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A Robust and Non-Fragile Observer Design for Nonlinear Fractional-Order Systems

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Abstract: The challenge of developing observers for classical integer-order systems that are both resilient and non-fragile has received a lot of attention in the literature. However, only a few articles have addressed the topic of developing observers for Fractional-Order (FO) systems that are both H-infinity (H_∞) and non-fragile. The current work handles the Caputo fractional-order systems as the first work, to our knowledge, which treats such problems. The authors provide a novel result for building non-fragile and robust observers for nonlinear Caputo fractional-order systems. For this, the H_∞ performance method is utilized. Simulations for a numerical example confirm the efficacy of the suggested technique. The primary advantage of the current work is that it is the first to address the Caputo fractional-order system problem.

Keywords: H_∞ performance; non-fragile; fractional-order calculus; Caputo derivative; observer; Lipschitz condition

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1. Introduction

A Fractional-Order System (FOS) is one among nonlinear systems that may be characterized utilizing a non-integer derivative [1]. Such systems are referred to as fractional dynamics. Derivatives and integrals of fractional-order systems are utilized to depict events that may be investigated in a variety of ways, for example, power-law long-range dependence, power-law nonlocality [2], or fractal features. Many areas, including biology, physics, viscoelasticity, electrochemistry, and chaotic systems, have employed fractional-order calculus to explore system dynamics [1]. In recent decades, advances in science, mathematics [3], and engineering have boosted the use of fractional calculus in many areas of control theory, such as stability [4], finite-time stability (FTS) [5], stabilization [6], observer design, and fault estimation [7].

In systems engineering, observers are the crucial component for generating system states, either totally (complete observers) or substantially (reduced-order observers). Researchers have utilized these structures for additional reasons besides predicting states, such as identifying defects [8] and solving control issues [8,9]. Typically, data from real-world applications have two types of distributions: symmetric and asymmetric. For instance, social networks and protein networks frequently exhibit symmetric relationships. In other words, there is a clear asymmetry between the probability distributions of accidents and typical scenarios in traffic statistics. Consequently, it is necessary to examine symmetry and asymmetry issues while analyzing incomplete huge data. By removing physical sensors and replacing them with software sensors, the inclusion of the observer in symmetric and asymmetric distributions may minimize the complexity of networks and improve the efficiency of certain calculations. In this situation, the usage of observers can be a useful tool for resolving several issues, for example [10,11].

On the other hand, a state observer is used to rebuild the states of a dynamic system and has numerous essential applications, including feedback control implementation, system supervision, gas-fired furnace systems, and fault detection. A notion for non-fragile control is presented as a solution to the dilemma of how to build a feedback control that is insensitive to errors in its gains. The relevance of this method relative to others lies in the fact that the error system has a higher margin of stability than the classical system, allowing the observer to achieve favorable performance. It should be noted that spectators come in a variety of shapes and sizes. Non-fragile observers are one of these categories, and they have a positive impact on performance. The estimation is non-fragile if the error does not diverge when the observer gain is affected by an external disturbing element. Alternatively, whenever the system under investigation includes a perturbation vector, the design of the observer is ineffective at quickly reconstructing the states, mandating the use of special techniques. The H_∞ observer is one of these approaches.

H_∞ and/or non-fragile approaches for classical systems have been a topic of considerable research for decades. The construction of such methods for delayed time—variant switching systems was the subject of [12]. Another work [13] focuses on developing an observer-based control which is non-fragile. On the other hand, the authors proposed in [14] a sliding mode discrete time-delay H_∞ observer. Subsequently in [15], for linear systems with unknown inputs, certain experts developed an integrated H_∞ observer. A further noteworthy article [16] offered a H_∞ observer technique for linear parameter varying systems with uncertainties and validated their findings for a battery-powered wheeled vehicle.

It is worth noting that integer-order calculation is inadequate for examining a wide array of different systems whose behaviors are better described by fractional order. Fractional-order calculus, for example, has been effectively used to simulate electrochemical systems [17] and heat transfer systems [18]. In a related vein, in recent years, the application of fractional-order equations in stability theory has increased dramatically [19,20]. For fractional-order systems, dealing with non-fragility and/or H_∞ observer development is a significant topic of inquiry compared to the integer-order situation. In fact, scholars have only published a few papers in this area. Furthermore, only a few publications have addressed the problem of constructing observers for fractional-order systems that are both H_∞ and non-fragile [21,22].

