2 + h^\Delta(r_1)\Psi(\sigma(r_1), h(r_1)) - u^\Delta(r_1)$ $\Psi(\sigma(r_1), u(r_1))$;

(ii) $\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\Delta r_2\right]^\nabla = \int_{u(r_1)}^{h(r_1)} \Psi^\nabla(r_1, r_2)\Delta r_2 + h^\nabla(r_1)\Psi(\rho(r_1), h(r_1)) - u^\nabla(r_1)$ $\Psi(\rho(r_1), u(r_1))$;

(iii) $\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\nabla r_2\right]^\Delta = \int_{u(r_1)}^{h(r_1)} \Psi^\Delta(r_1, r_2)\nabla r_2 + h^\Delta(r_1)\Psi(\sigma(r_1), h(r_1)) - u^\Delta(r_1)$ $\Psi(\sigma(r_1), u(r_1))$;

(iv) $\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\nabla r_2\right]^\nabla = \int_{u(r_1)}^{h(r_1)} \Psi^\nabla(r_1, r_2)\nabla r_2 + h^\nabla(r_1)\Psi(\rho(r_1), h(r_1)) - u^\nabla(r_1)$ $\Psi(\rho(r_1), u(r_1))$.

In this work, by using the results of Theorem 1 (iii), we establish the delayed time scale case of the inequalities proved in [17]. Further, these results are proved here extend some known results in [18–20]. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

2. Fundamental Result

Here we introduce basic result.

Lemma 1. *Suppose $\mathbb{T}_1$, $\mathbb{T}_2$ are two times scales and $a \in C(\Omega = \mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R}_+)$ is nondecreasing with respect to $(\vartheta, t) \in \Omega$. Assume that τ, ϕ, $f \in C(\Omega, \mathbb{R}_+)$, $\varphi_1 \in C^1(\mathbb{T}_1, \mathbb{T}_1)$ and $\varphi_2 \in$*

$C^1(\mathbb{T}_2, \mathbb{T}_2)$ be nondecreasing functions with $\varphi_1(\vartheta) \le \vartheta$ on $\mathbb{T}_1$, $\varphi_2(t) \le t$ on $\mathbb{T}_2$. Furthermore, suppose Θ, $\zeta \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\Theta, \zeta\}(\phi) > 0$ for $\phi > 0$, and $\lim_{\phi \to +\infty} \Theta(\phi) = +\infty$. If $\phi(\vartheta, t)$ satisfies

$$\Theta(\phi(\vartheta,t)) \le a(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau(\varsigma,\eta) f(\varsigma,\eta) \zeta(\phi(\varsigma,\eta)) \nabla\eta \nabla\varsigma, \tag{4}$$

for $(\vartheta, t) \in \Omega$, then

$$\phi(\vartheta,t) \le \Theta^{-1}\left\{ \Upsilon^{-1}\left[\Upsilon(a(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta \Delta\varsigma \right] \right\}, \tag{5}$$

for $0 \le \vartheta \le \vartheta_1, 0 \le t \le t_1$, where

$$\Upsilon(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta(\Theta^{-1}(\varsigma))}, v \ge v_0 > 0,\ \Upsilon(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{\zeta(\Theta^{-1}(\varsigma))} = +\infty, \tag{6}$$

and $(\vartheta_1, t_1) \in \Omega$ is chosen so that

$$\left(\Upsilon(a(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta \Delta\varsigma \right) \in \mathrm{Dom}\left(\Upsilon^{-1}\right).$$

Proof. Suppose that $a(\vartheta, t) > 0$. Fixing an arbitrary $(\vartheta_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\vartheta, t)$ by

$$\psi(\vartheta,t) = a(\vartheta_0,t_0) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau(\varsigma,\eta) f(\varsigma,\eta) \zeta(\phi(\varsigma,\eta)) \nabla\eta \nabla\varsigma, \tag{7}$$

for $0 \le \vartheta \le \vartheta_0 \le \vartheta_1, 0 \le t \le t_0 \le t_1$, then $\psi(\vartheta_0, t) = \psi(\vartheta, t_0) = a(\vartheta_0, t_0)$ and

$$\phi(\vartheta,t) \le \Theta^{-1}(\psi(\vartheta,t)). \tag{8}$$

Taking Δ-derivative for (7) with employing Theorem 1 (iii), we have

$$\begin{aligned}
\psi^{\Delta_\vartheta}(\vartheta,t) &= \varphi_1^{\Delta}(\vartheta) \int_{t_0}^{\varphi_2(t)} \tau(\varphi_1(\vartheta),\eta) f(\varphi_1(\vartheta),\eta) \zeta(\phi(\varphi_1(\vartheta),\eta)) \nabla\eta \\
&\le \varphi_1^{\Delta}(\vartheta) \int_{t_0}^{\varphi_2(t)} \tau(\varphi_1(\vartheta),\eta) f(\varphi_1(\vartheta),\eta) \zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) \nabla\eta \\
&\le \zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\varphi_2(t)))\Big) \varphi_1^{\Delta}(\vartheta) \int_{t_0}^{\varphi_2(t)} \tau(\varphi_1(\vartheta),\eta) f(\varphi_1(\vartheta),\eta) \nabla\eta.
\end{aligned} \tag{9}$$

