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A General Picard-Mann Iterative Method for Approximating Fixed Points of Nonexpansive Mappings with Applications

Rahul Shukla [1], Rajendra Pant [2] and Winter Sinkala [1,*]

[1] Department of Mathematical Sciences and Computing, Walter Sisulu University, Mthatha 5117, South Africa
[2] Department of Mathematics & Applied Mathematics, Kingsway Campus, University of Johannesburg, Auckland Park 2006, South Africa
* Correspondence: wsinkala@wsu.ac.za

Abstract: Fixed point theory provides an important structure for the study of symmetry in mathematics. In this article, a new iterative method (general Picard–Mann) to approximate fixed points of nonexpansive mappings is introduced and studied. We study the stability of this newly established method which we find to be summably almost stable for contractive mappings. A number of weak and strong convergence theorems of such iterative methods are established in the setting of Banach spaces under certain geometrical assumptions. Finally, we present a number of applications to address various important problems (zero of an accretive operator, mixed equilibrium problem, convex optimization problem, split feasibility problem, periodic solution of a nonlinear evolution equation) appearing in the field of nonlinear analysis.

Keywords: nonexpansive mapping; iterative method; Opial property

Citation: Shukla, R.; Pant, R.; Sinkala, W. A General Picard-Mann Iterative Method for Approximating Fixed Points of Nonexpansive Mappings with Applications. *Symmetry* **2022**, *14*, 1741. https://doi.org/10.3390/sym14081741

Academic Editor: Alexander Zaslavski

Received: 29 July 2022
Accepted: 16 August 2022
Published: 21 August 2022

1. Introduction

Let $\mathcal{K}$ be a nonempty subset of Banach space $\mathcal{M}$. A mapping $T : \mathcal{K} \to \mathcal{K}$ is said to be nonexpansive if $\|T(\varrho) - T(\nu)\| \leq \|\varrho - \nu\|$ for all $\varrho, \nu \in \mathcal{K}$. A point $\varrho^{\dagger} \in \mathcal{K}$ is said to be a fixed point of T if $T(\varrho^{\dagger}) = \varrho^{\dagger}$. To reckon fixed points of nonlinear mappings, various iterative methods have been used by several mathematicians. The simplest and most popular iterative method was developed by Charles Emile Picard (1856–1941) and is defined as:

$$\begin{cases} \varrho_1 = \varrho \in \mathcal{K} \\ \varrho_{n+1} = T(\varrho_n), \quad n \in \mathbb{N}, \end{cases} \tag{1}$$

and is known as Picard iterative method [1]. It is mostly used to obtain fixed points of contractive mappings. In general, the contractive condition is strong enough, not only to guarantee the existence of a unique fixed point, but also to approximate that fixed point by the Picard method. However, for nonexpansive mappings, the Picard iterative method need not converge to a fixed point. This can be seen by considering an anti-clockwise rotation of the unit disc of $\mathbb{R}^2$ about the origin through an angle of, say, $\frac{\pi}{4}$. This is a nonexpansive symmetric mapping which has the origin as the unique fixed point. However, the sequence fails to converge with any initial guess other than the original.

Krasnosel'skiĭ [2] resolved this problem and considered a new method known as the Krasnosel'skiĭ iterative method. Schaefer [3] improved the Krasnosel'skiĭ iterative method by introducing a parameter as a controlling object. Mann [4] proposed a more general iterative method to approximate fixed points of nonexpansive mappings. He considered the sequence of parameters as controlling objects. Many well-known algorithms in signal processing and image recovery are iterative in nature. For particular choices of nonexpasive mappings, a wide variety of iteration algorithms used in signal processing and image recovery among others are special cases of the Krasnosel'skiĭ and Mann methods (cf. [5–12]).

In the last two decades, a number of iterative methods (from one step to four steps) have been considered and studied by researchers in order to improve the speed of convergence; see [4,13–22]. Motivated by these results, we consider a new iterative method (general Picard–Mann, in short GPM) to approximate fixed points of nonexpansive mappings in the setting of Banach spaces. It turns out that this method is highly efficient and an improvement over many other methods reported in the literature. Some new algorithms are suggested to find zeros of accretive operators, constrained convex optimization problems, generalized mixed equilibrium problems, split feasibility problems and periodic solutions of nonlinear evolution equations.

The rest of the paper is organized as follows: In Section 2, we present some existing results from the literature which are utilized in the rest part of paper. In Section 3, we define a general Picard–Mann iterative method. Moreover, we present stability results for the GPM iterative method and show that this method is summably almost stable for contractive mappings. Section 4 is devoted to weak and strong convergence results. We show that the sequence defined by GPM converges weakly and strongly to fixed points of nonexpansive mappings under different geometric conditions on Banach spaces. In Section 5, we discuss some applications.

2. Preliminaries

$\to$ denotes strong convergence, $\rightharpoonup$ denotes weak convergence and $\omega_w(\varrho_n)$ denotes the cluster points (ω-limit) of a sequence $\{\varrho_n\}$, that is, $\omega_w(\varrho_n) := \{\varrho : \exists\, \varrho_{n_k} \rightharpoonup \varrho\}$. Let $\mathcal{M}$ be a Banach space with $\mathcal{M}^*$ its dual. The value of $f \in \mathcal{M}^*$ at $\varrho \in \mathcal{M}$ is denoted by $\langle \varrho, f\rangle$. The normalized duality mapping $J : \mathcal{M} \to 2^{\mathcal{M}^*}$ is defined as

$$J(\varrho) := \{f \in \mathcal{M}^* : \langle \varrho, f\rangle = \|\varrho\|^2 = \|f\|^2\}.$$

A Banach space $\mathcal{M}$ is called smooth if, for every $\varrho, \nu \in \mathcal{S} := \{\varrho \in \mathcal{M} : \|\varrho\| = 1\}$, the limit

$$\lim_{s\to 0} \frac{\|\varrho + s\nu\| - \|\varrho\|}{s} \tag{2}$$

exists. The norm of $\mathcal{M}$ is a Fréchet differentiable norm if, for every $\varrho \in \mathcal{S}$, the limit (2) exists and is attained uniformly for $\nu \in \mathcal{S}$. A Banach space $\mathcal{M}$ has the Kadec-–Klee property (or, KK-property) if, for any sequence $\{w_n\}$, we have the following:

$$w_n \rightharpoonup w \text{ and } \|w_n\| \to \|w\| \text{ imply } w_n \to w.$$

In [23], (Remark 1) it is proved that if a reflexive Banach space $\mathcal{M}$ has a Fréchet differentiable norm, then the dual space of $\mathcal{M}$ has the KK-property.

The definition of a uniformly convex Banach space (in short, UCBS) can be found in [24].

A Banach space $\mathcal{M}$ has the Opial property [25] if, for every weakly convergent sequence, $\{\varrho_n\}$ in $\mathcal{M}$ with a weak limit w,

$$\liminf_{n\to\infty} \|\varrho_n - w\| < \liminf_{n\to\infty} \|\varrho_n - \nu\|$$

for all $\nu \in \mathcal{M}$ with $\nu \neq w$. All finite dimensional Banach spaces, Hilbert spaces and ℓ^p $(1 < p < \infty)$ have the Opial property; see [24].

