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Applications on Double ARA–Sumudu Transform in Solving Fractional Partial Differential Equations

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Abstract: In this article, we apply the double ARA–Sumudu transformation (DARA-ST) to the nonlocal fractional Caputo derivative operator. We achieve interesting results and implement them to solve certain classes of fractional partial differential equations (FPDEs). Several physical applications are discussed and analyzed, such as telegraph, Klein–Gordon and Fokker–Planck equations. The new technique with DARA-ST is efficient and accurate in examining exact solutions of FPDEs. In order to show the applicability of the presented method, some numerical examples and figures are illustrated. A symmetry analysis is used to verify the results.

Keywords: double ARA–Sumudu transform; ARA transform; Sumudu transform; fractional partial differential equations; conformable fractional derivative

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1. Introduction

The fractional calculus generalizes the operations of differentiation and integration to noninteger orders. The fractional calculus has become an important tool for the study of some physical phenomena, engineering and science, such as electromagnetics, viscoelasticity, fluid mechanics, electrochemistry, biological population models, optics and signal processing. Furthermore, fractional calculus processes have become one of the most useful approaches in a variety of applied sciences to deal with certain properties of (long) memory effects. There are many definitions of fractional derivatives, such as Riemann–Liouville, Caputo, Caputo–Fabrizio, Atangana–Baleanu, conformable and the generalized fractional derivative [1–6].

Fractional partial differential equations appear in various applications of science, such as chemistry, physics, engineering and mathematics, which is why researchers have established many techniques for solving such equations such as the homotopy perturbation method, variation iteration method, Adomian decomposition method, finite difference method and others [7–15].

A new approach in this area has recently emerged, including combining some previous methods with integral transforms, such as Laplace transform, Sumudu transform, Elzaki transform and ARA transform. These composites generated some new methods, such as Laplace decomposition method, Laplace variation iteration method, Sumudu decomposition method, Sumudu homotopy perturbation method, Elzaki variation iteration method, Elzaki project differential transform method, Elzaki homotopy perturbation method, Elzaki decomposition method, ARA residual power series method, etc. [16–24]. The previous methods can be implemented to solve linear and nonlinear FPDEs.

The method of double integral transforms is a hot topic in recent research, and it basically depends on applying a single transformation twice on functions of two variables or applying two different transformations on the same function. This new approach is a powerful tool for solving PDEs. Although double integral transformations, their properties and theorems are recent studies, they have attracted the interest of many mathematicians.

Therefore, many researchers have studied new combinations, such as double Laplace transform, double Sumudu transform, double Elzaki transform, double Laplace-Sumudu transform and others [25–32].

Sumudu and ARA integral transformations are efficient tools for solving FDPEs [33–35], these transformations can be combined with other iteration methods to solve nonlinear problems.

Recently, a new combination between ARA transform and Sumudu transform was introduced in [35] and it is given by

$$\mathcal{G}_x S_t[f(x,t)] = F(s,u) = \frac{s}{u}\int_0^{\infty}\int_0^{\infty} e^{-sx-\frac{t}{u}} f(x,t)dxdt,\ s>0, u>0,$$

where $f(x,t)$ is a continuous function of two variables $x>0$ and $t>0$.

In this article, we implement DARA-ST to solve families of FPDEs of the form

$$\begin{gathered} A\ D_x^{\alpha} f(x,t) + B\ D_t^{\beta} f(x,t) + C\ Lf(x,t) = z(x,t), \quad x,t \geq 0, \\ n-1 < \alpha \leq n\,, \quad m-1 < \beta \leq m \text{ and } m,n \in \mathbb{N}, \end{gathered} \tag{1}$$

with the initial conditions (ICs)

$$\frac{\partial^j f(x,0)}{\partial t^j} = g_j(x), \quad j = 0,1,\cdots,m-1 \tag{2}$$

and the boundary conditions (BCs)

$$\frac{\partial^i f(0,t)}{\partial x^i} = h_i(t), \quad i = 0,1,\cdots,n-1, \tag{3}$$

where A, B and C are real constants, D_x^{α} and D_t^{β} are the fractional Caputo's derivatives with respect to x and t, respectively, L is a linear operator and $z(x,t)$ is the source function.

The main motivation of the present study is to expand the applications of DARA-ST by using it to solve FPDEs. We show the efficiency of the proposed method by applying the DARA-ST to several interesting applications to obtain the exact solutions and analyze the results. The novelty of this work arises from the establishment of a new simple formula for solving PDEs of fractional orders. The simplicity and applicability of this new formula is illustrated by handling some applications, where we use the new approach to solve some important FPDEs.

This article is organized as follows: in the next two sections, we present some basic definitions and theorems related to our work. A new algorithm for solving families of FPDEs using DARA-ST is presented in Section 4. Several examples are given in Section 5 to demonstrate the proposed technique. We illustrate the numerical evaluations of the results in Section 6. Finally, our results are discussed in Section 7.

2. Sumudu and ARA Transformations

In this section, we introduce the definition of Sumudu and ARA transforms with some properties.

Definition 1 [33]. *Sumudu transform of the function $f(x)$ is defined as*

$$S[f(x)] = \frac{1}{u}\int_0^{\infty} e^{-\frac{x}{u}} f(x)dx,\ u>0.$$

Definition 2 [34]. *ARA transform of order n of the function $f(x)$ is given by*

$$\mathcal{G}_n[f(x)](s) = F(n,s) = s\int_0^t t^{n-1}e^{-sx}f(x)dx,\ s>0,\ n\in\mathbb{N}$$

and the ARA transform of the function $f(x)$ of order one is defined as

$$\mathcal{G}_1[f(x)] = s\int_0^\infty e^{-sx}f(x)dx,\ s>0.$$

For simplicity, let us denote $\mathcal{G}_1[f(x)]$ by $\mathcal{G}[f(x)]$.

Theorem 1 [33]. *(The sufficient condition for the existence of Sumudu transform).*
If the function $f(x)$ is a piecewise continuous in every finite interval $0\le x\le\alpha$ and satisfies

$$|f(x)|\le Me^{\alpha x},\ M>0,$$

then Sumudu transform exists for all $\frac{1}{u}>\alpha$.

Theorem 2 [34]. *(The sufficient condition for the existence of ARA transform).*
If the function $f(x)$ is a piecewise continuous in every finite interval $0\le x\le\beta$ and satisfies

$$\left|x^{n-1}f(x)\right|\le Ke^{\beta x},\ K>0,$$

then ARA transform exists for all $s>\beta$.

Table 1 presents the fundamental properties of ARA and Sumudu transforms.

Table 1. ARA and Sumudu transforms for some functions.