Inequality (9) can be written in the form

$$\frac{\psi^{\Delta_\vartheta}(\vartheta,t)}{\zeta(\Theta^{-1}(\psi(\vartheta,t)))} \le \varphi_1^{\Delta}(\vartheta) \int_{t_0}^{\varphi_2(t)} \tau(\varphi_1(\vartheta),\eta) f(\varphi_1(\vartheta),\eta) \nabla\eta. \tag{10}$$

Taking Δ-integral for Inequality (10), obtains

$$\begin{aligned}
\Upsilon(\psi(\vartheta,t)) &\le \Upsilon(\psi(\vartheta_0,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta \Delta\varsigma \\
&\le \Upsilon(a(\vartheta_0,t_0)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta \Delta\varsigma.
\end{aligned}$$

Since $(\vartheta_0, t_0) \in \Omega$ is chosen arbitrary,

$$\psi(\vartheta,t) \leq \Upsilon^{-1}\left[\Upsilon(a(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta\Delta\varsigma\right]. \tag{11}$$

From (11) and (8) we obtain the desired result (5). We carry out the above procedure with $\epsilon > 0$ instead of $a(\vartheta,t)$ when $a(\vartheta,t) = 0$ and subsequently let $\epsilon \to 0$. □

Remark 1. *If we take $\mathbb{T} = \mathbb{R}$, $\vartheta_0 = 0$ and $t_0 = 0$ in Lemma 1, then, inequality (4) becomes the inequality obtained in ([17] Lemma 2.1).*

3. Main Results

In the following theorems, with the help of Leibniz integral rule on time scales, Theorem 1 (item (iii)), and employing Lemma 1, we establish some new dynamic of Gronwall-Bellman-Pachpatte-type on time scales.

Theorem 2. *Let ϕ, a, f, φ_1 and φ_2 be as in Lemma 1. Let $\tau_1, \tau_2 \in C(\Omega, \mathbb{R}_+)$. If $\phi(\vartheta,t)$ satisfies*

$$\begin{aligned}\Theta(\phi(\vartheta,t)) \quad \leq \quad & a(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta)) \\ & + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\phi(\chi,\eta))\Delta\chi\Big]\nabla\eta\nabla\varsigma,\end{aligned} \tag{12}$$

for $(\vartheta,t) \in \Omega$, then

$$\phi(\vartheta,t) \leq \Theta^{-1}\left\{\Upsilon^{-1}\left(p(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta\Delta\varsigma\right)\right\}, \tag{13}$$

for $0 \leq \vartheta \leq \vartheta_1, 0 \leq t \leq t_1$, where Υ is defined by (6) and

$$p(\vartheta,t) = \Upsilon(a(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right)\nabla\eta\nabla\varsigma, \tag{14}$$

and $(\vartheta_1, t_1) \in \Omega$ is chosen so that

$$\left(p(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta\Delta\varsigma\right) \in \mathrm{Dom}\left(\Upsilon^{-1}\right).$$

Proof. By the same steps of the proof of Lemma 1 we can obtain (13), with suitable changes. □

Remark 2. *If we take $\tau_2(\vartheta,t) = 0$, then Theorem 2 reduces to Lemma 1.*

Corollary 1. *Let the functions ϕ, f, τ_1, τ_2, a, φ_1 and φ_2 be as in Theorem 2. Further suppose that $q > p > 0$ are constants. If $\phi(\vartheta,t)$ satisfies*

$$\begin{aligned}\phi^q(\vartheta,t) \quad \leq \quad & a(\vartheta,t) + \frac{q}{q-p}\int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\phi^p(\varsigma,\eta) \\ & + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\phi^p(\chi,\eta)\Delta\chi\Big]\nabla\eta\nabla\varsigma,\end{aligned} \tag{15}$$

for $(\vartheta,t) \in \Omega$, then

$$\phi(\vartheta,t) \leq \left\{p(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta\Delta\varsigma\right\}^{\frac{1}{q-p}}, \tag{16}$$

where

$$p(\vartheta,t)=(a(\vartheta,t))^{\frac{q-p}{q}}+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right)\nabla\eta\nabla\varsigma.$$

Proof. In Theorem 2, by letting $\Theta(\phi)=\phi^q,\zeta(\phi)=\phi^p$ we have

$$\Upsilon(v)=\int_{v_0}^{v}\frac{\Delta\varsigma}{\zeta(\Theta^{-1}(\varsigma))}=\int_{v_0}^{v}\frac{\Delta\varsigma}{\varsigma^{\frac{p}{q}}}\geq\frac{q}{q-p}\left(v^{\frac{q-p}{q}}-v_0^{\frac{q-p}{q}}\right),v\geq v_0>0,$$

and

$$\Upsilon^{-1}(v)\geq\left\{v_0^{\frac{q-p}{q}}+\frac{q-p}{q}v\right\}^{\frac{1}{q-p}},$$

we obtain the inequality (16). □

Theorem 3. *Under the hypotheses of Theorem 2. Suppose $\Theta,\zeta,\varpi\in C(\mathbb{R}_+,\mathbb{R}_+)$ to be nondecreasing functions with $\{\Theta,\zeta,\varpi\}(\phi)>0$ for $\phi>0$ and $\phi(\vartheta,t)$ satisfies*