Lemma 1 ([26], (p. 484))**.** *Let $\mathcal{M}$ be a UCBS and $0 < a \le p_n \le b < 1$, $\forall\, n \in \mathbb{N}$. Let $\{\varrho_n\}$ and $\{\nu_n\}$ be two sequences in $\mathcal{M}$ such that $\limsup_{n\to\infty} \|\varrho_n\| \le r$, $\limsup_{n\to\infty} \|\nu_n\| \le r$ and $\lim_{n\to\infty} \|p_n\varrho_n + (1-p_n)\nu_n\| = r$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|\varrho_n - \nu_n\| = 0$.*

Lemma 2 ([27])**.** *For given $r > 0$, a Banach space $\mathcal{M}$ is uniformly convex if and only if there exists a strictly increasing continuous function $\varphi : [0,\infty) \to [0,\infty)$, $\varphi(0) = 0$, in such a way that*

$$\|\mu\varrho + (1-\mu)\nu\|^2 \le \mu\|\varrho\|^2 + (1-\mu)\|\nu\|^2 - \mu(1-\mu)\varphi(\|\varrho - \nu\|) \tag{3}$$

for all $\varrho, \nu \in \mathcal{M}$, $\|\varrho\| \leq r$, $\|\nu\| \leq r$ *and* $\mu \in [0,1]$.

Proposition 1. *Let $\mathcal{M}$ be a UCBS and $\mathcal{K}$ a closed convex subset of $\mathcal{M}$ such that $\mathcal{K} \neq \emptyset$.*

(i) *Ref. [28] (Demiclosedness principle) Let $T : \mathcal{K} \to \mathcal{K}$ be a nonexpansive mapping. If $\{\varrho_n\}$ is a sequence in $\mathcal{M}$ such that $\{\varrho_n\}$ weakly converges to ϱ and $\lim_{n\to\infty} \|\varrho_n - T(\varrho_n)\| = 0$, then $T(\varrho) = \varrho$. That is, $I - T$ is demiclosed at zero.*

(ii) *Ref. [29] If $\mathcal{K}$ is bounded, then there exists a continuous, strictly increasing and convex function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ (depending only on the diameter of $\mathcal{K}$) with $\chi(0) = 0$ such that for every nonexpansive mapping $T : \mathcal{K} \to \mathcal{K}$, for all $\varrho, \nu \in \mathcal{K}$ and $\beta \in [0,1]$, the following inequality holds:*

$$\chi(\|\beta T(\varrho) + (1-\beta)T(\nu) - T\{\beta\varrho + (1-\beta)\nu\}\|) \leq \|\varrho - \nu\| - \|T(\varrho) - T(\nu)\|.$$

Definition 1 ([30]). *Let $\mathcal{M}$ be a norm space and $\mathcal{K}$ a subset of $\mathcal{M}$ such that $\mathcal{K} \neq \emptyset$. A mapping $T : \mathcal{K} \to \mathcal{K}$ satisfies condition (I) if there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ and $g(s) > 0$ for all $s \in (0,\infty)$ such that $\|\varrho - T(\varrho)\| \geq g(d(\varrho, F(T)))$ for all $\varrho \in \mathcal{K}$, where $F(T) := \{\varrho \in \mathcal{K} : T(\varrho) = \varrho\}$. If mapping T is nonexpansive with $F(T) \neq \emptyset$ and demicompact, then T must satisfy condition (I).*

Definition 2. *Let $\mathcal{M}$ be a normed space. A mapping $T : \mathcal{M} \to \mathcal{M}$ is said to be a contraction if there exists a number $k \in [0,1)$ such that, for all $\varrho, \nu \in \mathcal{M}$,*

$$\|T(\varrho) - T(\nu)\| \leq k\|\varrho - \nu\|.$$

3. A General Picard-Mann Iterative Method

In this section, we propose a new iterative method (a general Picard–Mann) which we define as follows:

$$\begin{cases} \varrho_1 = \varrho \in C \\ \varrho_{n+1} = T^k\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\}, \quad n \in \mathbb{N}, \end{cases} \tag{4}$$

where k is a fixed natural number and $\{\alpha_n\}$ is a sequence in $[0,1]$.

Remark 1.

(i) *For $k = 1$, the iterative method (4) becomes normal S-iterative method [17] (or Picard–Mann hybrid iterative method [31]).*

(ii) *For $k = 2$, the iterative method (4) becomes M-iterative method [19] (or F^*-iterative method [21]).*

(iii) *For $k = 3$, the iterative method (4) becomes F-iterative method [22].*

Stability Results

Now, we discuss the stability results for the GPM method (4). A fixed point iteration method is numerically stable if a small perturbation (due to rounding errors, approximation, etc.) during computations, will produce small changes in the approximate value of the fixed point computed by means of this method; see [32]. The stability of an iterative method plays a vital role in fractal geometry, computational analysis, game theory and others.

Let $\mathcal{M}$ be a Banach space and $\mathcal{K}$ a convex subset of $\mathcal{M}$ such that $\mathcal{K} \neq \emptyset$. Let $T : \mathcal{K} \to \mathcal{K}$ be a mapping with $F(T) \neq \emptyset$. For given $\varrho_1 \in \mathcal{K}$ the fixed point iteration method generates a sequence $\{\varrho_n\}$ in $\mathcal{K}$ as follows:

$$\varrho_{n+1} = f(T, \varrho_n) \tag{5}$$

where f is some function. Harder and Hicks [33] considered the following definition:

Definition 3. *Let $\mathcal{K}$ and $\mathcal{M}$ be same as defined above. Let $T : \mathcal{K} \to \mathcal{K}$ be a mapping. Suppose that the method (5) strongly converges to a fixed point $\varrho^\dagger$ of T. Let $\{\nu_n\}$ be an arbitrary sequence in $\mathcal{K}$ and define*

$$\varepsilon_n = \|\nu_{n+1} - f(T, \nu_n)\|. \tag{6}$$

Then, the fixed point iterative method (5) is said to be T-stable (or stable with respect to T) if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ implies that } \lim_{n\to\infty} \nu_n = \varrho^\dagger.$$

Osilike [34] considered the following concept of almost stability.

Definition 4. *Let $\mathcal{K}$, $\mathcal{M}$ and T be same as in Definition (3). Suppose that the method (5) strongly converges to a fixed point $\varrho^\dagger$ of T. Let $\{\nu_n\}$ and $\{\varepsilon_n\}$ be sequences defined in (6). Then, the iterative method (5) is said to be almost T-stable (or almost stable with respect to T) if and only if*

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty \text{ implies that } \lim_{n\to\infty} \nu_n = \varrho^\dagger.$$

Berinde [35] considered the weaker concept of stability, called summably almost T-stable.

Definition 5. *Let $\mathcal{K}$, $\mathcal{M}$ and T be same as in Definition (3). Suppose that the method (5) strongly converges to a fixed point $\varrho^\dagger$ of T. Let $\{\nu_n\}$ and $\{\varepsilon_n\}$ be sequences defined in (6). Then, the iterative method (5) is said to be summably almost T-stable (or summably almost stable with respect to T) if and only if*

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty \text{ implies that } \sum_{n=1}^{\infty} \|\nu_n - \varrho^\dagger\| < \infty.$$

Any almost stable iteration procedure is also summably almost stable, but the reverse implication does not hold in general.

Now, we show that iterative method (4) is summably almost stable for contractive type mappings.

Theorem 1. *Let $\mathcal{M}$ be a Banach space and $\mathcal{K}$ a closed convex subset of $\mathcal{M}$ such that $\mathcal{K} \neq \emptyset$. Let $T : \mathcal{K} \to \mathcal{K}$ be a contraction mapping with a fixed point $\varrho^\dagger$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$, for given $\varrho_1 \in \mathcal{K}$ and for fixed $k \in \mathbb{N}$, the sequence $\{\varrho_n\}$ defined by (4). Suppose $\{\nu_n\}$ is an arbitrary sequence in $\mathcal{K}$ and define*

$$\varepsilon_n = \|\nu_{n+1} - T^k\{(1-\alpha_n)\nu_n + \alpha_n T(\nu_n)\}\|.$$

Then, we have following:

(1) The sequence $\{\varrho_n\}$ strongly converges to the fixed point $\varrho^\dagger$.