$f(x)$	$\mathcal{G}[f(x)]=F(s)$	$S[f(x)]=F(u)$
1	1	1
x^a	$\frac{\Gamma(a+1)}{s^a}$	$\Gamma(a+1)\ u^a$
e^{ax}	$\frac{s}{s-a}$	$\frac{1}{1-au}$
$\sin ax$	$\frac{a\ s}{s^2+a^2}$	$\frac{a\ u}{1+a^2u^2}$
$\cos ax$	$\frac{s^2}{s^2+a^2}$	$\frac{1}{1+a^2u^2}$
$\sinh ax$	$\frac{as}{s^2-a^2}$	$\frac{a\ u}{1-a^2u^2}$
$\cosh ax$	$\frac{s^2}{s^2-a^2}$	$\frac{1}{1-a^2u^2}$
$f^{(n)}(x)$	$\mathcal{G}_1[f(t)](s)-\sum_{j=1}^{n}s^{n-j}f^{(j-1)}(0)$	$\frac{G(u)}{u^n}-\frac{f(0)}{u^n}-\ldots-\frac{f^{(n-1)}(0)}{u}$
$(f*g)(x)$	$\frac{\mathcal{G}_1[f(x)]\mathcal{G}_1[g(x)]}{s}$	$u\ S(f(x))S(g(x))$

3. Basic Definitions and Theorems of DARA-ST

In this section, we present the definition of DARA-ST of functions of two variables and the existence conditions and some basic properties of the new double transform are introduced.

Basic Definitions

Definition 3 [35]**.** *The DARA-ST of the continuous function $f(x,t)$ of two variables $x>0$ and $t>0$ is given by*

$$\mathcal{G}_x S_t[f(x,t)] = F(s,u) = \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}} f(x,t)dxdt,\ s>0, u>0,$$

provided the integral exists.

Clearly, the DARA-ST is linear, since

$$\mathcal{G}_x S_t[a\, f(x,t) + b\, g(x,t)] = a\, \mathcal{G}_x S_t[\, f(x,t)] + b\, \mathcal{G}_x S_t[\, g(x,t)],$$

where a and b are constants.

The inverse DARA-ST is given by

$$\mathcal{G}_x^{-1} S_t^{-1}[F(s,u)] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{sx}}{s} ds\ \frac{1}{2\pi i}\int_{\omega-i\infty}^{\omega+i\infty} \frac{e^{\frac{t}{u}}}{u} F(s,u)du\ .$$

Definition 4. *The Caputo derivatives of orders α and β of the function $f(x,t)$ with respect to x and t, respectively, are given by*

$$D_x^\alpha f(x,t) = \frac{\partial^\alpha f(x,t)}{\partial x^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-\varsigma)^{n-\alpha-1}\frac{\partial^n f(\varsigma,t)}{\partial\varsigma^n}d\varsigma, & n-1<\alpha<n, \\ \frac{\partial^n f}{\partial x^n}, & n=\alpha, \end{cases},\ n\in\mathbb{N},$$

$$D_t^\beta f(x,t) = \frac{\partial^\beta f(x,t)}{\partial t^\beta} = \begin{cases} \frac{1}{\Gamma(m-\beta)}\int_0^t (t-\tau)^{m-\beta-1}\frac{\partial^m f(x,\tau)}{\partial\tau^m}d\tau, & m-1<\beta<m, \\ \frac{\partial^m f}{\partial x^m}, & m=\beta, \end{cases},\ m\in\mathbb{N}.$$

Definition 5. *The Mittag-Leffler function is defined by*

$$E_{\alpha,\beta}(x) = \sum_{k=0}^\infty \frac{x^k}{\Gamma(k\alpha+\beta)},\ x\in\mathbb{C},\ Re(\alpha)>0,\ Re(\beta)>0.$$

The single ARA transform of $x^{\beta-1}E_{\alpha,\beta}(\lambda x^\alpha)$ takes the value

$$\mathcal{G}_x\left[x^{\beta-1}E_{\alpha,\beta}(\lambda x^\alpha)\right] = \frac{s^{\alpha-\beta+1}}{s^\alpha-\lambda},\ |\lambda|<|s^\alpha|.$$

The single Sumudu transform of $t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)$ takes the value

$$S_t\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha)\right] = \frac{u^{\beta-1}}{1-\lambda u^\alpha},\ |\lambda|<|u^\alpha|.$$

Definition 6. *A function $f(x,t)$ defined on $[0,X]\times[0,T]$ is called a function of exponential orders λ and γ as $x\to\infty$ and $t\to\infty$, if $\exists M>0$ such that $\forall x>X$ and $t>T$, we have*

$$|f(x,t)| \le Me^{\lambda x+\gamma t}.$$

Table 2 illustrates the values of DARA-ST for some basic functions.

Table 2. DARA-ST for some functions [35].

$f(x,t)$	$\mathcal{G}_x S_t[f(x,t)]=F(s,u)$
1	1
$x^a t^b$,	$s^{-a}\Gamma(a+1)u^b\Gamma(b+1)$
e^{ax+bt}	$\frac{s}{(s-a)(1-bu)}$
$e^{i\ (ax+bt)}$	$\frac{i\ s}{(s-i\ a)(bu+i)}$
$\sin(ax+bt)$	$\frac{s\ (a+b\ s\ u)}{(a^2+s^2)(b^2\ u^2+1)}$
$\cos(ax+bt)$	$\frac{s\ (s-abu)}{(a^2+s^2)(b^2\ u^2+1)}$
$\sinh(ax+bt)$	$\frac{s\ (a+b\ s\ u)}{(a^2-s^2)(b^2\ u^2-1)}$
$\cosh(ax+bt)$	$\frac{s\ (s+abu)}{(a^2-s^2)(b^2\ u^2-1)}$
$J_0\left(c\sqrt{xt}\right)$, J_0 is the zero order Bessel function	$\frac{4s}{4s+c^2u}$
$f(x-\delta,t-\epsilon)$ $H(x-\delta,t-\epsilon)$	$e^{-s\delta-\frac{\epsilon}{y}}F(s,u)$
$(f{*}{*}k)(x,t)$	$\left(\frac{u}{s}\right)F(s,u)K(s,u)$

Theorem 3 [35]**.** *(Existence condition). Let $f(x,t)$ be a continuous function on the region $[0,X)\times[0,T)$. If $f(x,t)$ is exponential orders λ and γ, then DARA-ST of $f(x,t)$ exists, for $Re[s]>\lambda$ and $Re\left[\frac{1}{u}\right]>\gamma$.*

Proof of Theorem 1. The DARA-ST definition yields that

$$\begin{aligned}|F(s,u)| &= \left|\frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}f(x,t)dx\,dt\right| \leq \frac{s}{u}\int_0^\infty\int_0^\infty e^{-sx-\frac{t}{u}}|f(x,t)|dx\,dt \\ &\leq \frac{M\,s}{u}\int_0^\infty e^{-(s-\lambda)x}dx\int_0^\infty e^{-(\frac{1}{u}-\gamma)t}dt = \frac{Ms}{u(s-\lambda)\left(\frac{1}{u}-\gamma\right)} \\ &= \frac{Ms}{(s-\lambda)(1-u\gamma)},\ \mathrm{Re}[s]>\lambda \text{ and } \mathrm{Re}\left[\frac{1}{u}\right]>\gamma.\end{aligned}$$