$$\begin{aligned}\Theta(\phi(\vartheta,t)) \leq\ & a(\vartheta,t)+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta))\varpi(\phi(\varsigma,\eta))\\ &+\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\zeta(\phi(\chi,\eta))\Delta\chi\Big]\nabla\eta\nabla\varsigma,\end{aligned}\tag{17}$$

for $(\vartheta,t)\in\Omega$, then

$$\phi(\vartheta,t)\leq\Theta^{-1}\left\{\Upsilon^{-1}\left(F^{-1}\left[F(p(\vartheta,t))+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\right]\right)\right\},\tag{18}$$

for $0\leq\vartheta\leq\vartheta_1,0\leq t\leq t_1$, where Υ and p are as in (6) and (14) respectively, and

$$F(v)=\int_{v_0}^{v}\frac{\Delta\varsigma}{\varpi(\Theta^{-1}(\Upsilon^{-1}(\varsigma)))},v\geq v_0>0,\qquad F(+\infty)=+\infty,\tag{19}$$

and $(\vartheta_1,t_1)\in\Omega$ is chosen so that

$$\left[F(p(\vartheta,t))+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\right]\in\mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\vartheta,t)>0$. Fixing an arbitrary $(\vartheta_0,t_0)\in\Omega$, we define a positive and nondecreasing function $\psi(\vartheta,t)$ by

$$\begin{aligned}\psi(\vartheta,t) =\ & a(\vartheta_0,t_0)+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta))\varpi(\phi(\varsigma,\eta))\\ &+\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\zeta(\phi(\chi,\eta))\Delta\chi\Big]\nabla\eta\nabla\varsigma,\end{aligned}\tag{20}$$

for $0\leq\vartheta\leq\vartheta_0\leq\vartheta_1,0\leq t\leq t_0\leq t_1$, then $\psi(\vartheta_0,t)=\psi(\vartheta,t_0)=a(\vartheta_0,t_0)$ and

$$\phi(\vartheta,t)\leq\Theta^{-1}(\psi(\vartheta,t)).\tag{21}$$

Taking Δ-derivative for (20) with employing Theorem 1 (*iii*), gives

$$\begin{aligned}
\psi^{\Delta_\vartheta}(\vartheta,t) &= \varphi_1^\Delta(\vartheta)\int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)[f(\varphi_1(\vartheta),\eta)\zeta(\phi(\varphi_1(\vartheta),\eta))\varpi(\phi(\varphi_1(\vartheta),\eta)) \\
&\qquad + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\zeta(\phi(\chi,\eta))\Delta\chi\Big]\nabla\eta \\
&\leq \varphi_1^\Delta(\vartheta)\int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)\Big[f(\varphi_1(\vartheta),\eta)\zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big)\varpi\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) \\
&\quad + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\zeta\Big(\Theta^{-1}(\psi(\chi,\eta))\Big)\Delta\chi\Big]\nabla\eta \qquad (22) \\
&\leq \varphi_1^\Delta(\vartheta).\zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\varphi_2(t)))\Big)\times \\
&\quad \int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)\Big[f(\varphi_1(\vartheta),\eta)\varpi\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta.
\end{aligned}$$

From (22), we have

$$\begin{aligned}
\frac{\psi^{\Delta_\vartheta}(\vartheta,t)}{\zeta(\Theta^{-1}(\psi(\vartheta,t)))} &\leq \varphi_1^\Delta(\vartheta)\int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)\Big[f(\varphi_1(\vartheta),\eta)\varpi\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) \\
&\qquad + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta. \qquad (23)
\end{aligned}$$

Taking Δ-integral for (23), gives

$$\begin{aligned}
\Upsilon(\psi(\vartheta,t)) &\leq \Upsilon(\psi(\vartheta_0,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Theta^{-1}(\psi(\varsigma,\eta))\Big) \\
&\quad + \int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta\Delta\varsigma \\
&\leq \Upsilon(a(\vartheta_0,t_0)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Theta^{-1}(\psi(\varsigma,\eta))\Big) \\
&\quad + \int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta\Delta\varsigma.
\end{aligned}$$

Since $(\vartheta_0,t_0)\in\Omega$ is chosen arbitrarily, the last inequality can be rewritten as

$$\Upsilon(\psi(\vartheta,t)) \leq p(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\varpi\Big(\Theta^{-1}(\psi(\varsigma,\eta))\Big)\nabla\eta\Delta\varsigma. \qquad (24)$$

Since $p(\vartheta,t)$ is a nondecreasing function, an application of Lemma 1 to (24) gives us

$$\psi(\vartheta,t) \leq \Upsilon^{-1}\left(F^{-1}\left[F(p(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\right]\right). \qquad (25)$$

From (21) and (25) we obtain the desired inequality (18).