(2) $\sum\limits_{n=1}^{\infty} \varepsilon_n < \infty$ implies that $\sum\limits_{n=1}^{\infty} \|\nu_n - \varrho^\dagger\| < \infty$, so that $\{\varrho_n\}$ is summably almost T-stable

(3) $\lim\limits_{n\to\infty} \nu_n = \varrho^\dagger$ implies $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Proof. By (4) and let $\varrho^\dagger \in F(T)$, we have

$$\begin{aligned} \|\varrho_{n+1} - \varrho^\dagger\| &= \|T^k\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - \varrho^\dagger\| \\ &\leq \theta^k \|(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n) - \varrho^\dagger\| \leq \theta^k \|\varrho_n - \varrho^\dagger\| \end{aligned} \tag{7}$$

by successively induction, we obtain

$$\|\varrho_{n+1} - \varrho^\dagger\| \leq \left(\theta^k\right)^n \|\varrho_1 - \varrho^\dagger\|.$$

Since $\theta^k < 1$, $\{\varrho_n\}$ strongly converges to $\varrho^\dagger$.
Now, we prove (2), for each $\varrho^\dagger \in F(T)$, we have

$$\|T(\nu_n) - \varrho^\dagger\| \leq \theta\|\nu_n - \varrho^\dagger\|$$

and, for fixed $k \in \mathbb{N}$

$$\|T^k(\nu_n) - \varrho^\dagger\| \leq \theta^k\|\nu_n - \varrho^\dagger\|.$$

By the triangle inequality,

$$\begin{aligned}
\|\nu_{n+1} - \varrho^\dagger\| &\leq \|\nu_{n+1} - T^k\{(1-\alpha_n)\nu_n + \alpha_n T(\nu_n)\}\| + \|T^k\{(1-\alpha_n)\nu_n + \alpha_n T(\nu_n)\} - \varrho^\dagger\| \\
&\leq \theta^k\|(1-\alpha_n)\nu_n + \alpha_n T(\nu_n) - \varrho^\dagger\| + \varepsilon_n \\
&\leq \theta^k\{(1-\alpha_n)\|\nu_n - \varrho^\dagger\| + \alpha_n\theta\|\nu_n - \varrho^\dagger\|\} + \varepsilon_n \\
&\leq \theta^k\|\nu_n - \varrho^\dagger\| + \varepsilon_n.
\end{aligned}$$

In view of assumption $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and [35], (Lemma 1) it implies that $\sum_{n=0}^{\infty} \|\nu_n - \varrho^\dagger\| < \infty$.
Finally, we prove (3). Suppose $\lim_{n\to\infty} \nu_n = \varrho^\dagger$. Now,

$$\begin{aligned}
\varepsilon_n &= \|\nu_{n+1} - T^k\{(1-\alpha_n)\nu_n + \alpha_n T(\nu_n)\}\| \\
&\leq \|\nu_{n+1} - \varrho^\dagger\| + \left\|T^k\{(1-\alpha_n)\nu_n + \alpha_n T(\nu_n)\} - \varrho^\dagger\right\| \\
&\leq \|\nu_{n+1} - \varrho^\dagger\| + \|\nu_n - \varrho^\dagger\| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This completes the proof. □

4. Convergence Results for Nonexpansive Mappings

In this section, we present some convergence results for the sequence generated by iterative method (4).

Lemma 3. *Let $\mathcal{M}$ be a Banach space and $\mathcal{K}$ a closed convex subset of $\mathcal{M}$ such that $\mathcal{K} \neq \emptyset$. Let $T : \mathcal{K} \to \mathcal{K}$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\varrho_n\}$ be a sequence defined by (4). The following assertions hold:*

(1) *If $p^\dagger \in F(T)$, then $\lim_{n\to\infty} \|\varrho_n - p^\dagger\|$ exists;*
(2) *$\lim_{n\to\infty} d(\varrho_n, F(T))$ exists, where $d(\varrho, F(T))$ denotes the distance from ϱ to $F(T)$.*
(3) *For all $\beta \in [0,1]$ and $\varrho^\dagger, \nu^\dagger \in F(T)$, the limit $\lim_{n\to\infty} \|\beta\varrho_n + (1-\beta)\varrho^\dagger - \nu^\dagger\|$ exists.*
(4) *In addition, if $\mathcal{M}$ is uniformly convex and the dual space $\mathcal{M}^*$ of $\mathcal{M}$ has the KK-property, then $\omega_w(\varrho_n)$ is a singleton.*

Proof. From (4) and let $p^\dagger \in F(T)$,

$$\begin{aligned}
\|\varrho_{n+1} - p^\dagger\| &= \|T^k\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - p^\dagger\| \\
&\leq \|(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n) - p^\dagger\| \\
&\leq (1-\alpha_n)\|\varrho_n - p^\dagger\| + \alpha_n\|T(\varrho_n) - p^\dagger\| \\
&\leq \|\varrho_n - p^\dagger\|. \qquad (8)
\end{aligned}$$

Therefore, the sequence $\{\|\varrho_n - p^\dagger\|\}$ is a nonincreasing and bounded. Thus, $\lim_{n\to\infty} \|\varrho_n - p^\dagger\|$ exists for each $p^\dagger \in F(T)$. Therefore, $\lim_{n\to\infty} d(\varrho_n, F(T))$ exists.

From (1), the sequence $\{\varrho_n\}$ is bounded. Let $\varrho^\dagger, \nu^\dagger \in F(T)$ and set

$$\zeta_n(\beta) := \|\beta\varrho_n + (1-\beta)\varrho^\dagger - \nu^\dagger\|.$$

Then, $\lim_{n\to\infty} \zeta_n(0) = \|\varrho^\dagger - \nu^\dagger\|$ and $\lim_{n\to\infty} \zeta_n(1) = \|\varrho_n - \nu^\dagger\|$ exists. Now, we need to check the case $\beta \in (0,1)$. Now, we define a mapping $T_n : \mathcal{K} \to \mathcal{K}$ by

$$T_n(\varrho) := T^k\{(1-\alpha_n)\varrho + \alpha_n T(\varrho)\}, \quad \forall \ \ n \in \mathbb{N} \ \ \text{and} \quad \forall \ \ \varrho \in \mathcal{K}.$$

Then, T_n is a nonexpansive mapping. Indeed, $\forall\ \varrho, \nu \in \mathcal{K}$

$$\begin{aligned} \|T_n(\varrho) - T_n(\nu)\| &= \|T^k\{(1-\alpha_n)\varrho + \alpha_n T(\varrho)\} - T^k\{(1-\alpha_n)\nu + \alpha_n T(\nu)\}\| \\ &\leq \|\{(1-\alpha_n)\varrho + \alpha_n T(\varrho)\} - \{(1-\alpha_n)\nu + \alpha_n T(\nu)\}\| \\ &\leq (1-\alpha_n)\|\varrho - \nu\| + \alpha_n\|T(\varrho) - T(\nu)\| \\ &\leq \|\varrho - \nu\|. \end{aligned}$$

Moreover, $\varrho_{n+1} = T_n(\varrho_n)$ and $F(T) \subseteq \bigcap_{n=1}^{\infty} F(T_n)$. Let $V_{n,m} : \mathcal{K} \to \mathcal{K}$ be the mapping defined as

$$V_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_{n+1}T_n.$$

It can be observed that $\varrho_{n+m} = V_{n,m}(\varrho_n)$ and $F(T) \subseteq \bigcap_{n=1}^{\infty} F(V_{n,m})$. Moreover,

$$\|V_{n,m}(\varrho) - V_{n,m}(\nu)\| \leq \|\varrho - \nu\| \text{ for all } \varrho, \nu \in \mathcal{K}.$$

Set

$$\xi_{n,m}(\beta) := \|\beta V_{n,m}(\varrho_n) + (1-\beta)\varrho^\dagger - V_{n,m}\{\beta\varrho_n + (1-\beta)\varrho^\dagger\}\|.$$