The proof is completed. □

Theorem 4 *(Derivative properties)* [35]**.** If $F(s,u)=\mathcal{G}_x S_t[f(x,t)]$, then

i. $\mathcal{G}_x S_t\left[\frac{\partial f(x,t)}{\partial x}\right]=sF(s,u)-sS_t[f(0,t)]$.

ii. $\mathcal{G}_x S_t\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{1}{u}F(s,u)-\frac{1}{u}\mathcal{G}_x[f(x,0)]$.

iii. $\mathcal{G}_x S_t\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=s^2F(s,u)-s^2S_t[f(0,t)]-sS_t[f_x(0,t)]$.

iv. $\mathcal{G}_x S_t\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{1}{u^2}F(s,u)-\frac{1}{u^2}\mathcal{G}_x[f(x,0)]-\frac{1}{u}\mathcal{G}_x[f_t(x,0)]$.

v. $\mathcal{G}_x S_t\left[\frac{\partial^2 f(x,t)}{\partial x\partial t}\right]=\frac{s}{u}(F(s,u)-S_t[f(0,t)]-\mathcal{G}_x[f(x,0)]+f(0,0))$.

Theorem 5 [35]**.** *(Convolution theorem).*
If $\mathcal{G}_x S_t[f(x,t)]=F(s,u)$ and $\mathcal{G}_x S_t[k(x,t)]=K(s,u)$, then

$$\mathcal{G}_x S_t[(f**k)(x,t)]=\left(\frac{u}{s}\right)F(s,u)K(s,u),$$

where

$$(f**k)(x,t)=\int_0^x\int_0^t f(x-\delta,t-\epsilon)k(\delta,\epsilon)d\delta d\epsilon.$$

4. Algorithm of DARA-ST Method

In this section, we present the technique of using DARA-ST to solve families of FPDEs. In order to achieve our goal, we have to calculate DARA-ST for the nonlocal Caputo fractional derivative in the following lemma.

4.1. DARA-ST of Fractional Derivatives

Lemma 1. *The DARA-ST for Caputo fractional derivatives can expressed as*

i. $\mathcal{G}_x S_t[D_x^{\alpha} f(x,t)] = s^{\alpha} F(s,u) - \sum_{i=0}^{n-1} s^{\alpha-i} S_t\left[\frac{\partial^i f(0,t)}{\partial x^i}\right], \ n-1 < \alpha \le n.$

ii. $\mathcal{G}_x S_t\left[D_t^{\beta} f(x,t)\right] = \frac{F(s,u)}{u^{\beta}} - \sum_{j=0}^{m-1} u^{j-\beta} \mathcal{G}_x\left[\frac{\partial^j f(x,0)}{\partial t^j}\right], \qquad m-1 < \beta \le m.$

Proof of Lemma 1. i. Applying DARA-ST on $D_x^{\alpha} f(x,t)$, we obtain

$$\mathcal{G}_x S_t[D_x^{\alpha} f(x,t)] = \mathcal{G}_x S_t\left[\frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\varsigma)^{n-\alpha-1} \frac{\partial^n f(\varsigma,t)}{\partial \varsigma^n} d\varsigma\right],$$

from the definition of the convolution, we have

$$\begin{aligned} \mathcal{G}_x S_t[D_x^{\alpha} f(x,t)] &= \mathcal{G}_x S_t\left[\frac{1}{\Gamma(n-\alpha)}\left(x^{n-\alpha-1} * \frac{\partial^n f(x,t)}{\partial x^n}\right)\right] \\ &= S_t\left[\frac{1}{\Gamma(n-\alpha)} \mathcal{G}_x\left[x^{n-\alpha-1} * \frac{\partial^n f(x,t)}{\partial x^n}\right]\right]. \end{aligned}$$

Using the convolution property of ARA transform in Table 1, we obtain

$$\mathcal{G}_x S_t[D_x^{\alpha} f(x,t)] = S_t\left[\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{s}\mathcal{G}_x\left[x^{n-\alpha-1}\right]\mathcal{G}_x\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]\right)\right].$$

Applying the derivative property of ARA transform in Table 1, we obtain

$$\begin{aligned} \mathcal{G}_x S_t[D_x^{\alpha} f(x,t)] \quad &= \tfrac{1}{\Gamma(n-\alpha)} S_t\left[\tfrac{\Gamma(n-\alpha)}{s^{n-\alpha}}\left(s^n \mathcal{G}_x[f(x,t)] - s^n f(0,t) - \dots \right.\right. \\ &\left.\left. - s\tfrac{\partial^{n-1} f(0,t)}{\partial x^{n-1}}\right)\right]. \end{aligned}$$

After simple computations, we obtain

$$\begin{aligned} \mathcal{G}_x S_t[D_x^{\alpha} f(x,t)] &= s^{\alpha} F(s,u) - s^{\alpha} S_t[f(0,t)] - \dots - s^{\alpha-n+1} S_t\left[\tfrac{\partial^{n-1} f(0,t)}{\partial x^{n-1}}\right] \\ &= s^{\alpha} F(s,u) - \sum_{i=0}^{n-1} s^{\alpha-i} S_t\left[\tfrac{\partial^i f(0,t)}{\partial x^i}\right]. \end{aligned}$$

ii. Applying DARA-ST on $D_t^{\beta} f(x,t)$, we obtain

$$\mathcal{G}_x S_t\left[D_t^{\beta} f(x,t)\right] = \mathcal{G}_x S_t\left[\frac{1}{\Gamma(m-\beta)} \int_0^t (t-\tau)^{m-\beta-1} \frac{\partial^m f(x,\tau)}{\partial \tau^m} d\tau\right],$$

from the definition of the convolutions, we have

$$\begin{aligned} \mathcal{G}_x S_t\left[D_t^{\beta} f(x,t)\right] &= \mathcal{G}_x S_t\left[\tfrac{1}{\Gamma(m-\beta)}\left(t^{m-\beta-1} * \tfrac{\partial^m f(x,t)}{\partial t^m}\right)\right] \\ &= \mathcal{G}_x\left[\tfrac{1}{\Gamma(m-\beta)} S_t\left[t^{m-\beta-1} * \tfrac{\partial^m f(x,t)}{\partial t^m}\right]\right]. \end{aligned}$$

Using the convolution property of Sumudu transform in Table 1, we obtain

$$\mathcal{G}_x S_t\left[D_t^{\beta} f(x,t)\right] = \mathcal{G}_x\left[\frac{1}{\Gamma(m-\beta)}\left(u\ S_t\left[t^{m-\beta-1}\right]\ S_t\left[\frac{\partial^m f(x,t)}{\partial t^m}\right]\right)\right].$$

Applying the derivative property of Sumudu transform in Table 1, we obtain

$$\begin{aligned}\mathcal{G}_x S_t\left[D_t^{\beta} f(x,t)\right] &= \tfrac{1}{\Gamma(m-\beta)}\mathcal{G}_x[\Gamma(m-\beta)u\ u^{m-\beta-1}(\tfrac{S_t[f(x,t)]}{u^m} - \tfrac{f(x,0)}{u^m} - \dots \\ &-\tfrac{1}{u}\tfrac{\partial^{m-1} f(x,0)}{\partial t^{m-1}})].\end{aligned}$$

After simple computations, we obtain

$$\begin{aligned}\mathcal{G}_x S_t\left[D_t^{\beta} f(x,t)\right] &= u^{-\beta} F(s,u) - u^{-\beta}\mathcal{G}_x[f(x,0)] - \dots - u^{m-\beta-1}\mathcal{G}_x\left[\tfrac{\partial^{m-1} f(x,0)}{\partial t^{m-1}}\right] \\ &== \tfrac{F(s,u)}{u^{\beta}} - \sum_{j=0}^{m-1} u^{j-\beta}\mathcal{G}_x\left[\tfrac{\partial^j f(x,0)}{\partial t^j}\right].\end{aligned}$$

□.