Now we take the case $a(\vartheta,t)=0$ for some $(\vartheta,t)\in\Omega$. Let $a_\epsilon(\vartheta,t)=a(\vartheta,t)+\epsilon$, for all $(\vartheta,t)\in\Omega$, where $\epsilon>0$ is arbitrary, then $a_\epsilon(\vartheta,t)>0$ and $a_\epsilon(\vartheta,t)\in C(\Omega,\mathbb{R}_+)$ be nondecreasing with respect to $(\vartheta,t)\in\Omega$. We carry out the above procedure with $a_\epsilon(\vartheta,t)>0$ instead of $a(\vartheta,t)$, and we get

$$\phi(\vartheta,t) \leq \Theta^{-1}\left\{\Upsilon^{-1}\left(F^{-1}\left[F(p_\epsilon(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\right]\right)\right\},$$

where

$$p_\epsilon(\vartheta,t) = \Upsilon(a_\epsilon(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right)\nabla\eta\nabla\varsigma.$$

Letting $\epsilon\to 0^+$, we obtain (18). The proof is complete. $\square$

Remark 3. *If we take* $\mathbb{T} = \mathbb{R}$, $\vartheta_0 = 0$ *and* $t_0 = 0$ *in Theorem 3, then, inequality (17) becomes the inequality obtained in ([17] Theorem 2.2 (A_2)).*

Corollary 2. *Let the functions* ϕ, a, f, τ_1, τ_2, φ_1 *and* φ_2 *be as in Theorem 2. Further suppose that* q, p *and* r *are constants with* $p > 0$, $r > 0$ *and* $q > p + r$. *If* $\phi(\vartheta, t)$ *satisfies*

$$\begin{aligned} \phi^q(\vartheta,t) \quad &\leq \quad a(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\phi^p(\varsigma,\eta)\phi^r(\varsigma,\eta) \\ &\quad + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\phi^p(\chi,\eta)\Delta\chi\bigg]\nabla\eta\nabla\varsigma, \end{aligned} \tag{26}$$

for $(\vartheta, t) \in \Omega$, *then*

$$\phi(\vartheta,t) \leq \left\{ [p(\vartheta,t)]^{\frac{q-p-r}{q-p}} + \frac{q-p-r}{q} \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta) \nabla\eta\Delta\varsigma \right\}^{\frac{1}{q-p-r}}, \tag{27}$$

where

$$p(\vartheta,t) = (a(\vartheta,t))^{\frac{q-p}{q}} + \frac{q-p}{q} \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) \left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi \right) \nabla\eta\nabla\varsigma.$$

Proof. An application of Theorem 3 with $\Theta(\phi) = \phi^q, \zeta(\phi) = \phi^p$, and $\varpi(\phi) = \phi^r$ yields the desired inequality (27). □

Theorem 4. *Under the hypotheses of Theorem 3. If* $\phi(\vartheta, t)$ *satisfies*

$$\begin{aligned} \Theta(\phi(\vartheta,t)) \quad &\leq \quad a(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta))\varpi(\phi(\varsigma,\eta)) \\ &\quad + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\phi(\chi,\eta))\varpi(\phi(\chi,\eta))\Delta\chi\bigg]\nabla\eta\nabla\varsigma, \end{aligned} \tag{28}$$

for $(\vartheta, t) \in \Omega$, *then*

$$\phi(\vartheta,t) \leq \Theta^{-1}\left\{ \Upsilon^{-1}\left(F^{-1}\left[p_0(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta)\nabla\eta\Delta\varsigma \right] \right) \right\}, \tag{29}$$

for $0 \leq \vartheta \leq \vartheta_1$, $0 \leq t \leq t_1$ *where*

$$p_0(\vartheta,t) = F(\Upsilon(a(\vartheta,t))) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) \left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi \right) \nabla\eta\nabla\varsigma,$$

and $(\vartheta_1, t_1) \in \Omega$ *is chosen so that*

$$\left[p_0(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta)\nabla\eta\Delta\varsigma \right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\vartheta, t) > 0$. Fixing an arbitrary $(\vartheta_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\vartheta, t)$ by

$$\begin{aligned} \psi(\vartheta,t) \quad &= \quad a(\vartheta_0,t_0) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta))\varpi(\phi(\varsigma,\eta)) \\ &\quad + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\phi(\chi,\eta))\varpi(\phi(\chi,\eta))\Delta\chi\bigg]\nabla\eta\nabla\varsigma \end{aligned}$$

for $0 \le \vartheta \le \vartheta_0 \le \vartheta_1$, $0 \le t \le t_0 \le t_1$, then $\psi(\vartheta_0, t) = \psi(\vartheta, t_0) = a(\vartheta_0, t_0)$, and

$$\phi(\vartheta, t) \le \Theta^{-1}(\psi(\vartheta, t)). \tag{30}$$

By the same steps as the proof of Theorem 3, we obtain

$$\begin{aligned} \psi(\vartheta, t) \quad \le \quad & \Upsilon^{-1}\Big\{\Upsilon(a(\vartheta_0, t_0)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta)\Big[f(\varsigma, \eta)\varpi\Big(\Theta^{-1}(\psi(\varsigma, \eta))\Big) \\ & + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\varpi\Big(\Theta^{-1}(\psi(\chi, \eta))\Big)\Delta\chi\Big]\nabla\eta\Delta\varsigma\Big\}. \end{aligned}$$