Now, from Proposition 1, there exists a strictly increasing continuous convex function $\chi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\chi(0) = 0$ such that

$$\begin{aligned} \chi(\xi_{n,m}(\beta)) &= \chi\Big(\|\beta V_{n,m}(\varrho_n) + (1-\beta)\varrho^\dagger - V_{n,m}\{\beta\varrho_n + (1-\beta)\varrho^\dagger\}\|\Big) \\ &\leq \|\varrho_n - \varrho^\dagger\| - \|V_{n,m}(\varrho_n) - V_{n,m}(\varrho^\dagger)\| \\ &= \|\varrho_n - \varrho^\dagger\| - \|\varrho_{n+m} - \varrho^\dagger\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|\varrho_n - \varrho^\dagger\|$ exists, the last difference is zero. Therefore, $\lim_{n,m\to\infty} \chi(\xi_{n,m}(\beta)) = 0$ and $\lim_{n,m\to\infty} \xi_{n,m}(\beta) = 0$. Now, we have

$$\begin{aligned} \zeta_{n+m}(\beta) &= \|\beta\varrho_{n+m} + (1-\beta)\varrho^\dagger - \nu^\dagger\| \\ &= \|\beta V_{n,m}(\varrho_n) + (1-\beta)\varrho^\dagger - \nu^\dagger\| \\ &\leq \xi_{n,m}(\beta) + \|V_{n,m}\{\beta\varrho_n + (1-\beta)\varrho^\dagger\} - \nu^\dagger\| \\ &\leq \xi_{n,m}(\beta) + \|\beta\varrho_n + (1-\beta)\varrho^\dagger - \nu^\dagger\| \\ &\leq \xi_{n,m}(\beta) + \zeta_n(\beta). \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{n\to\infty} \zeta_n(\beta) &\leq \lim_{n\to\infty} \xi_{n,m}(\beta) + \liminf_{n\to\infty} \zeta_n(\beta) \\ &\leq \liminf_{n\to\infty} \zeta_n(\beta). \end{aligned}$$

That is, there exists $\lim_{n\to\infty} \|(1-\beta)\varrho_n + \beta\varrho^\dagger - \nu^\dagger\|$ for all $\beta \in (0,1)$. From [23], (Lemma 3.2) we conclude that $\omega_w(\varrho_n)$ is a singleton. This completes the proof. □

Theorem 2. *Let $\mathcal{M}$ be a UCBS, $\mathcal{K}$ and T be same as in Lemma 3. Let $\{\varrho_n\}$ be a sequence defined by (4) with*

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty. \tag{9}$$

Assume that either of the following assumptions hold:

(a) *$\mathcal{M}$ satisfies the Opial's property;*

(b) *$\mathcal{M}^*$ has the KK-property.*

Then, $\{\varrho_n\}$ weakly converges to a fixed point of T.

Proof. In view of Lemma 3, both sequences $\{\varrho_n - \varrho^\dagger\}$ and $\{T(\varrho_n) - \varrho^\dagger\}$ are bounded, so these are contained in $\mathcal{B}_s := \{\varrho \in \mathcal{M} : \|\varrho\| \leq s\}$ for sufficiently large $s > 0$. In view of Lemma 2, there exists a continuous, convex and strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that (3) holds. Thus, we have

$$\begin{aligned}
\|\varrho_{n+1} - \varrho^\dagger\|^2 &= \|T^k\{(1-\alpha_n)\varrho + \alpha_n T(\varrho)\} - \varrho^\dagger\|^2 \\
&\leq \|\alpha_n(\varrho_n - \varrho^\dagger) + (1-\alpha_n)(T(\varrho_n) - \varrho^\dagger)\|^2 \\
&\leq \alpha_n\|\varrho_n - \varrho^\dagger\|^2 + (1-\alpha_n)\|T(\varrho_n) - \varrho^\dagger\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\varphi(\|\varrho_n - T(\varrho_n)\|) \\
&\leq \|\varrho_n - \varrho^\dagger\|^2 - \alpha_n(1-\alpha_n)\varphi(\|\varrho_n - T(\varrho_n)\|).
\end{aligned}$$

So,

$$\alpha_n(1-\alpha_n)\varphi(\|\varrho_n - T(\varrho_n)\|) \leq \|\varrho_n - \varrho^\dagger\|^2 - \|\varrho_{n+1} - \varrho^\dagger\|^2. \tag{10}$$

This implies that

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)\varphi(\|\varrho_n - T(\varrho_n)\|) < \infty.$$

In particular, $\lim_{n\to\infty} \alpha_n(1-\alpha_n)\varphi(\|\varrho_n - T(\varrho_n)\|) = 0$. Due to (9), $\liminf_{n\to\infty} \varphi(\|\varrho_n - T(\varrho_n)\|) = 0$. Therefore,

$$\liminf_{n\to\infty} \|\varrho_n - T(\varrho_n)\| = 0. \tag{11}$$

However,

$$\begin{aligned}
\|T(\varrho_{n+1}) - \varrho_{n+1}\| &= \|T(\varrho_{n+1}) - T^k\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\}\| \\
&\leq \|\varrho_{n+1} - T^{k-1}\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\}\| \\
&= \|T^k\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - T^{k-1}\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\}\| \\
&\leq \|T\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - \{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\}\| \\
&= \|T\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - (1-\alpha_n)\varrho_n - \alpha_n T(\varrho_n) \\
&\quad -T(\varrho_n) + T(\varrho_n)\| \\
&= \|T\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - T(\varrho_n) - (1-\alpha_n)\varrho_n \\
&\quad +(1-\alpha_n)T(\varrho_n)\| \\
&\leq \|T\{(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n)\} - T(\varrho_n)\| + (1-\alpha_n)\|\varrho_n - T(\varrho_n)\| \\
&\leq \|(1-\alpha_n)\varrho_n + \alpha_n T(\varrho_n) - \varrho_n\| + (1-\alpha_n)\|\varrho_n - T(\varrho_n)\| \\
&\leq \alpha_n\|\varrho_n - T(\varrho_n)\| + (1-\alpha_n)\|\varrho_n - T(\varrho_n)\| \\
&= \|\varrho_n - T(\varrho_n)\|.
\end{aligned}$$

Thus, sequence $\{\|\varrho_n - T(\varrho_n)\|\}$ is nonincreasing. Therefore, $\lim_{n\to\infty} \|\varrho_n - T(\varrho_n)\|$ exists. From (11), we obtain

$$\lim_{n\to\infty} \|\varrho_n - T(\varrho_n)\| = 0. \tag{12}$$

Since $\mathcal{M}$ is uniformly convex, $\mathcal{M}$ is reflexive. Since $\mathcal{M}$ is reflexive, there exists a subsequence $\{\varrho_{n_j}\}$ of $\{\varrho_n\}$ and $\{\varrho_{n_j}\}$ weakly converges to a point $p \in \mathcal{K}$. From the demiclosedness principle of $I - T$ (Proposition (1)), we notice that

$$p \in \omega_w(\varrho_n) \subset F(T).$$

Now, we show that $\omega_w(\varrho_n)$ is a singleton; this implies that there is a unique weak limit for each subsequences of $\{\varrho_n\}$, and $\{\varrho_n\}$ weakly converges to a fixed point of T.