4.2. Solving FPDEs by DARA-ST

In this section, we apply DARA-ST to obtain solutions of some FPDEs. We consider the initial boundary value problems (1)–(3). To obtain the solution by the new approach, we apply DARA-ST on both sides of Equation (1), to obtain

$$\mathcal{G}_x S_t\left[A D_x^{\alpha} f(x,t)\right] + \mathcal{G}_x S_t[B\ D_t^{\beta} f(x,t)] + \mathcal{G}_x S_t[C\ L[f(x,t)]] = \mathcal{G}_x S_t[z(x,t)],$$

which implies

$$\begin{aligned}A&\left(s^{\alpha} F(s,u) - \sum_{i=0}^{n-1} s^{\alpha-i} S_t\left[\tfrac{\partial^i f(0,t)}{\partial x^i}\right]\right) \\ &+B\left(u^{-\beta} F(s,u) - \sum_{j=0}^{m-1} u^{-\beta+j}\mathcal{G}_x\left[\tfrac{\partial^j f(x,0)}{\partial t^j}\right]\right) \\ &+C\mathcal{G}_x S_t[\ L[f(x,t)]] = Z(s,u).\end{aligned} \tag{4}$$

Furthermore, we apply the single ARA transform to the ICs (3), and the single Sumudu transform to the BCs (2), to obtain

$$\mathcal{G}_x\left[\frac{\partial^j f(x,0)}{\partial t^j}\right] = \mathcal{G}\left[g_j(x)\right] = G_j(s),\ \forall j = 1,2,\cdots,m-1, \tag{5}$$

$$S_t\left[\frac{\partial^i f(0,t)}{\partial x^i}\right] = S[h_i(t)] = H_i(u),\ \forall i = 1,2,\cdots,n-1. \tag{6}$$

Simplifying Equation (4), and substituting the values in Equations (5) and (6), we have

$$\begin{aligned}F(s,u) = \tfrac{1}{As^{\alpha}+Bu^{-\beta}}&(A\sum_{i=0}^{n-1} s^{\alpha-i} H_i(u) + B\sum_{j=0}^{m-1} u^{-\beta+j} G_j(s) \\ &-C\mathcal{G}_x S_t[\ L[f(x,t)]] + Z(s,u)).\end{aligned} \tag{7}$$

Running the inverse DARA-ST, $\mathcal{G}_x^{-1} S_t^{-1}$ on both sides of Equation (7), we obtain

$$\begin{aligned}f(x,t) = \mathcal{G}_x^{-1} S_y^{-1}[\tfrac{1}{As^{\alpha}+Bu^{-\beta}}&(A\sum_{i=0}^{n-1} s^{\alpha-1} H_i(u) + B\sum_{j=0}^{m-1} u^{-\beta+j} G_j(s) \\ &-C\mathcal{G}_x S_t[\ L[f(x,t)]] + Z(s,u))],\end{aligned} \tag{8}$$

which is the solution of the target problem.

5. Illustrative Examples

In this section, we introduce some famous PDEs in mathematical physics such as Reaction–diffusion, advection–diffusion, telegraph equation, wave equation, Klein–Gordon and Fokker–Planck, we apply the new double transform on these equations and use it to obtain the solution of these problems and we implement the obtained formula in Equation (8) to solve FPDEs, to handle these problems using the new approach. The main goal here is to illustrate the applicability and ease of use of the new double transform.

5.1. Fractional Reaction–Diffusion Equation

Consider the fractional reaction–diffusion equation

$$A\, D_x^{\alpha} f(x,t) - D_t^{\beta} f(x,t) + C\, f(x,t) = 0, 1 < \alpha \leq 2, 0 < \beta \leq 1, \tag{9}$$

with the IC

$$f(x,0) = g_0(x), \tag{10}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{11}$$

Applying the single ARA transform on $g_0(x)$ in Equation (10), we obtain

$$G_0(s) = \mathcal{G}_x[g_0(x)].$$

Applying the single Sumudu transform on $h_0(t)$ and $h_1(t)$ in Equation (11), we obtain

$$H_0(u) = S_t[h_0(t)],$$

$$H_1(u) = S_t[h_1(t)].$$

Substituting $B = -1$, $L(f(x,t)) = f(x,t)$, $z(x,t) = 0$, $n = 2$, $m = 1$ and the functions $G_0(s)$, $H_0(s)$, $H_1(s)$ in the general formula in Equation (8), after simple computations, we obtain

$$f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{1}{As^{\alpha} - u^{-\beta} + C}\left(As^{\alpha} H_0(u) + A\, s^{\alpha-1} H_1(u) - u^{-\beta} G_0(s)\right)\right]. \tag{12}$$

Example 1. *Consider the heat diffusion equation*

$$f_{xx}(x,t) - D_t^{\beta} f(x,t) = 0, \quad 0 < \beta \leq 1, \tag{13}$$

with the IC

$$f(x,0) = \sin x, \tag{14}$$

and the BCs

$$f(0,t) = 0, \quad f_x(0,t) = E_{\beta}\left(-t^{\beta}\right). \tag{15}$$

Solution. Putting $A = 1$, $C = 0$, $\alpha = 2$, $G_0(s) = \frac{s}{s^2+1}$, $H_0(u) = 0$ and $H_1(u) = \frac{1}{1+u^{\beta}}$ in Equation (12), we obtain the solution of (13) as follows

$$f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{s}{(s^2+1)(1+u^{\beta})}\right] = \sin x E_{\beta}\left(-t^{\beta}\right).$$

Figure 1 represent the solution $f(x,t)$ of the heat diffusion Equation (13) with the IC (14) and the BCs (15).