We define a nonnegative and nondecreasing function $v(\vartheta, t)$ by

$$\begin{aligned} v(\vartheta, t) \quad = \quad & \Upsilon(a(\vartheta_0, t_0)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta)\Big[\Big[f(\varsigma, \eta)\varpi\Big(\Theta^{-1}(\psi(\varsigma, \eta))\Big)\Big] \\ & + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\varpi\Big(\Theta^{-1}(\psi(\chi, \eta))\Big)\Delta\chi\Big]\nabla\eta\nabla\varsigma, \end{aligned}$$

then $v(\vartheta_0, t) = v(\vartheta, t_0) = \Upsilon(a(\vartheta_0, t_0))$,

$$\psi(\vartheta, t) \le \Upsilon^{-1}[v(\vartheta, t)], \tag{31}$$

and then, employing Theorem 1 (*iii*), we have

$$\begin{aligned} v^{\Delta\vartheta}(\vartheta, t) \quad \le \quad & \varphi_1^{\Delta}(\vartheta)\int_{t_0}^{\varphi_2(t)} \tau_1(\varphi_1(\vartheta), \eta)\Big[f(\varphi_1(\vartheta), \eta)\varpi\Big(\Theta^{-1}\Big(\Upsilon^{-1}(v(\varphi_1(\vartheta), t))\Big)\Big) \\ & \qquad + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \tau_2(\chi, \eta)\varpi\Big(\Theta^{-1}\Big(\Upsilon^{-1}(v(\chi, t))\Big)\Big)\Delta\chi\Big]\nabla\eta \\ \le \quad & \varphi_1^{\Delta}(\vartheta)\varpi\Big(\Theta^{-1}\Big(\Upsilon^{-1}(v(\varphi_1(\vartheta), \varphi_2(t)))\Big)\Big)\int_{t_0}^{\varphi_2(t)} \tau_1(\varphi_1(\vartheta), \eta)[f(\varphi_1(\vartheta), \eta) \\ & + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \tau_2(\chi, \eta)\Delta\chi\Big]\nabla\eta, \end{aligned}$$

or

$$\begin{aligned} \frac{v^{\Delta\vartheta}(\vartheta, t)}{\varpi(\Theta^{-1}(\Upsilon^{-1}(v(\vartheta, t))))} \quad \le \quad & \varphi_1^{\Delta}(\vartheta)\int_{t_0}^{\varphi_2(t)} \tau_1(\varphi_1(\vartheta), \eta)[f(\varphi_1(\vartheta), \eta) \\ & + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \tau_2(\chi, \eta)\Delta\chi\Big]\nabla\eta. \end{aligned}$$

Taking Δ-integral for the above inequality, gives

$$F(v(\vartheta, t)) \le F(v(\vartheta_0, t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta)\Big[f(\varsigma, \eta) + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi\Big]\nabla\eta\Delta\varsigma,$$

or

$$\begin{aligned} v(\vartheta, t) \quad \le \quad & F^{-1}\Big\{F(\Upsilon(a(\vartheta_0, t_0))) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta)[f(\varsigma, \eta) \\ & + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi\Big]\nabla\eta\Delta\varsigma\Big\}. \end{aligned} \tag{32}$$

From (30)–(32), and since $(\vartheta_0, t_0) \in \Omega$ is chosen arbitrarily, we obtain the desired inequality (29). If $a(\vartheta, t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\vartheta, t)$ and subsequently let $\epsilon \to 0$. The proof is complete. $\square$

Remark 4. *If we take $\mathbb{T} = \mathbb{R}$ and $\vartheta_0 = 0$ and $t_0 = 0$ in Theorem 4, then, inequality (28) becomes the inequality obtained in ([17] Theorem 2.2 (A_3)).*

Corollary 3. *Under the hypothesise of Corollary 2. If $\phi(\vartheta, t)$ satisfies*

$$\begin{aligned} \phi^q(\vartheta, t) \quad &\leq \quad a(\vartheta, t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta)[f(\varsigma, \eta)\phi^p(\varsigma, \eta)\phi^r(\varsigma, \eta) \\ &\quad + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\phi^p(\chi, \eta)\phi^r(\chi, \eta)\Delta\chi\Big]\nabla\eta\nabla\varsigma, \end{aligned} \tag{33}$$

for $(\vartheta, t) \in \Omega$, then

$$\phi(\vartheta, t) \leq \left\{ p_0(\vartheta, t) + \frac{q-p-r}{q} \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta) f(\varsigma, \eta)\nabla\eta\Delta\varsigma \right\}^{\frac{1}{q-p-r}}, \tag{34}$$

where

$$p_0(\vartheta, t) = (a(\vartheta, t))^{\frac{q-p-r}{q}} + \frac{q-p-r}{q} \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta) \left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi \right) \nabla\eta\nabla\varsigma.$$

Proof. An application of Theorem 4 with $\Theta(\phi) = \phi^q, \zeta(\phi) = \phi^p$, and $\varpi(\phi) = \phi^r$ yields the desired inequality (34). □