This approach estimates the states of nonlinear fractional-order systems using the Caputo derivative. The results are surprising. It is possible to employ the non-fragile observer framework in an effort to allow a system-modeling approach adaptable to possible observer gain in an input disturbance-free and input disturbance-dependent scenario, included in a generalized theorem. An H_∞ performance is supplied when the system description incorporates unknown input disturbances, and the H_∞ observer is demonstrated to properly predict the states.

Based on the above discussions, the novelty and implications of this research can be listed as follows.

- As stated above, only a few papers have addressed the issue of designing H_∞ and non-fragile observers for FO systems. The key benefit of the current study is that it is the first to address the problem of the Caputo fractional-order system.
- The suggested strategy exploits a variety of mathematical properties and unique situations. The authors feel that this increases the value and significance of the current work.

The rest of the paper is generally divided into three categories. Preliminaries and fundamental principles of the FO calculus are provided in Section 2. The paper's core challenge is then mathematically stated. The non-fragile state estimate technique is shown in Section 3. The disturbance-free as well as the disturbance-dependent cases are also discussed. Finally, in Section 4, a numerical simulation investigation is carried out to corroborate the findings.

2. Some Preliminaries and a Description of the Problem

2.1. Preliminaries

This section reviews numerous essential concepts and lemmas in fractional calculus. We begin with the following definition of the fractional integral of Riemann–Liouville.

Definition 1 ([1]). *Let us say that we have* $[a,\ b] \subset \mathbb{R}$*. A fractional integral of* Riemann–Liouville *of a function* $x \in L^1([a,\ b])$ *with order* $\alpha > 0$ *is described this way:*

$$I_a^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} x(\tau) d\tau,\ t \in [a,b]$$

A generalizing factorial for non-integer parameters is represented by the Gamma function $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1} dt$*.*

The Caputo fractional derivative is defined as follows:

Definition 2 ([1]). *For* $x(t)$ *an absolutely continuous function and* $0 < \alpha < 1$*, the Caputo fractional derivative is defined as:*

$${}^C D_{a,t}^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \frac{d}{d\tau} x(\tau) d\tau$$

The definition that follows describes a commonly used function in the solution of fractional-order problems. This function may be thought of as a broadening of the exponential function.

Definition 3 describes the Mittag-Leffler function.

Definition 3 ([1]). *The two-parameter Mittag-Leffler function is given as:*

$$E_{\alpha,\mu}(\xi) = \sum_{k=0}^{+\infty} \frac{\xi^k}{\Gamma(k\alpha + \mu)},$$

with $\alpha > 0$*,* $\mu > 0$*,* $\xi \in \mathbb{C}$*. When* $\mu = 1$*, one has* $E_\alpha(\xi) = E_{\alpha,1}(\xi)$*; in addition,* $E_{1,1}(\xi) = e^\xi$*.*

In what follows, an important Lemma is used to demonstrate the stability of fractional-order systems.

Lemma 1 ([23]). *Let* $P \in \mathbb{R}^{n\times n}$ *be a constant symmetric and positive definite matrix and* $\in [0,1]$*. As a result, the following link exists:*

$$\frac{1}{2} {}^C D_{t_0,t}^\alpha \Big(x^T(t) P x(t) \Big) \ \leq\ x^T(t) P^C D_{t_0,t}^\alpha x(t).$$

The authors use a Caputo derivative order to solve a system of fractional differential equations when $0 < \alpha \leq 1$:

$${}^C D_{t_0,t}^\alpha x(t) = \psi(t,x)\ ;\ t > t_0 x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$, $\psi : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a smooth nonlinear function such that we have the existence and uniqueness of the global solution, and ψ is supposed to satisfy $\psi(t,\ 0) = 0$ for every $t\ \geq\ 0$.

The following theorem is essential for the Mittag-Leffler global stability of Caputo fractional order systems.