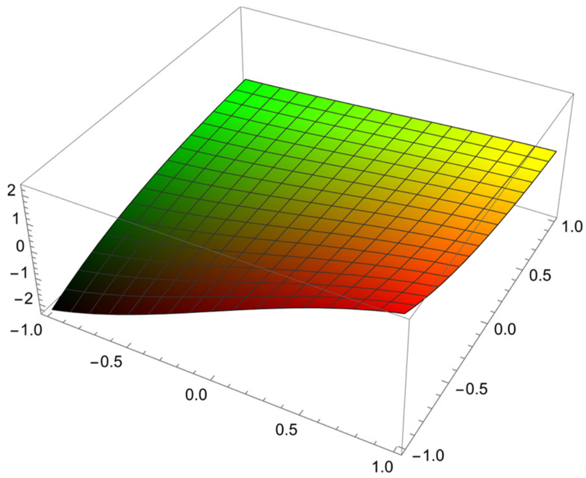


Figure 1. The surface graph of the solution $f(x,t)$ for the heat diffusion equation at $\alpha = 1$ for the problem in Example 1.

5.2. Fractional Advection–Diffusion Equation

Consider the fractional advection–diffusion equation

$$A\,D_x^{\alpha} f(x,t) - D_t^{\beta} f(x,t) + C\, f_x(x,t) = 0,\ 1 < \alpha \le 2,\ 0 < \beta \le 1, \tag{16}$$

with the IC

$$f(x,0) = g_0(x), \tag{17}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{18}$$

Applying the single ARA transform on $g_0(x)$ in Equation (17), we obtain

$$G_0(s) = \mathcal{G}_x[g_0(x)].$$

Applying the single Sumudu transform on $h_0(t)$ and $h_1(t)$ in Equation (18), we obtain

$$H_0(u) = S_t[h_0(t)],$$

$$H_1(u) = S_t[h_1(t)].$$

Substituting $B = -1$, $L[f(x,t)] = f_x(x,t)$, $z(x,t) = 0$, $n = 2$, $m = 1$ and the functions $G_0(s)$, $H_0(u)$, $H_1\ (u)$ in the general formula in Equation (8) and after simple computations, we obtain

$$\begin{aligned} f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}\Big[\tfrac{1}{As^{\alpha} - u^{-\beta} + Cs}\big(As^{\alpha} H_0(u) + A\, s^{\alpha-1} H_1(u) - u^{-\beta} G_0(s) \\ + CsH_0(u)\big)\Big]. \end{aligned} \tag{19}$$

Example 2. *Consider the fractional advection–diffusion equation*

$$f_{xx}(x,t) - D_t^{\beta} f(x,t) - f_x(x,t) = 0, \quad 0 < \beta \le 1, \tag{20}$$

with the IC

$$f(x,0) = e^{-x}, \tag{21}$$

and the BCs

$$f(0,t) = E_{\beta}\left(2t^{\beta}\right), \quad f_x(0,t) = -E_{\beta}\left(2t^{\beta}\right). \tag{22}$$

Solution. Putting $A = 1, \alpha = 2, C = -1, G_0(s) = \frac{s}{s+1}$, $H_0(u) = \frac{1}{1-2u^\beta}$ and $H_1(u) = \frac{-1}{1-2u^\beta}$ in Equation (19), we obtain the solution of (20) as follows

$$f(x,t) = \mathcal{G}_x^{-1} S_t^{-1} \left[\frac{u^2}{s^2-u^{-\beta}-s} \left(\frac{s^2}{1-2u^\beta} - \frac{s}{1-2u^\beta} - u^{-\beta}\frac{s}{s+1} - \frac{s}{1-2u^\beta} \right) \right]$$
$$= \mathcal{G}_x^{-1} S_t^{-1} \left[\frac{1}{s^2-u^{-\beta}-s} \left(\frac{s\left(s^2-u^{-\beta}-s\right)}{(s+1)\left(1-2u^\beta\right)} \right) \right]$$
$$= \mathcal{G}_x^{-1} S_t^{-1} \left[\frac{s}{(s+1)\left(1-2u^\beta\right)} \right] = e^{-x} E_\beta\left(2t^\beta\right).$$

Figure 2 represent the solution $f(x,t)$ of the advection–diffusion Equation (20) with the IC (21) and the BCs (22).

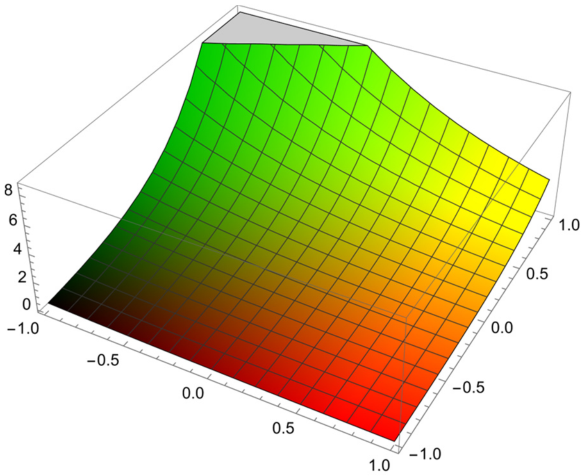


Figure 2. The surface graph of the solution $f(x,t)$ for the advection–diffusion equation at $\alpha = 1$ for the problem in Example 2.

5.3. Fractional Telegraph Equation

Consider the fractional telegraph equation

$$A\ D_x^\alpha f(x,t) + BD_t^\beta f(x,t) + C_0 f(x,t) + C_1 f_t(x,t) = 0, 1 < \alpha, \beta \leq 2, \tag{23}$$

with the ICs

$$f(x,0) = g_0(x), f_t(x,0) = g_1(x) \tag{24}$$

and the BCs

$$f(0,t) = h_0(t), f_x(0,t) = h_1(t). \tag{25}$$

Applying the single ARA transform on $g_0(x)$ and $g_1(x)$ in Equation (24), we obtain

$$G_0(s) = \mathcal{G}_x[g_0(x)],$$

$$G_1(s) = \mathcal{G}_x[g_1(x)].$$

Applying the single Sumudu transform on $h_0(t)$ and $h_1(t)$ in Equation (25), we obtain

$$H_0(u) = S_t[h_0(t)],$$

$$H_1(u) = S_t[h_1(t)].$$

Substituting $L[f(x,t)] = C_0 f(x,t) + C_1 f_t(x,t) = 0$, $z(x,t) = 0$, $n = m = 2$, and the functions $G_0(s)$, $G_1\ (s)$, $H_0(u)$, $H_1\ (u)$ in the general formula in Equation (8) and after simple computations, we obtain

$$\begin{aligned} f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}[\tfrac{1}{As^\alpha + Bu^{-\beta} + C_0 + C_1 u^{-1}} (As^\alpha H_0(u) + A\, s^{\alpha-1} H_1(u) \\ -Bu^{-\beta} G_0(s) + Bu^{-\beta+1} G_1(s) + C_1 u^{-1} G_0(s))]. \end{aligned} \tag{26}$$