Theorem 5. *Under the hypotheses of Theorem 3. If $\phi(\vartheta, t)$ satisfies*

$$\begin{aligned} \Theta(\phi(\vartheta, t)) \quad &\leq \quad a(\vartheta, t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta)\varpi(\phi(\varsigma, \eta)) \times \\ &\quad \left[f(\varsigma, \eta)\zeta(\phi(\varsigma, \eta)) + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi \right] \nabla\eta\nabla\varsigma, \end{aligned} \tag{35}$$

for $(\vartheta, t) \in \Omega$, then

$$\phi(\vartheta, t) \leq \Theta^{-1}\left\{ \Upsilon_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\vartheta, t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta) f(\varsigma, \eta)\nabla\eta\Delta\varsigma \right] \right) \right\}, \tag{36}$$

for $0 \leq \vartheta \leq \vartheta_2, 0 \leq t \leq t_2$, where

$$\Upsilon_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varpi(\Theta^{-1}(\varsigma))}, v \geq v_0 > 0, \Upsilon_1(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{\varpi(\Theta^{-1}(\varsigma))} = +\infty, \tag{37}$$

$$F_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta\left[\Theta^{-1}\left(\Upsilon_1^{-1}(\varsigma)\right)\right]}, v \geq v_0 > 0, F_1(+\infty) = +\infty, \tag{38}$$

$$p_1(\vartheta, t) = \Upsilon_1(a(\vartheta, t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta) \left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi \right) \nabla\eta\nabla\varsigma, \tag{39}$$

and $(\vartheta_2, t_2) \in \Omega$ is chosen so that

$$\left[F_1(p_1(\vartheta, t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma, \eta) f(\varsigma, \eta)\nabla\eta\Delta\varsigma \right] \in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. Suppose that $a(\vartheta,t)>0$. Fixing an arbitrary $(\vartheta_0,t_0)\in\Omega$, we define a positive and nondecreasing function $\psi(\vartheta,t)$ by

$$\begin{aligned}\psi(\vartheta,t) &= a(\vartheta_0,t_0)+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\varpi(\phi(\varsigma,\eta))[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta)) \\ &\quad +\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta\nabla\varsigma,\end{aligned}$$

for $0\le\vartheta\le\vartheta_0\le\vartheta_2, 0\le t\le t_0\le t_2$, then $\psi(\vartheta_0,t)=\psi(\vartheta,t_0)=a(\vartheta_0,t_0)$,

$$\phi(\vartheta,t)\le\Theta^{-1}(\psi(\vartheta,t)). \tag{40}$$

Employing Theorem 1 (*iii*)

$$\begin{aligned}\psi^{\Delta_\vartheta}(\vartheta,t) &\le \varphi_1^{\Delta}(\vartheta)\int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)\eta\Big[\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big]\Big[f(\varphi_1(\vartheta),\eta)\zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) \\ &\quad+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta \\ &\le \varphi_1^{\Delta}(\vartheta)\eta\Big[\Theta^{-1}(\psi(\varphi_1(\vartheta),\varphi_2(t)))\Big]\int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)\Big[f(\varphi_1(\vartheta),\eta)\zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) \\ &\quad+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta,\end{aligned}$$

then

$$\begin{aligned}\frac{\psi^{\Delta_\vartheta}(\vartheta,t)}{\eta[\Theta^{-1}(\psi(\vartheta,t))]} &\le \varphi_1^{\Delta}(\vartheta)\int_{t_0}^{\varphi_2(t)}\tau_1(\varphi_1(\vartheta),\eta)\Big[f(\varphi_1(\vartheta),\eta)\zeta\Big(\Theta^{-1}(\psi(\varphi_1(\vartheta),\eta))\Big) \\ &\quad+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta.\end{aligned}$$

Taking Δ-integral for the above inequality, gives

$$\begin{aligned}\Upsilon_1(\psi(\vartheta,t)) &\le \Upsilon_1(\psi(0,t))+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Theta^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad+\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta\Delta\varsigma,\end{aligned}$$

then

$$\begin{aligned}\Upsilon_1(\psi(\vartheta,t)) &\le \Upsilon_1(a(\vartheta_0,t_0))+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Theta^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad+\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\nabla\eta\Delta\varsigma.\end{aligned}$$

Since $(\vartheta_0,t_0)\in\Omega$ is chosen arbitrary, the last inequality can be restated as

$$\Upsilon_1(\psi(\vartheta,t))\le p_1(\vartheta,t)+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\zeta\Big(\Theta^{-1}(\psi(\varsigma,\eta))\Big)\nabla\eta\Delta\varsigma. \tag{41}$$

It is easy to observe that $p_1(\vartheta,t)$ is a positive and nondecreasing function for all $(\vartheta,t)\in\Omega$, then an application of Lemma 1 to (41) yields the inequality

$$\psi(\vartheta,t)\le\Upsilon_1^{-1}\Big(F_1^{-1}\Big[F_1(p_1(\vartheta,t))+\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\Big]\Big). \tag{42}$$

From (42) and (40) we get the desired inequality (36).

If $a(\vartheta,t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\vartheta,t)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Remark 5. *If we take $\mathbb{T} = \mathbb{R}$ and $\vartheta_0 = 0$ and $t_0 = 0$ in Theorem 5, then, inequality (36) becomes the inequality obtained in ([17] Theorem 2.7).*

