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Some Generalization of the Method of Stability Investigation for Nonlinear Stochastic Delay Differential Equations

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Abstract: It is known that the method of Lyapunov functionals is a powerful method of stability investigation for functional differential equations. Here, it is shown how the previously proposed method of stability investigation for nonlinear stochastic differential equations with delay and a high order of nonlinearity can be extended to nonlinear mathematical models of a much more general form. An important feature is the combination of the method of Lyapunov functionals with the method of Linear Matrix Inequalities (LMIs). Some examples of applications of the proposed method of stability research to known mathematical models are given.

Keywords: stability of equilibria; nonlinear delay differential equations; stochastic perturbations; asymptotic mean square stability; stability in probability; Linear Matrix Inequality (LMI)

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Citation: Shaikhet, L. Some Generalization of the Method of Stability Investigation for Nonlinear Stochastic Delay Differential Equations. *Symmetry* **2022**, *14*, 1734. https://doi.org/10.3390/sym14081734

Academic Editor: Sergei D. Odintsov

Received: 26 July 2022
Accepted: 15 August 2022
Published: 19 August 2022

1. Introduction

It is known that after the works of Krasovskii N.N. [1–3], the method of Lyapunov functionals or the so-called method of Lyapunov–Krasovskii functionals is one of the most powerful methods of stability investigation for functional differential equations (see, for instance [4–8] and the references therein). The special procedure of Lyapunov functionals construction allows for the construction of different Lyapunov functionals for one differential equation with delay and, as a result, obtains different stability conditions for the considered equation [4].

The aim of this paper is to show how the application of the method proposed in [9] for studying the stability of nonlinear stochastic functional differential equations with a high order of nonlinearity can be extended to mathematical models of a much more general form.

1.1. Statement of the Problem

Consider the nonlinear differential equation with distributed delays:

$$\begin{aligned} \dot{x}(t) &= a + Ax(t) + \sum_{i=1}^{k} \int_0^\infty B_i x(t-s)dK_i(s) + \sum_{i=1}^{m} \int_0^\infty f_i(x(t), x(t-s))dF_i(s), \\ x(s) &= \phi(s), \quad s \le 0, \end{aligned} \tag{1}$$

where $a, x(t) \in \mathbf{R}^n$, $A, B_i \in \mathbf{R}^{n\times n}$, $f_i(x_1, x_2) \in \mathbf{R}^n$ are nonlinear differentiable functions, and $K_i(s)$ and $F_i(s)$ are scalar right-continuous nondecreasing functions of bounded variation on $[0, \infty)$, such that

$$K_i = \int_0^\infty dK_i(s) < \infty, \qquad F_i = \int_0^\infty dF_i(s) < \infty, \tag{2}$$

and the integrals are understood in the Stieltjes sense.

From (1) and (2), it follows that the equilibrium x^* of Equation (1) is defined by the equation

$$a + \left(A + \sum_{i=1}^{k} K_i B_i\right) x^* + \sum_{i=1}^{m} F_i f_i(x^*, x^*) = 0. \tag{3}$$

We will investigate the stability of Equation (1) equilibrium x^* under stochastic perturbations of the white-noise type that are directly proportional to the deviation of the solution $x(t)$ from the equilibrium x^* and immediately influence the derivative. In doing this, Equation (1) takes the form of Ito's stochastic differential equation [4,10]

$$\begin{aligned} dx(t) = &\left(a + Ax(t) + \sum_{i=1}^{k} \int_0^\infty B_i x(t-s) dK_i(s) + \sum_{i=1}^{m} \int_0^\infty f_i(x(t), x(t-s)) dF_i(s)\right) dt \\ &+ \sum_{j=1}^{l} C_j (x(t) - x^*) dw_j(t), \qquad x(s) = \phi(s) \in H_2, \quad s \le 0, \end{aligned} \tag{4}$$

where $C_j \in \mathbf{R}^{n \times n}$ and $w_j(t)$, $j = 1, \ldots, l$, are mutually independent standard Wiener processes on the completed probability space $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ with a nondecreasing family of σ-algebras $\mathfrak{F}_t$, and H_2 is the space of $\mathfrak{F}_0$-adapted stochastic processes $\phi(s)$, $s \le 0$, with continuous trajectories.

Note that stochastic perturbations of the type (4) were firstly used in [11] and later in many other research works (see, for instance, [4] and references therein). In this, the equilibrium x^* of Equation (1) is also the solution of the stochastic differential Equation (4).