First, we suppose that (a) is true; that is, $\mathcal{M}$ has Opial's property. We assume that $\{\varrho_n\}$ does not converge weakly to p, i.e., let $\{\varrho_{n_i}\}$ and $\{\varrho_{m_j}\}$ be subsequences of $\{\varrho_n\}$ such that $\varrho_{n_i} \rightharpoonup p$ and $\varrho_{m_j} \rightharpoonup q$, respectively, then $p, q \in \omega_w(\varrho_n)$. If $p \neq q$, we reach the following contradiction:

$$\begin{aligned}
\lim_{n\to\infty} \|\varrho_n - p\| &= \lim_{i\to\infty} \|\varrho_{n_i} - p\| < \lim_{i\to\infty} \|\varrho_{n_i} - q\| \\
&= \lim_{n\to\infty} \|\varrho_n - q\| = \lim_{j\to\infty} \|\varrho_{m_j} - q\| \\
&< \lim_{j\to\infty} \|\varrho_{m_j} - p\| = \lim_{n\to\infty} \|\varrho_n - p\|.
\end{aligned}$$

Suppose (b) is true ($\mathcal{M}^*$ has the KK-property); from Lemma 3, it is guaranteed that $\omega_w(\varrho_n)$ is a singleton. Therefore, in both cases it is showed that $\omega_w(\varrho_n)$ is a singleton. This completes the proof. □

Theorem 3. *Let $\mathcal{K}$, T, $\{\varrho_n\}$ be same as in Theorem 2 and $\mathcal{M}$ is UCBS. If the range of $\mathcal{K}$ under T is contained in a compact subset of $\mathcal{M}$. Then, $\{\varrho_n\}$ strongly converges to a fixed point of T.*

Proof. Since the range of $\mathcal{K}$ under T is contained in a compact set, there exists a subsequence $\{T(\varrho_{n_j})\}$ of $\{T(\varrho_n)\}$ that strongly converges to $p^\dagger \in \mathcal{K}$. By the triangle inequality, we obtain

$$\|\varrho_{n_j} - p^\dagger\| \leq \|\varrho_{n_j} - T(\varrho_{n_j})\| + \|T(\varrho_{n_j}) - p^\dagger\|$$

and by (12), the subsequence $\{\varrho_{n_j}\}$ strongly converges to $\varrho^\dagger$. By the triangle inequality and nonexpansiveness of T

$$\|\varrho_{n_j} - T(p^\dagger)\| \leq \|\varrho_{n_j} - T(\varrho_{n_j})\| + \|\varrho_{n_j} - p^\dagger\|.$$

Taking $j \to \infty$,

$$\limsup_{j\to\infty} \|\varrho_{n_j} - T(p^\dagger)\| \leq \lim_{j\to\infty} \|\varrho_{n_j} - T(\varrho_{n_j})\| + \limsup_{j\to\infty} \|\varrho_{n_j} - p^\dagger\|,$$

and, we have $T(p^\dagger) = p^\dagger$. Lemma 3 ensures that $\lim\limits_{n\to\infty} \|\varrho_n - p^\dagger\|$ exists. Therefore, $p^\dagger$ is a strong limit of the sequence $\{\varrho_n\}$. □

Theorem 4. *Let $\mathcal{K}$, T, $\{\varrho_n\}$ be same as in Theorem 2 and $\mathcal{M}$ be a UCBS. Then, the sequence $\{\varrho_n\}$ strongly converges to a fixed point of T if $\liminf\limits_{n\to\infty} d(\varrho_n, F(T)) = 0$.*

Proof. Let $\liminf\limits_{n\to\infty} d(\varrho_n, F(T)) = 0$. From Lemma 3, $\lim\limits_{n\to\infty} d(\varrho_n, F(T))$ exists, so

$$\lim_{n\to\infty} d(\varrho_n, F(T)) = 0.$$

Let $\{\varrho_{n_j}\}$ be a subsequence of sequence $\{\varrho_n\}$ such that $\|\varrho_{n_j} - z_j\| \leq \dfrac{1}{2^j}, \forall\, j \in \mathbb{N}$, where $\{z_j\}$ is a sequence in $F(T)$. From Lemma 3, we have

$$\|\varrho_{n_{j+1}} - z_j\| \leq \|\varrho_{n_j} - z_j\| \leq \frac{1}{2^j}. \tag{13}$$

By the triangle inequality and (13),

$$\begin{aligned} \|z_{j+1} - z_j\| &\leq \|z_{j+1} - \varrho_{n_{j+1}}\| + \|\varrho_{n_{j+1}} - z_j\| \\ &< \frac{1}{2^{j+1}} + \frac{1}{2^j} < \frac{1}{2^{j-1}}. \end{aligned}$$

Following the standard argument, it can be easily shown that $\{z_j\}$ is a Cauchy sequence in $F(T)$. Since $F(T)$ is closed [24], the sequence $\{z_j\}$ converges to a point $z \in F(T)$. By the triangle inequality

$$\|\varrho_{n_j} - z\| \leq \|\varrho_{n_j} - z_j\| + \|z_j - z\|.$$

Letting $j \to \infty$ it follows that $\{\varrho_{n_j}\}$ strongly converges to z. By Lemma 3, $\lim_{n\to\infty} \|\varrho_n - z\|$ exists; thus, the sequence $\{\varrho_n\}$ strongly converges to z. □

Theorem 5. *Let $\mathcal{K}$, $\mathcal{M}$, T and $\{\varrho_n\}$ be same as in Theorem 4. If T satisfies condition (I), then $\{\varrho_n\}$ strongly converges to a fixed point of T.*

Proof. From Theorem 2,

$$\lim_{n\to\infty} \|\varrho_n - T(\varrho_n)\| = 0. \tag{14}$$

Since T satisfies condition (I),

$$\|\varrho_n - T(\varrho_n)\| \geq g(d(\varrho_n, F(T)))$$

From (14), we obtain

$$\lim_{n\to\infty} g(d(\varrho_n, F(T))) = 0$$

Since the function $g : [0,\infty) \to [0,\infty)$ is a nondecreasing with $g(s) > 0 \; \forall \; s \in (0,\infty)$ and $g(0) = 0$, $\lim_{n\to\infty} d(\varrho_n, F(T)) = 0$. Thus, all conditions of Theorem 4 are fulfilled and $\{\varrho_n\}$ strongly converges to a point in $F(T)$. □

5. Applications

In this section, we discuss some useful applications of our results.

5.1. Zero of an Accretive Operator

Let $\mathfrak{B}$ be an operator having domain $D(\mathfrak{B})$ and range $R(\mathfrak{B})$ in a Banach space $\mathcal{M}$. The operator $\mathfrak{B}$ is called as accretive if there exists $j(\varrho - \nu) \in J(\varrho - \nu)$ such that

$$\langle \mathfrak{B}(\varrho) - \mathfrak{B}(\nu), j(\varrho - \nu) \rangle \geq 0, \qquad \forall \quad \varrho, \nu \in D(\mathfrak{B}),$$

where J is the duality mapping from $\mathcal{M}$ to $\mathcal{M}^*$ (dual space of $\mathcal{M}$). An operator $\mathfrak{B}$ is m-accretive if

$$R(\mathfrak{B} + \lambda\mathfrak{B}) = \mathcal{M}, \qquad \forall \quad \lambda > 0.$$

We denote by $\mathcal{F}$ the set of zeros of $\mathfrak{B}$, that is,

$$\mathcal{F} := \mathfrak{B}^{-1}(0) = \{\varrho \in D(\mathfrak{B}) : 0 \in \mathfrak{B}(\varrho)\}.$$

For any $r > 0$, denote by J_r the resolvent of $\mathfrak{B}$ and defined as

$$J_r = (I + r\mathfrak{B})^{-1}.$$

It is well-known that J_r is a nonexpansive mapping from $\mathcal{M}$ to $\mathcal{C} := \overline{D(\mathfrak{B})}$. For any $r > 0$, $F(J_r) = \mathcal{F}$.

It is well-known fact that many pivotal problems originating in different fields can be modelled as an initial value problem defined below

$$\frac{du}{dt} + \mathfrak{B}(u) = 0, \quad u(0) = u_0, \tag{15}$$

where $\mathfrak{B}$ is an accretive operator on $\mathcal{M}$. Some important models such as Schrödinger, heat and wave equations are examples of evolution equations (cf. [36]). In [37], Browder showed that (15) is solvable if $\mathfrak{B}$ is locally Lipschitzian and accretive operator on $\mathcal{M}$. Many researchers considered the solution of (15) under various conditions on the operator $\mathfrak{B}$.