Theorem 1 ([24]). *Assume that $x = 0$ is the equilibrium of system (1):*

Assume that $V : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz with respect to x and a continuously differentiable function and such that:

$$\mu_1 ||x||^c \leq V(t, x) \leq \mu_2 ||x||^{cd}, \\ {}^C D^{\alpha}_{t_0,t} V(t, x) \leq \mu_3 ||x||^{cd},$$

where $t \geq t_0$, $x \in \mathbb{R}^n$, $\alpha \in (0, 1)$, and μ_1, μ_2, μ_3, c and d consist of arbitrarily positive numbers. Then, $x = 0$ is globally Mittag-Leffler stable.

The below Lemma is used as a tool for the proof of the main results, named the Schur complement Lemma.

Lemma 2 ([25]). *Consider a set of constant matrices M, N, and$Q$, of appropriate dimensions, where $M$and$Q$are symmetric, then:*

$$\begin{cases} Q > 0 \\ M + \mathrm{N}^{\mathrm{T}}\mathrm{Q}^{-1}N < 0 \end{cases} \text{if and only if} \begin{bmatrix} \mathrm{M} & \mathrm{N}^{\mathrm{T}} \\ \mathrm{N} & -\mathrm{Q} \end{bmatrix} < 0$$

2.2. Problem Statement

Consider the following nonlinear Caputo fractional-order system with $\in [0, 1]$:

$$\begin{cases} {}^C D^{\alpha}_{t_0,t} x(t) = Ax(t) + \varphi(x, u) + \omega(t) \\ y(t) = Cx(t), \end{cases} \tag{2}$$

where $\omega \in \mathbb{R}^n$ is an input disturbance vector, $y \in \mathbb{R}^m$ is the output vector, $u \in \mathbb{R}^q$ is the input vector, and $x \in \mathbb{R}^n$ is the state vector. $C \in \mathbb{R}^{m\times n}$and $A \in \mathbb{R}^{n\times n}$ are two known constant matrices and $\varphi(x, u)$ is the nonlinearity in system (2). The input disturbance $\omega(t)$ is considered such that the following condition (3) is satisfied:

$$\int_{t_0}^{+\infty} \omega^T(t)\omega(t)dt < +\infty \tag{3}$$

Define the non-fragile Caputo fractional-order observer, given by (4):

$$\begin{cases} {}^C D^{\alpha}_{t_0,t} \hat{x}(t) = A\hat{x}(t) + \varphi(\hat{x}, u) + (L - \Delta L(t))(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t), \end{cases} \tag{4}$$

where L is the observer gain matrix, $\hat{y}(t)$ is the output vector estimation, $\hat{x}(t)$ is the state estimate vector, and the term $\Delta L(t)$ denotes a supplementary perturbation on the gain of the observer. Let $e(t) = x(t) - \hat{x}(t)$ be the estimation error. The following formula therefore governs the error dynamics:

$${}^C D^{\alpha}_{t_0,t} e(t) = (A - LC)e(t) + \Delta L(t)Ce(t) + \Delta\varphi(t) + \omega(t) \tag{5}$$

where $\Delta\varphi(t) = \varphi(x, u) - \varphi(\hat{x}, u)$. The authors define the H_∞ performance metric for system (5), given a positive scalar, as:

$$J_\infty = \int_{t_0}^{+\infty} \left(e(t)^T e(t) - \frac{\gamma}{2}\omega(t)^T \omega(t) \right) dt, \tag{6}$$

where γ is a positive constant.

To proceed with the development of the main result, we define the H_∞ performance criteria by the following definition.

Definition 4. *System (5) is Mittag-Leffler stable under an H_∞ performance γ, if it is Mittag-Leffler stable as long as $\omega = 0$, as well as fulfilling $\int_{t_0}^{+\infty} e^T(t)e(t)dt < \gamma \int_{t_0}^{+\infty} \omega^T(t)\omega(t)dt$, when the initial condition equals to zero.*

In the present paper, the following two conditions are taken as assumptions:

Assumption 1. *The function $\varphi(x, u)$ is Lipschitz in $\mathbb{R}^n$, if $v \in \mathbb{R}^+\backslash\{0\}$ exists such that, $\forall x_1, x_2 \in \mathbb{R}^n$:*

$$||\varphi(x_1, u) - \varphi(x_2, u)|| \le v||x_1 - x_2||, \tag{7}$$

Assumption 2. *The disturbance $\Delta L(t)$ fulfills the following conditions: $\Delta L(t) = E_l F_l(t) H_l$ where E_l, H_l are known real constant matrices and $F_l(t)$ is an unknown real time-varying matrix where $F_l(t)^T F_l(t) \le I$, with I as the identity matrix.*

3. Estimation of Non-Fragile States

Theorem 2 is developed to establish a required hypothesis for the state estimate error origin $e = 0$ to be Mittag-Leffler stable.

Theorem 2. *Taking into consideration systems (2) and (4), in terms of circumstances (**Assumption 1**), (3), and (**Assumption 2**). If there exists X and $P = P^T > 0$ and scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ in a such way that the LMI (8) is feasible, so the origins of the error $e = 0$ is globally Mittag-Leffler stable, with H_∞ performance $\gamma > 0$.*

$$\begin{pmatrix} \delta_2 & PE_l & P & 0 \\ * & -\varepsilon_1 I & 0 & 0 \\ * & * & -\frac{1}{\beta+\varepsilon_2{}^{-1}}I & 0 \\ * & * & * & (\varepsilon_2 - \frac{\gamma}{2})I \end{pmatrix} < 0 \tag{8}$$

where $\delta_2 = PA + A^T P - XC - C^T X^T + \left(\frac{v^2}{\beta} + 1\right)I + \varepsilon_1 C^T H_l{}^T H_l C$ *and* $X = PL$.

Proof. We consider

$$V(e) = e^T P e$$

Using Lemma 1, we have:

$${}^C D^{\alpha}_{t_0,t} V(e) \le 2e^T P T^{\alpha}_{t_0} e \le e^T\Big((A - LC)^T P + P(A - LC)\Big)e + 2e^T PE_l F_l(t) H_l Ce + 2e^T P\Delta\varphi + 2e^T P\omega \tag{9}$$

The following property, which is true for all scalars $\beta > 0$, can be used:

$$2e^T P\Delta\varphi \le \beta e^T PPe + \frac{1}{\beta}\Delta\varphi^T\Delta\varphi$$

One has (C1) as a starting point:

$$\Delta\varphi^T\Delta\varphi \le v^2 e^T e \tag{10}$$

For each positive scalar $\varepsilon > 0$, it is true:

$$2e^T PE_l F_l(t) H_l Ce \le \varepsilon^{-1} e^T PE_l E_l{}^T Pe + \varepsilon e^T C^T H_l{}^T H_l Ce \tag{11}$$

For each positive scalar $\varepsilon_2 > 0$, it is true:

$$2e^T P\omega \le \varepsilon_2{}^{-1} e^T PPe + \varepsilon_2 \omega^T\omega \tag{12}$$

- ***Case 1*** **(**$\omega(t) = 0$**):**

  From (9) through (11), we get:

$$^{C}D^{\alpha}_{t_0,t}V \ \leq \ e^{T}\delta e \tag{13}$$

where $\delta = (A - LC)^{T}P + P(A - LC) + \varepsilon C^{T}{H_l}^{T}H_lC + \varepsilon^{-1}PE_l{E_l}^{T}P + \beta PP + \frac{v^2}{\beta}I$.

Lemma (Lemma 2) may be used to show that $\delta < 0$ is equivalent to:

$$\begin{pmatrix} \delta_1 & PE_l & P \\ * & -\varepsilon I & 0 \\ * & * & -\beta^{-1}I \end{pmatrix} < 0 \tag{14}$$

where $\delta_1 = PA + A^{T}P - XC - C^{T}X^{T} + \frac{v^2}{\beta}I + \varepsilon C^{T}{H_l}^{T}H_lC$ and $X = PL$.

Now condition (14) ensures that $^{C}D^{\alpha}_{t_0,t}V(t, e(t)) \ \leq -\mu_3 e^2$ with $\mu_3 = \lambda_{min}(\delta)$. So, by applying Theorem 1, it may be demonstrated that $e = 0$ is globally Mittag-Leffler stable.