Example 3. *Consider the telegraph equation*

$$D_x^\alpha(x,t) - f_{tt}(x,t) - f_t(x,t) - f(x,t) = 0, \quad 1 < \alpha \leq 2, \tag{27}$$

with the ICs

$$f(x,0) = E_\alpha(x^\alpha) + xE_{\alpha,2}(x^\alpha), \quad f_t(x,0) = -E_\alpha(x^\alpha) - xE_{\alpha,2}(x^\alpha) \tag{28}$$

and the BCs

$$f(0,t) = e^{-t}, \quad f_x(0,t) = e^{-t}. \tag{29}$$

Solution. Putting $A = 1$, $B = -1$, $\beta = 2$, $C_0 = -1$, $C_1 = -1$, $G_0(s) = \left(1 + \frac{1}{s}\right)\frac{s^\alpha}{s^\alpha - 1}$, $G_1(s) = -\left(1 + \frac{1}{s}\right)\frac{s^\alpha}{s^\alpha - 1}$, $H_0(u) = \frac{1}{1+u}$ and $H_1(u) = \frac{1}{1+u}$ in Equation (26), we obtain the solution of (27) as follows

$$\begin{aligned} f(x,t) = \mathcal{G}_x^{-1} S_t^{-1} \quad & \left[\frac{u^2}{s^\alpha - u^2 - u - 1}\left(\frac{s^\alpha}{1+u} + \frac{s^{\alpha-1}}{1+u} + u^{-2}\left(1 + \frac{1}{s}\right)\frac{s^\alpha}{s^\alpha - 1}\right)\right] \\ & = \mathcal{G}_x^{-1} S_t^{-1}\left[\left(\frac{1}{1+u}\right)\left(1 + \frac{1}{s}\right)\left(\frac{s^\alpha}{s^\alpha - 1}\right)\right] \\ & = e^{-t}(E_\alpha(x^\alpha) + xE_{\alpha,2}(x^\alpha)). \end{aligned}$$

Figure 3 represent the solution $f(x,t)$ of the telegraph Equation (27) with the ICs (28) and the BCs (29).

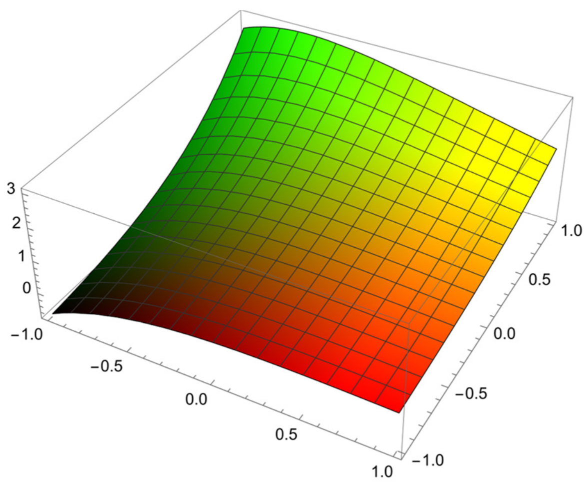


Figure 3. The surface graph of the solution $f(x,t)$ for the telegraph equation at $\alpha = 1$ for the problem in Example 3.

5.4. Fractional Wave Equation

Consider the fractional wave equation

$$A\ D_x^\alpha f(x,t) - D_t^\beta f(x,t) = 0, \quad 1 < \alpha, \beta \leq 2, \tag{30}$$

with the ICs

$$f(x,0) = g_0(x), \quad f_t(x,0) = g_1(x) \tag{31}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{32}$$

Applying the single ARA transform on $g_0(x)$ and $g_1(x)$ in Equation (31), we obtain

$$G_0(s) = \mathcal{G}_x[g_0(x)],$$

$$G_1(s) = \mathcal{G}_x[g_1(x)].$$

Applying the single Sumudu transform on $h_0(t)$ and $h_1(t)$ in Equation (32), we obtain

$$H_0(u) = S_t[h_0(t)],$$

$$H_1(u) = S_t[h_1(t)].$$

Substituting $B = -1$, $C = 0$, $z(x,t) = 0$, $n = m = 2$, and the functions $G_0(s)$, $G_1\ (s)$, $H_0(u)$, $H_1\ (u)$ in the general formula in Equation (8) and after simple computations, we obtain

$$f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}[\tfrac{1}{As^\alpha - u^{-\beta}}(As^\alpha H_0(u) + A\ s^{\alpha-1} H_1(u) - u^{-\beta} G_0(s) \\ -u^{-\beta+1} G_1(s))]. \tag{33}$$

5.5. Fractional Klein–Gordon Equation

Consider the fractional Klein–Gordon equation

$$D_x^\alpha f(x,t) - D_t^\beta f(x,t) + Cf(x,t) = z(x,t), \quad 1 < \alpha, \beta \le 2, \tag{34}$$

with the ICs

$$f(x,0) = g_0(x), \quad f_t(x,0) = g_1(x) \tag{35}$$

and the BCs

$$f(0,t) = h_0(t), \quad f_x(0,t) = h_1(t). \tag{36}$$

Applying the single ARA transform on $g_0(x)$ and $g_1(x)$ in Equation (35), we obtain

$$G_0(s) = \mathcal{G}_x[g_0(x)],$$

$$G_1(s) = \mathcal{G}_x[g_1(x)].$$

Applying the single Sumudu transform on $h_0(t)$ and $h_1(t)$ in Equation (36), we obtain

$$H_0(u) = S_t[h_0(t)],$$

$$H_1(u) = S_t[h_1(t)].$$

Substituting $A = 1,\ B = -1$, $L[f(x,t)] = f(x,t)$, $n = m = 2$, and the functions $G_0(s)$, $G_1\ (s)$, $H_0(u)$, $H_1\ (u)$ in the general formula in Equation (8) and after simple computations, we obtain

$$f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}[\tfrac{1}{s^\alpha - u^{-\beta} + C}(s^\alpha H_0(u) + s^{\alpha-1} H_1(u) - u^{-\beta} G_0(s) \\ -u^{-\beta+1} G_1(s) + Z(s,u))]. \tag{37}$$

Example 4. *Consider the fractional Klein–Gordon equation*

$$f_{xx}(x,t) - D_t^\beta f(x,t) + f(x,t) = 0, \quad 1 < \beta \le 2, \tag{38}$$

with the ICs

$$f(x,0) = \sin x + 1, \quad f_t(x,0) = 0, \tag{39}$$

and the BCs

$$f(0,t) = E_\beta\left(t^\beta\right), \quad f_x(0,t) = 1. \tag{40}$$

Solution. Putting $C = 1, \alpha = 2$, $z(x,t) = 0$, $G_0(s) = \frac{s}{s^2+1} + 1$, $G_1\ (s) = 0$, $H_0(u) = \frac{1}{1-u^\beta}$, $H_1\ (u) = 1$ in Equation (37), we obtain

$$\begin{aligned} f(x,t) = \mathcal{G}_x^{-1} S_t^{-1} \quad & \left[\frac{1}{s^2-u^{-\beta}+1}\left(\frac{s^\alpha}{1-u^\beta} + s - \frac{u^{-\beta}s}{s^2+1} - u^{-\beta}\right)\right] \\ & = \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{s - su^\beta + s^2 + 1}{(s^2+1)\left(1-u^\beta\right)}\right] = \mathcal{G}_x^{-1} S_t^{-1}\left[\frac{s}{s^2+1} + \frac{1}{1-u^\beta}\right] \\ & = \sin x + E_\beta\left(t^\beta\right). \end{aligned} \tag{41}$$

Figure 4 represent the solution $f(x,t)$ of the Klein–Gordon (38) with the ICs (39) and the BCs (40).