It can be seen that $\frac{du}{dt} = 0$ in (15), whenever u is not depending on t, then (15) reduces to $\mathfrak{B}(u) = 0$. Therefore, the zero of accretive operators is equivalent to the equilibrium points of the system (15), see [37]. Thus, the equilibrium points of the system described by (15) correspond to approximating zeros of accretive operators; see [37] and references therein.

Now, we consider a problem of finding zeros of an m-accretive operator $\mathfrak{B}$ in $\mathcal{M}$:

$$\text{Find } \varrho \in D(\mathfrak{B}) \text{ such that } 0 \in \mathfrak{B}(\varrho).$$

Lemma 4. *[38] Let $c^* \geq c_* > 0$. Then, for all $\varrho \in \mathcal{M}$,*

$$\|J_{c_*}(\varrho) - \varrho\| \leq 2\|J_{c^*}(\varrho) - \varrho\|.$$

In particular, if $c_n \geq c_* > 0$ for all $n \geq 0$, and $\{\varrho_n\}$ is any sequence in $\mathcal{M}$, then

$$\|J_{c_*}(\varrho_n) - \varrho_n\| \leq 2\|J_{c_n}(\varrho_n) - \varrho_n\|.$$

Theorem 6. *Let $\mathcal{M}$ be a UCBS and $\mathfrak{B}$ an m-accretive operator in $\mathcal{M}$ such that $\mathcal{F} \neq \emptyset$. For fixed $k \in \mathbb{N}$, let $\{\varrho_n\}$ be a sequence defined as follows:*

$$\begin{cases} \varrho_1 = \varrho \in C \\ \varrho_{n+1} = J^k_{c_n}\{(1-\alpha_n)\varrho_n + \alpha_n J_{c_n}(\varrho_n)\}, \quad n \in \mathbb{N}, \end{cases} \tag{16}$$

where $\{\alpha_n\}$ is a sequences in $[a, b]$ with $a, b \in (0, 1)$ and $\{c_n\}$ is a sequence of positive numbers satisfying the following condition:

$$0 < c_* < c_n < c^* < \infty.$$

Assume that either of following assumptions hold:

(a) $\mathcal{M}$ has the Opial's property;
(b) $\mathcal{M}^*$ has the KK-property.

Then, the sequence $\{\varrho_n\}$ weakly converges to a point of $\mathcal{F}$.

Proof. Let $p^\dagger \in \mathcal{F}$. By (16), we obtain

$$\begin{aligned} \|\varrho_{n+1} - p^\dagger\| &= \|J^k_{c_n}\{(1-\alpha_n)\varrho_n + \alpha_n J_{c_n}(\varrho_n)\} - p^\dagger\| \\ &\leq \|(1-\alpha_n)\varrho_n + \alpha_n J_{c_n}(\varrho_n) - p^\dagger\| \\ &= \|(1-\alpha_n)(\varrho_n - p^\dagger) + \alpha_n(J_{c_n}(\varrho_n) - p^\dagger)\| \\ &\leq \|\varrho_n - p^\dagger\|. \end{aligned}$$

Thus, $\{\varrho_n\}$ is bounded and $\lim_{n\to\infty} \|\varrho_n - p^\dagger\|$ exists. Call it r. That is

$$\lim_{n\to\infty} \|\varrho_n - p^\dagger\| = r. \tag{17}$$

Using nonexpansiveness of J_{c_n} and (17)

$$\limsup_{n\to\infty} \|J_{c_n}(\varrho_n) - p^\dagger\| \le r. \tag{18}$$

Now, by (16) and (17), we have

$$\begin{aligned} r = \lim_{n\to\infty} \|\varrho_{n+1} - p^\dagger\| &= \limsup_{n\to\infty} \|J_{c_n}^k\{(1-\alpha_n)\varrho_n + \alpha_n J_{c_n}(\varrho_n)\} - p^\dagger\| \\ &\le \limsup_{n\to\infty} \|(1-\alpha_n)\varrho_n + \alpha_n J_{c_n}(\varrho_n) - p^\dagger\| \\ &\le \lim_{n\to\infty} \|\varrho_n - p^\dagger\| = r. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|(1-\alpha_n)(\varrho_n - p^\dagger) + \alpha_n(J_{c_n}(\varrho_n) - p^\dagger)\| = r. \tag{19}$$

From (17)–(19) and Lemma 1, it follows that

$$\lim_{n\to\infty} \|\varrho_n - J_{c_n}(\varrho_n)\| = 0. \tag{20}$$

By (20) and Lemma 4, we obtain

$$\lim_{n\to\infty} \|\varrho_n - J_{c_*}(\varrho_n)\| \le 2 \lim_{n\to\infty} \|\varrho_n - J_{c_n}(\varrho_n)\| = 0. \tag{21}$$

From (21) and demiclosedness principle, it follows that $\omega_w(\varrho_n) \subset F(J_{c_*}) = \mathcal{F}$. Following the last part of the proof of Theorem 2, we can conclude that the sequence $\{\varrho_n\}$ weakly converges to a point in $\mathcal{F}$. $\square$

5.2. Generalized Mixed Equilibrium Problem

Let $\mathcal{C}$ be a closed convex subset of $\mathcal{H}$ and $\varphi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a bifunction satisfying certain conditions. Consider the following problem which is known as equilibrium problem (or EP), see [39]:

$$\text{Find } u \in \mathcal{C} \text{ such that } \varphi(u,\nu) \ge 0 \text{ for all } \nu \in \mathcal{C}. \tag{22}$$

Zhang [40] generalized EP and called it generalized mixed equilibrium problem:

$$\text{Find } u \in \mathcal{C} \text{ such that } \varphi(u,\nu) + \langle \Theta(u), \nu - u\rangle - \phi(u) + \phi(\nu) \ge 0 \tag{23}$$

for all $\nu \in \mathcal{C}$, where $\Theta : \mathcal{C} \to \mathcal{H}$ is a nonlinear mapping and $\phi : \mathcal{C} \to \mathbb{R} \cup \{\infty\}$ is a real valued function. We denote the set of solutions of (23) by $GMEP(\varphi, \Theta, \phi)$, that is,

$$GMEP(\varphi,\Theta,\phi) = \{u \in \mathcal{C} : \varphi(u,\nu) + \langle \Theta(u), \nu - u\rangle - \phi(u) + \phi(\nu) \ge 0\} \tag{24}$$

for all $\nu \in \mathcal{C}$.

- The problem (23) reduced to mixed equilibrium problem (in short, MEP) [41] if $\Theta = \emptyset$.
- The problem (23) is equivalent to mixed variational inequality problem of Browder type [42] if $\varphi = \emptyset$.
- The problem (23) is known as equilibrium problem (22) if $\Theta = \phi = \emptyset$.

In order to solve problem (23), we suppose that the bifunction $\varphi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ satisfies the following assumptions:

(X1) $\varphi(\varrho, \varrho) = 0$ for all $\varrho \in \mathcal{C}$;
(X2) φ is monotone, that is, $\varphi(\varrho,\nu) + \varphi(\nu,\varrho) \ge 0$ for all $\varrho, \nu \in \mathcal{C}$;
(X3) for all $\varrho, \nu, w \in \mathcal{C}$, $\lim_{t\to 0} \varphi(tw + (1-t)\varrho, \nu) \le \varphi(\varrho,\nu)$;
(X4) for each $\varrho \in \mathcal{C}$; $\nu \mapsto \varphi(\varrho,\nu)$ is a convex and lower semicontinuous;

(X5) for fixed $r > 0$ and $w \in \mathcal{C}$, there exists a bounded subset $\mathcal{K}$ of $\mathcal{H}$ and $\varrho \in \mathcal{C} \cap \mathcal{K}$ such that for all $\nu \in \mathcal{C} \backslash \mathcal{K}$

$$\varphi(w, \varrho) + \frac{1}{r}\langle \nu - \varrho, \varrho - w \rangle \geq 0.$$

It is shown in [40] that if $\varphi(\varrho, \nu)$ satisfies (X1)–(X4), then for the function

$$H_1(\varrho, \nu) := \varphi(\varrho, \nu) + \langle \Theta(\varrho), \nu - \varrho \rangle - \phi(\varrho) + \phi(\nu)$$

assumptions (X1)–(X4) still hold and $\mathrm{GMEP}(\varphi, \Theta, \phi)$ is closed and convex.