Theorem 6. *Under the hypotheses of Theorem 3 and let p be a nonnegative constant. If $\phi(\vartheta,t)$ satisfies*

$$\begin{aligned}\Theta(\phi(\vartheta,t)) \quad \leq \quad & a(\vartheta,t) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)\phi^p(\varsigma,\eta) \times \\ & \left[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta)) + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right]\nabla\eta\nabla\varsigma, \end{aligned} \tag{43}$$

for $(\vartheta,t) \in \Omega$, then

$$\phi(\vartheta,t) \leq \Theta^{-1}\left\{\Upsilon_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\right]\right)\right\}, \tag{44}$$

for $0 \leq \vartheta \leq \vartheta_2, 0 \leq t \leq t_2$, where

$$\Upsilon_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{[\Theta^{-1}(\varsigma)]^p}, v \geq v_0 > 0, \Upsilon_1(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{[\Theta^{-1}(\varsigma)]^p} = +\infty, \tag{45}$$

and F_1, p_1 are as in Theorem 5 and $(\vartheta_2,t_2) \in \Omega$ is chosen so that

$$\left[F_1(p_1(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\Delta\varsigma\right] \in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. An application of Theorem 5, with $\varpi(\phi) = \phi^p$ yields the desired inequality (44). □

Remark 6. *Taking $\mathbb{T} = \mathbb{R}$. The inequality established in Theorem 6 generalizes ([20] Theorem 1) (with $p = 1$, $a(\vartheta,t) = b(\vartheta) + c(t)$, $\vartheta_0 = 0$, $t_0 = 0$, $\tau_1(\varsigma,\eta)f(\varsigma,\eta) = h(\varsigma,\eta)$, and $\tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right) = g(\varsigma,\eta)$).*

Corollary 4. *Under the hypotheses of Theorem 6 and $q > p > 0$ be constants. If $\phi(\vartheta,t)$ satisfies*

$$\begin{aligned}\phi^q(\vartheta,t) \quad \leq \quad & a(\vartheta,t) + \frac{p}{p-q}\int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)\phi^p(\varsigma,\eta) \times \\ & \left[f(\varsigma,\eta)\zeta(\phi(\varsigma,\eta)) + \int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right]\nabla\eta\nabla\varsigma, \end{aligned} \tag{46}$$

for $(\vartheta,t) \in \Omega$, then

$$\phi(\vartheta,t) \leq \left\{F_1^{-1}\left[F_1(p_1(\vartheta,t)) + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\nabla\eta\nabla\varsigma\right]\right\}^{\frac{1}{q-p}}, \tag{47}$$

for $0 \leq \vartheta \leq \vartheta_2$, $0 \leq t \leq t_2$, where

$$p_1(\vartheta,t) = [a(\vartheta,t)]^{\frac{q-p}{q}} + \int_{\vartheta_0}^{\varphi_1(\vartheta)} \int_{t_0}^{\varphi_2(t)} \tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right)\nabla\eta\nabla\varsigma,$$

and F_1 is defined in Theorem 5.

Proof. An application of Theorem 6 with $\Theta(\phi(\vartheta,t)) = \phi^p$ to (46) yields the inequality (47); to save space we omit the details. □

Remark 7. *Taking* $\mathbb{T} = \mathbb{R}$, $\vartheta_0 = 0$, $t_0 = 0$, $a(\vartheta,t) = b(\vartheta) + c(t)$, $\tau_1(\varsigma,\eta)f(\varsigma,\eta) = h(\varsigma,\eta)$, *and* $\tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right) = g(\varsigma,\eta)$ *in Corollary 4 we obtain ([21] Theorem 1).*

Remark 8. *Taking* $\mathbb{T} = \mathbb{R}$, $\vartheta_0 = 0$, $t_0 = 0$, $a(\vartheta,t) = c^{\frac{p}{p-q}}$, $\tau_1(\varsigma,\eta)f(\varsigma,\eta) = h(\eta)$, *and* $\tau_1(\varsigma,\eta)\left(\int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right) = g(\eta)$ *and keeping t fixed in Corollary 4, we obtain ([22] Theorem 2.1).*

4. Application

In this following, we discus the boundedness of the solutions of the initial boundary value problem for partial delay dynamic equation of the form

$$(\Lambda^q)^{\nabla_\vartheta\nabla_t}(\vartheta,t) = A\left(\vartheta,t,\Lambda(\vartheta - h_1(\vartheta), t - h_2(t)), \int_{\vartheta_0}^{\vartheta} B(\varsigma,t,\Lambda(\varsigma - h_1(\varsigma),t))\Delta\varsigma\right), \tag{48}$$

$$\Lambda(\vartheta,t_0) = a_1(\vartheta), \Lambda(\vartheta_0,t) = a_2(t), a_1(\vartheta_0) = a_{t_0}(0) = 0,$$

for $(\vartheta,t) \in \Omega$, where $\Lambda, b \in C(\Omega,\mathbb{R}_+), A \in C(\Omega \times R^2, R), B \in C(\zeta \times R, R)$ and $h_1 \in C^1(\mathbb{T}_1,\mathbb{R}_+), h_2 \in C^1(\mathbb{T}_2,\mathbb{R}_+)$ are nondecreasing functions such that $h_1(\vartheta) \le \vartheta$ on $\mathbb{T}_1$, $h_2(t) \le t$ on $\mathbb{T}_2$, and $h_1^\Delta(\vartheta) < 1, h_2^\Delta(t) < 1$.