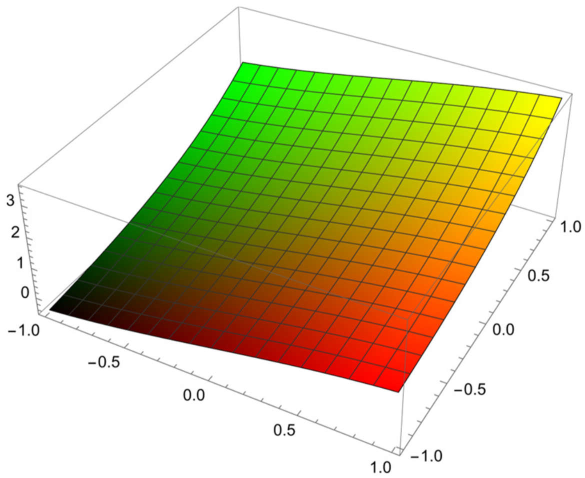


Figure 4. The surface graph of the solution $f(x,t)$ for the Klein–Gordon equation at $\alpha = 1$ for the problem in Example 4.

5.6. Fractional Fokker–Planck Equation

Consider the fractional Fokker–Planck equation

$$D_x^\alpha f(x,t) - D_t^\beta f(x,t) + f_x(x,t) = 0, 1 < \alpha \leq 2,\ 0 < \beta \leq 1, \tag{42}$$

with the IC

$$f(x,0) = g_0(x), \tag{43}$$

and the BCs

$$f(0,t) = h_0(t), f_x(0,t) = h_1(t). \tag{44}$$

Applying the single ARA transform on $g_0(x)$ in Equation (43), we obtain

$$G_0(s) = \mathcal{G}_x[g_0(x)].$$

Applying the single Sumudu transform on $h_0(t)$ and $h_1(t)$ in Equation (44), we obtain

$$H_0(u) = S_t[h_0(t)],$$

$$H_1(u) = S_t[h_1(t)].$$

Substituting $A = 1,\ B = -1, C = 1, L[f(x,t)] = f_x(x,t), z(x,t) = 0, n = 2,\ m = 1,$ and the functions $G_0(s)$, $H_0(u)$, $H_1\ (u)$ in the general formula in Equation (8) and after simple computations, we obtain

$$f(x,t) = \mathcal{G}_x^{-1} S_t^{-1}[\tfrac{1}{s^\alpha - u^{-\beta} + s}(s^\alpha H_0(u) + s^{\alpha-1} H_1(u) - u^{-\beta} G_0(s) \\ + s H_0(u))]. \tag{45}$$

Example 5. *Consider the fractional Fokker–Planck equation*

$$f_{xx}(x,t) - D_t^\beta f(x,t) + f_x(x,t) = 0, 0 < \beta \leq 1, \tag{46}$$

with the IC

$$f(x,0) = x, \tag{47}$$

and the BCs

$$f(0,t) = \frac{t^\beta}{\Gamma(1+\beta)}, f_x(0,t) = 1. \tag{48}$$

Solution. Putting $\alpha = 2$, $G_0(s) = \frac{1}{s}$, $H_0(u) = u^\beta$, $H_1\ (u) = 1$ in Equation (45), we obtain the solution of (46) as follows

$$\begin{aligned} f(x,t) = \mathcal{G}_x^{-1} S_t^{-1} \quad & \left[\tfrac{1}{s^2 - u^{-\beta} + s}\left(s^2 u^\beta + s - \tfrac{u^{-\beta}}{s} + s u^\beta\right)\right] \\ & = \mathcal{G}_x^{-1} S_t^{-1}\left[\tfrac{1}{s^2 - u^{-\beta} + s}\left(\left(s^2 - u^{-\beta} + s\right)\left(\tfrac{1}{s} + u^\beta\right)\right)\right] \\ & = \mathcal{G}_x^{-1} S_t^{-1}\left[\tfrac{1}{s} + u^\beta\right] = x + \tfrac{t^\beta}{\Gamma(1+\beta)}. \end{aligned} \tag{49}$$

Figure 5 represent the solution $f(x,t)$ of the Fokker–Planck (46) with the IC (47) and the BCs (48).

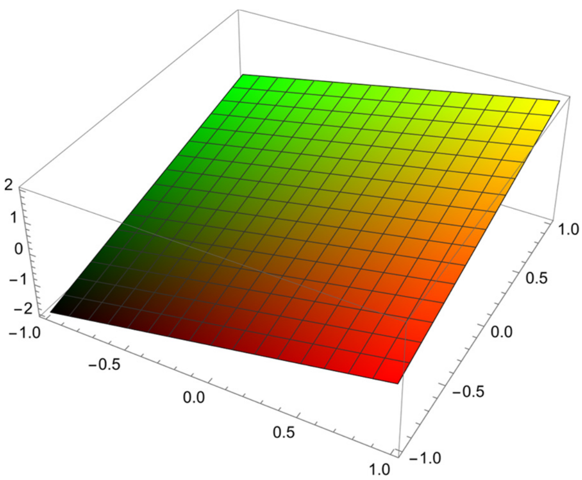


Figure 5. The surface graph of the solution $f(x,t)$ for the Fokker–Planck equation at $\alpha = 1$ for the problem in Example 5.

6. Numerical Simulations

In this section, we illustrate the numerical evaluation of the solutions obtained by solving the given FPDEs. We also discuss the numerical behavior of the results when solving FPDEs, then we compare it with the solution of the equation of integer order.

The solutions of Examples 1, 2 and 5 are simply computed when $\beta = 1$. We examine the numerical solutions of different values of $\beta = 0.95,\ 0.85$ and 0.75. As a result, we notice that, with choosing different values of β, the obtained fractional solutions are in coordination with the closed form of the solution when $\beta = 1$, as illustrated in Figure 6.

Moreover, it clearly implies that as β approaches 1, the obtained solutions of the FPDEs approach the exact solutions obtained in the integer case.