- ***Case 2*** **(**$\omega(t) \neq 0$**):**

  Taking (9), (10), (11), and (12) into account, the following upper bound may be obtained for $^{C}D^{\alpha}_{t_0,t}V(e)$:

$$\begin{aligned} ^{C}D^{\alpha}_{t_0,t}V(e) \ \leq \ e^{T}\Big[&(A - LC)^{T}P + P(A - LC) + \beta PP + \tfrac{v^2}{\beta}I \\ &+{\varepsilon_1}^{-1}PE_l{E_l}^{T}\ P + \varepsilon_1 C^{T}{H_l}^{T}H_lC + {\varepsilon_2}^{-1}PP\Big]e + \varepsilon_2\omega^{T}\omega \end{aligned} \tag{15}$$

As a result, we may write the following inequality:

$$^{C}D^{\alpha}_{t_0,t}V(e) + e^{T}e - \frac{\gamma}{2}\omega^{T}\omega \ \leq \ \xi^{T}\Psi\xi, \tag{16}$$

where $\xi^{T} = \left[e^{T}\omega^{T}\right]$, $\Psi = \begin{pmatrix} \psi_1 & 0 \\ * & (\varepsilon_2 - \frac{\gamma}{2})I \end{pmatrix}$ with:

$$\begin{aligned} \psi_1 = (A - LC)^{T}P \ &+ P(A - LC) + \left(\beta + {\varepsilon_2}^{-1}\right)PP + \left(\tfrac{v^2}{\beta} + 1\right)I \\ &+\varepsilon_1 C^{T}{H_l}^{T}H_lC + {\varepsilon_1}^{-1}PE_l{E_l}^{T}P \end{aligned}$$

Let us now consider the situation:

$$\Psi \ \leq \ 0, \tag{17}$$

So:

$$^{C}D^{\alpha}_{t_0,t}V(e) + e^{T}e - \frac{\gamma}{2}\omega^{T}\omega \ \leq \ 0$$

And when it has been integrated, it maintains that:

$$I^{1}_{t_0}\,{}^{C}D^{\alpha}_{t_0,t}V(e) + I^{1}_{t_0}e^{T}(t)e(t) - \frac{\gamma}{2}I^{1}_{t_0}\omega^{T}(t)\omega(t) \ \leq \ 0, \tag{18}$$

Now, based on the work of [23], and using the following property:

$$I^{1}_{t_0}\,{}^{C}D^{\alpha}_{t_0,t}V(e) = I^{1-\alpha}_{t_0}I^{\alpha}_{t_0}\,{}^{C}D^{\alpha}_{t_0,t}V(e)$$

One gets:

$$I^{1}_{t_0}\,{}^{C}D^{\alpha}_{t_0,t}V(e) = I^{1-\alpha}_{t_0}(V(e(t)) - V(e(t_0)))$$

Using the initial condition of zero, one obtains $V(e(t_0)) = 0$, and then:

$$I^{1}_{t_0}\,{}^{C}D^{\alpha}_{t_0,t}V(e) = I^{1-\alpha}_{t_0}(V(e(t))) \tag{19}$$

Now, since $V(e(t)) \geq 0$ for any $t \geq t_0$, then from (19), we get: $I^1_{t_0}\,{}^C D^{\alpha}_{t_0,t} V(e(t)) \geq 0$ for any $t \geq t_0$. Then, using (18), one can write, for any $t \geq t_0$:

$$\int_{t_0}^{t} \left(e^T(t)e(t) - \frac{\gamma}{2}\omega^T(t)\omega(t) \right) ds \leq 0$$

Letting $t \to +\infty$, we obtain:

$$J_\infty = \int_{t_0}^{+\infty} \left(e^T(t)e(t) - \frac{\gamma}{2}\omega^T(t)\omega(t) \right) dt \leq 0,$$

Hence, $\int_{t_0}^{+\infty} e^T(t)e(t)dt < \gamma \int_{t_0}^{+\infty} \omega^T(t)\omega(t)dt$. As a result of applying Definition 4, it is possible to deduce that system (5) is Mittag-Leffler stable under a H_∞ performance $\gamma > 0$. Using Lemma 2, we can see that (17) Equation (8). The proof is now complete. □