Theorem 7. *Assume that the functions a_1, a_2, A, B in (48) satisfy the conditions*

$$|a_1(\vartheta) + a_2(t)| \le a(\vartheta,t), \tag{49}$$

$$|A(\varsigma,\eta,\Lambda,\phi)| \le \frac{q}{q-p}\tau_1(\varsigma,\eta)\left[f(\varsigma,\eta)|\Lambda|^p + |\phi|\right], \tag{50}$$

$$|B(\chi,\eta,\Lambda)| \le \tau_2(\chi,\eta)|\Lambda|^p, \tag{51}$$

where $a(\vartheta,t), \tau_1(\varsigma,\eta), f(\varsigma,\eta)$, and $\tau_2(\chi,\eta)$ are as in Theorem 2, $q > p > 0$ are constants. If $\Lambda(\vartheta,t)$ satisfies (48), then

$$|\Lambda(\vartheta,t)| \le \left\{p(\vartheta,t) + M_1M_2\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\bar{\tau}_1(\varsigma,\eta)\bar{f}(\varsigma,\eta)\nabla\eta\Delta\varsigma\right\}^{\frac{1}{q-p}}, \tag{52}$$

where

$$\begin{aligned} p(\vartheta,t) \quad &= \quad (a(\vartheta,t))^{\frac{q-p}{q}} \\ &\quad +M_1M_2\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\bar{\tau}_1(\varsigma,\eta)\left(M_1\int_{\vartheta_0}^{\varsigma}\bar{\tau}_2(\chi,\eta)\Delta\chi\right)\nabla\eta\nabla\varsigma, \end{aligned}$$

and

$$M_1 = \underset{\vartheta\in I_1}{Max}\frac{1}{1 - h_1^\Delta(\vartheta)}, \qquad M_2 = \underset{t\in I_2}{Max}\frac{1}{1 - h_2^\Delta(t)},$$

and $\bar{\tau}_1(\gamma,\xi) = \tau_1(\gamma + h_1(\varsigma), \xi + h_2(\eta)), \bar{\tau}_2(\mu,\xi) = \tau_2(\mu, \xi + h_2(\eta))$,
$\bar{f}(\gamma,\xi) = f(\gamma + h_1(\varsigma), \xi + h_2(\eta))$.

Proof. If $\Lambda(\vartheta,t)$ is any solution of (48), then

$$\Lambda^q(\vartheta,t) = a_1(\vartheta) + a_2(t)$$
$$+\int_{\vartheta_0}^{\vartheta}\int_{t_0}^{t} A\left(\varsigma,\eta,\Lambda(\varsigma - h_1(\varsigma), \eta - h_2(\eta)), \int_{\vartheta_0}^{\varsigma} B(\chi,\eta,\Lambda(\chi - h_1(\chi),\eta))\Delta\chi\right)\nabla\eta\nabla\varsigma. \tag{53}$$

Using the conditions (49)–(51) in (53) we obtain

$$|\Lambda(\vartheta,t)|^q \leq a(\vartheta,t)+\frac{q-p}{q}\int_{\vartheta_0}^{\vartheta}\int_{t_0}^{t}\tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)|\Lambda(\varsigma-h_1(\varsigma),\eta-h_2(\eta))|^p + \int_{\vartheta_0}^{\varsigma}\tau_2(\chi,\eta)|\Lambda(\chi,\eta)|^p\Delta\chi\Big]\nabla\eta\nabla\varsigma. \tag{54}$$

Now making a change of variables on the right side of (54), $\varsigma-h_1(\varsigma)=\gamma,\eta-h_2(\eta)=\xi,\vartheta-h_1(\vartheta)=\varphi_1(\vartheta)$ for $\vartheta\in\mathbb{T}_1, t-h_2(t)=\varphi_2(t)$ for $t\in\mathbb{T}_2$ we obtain the inequality

$$|\Lambda(\vartheta,t)|^q \leq a(\vartheta,t)+\frac{q-p}{q}M_1M_2\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\bar{\tau}_1(\gamma,\xi)\Big[\bar{f}(\gamma,\xi)|\Lambda(\gamma,\xi)|^p + M_1\int_{\vartheta_0}^{\gamma}\bar{\tau}_2(\mu,\xi)|\Lambda(\mu,\eta)|^p\Delta\mu\Big]\nabla\xi\Delta\gamma. \tag{55}$$

We can rewrite the inequality (55) as follows:

$$|\Lambda(\vartheta,t)|^q \leq a(\vartheta,t)+\frac{q-p}{q}M_1M_2\int_{\vartheta_0}^{\varphi_1(\vartheta)}\int_{t_0}^{\varphi_2(t)}\bar{\tau}_1(\varsigma,\eta)\Big[\bar{f}(\varsigma,\eta)|\Lambda(\varsigma,\eta)|^p + M_1\int_{\vartheta_0}^{\varsigma}\bar{\tau}_2(\chi,\eta)|\Lambda(\chi,\eta)|^p\Delta\chi\Big]\nabla\eta\Delta\varsigma. \tag{56}$$

As an application of Corollary 1 to (56) with $\phi(\vartheta,t)=|\Lambda(\vartheta,t)|$ we obtain the desired inequality (52). $\square$

5. Conclusions

In this important article, we proved some new two dimensional dynamic inequalities of the Gronwall–Bellman–Pachpatte-type by employing the Leibniz integral rule on time scales. We discussed many extensions of the delay dynamic inequalities proven in [4,17] and generalised a few of those inequalities to a generic time scale. We also looked at the qualitative characteristics of various different dynamic equations' time-scale solutions. Besides that, in order to obtain some new inequalities as special cases, we also extended our inequalities to discrete and continuous calculus. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

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