Lemma 5. *[40]. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C}$ a nonempty closed convex subset of $\mathcal{H}$. Let $\Theta : \mathcal{C} \to \mathcal{H}$ be a continuous and monotone mapping, $\varphi : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ a bifunction satisfying (X1)–(X4) and $\phi : \mathcal{C} \to \mathbb{R} \cup \{\infty\}$ a proper lower semicontinuous and convex function. For given $r > 0$ and $\varrho \in \mathcal{H}$, define a mapping $T_r^{\varphi} : \mathcal{H} \to \mathcal{C}$ by*

$$T_r^{\varphi}(\varrho) = \left\{ u \in \mathcal{C} : \varphi(u, \nu) + \langle \Theta(u), \nu - u \rangle - \phi(u) + \phi(\nu) + \frac{1}{r}\langle \nu - u, u - \varrho \rangle \geq 0 \right\} \tag{25}$$

for all $\nu \in \mathcal{C}$. Then

(a) For each $\varrho \in \mathcal{H}$, $T_r^{\varphi}(\varrho)$ is nonempty;
(b) T_r^{φ} is a single valued mapping;
(c) T_r^{φ} is firmly nonexpansive, that is, for all $\varrho, y \in \mathcal{C}$

$$\|T_r^{\varphi}(\varrho) - T_r^{\varphi}(\nu)\|^2 \leq \langle T_r^{\varphi}(\varrho) - T_r^{\varphi}(\nu), \varrho - \nu \rangle;$$

so T_r^{φ} is a nonexpansive.
(d) $F(T_r^{\varphi}) = GMEP(\varphi, \Theta, \phi)$;
(e) $GMEP(\varphi, \Theta, \phi)$ is closed and convex.

Theorem 7. *Let $\mathcal{C}$, $\mathcal{H}$, Θ, φ and ϕ be same as in Lemma (5). Suppose $GMEP(\varphi, \Theta, \phi) \neq \emptyset$. Let $\{\varrho_n\}$ be a sequence defined by*

$$\begin{cases} \varrho_1 \in \mathcal{H} \\ \varphi(u_n, \nu) + \langle \Theta(u_n), \nu - u_n \rangle - \phi_1(u_n) + \phi_1(\nu) + \frac{1}{r}\langle \nu - u_n, u_n - \varrho_n \rangle \geq 0 \text{ for all } \nu \in \mathcal{C} \\ \nu_n = (1 - \alpha_n)\varrho_n + \alpha_n u_n \\ \varphi(v_n, \nu) + \langle \Theta(v_n), \nu - v_n \rangle - \phi_1(v_n) + \phi_1(\nu) + \frac{1}{r}\langle \nu - v_n, v_n - \nu_n \rangle \geq 0 \text{ for all } \nu \in \mathcal{C} \\ \varrho_{n+1} = v_n, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a same sequence as in Theorem 2. Then, $\{\varrho_n\}$ weakly converges to a point in $GMEP(\varphi, \Theta, \phi)$.

Proof. Taking $T = T_r^{\varphi}$, $k = 1$ in Theorem 2 and in view of Lemma 5, we can easily obtain the desired result. □

5.3. Constrained Convex Optimization Problem

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{C}$ a closed convex subset of $\mathcal{H}$. Let $\varsigma : \mathcal{H} \to \mathbb{R}$ be a differentiable convex function. Consider the following minimization problem:

$$\min_{\varrho \in \mathcal{C}} \varsigma(\varrho). \tag{26}$$

It can be seen that $\varrho^* \in \mathcal{C}$ is a solution to the minimization problem (26) if and only if ϱ^* is a solution to the fixed point equation

$$\varrho^* = P_{\mathcal{C}}(I - \gamma \nabla \varsigma)\varrho^*, \tag{27}$$

where $\gamma > 0$ is any fixed number. It is known that if $\nabla\varsigma$ satisfies the Lipschitz condition, that is,

$$\|\nabla\varsigma(\varrho) - \nabla\varsigma(\nu)\| \le L\|\varrho - \nu\| \tag{28}$$

for all $\varrho, \nu \in \mathcal{H}$, where $L > 0$, then the mapping $P_{\mathcal{C}}(I - \gamma\nabla\varsigma)$ is $\frac{2+\gamma L}{4}$ is averaged for $0 < \gamma < \frac{2}{L}$. Hence, $T_\gamma = P_{\mathcal{C}}(I - \gamma\nabla\varsigma)$ is nonexpansive mapping. Now, we employ the iterative method (4) to solve the minimization problem (26).

Theorem 8. *Let $\varsigma : \mathcal{H} \to \mathbb{R}$ be a function defined above with the Lipschitz condition (28). Assume that the divergence condition (9) holds. For fixed $k \in \mathbb{N}$ and given $\varrho_1 \in \mathcal{H}$, the sequence $\{\varrho_n\}$ is defined as*

$$\varrho_{n+1} = [P_{\mathcal{C}}(I - \gamma\nabla\varsigma)]^k\{(1-\alpha_n)\varrho_n + P_{\mathcal{C}}(I - \gamma\nabla\varsigma)(\varrho_n)\}$$

where $0 < \gamma < \frac{2}{L}$. Then, $\{\varrho_n\}$ weakly converges to a solution of (26).

Again, we discuss a quadratic optimization problem on the trust region (see [43] for more details). Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint linear operator. Let $\delta > 0$ be a fixed constant and u a given point in $\mathcal{H}$. Consider the following problem:

$$\min_{\|\varrho\| \le \delta} \varsigma(\varrho) := \frac{1}{2}\langle A(\varrho), \varrho\rangle - \langle \varrho, u\rangle. \tag{29}$$

Take

$$\mathcal{C} := \{\varrho \in \mathcal{H} : \|\varrho\| \le \delta\}.$$

Then, $\mathcal{C}$ is a closed ball having radius δ with center at origin. Thus, the projection $P_{\mathcal{C}}$ can be defined as

$$P_\delta \equiv P_{\mathcal{C}} = \begin{cases} \varrho, & \text{if } \|\varrho\| \le \delta \\ \frac{\delta\varrho}{\|\varrho\|}, & \text{if } \|\varrho\| > \delta. \end{cases}$$

The gradient of ς is defined as

$$\nabla\varsigma(\varrho) = A(\varrho) - u$$

and $\nabla\varsigma$ is L-Lipschitz with $L = \|A\|$. We consider the following theorem.

Theorem 9. *Let A, P_δ, $\mathcal{H}$ and u be as defined above. Assume that the divergence condition (9) holds. For fixed $k \in \mathbb{N}$ and given $\varrho_1 \in \mathcal{H}$, the sequence $\{\varrho_n\}$ is defined as*

$$\varrho_{n+1} = [P_\delta(I - \gamma A + \gamma u)]^k\{(1-\alpha_n)\varrho_n + P_\delta(I - \gamma A + \gamma u)(\varrho_n)\}$$

where $0 < \gamma < \frac{2}{L}$. Then, $\{\varrho_n\}$ weakly converges to a solution of (29).