Remark 1. *The article* [26] *makes a similar development in the context of conformable fractional order systems.*

4. Numerical Illustration

This section applies the suggested non-fragile state estimation method to the numerical system (2) using the following configuration:

$$A = \begin{bmatrix} -4 & 0 & -5 \\ 1 & -5 & 0 \\ -1 & 0 & -2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \varphi(x,u) = \frac{1}{6}\begin{bmatrix} \sin(x_2)\sin(u) \\ \sin(x_1) \\ \cos(x_3) \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T,$$

The fractional-order derivative is taken as $\alpha = 0.7$. Regarding (7), the situation is satisfactory at $v = \frac{1}{6}$. The following $\Delta L(t) = E_l F_l(t) H_l$ is the structure of the disturbance on the gain of the observer $\Delta L(t)$ as demonstrated by the condition (C2), with:

$$E_l = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.1 \end{bmatrix},\ F_l(t) = \cos t,\ H_l = 0.3$$

This simulation part looks at both of the instances discussed in the preceding section (disturbance-free and disturbance-dependent).

4.1. Case 1 ($\omega(t) = 0$)

Solving Equation (14) provides:

$$\varepsilon = \beta = 0.998,\ X = \begin{pmatrix} -0.7525 \\ 0.1153 \\ -1.8247 \end{pmatrix} \text{ and } P = \begin{pmatrix} 0.3306 & -0.001 & -0.0236 \\ -0.001 & 0.1071 & 0.0004 \\ -0.0236 & 0.0004 & 0.3134 \end{pmatrix}$$

On the other hand, the observer gain matrix is calculated:
$L = \begin{pmatrix} -2.7028 \\ 1.0766 \\ -6.0263 \end{pmatrix}$. $u = \cos t$ is used to excite the system to implement the simulation.
The evolution of the state estimation errors $e_1,\ e_2$, and e_3 is depicted in Figure 1.

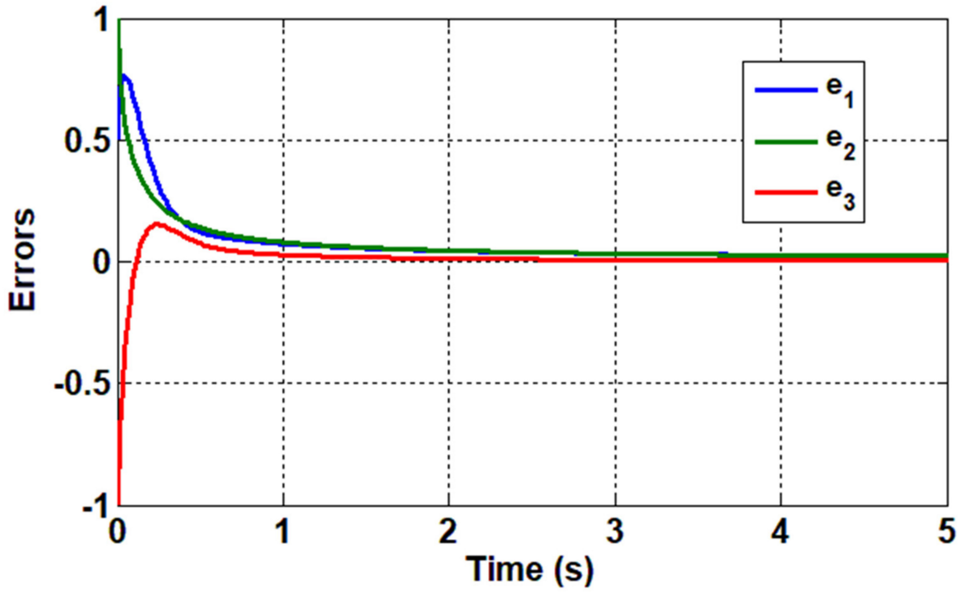

Figure 1. State estimation errors in the disturbance-free case.

As shown in Figure 1, the suggested non-fragile observer's effectiveness in the situation of no disturbances may be validated. $e_1 = 0$, $e_2 = 0$ and $e_3 = 0$ have a Mittag-Leffler stability that is clearly apparent.