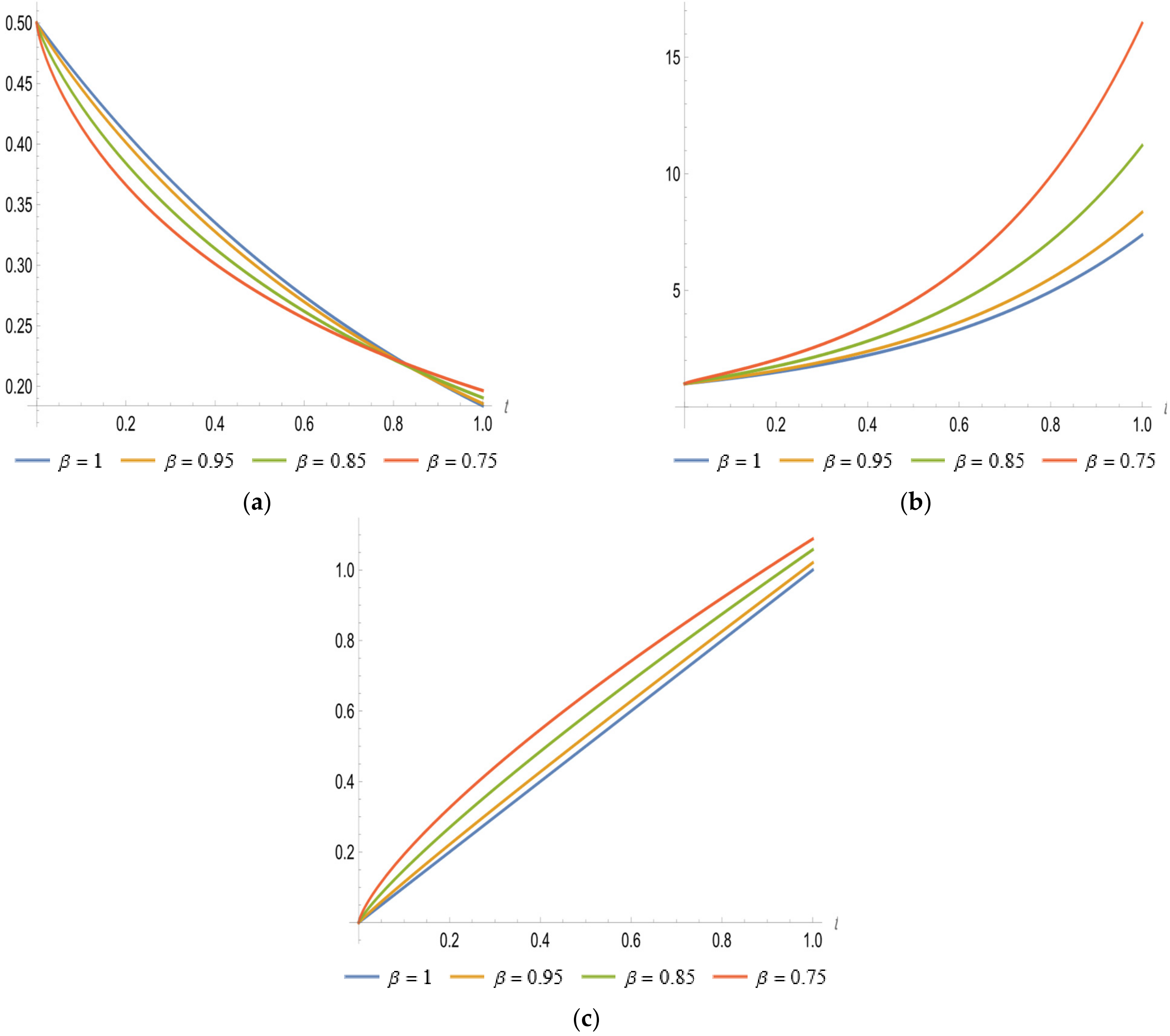
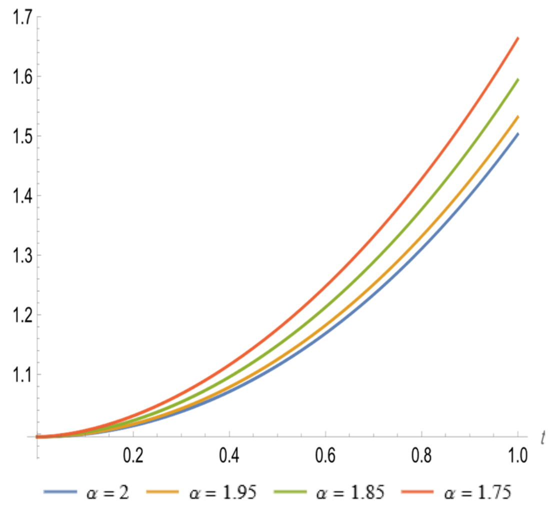
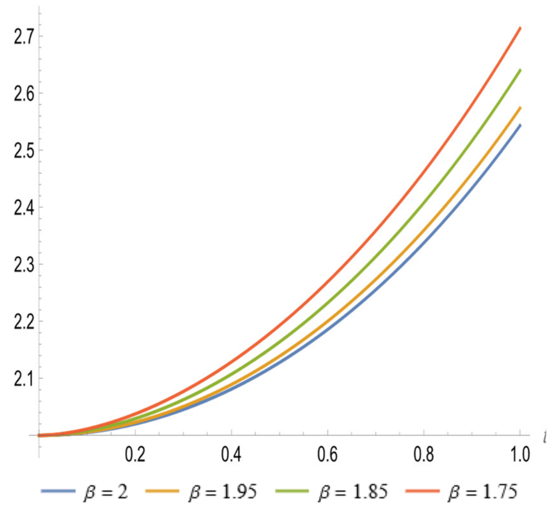


Figure 6. (**a**) Plots of the exact solution when $\beta = 1$ and different values of $\beta = 0.95,\ 0.85$ and 0.75 of Examples 1. (**b**) Plots of the exact solution when $\beta = 1$ and different values of $\beta = 0.95,\ 0.85$ and 0.75 of Examples 2. (**c**) Plots of the exact solution when $\beta = 1$ and different values of $\beta = 0.95,\ 0.85$ and 0.75 of Examples 5.

The solution of Example 3 when $\alpha = 2$ and the solution of Example 4 when $\beta = 2$ are simply computed. We examine the numerical solutions of different values of $\alpha = 1.75,\ 1.85,\ 1.95$ in Example 3 and $\beta = 1.75,\ 1.85,\ 1.95$ in Example 4. As a result, we note that, with choosing different values of α and β, the obtained fractional solutions are in coordination with the closed forms of the solutions when $\alpha = 2$ and $\beta = 2$, as illustrated in Figure 7.



(**a**)



(**b**)

Figure 7. (**a**) Plots of the exact solution when $\alpha = 2$ and different values of $\alpha = 1.95,\ 1.85$ and 1.75 of Example 3. (**b**) Plots of the exact solution when $\beta = 2$ and different values of $\beta = 1.95,\ 1.85$ and 1.75 of Example 4.

Moreover, we mention that as α and β approach the close integer orders, the obtained solutions of the FPDEs approach the exact solutions in the integer case.

7. Conclusions

In this research, DARA-ST is applied to the Caputo fractional derivative to obtain a new interesting formula, that is implemented to solve families of FPDEs. We have presented a new method to obtain exact solutions of these equations. We show the reliability and efficiency of the proposed method by presenting some interesting physical applications. In the future, we will pair DARA-ST with some iteration methods to solve nonlinear FPDEs, such as nonlinear telegraph equation, nonlinear wave equation, nonlinear Klein–Gordon and nonlinear Fokker–Planck. In addition, researchers can use new definitions of FC such as the generalized fractional derivative and others to search and obtain new results on transformations.

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