5.4. Split Feasibility Problem

Censor and Elfving [44] introduced the following problem (or split feasibility problem, in short SFP):

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite dimensional Hilbert spaces. Let $\Gamma : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is to find an element

$$\varrho^* \in \mathcal{C} \text{ such that } \Gamma(\varrho^*) \in \mathcal{Q}. \tag{30}$$

Let $\Omega := \{\varrho^* \in \mathcal{C} : \Gamma(\varrho^*) \in \mathcal{Q}\} = \mathcal{C} \cap \Gamma^{-1}(\mathcal{Q})$ be set of solutions of (30). The SFP has many important applications which appear in modeling inverse problems, medical image reconstruction and others; for more details, see [45]. A number of authors have extended the SFP from finite dimensional spaces to infinite dimensional Hilbert spaces.

Byrne [5] considered the following algorithm known as CQ to obtain the solution of (30):

$$\varrho_{n+1} = P_{\mathcal{C}}(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)\varrho_n$$

where $\Gamma^* : \mathcal{H}_2 \to \mathcal{H}_1$ is the adjoint of Γ and $\gamma > 0$.

In view of following analysis, we can use fixed point iterative method to solve the SFP (30) (see [46] for more details).

Suppose that $\varrho^* \in \Omega$ and $\gamma > 0$. Hence $\Gamma(\varrho^*) \in \mathcal{Q}$, which in turn follows the equation $(I - P_{\mathcal{Q}})\Gamma(\varrho^*) = 0$ which leads to the equation $\gamma(I - P_{\mathcal{Q}})\Gamma(\varrho^*) = 0$. Therefore, we have the fixed point equation

$$(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)\varrho^* = \varrho^*.$$

To ensure that $\varrho^* \in \mathcal{C}$, we can consider the following:

$$P_{\mathcal{C}}(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)\varrho^* = \varrho^*. \tag{31}$$

Proposition 2 ([46]). *For given $\varrho^* \in \mathcal{H}_1$, ϱ^* is a solution of the SFP* (30) *if and only if ϱ^* is a solution of the fixed point Equation* (31).

Theorem 10. *Let $\mathcal{H}_1, \mathcal{H}_2, \Gamma$, and Γ^* be as defined above. Assume that the SFP* (30) *is consistent and $0 < \gamma < \frac{2}{\|\Gamma\|^2}$. For fixed $k \in \mathbb{N}$ and let $\{\varrho_n\}$ be a sequence such that*

$$\begin{cases} \varrho_1 \in \mathcal{H} \\ \varrho_{n+1} = [P_{\mathcal{C}}(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)]^k\{(1 - \alpha_n)\varrho_n + \alpha_n P_{\mathcal{C}}(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)\varrho_n\} \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subseteq [0,1]$ satisfying the following condition:

$$\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty.$$

Then, the sequence $\{\varrho_n\}$ weakly converges to a point $p \in \Omega$.

Proof. It is shown in [46], (Theorem 3.6) that $P_{\mathcal{C}}(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)$ is α-averaged with $\alpha = \frac{2+\gamma\|\Gamma\|^2}{4} \in (0,1)$. Thus, take $T = P_{\mathcal{C}}(I - \gamma\Gamma^*(I - P_{\mathcal{Q}})\Gamma)$ and T is a nonexpansive mapping. Therefore, the required result follows from Theorem 2. □

5.5. Periodic Solution of a Nonlinear Evolution Equation

Browder [47] considered the following time-dependent nonlinear evolution equation,

$$\frac{d\varrho}{dr} + \Psi(r)\varrho = f(r, \varrho), \qquad r > 0 \tag{32}$$

where $\mathcal{H}$ is Hilbert space, $f : \mathbb{R} \times \mathcal{H} \to \mathcal{H}$ is a mapping and $\Psi(r)$ is a family of closed linear operators in $\mathcal{H}$.

Definition 6 ([47]). *Let $\varrho : \mathbb{R} \to \mathcal{H}$ be continuous under the strong topology. ϱ is called a mild solution of* (32) *on $\mathbb{R}^+$ with initial value*

$$\varrho(0) = \nu \tag{33}$$

if and only if

$$\varrho(r) = U(r,0)\nu + \int_0^r U(r,s)f(s,\varrho(s))ds, \quad \forall \quad r > 0,$$

where $\{U(r,s)\}_{r\geq s\geq 0}$ *is the evolution system for the homogeneous linear system*

$$\frac{d\varrho}{dr} + \Psi(r)\varrho = 0 \quad (r > s). \tag{34}$$

The following theorem proved the existence of periodic solution of (32).

Theorem 11 ([47])**.** *Assume that* $\Psi(r)$ *and* $f(r,\varrho)$ *are periodic in* r *with a common period* ξ *and the following conditions hold:*

(1) For each r*,*

$$\mathrm{Re}\langle f(r,\varrho) - f(r,\nu), \varrho - \nu\rangle \leq 0, \quad \forall \quad \varrho, \nu \in \mathcal{H}.$$

(2) For each r*,*

$$\mathrm{Re}\langle \Psi(r)\varrho, \varrho\rangle \geq 0, \quad \forall \quad \varrho \in D(\Psi(r)).$$

(3) For each initial value $\nu \in \mathcal{H}$*, there exists a mild solution* ϱ *of* (32) *on* $\mathbb{R}^+$*.*

(4) There exists some $R > 0$ *such that*

$$\mathrm{Re}\langle f(r,\varrho), \varrho\rangle < 0, \quad \forall \quad \|\varrho\| = R \text{ and } \forall \quad r \in [0,\xi].$$

Then, there exists an element $\nu \in \mathcal{H}$ *with* $\|\nu\| < R$ *such that the mild solution of* (32) *with initial condition* $\varrho(0) = \nu$*, is periodic of period* ξ*.*

We employ the iterative method (4) to obtain a periodic solution of (32). Let $T : \mathcal{H} \to \mathcal{H}$ be a mapping defined as follows:

$$T(\nu) = \varrho(\xi), \tag{35}$$

where ϱ is the solution of (32) satisfying the initial condition $\varrho(0) = \nu$. That is, let T be the mapping which assigns to each $\nu \in \mathcal{H}$ the value of ξ of the mild solution $\varrho(r)$ of (32) with $\varrho(0) = \nu$. T maps the closed ball $\mathcal{B} := \{\nu \in \mathcal{H} : \|\nu\| \leq R\}$ into itself due to the Assumption (4). It is noted that T is a nonexpansive mapping. Therefore, T has a fixed point, say ν, and the corresponding solution ϱ of (32) with $\varrho(0) = \nu$ is a desired periodic solution of (32) with period ξ. More precisely, finding a periodic solution ϱ of (32) is equivalent to finding a fixed point of T.

Now, we consider an iterative method approach to finding a periodic solution of (32).

Theorem 12. *Suppose that Assumptions (1)–(4) in Theorem 11 hold. For a given* $\nu_1 \in \mathcal{B}$*, define a sequence of functions* $\{\nu_n\}$ *as follows:*

$$\begin{cases} w_n = (1-\alpha_n)\nu_n + \alpha_n \varrho_n^{(1)}(\xi) \\ \nu_{n+1} = \varrho_n^{(2)}(\xi) \end{cases} \tag{36}$$

where $\{\alpha_n\}$ *is a sequence as in Theorem 2,* $\varrho_n^{(1)}$ *is a solution of* (32) *with* $\varrho_n^{(1)}(0) = \nu_n$ *and* $\varrho_n^{(2)}$ *is a solution of* (32) *with* $\varrho_n^{(2)}(0) = w_n$ *for each* $n \in \mathbb{N}$ *(that is* (35) *holds for* $\varrho = \varrho_n^{(1)}$*,* $\nu = \nu_n$ *and* $\varrho = \varrho_n^{(2)}$*,* $\nu = w_n$*). Then, the sequence* $\{\nu_n\}$ *weakly converges to a point in* $F(T)$ *and the corresponding mild solution of* (32) *with* $\varrho(0) = \nu$*, is a periodic of period* ξ*.*

Author Contributions: Conceptualization, R.S.; original draft preparation, R.S.; writing, review and editing, R.P. and W.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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