4.2. Case 2 ($\omega(t) \neq 0$)

In this part, the authors want to show Mittag-Leffler stability with an H_∞ effectiveness when the system is subjected to a perturbation vector that fulfills specific requirements (3). To create the simulation, the authors identify the following perturbation:

$$\omega(t) = 0.1 \begin{pmatrix} \frac{\sin t}{t^2+1} \\ \frac{\cos t}{t^2+1} \\ \frac{\sin t}{t^2+1} \end{pmatrix} \tag{20}$$

$\varepsilon_1 = 0.8201,\ \beta = 0.8236, \varepsilon_2 = 1.25, \gamma = 0.8236, X = \begin{pmatrix} 0.0264 \\ 0.2082 \\ -1.8899 \end{pmatrix}$ and $P = \begin{pmatrix} 0.2486 & 0.0002 & 0.0876 \\ 0.0002 & 0.2092 & -0.0005 \\ 0.0876 & -0.0005 & 0.2884 \end{pmatrix}$ are obtained by solving Equation (8) using the MATLAB LMI control toolbox. The observer gain matrix may then be calculated:

$$L = \begin{pmatrix} 2.7047 \\ 0.9743 \\ -7.3738 \end{pmatrix}$$

$e = 0$ is Mittag-Leffler stable with $\gamma = 0.8236$. For the simulation model to work, $u = \cos t$ is applied to the system. e_1, e_2, and e_3 are shown in Figure 2.

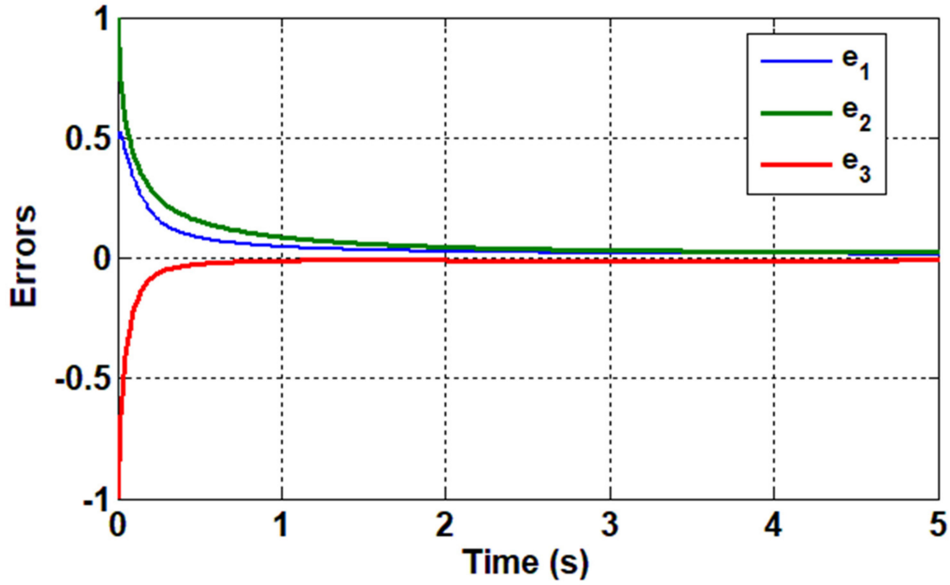

Figure 2. State estimation errors in the presence of disturbance $\omega(t)$.

As shown in Figure 2, the non-fragile observer's performance may be validated in the presence of a disturbance $\omega(t)$. On the contrary, the Mittag-Leffler stability of the error origins with H_∞ performance may be clearly shown.

5. Conclusions

The authors of this work have built a non-fragile observer for non-linear fractional-order systems using the Caputo fractional-order derivative concept. A general theorem has been established. This theorem considers both the disturbance-free and disturbance situations. The H_∞ theory was used to compensate for the influence of the input disturbance on the state estimation process; a H_∞ performance was developed, and the global Mittag-Leffler stability of the error origin, with a H_∞ performance $\gamma > 0$, was proven. At the end of the paper, a simulation analysis for a numerical example is presented to corroborate the theoretical conclusions and illustrate the usefulness of the recommended system. The current work is unique in that it is the first to address the problem of the Caputo fractional-order system.

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