Let us center Equation (4) at the equilibrium x^* using the new variable $y(t) = x(t) - x^*$. From (4) we have

$$\begin{aligned} dy(t) = &\left(a + A(x^* + y(t)) + \sum_{i=1}^{k} \int_0^\infty B_i (x^* + y(t-s)) dK_i(s) \right. \\ &\left. + \sum_{i=1}^{m} \int_0^\infty f_i(x^* + y(t), x^* + y(t-s)) dF_i(s)\right) dt + \sum_{j=1}^{l} C_j y(t) dw_j(t). \end{aligned} \tag{5}$$

It is clear that the stability of the equilibrium x^* of Equation (4) is equivalent to the stability of the zero solution of Equation (5).

Let $J_j f_i(x_1, x_2)$, $j = 1, 2$, be the Jacobian matrix of the function $f_i(x_1, x_2) \in \mathbf{R}^n$ with respect to the variable x_j. Using Taylor's expansion in the form

$$f_i(x_1^* + y_1, x_2^* + y_2) = f_i(x_1^*, x_2^*) + J_1 f_i(x_1^*, x_2^*) y_1 + J_2 f_i(x_1^*, x_2^*) y_2 + o(y_1) + o(y_2),$$

where $\lim_{|y_1| \to 0} \frac{o(y_1)}{|y_1|} = 0$, $\lim_{|y_2| \to 0} \frac{o(y_2)}{|y_2|} = 0$, via (2) we have

$$\begin{aligned} &\int_0^\infty f_i(x^* + y(t), x^* + y(t-s)) dF_i(s) \\ &= F_i f_i(x^*, x^*) + F_i J_1 f_i(x^*, x^*) y(t) \\ &\quad + \int_0^\infty J_2 f_i(x^*, x^*) y(t-s) dF_i(s) + o(y_1) + o(y_2). \end{aligned} \tag{6}$$

Substituting (6) into (5) and using (2) and (3), we obtain the linear approximation of the nonlinear Equation (5)

$$\begin{aligned} dz(t) = &\left[\left(A + \sum_{i=1}^{m} F_i D_{1i}\right) z(t) + \sum_{i=1}^{k} \int_0^\infty B_i z(t-s) dK_i(s) \right. \\ &\left. + \sum_{i=1}^{m} \int_0^\infty D_{2i} z(t-s) dF_i(s)\right] dt + \sum_{j=1}^{l} C_j z(t) dw_j(t), \end{aligned} \tag{7}$$

where
$$D_{1i} = J_1 f_i(x^*, x^*), \qquad D_{2i} = J_2 f_i(x^*, x^*). \tag{8}$$

Remark 1. *Using* (2), *the equalities*
$$\begin{aligned} \frac{d}{dt}\left(\int_0^\infty \int_{t-s}^t z(\tau)d\tau dK_i(s)\right) &= K_i z(t) - \int_0^\infty z(t-s)dK_i(s), \\ \frac{d}{dt}\left(\int_0^\infty \int_{t-s}^t z(\tau)d\tau dF_i(s)\right) &= F_i z(t) - \int_0^\infty z(t-s)dF_i(s), \end{aligned} \tag{9}$$
and (8), *transform Equation* (7) *into the form of a neutral-type equation*
$$\begin{aligned} dZ(t) &= (A+H)z(t)dt + \sum_{j=1}^{l} C_j z(t)dw_j(t), \qquad Z(t) = z(t) + G(t), \\ H &= \sum_{i=1}^{k} K_i B_i + \sum_{i=1}^{m} F_i(D_{1i} + D_{2i}), \qquad G(t) = \sum_{i=1}^{k} B_i \xi_i(t) + \sum_{i=1}^{m} D_{2i}\eta_i(t), \\ \xi_i(t) &= \int_0^\infty \int_{t-s}^t z(\tau)d\tau dK_i(s), \qquad \eta_i(t) = \int_0^\infty \int_{t-s}^t z(\tau)d\tau dF_i(s). \end{aligned} \tag{10}$$

1.2. Some Auxiliary Definitions and Statements

Definition 1 ([4])**.** *The zero solution of Equation* (5) *with the initial condition $y(s) = \phi(s)$, $s \le 0$, is called stable in probability if for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists $\delta > 0$ such that the solution $y(t,\phi)$ of Equation* (5) *satisfies the condition $\mathbf{P}\{\sup_{t\ge 0} |y(t,\phi)| > \varepsilon_1\} < \varepsilon_2$ for any initial function ϕ such that $\mathbf{P}\{\sup_{s\le 0} |\phi(s)| < \delta\} = 1$.*

Definition 2 ([4])**.** *The zero solution of Equation* (7) *with the initial condition $z(s) = \phi(s)$, $s \le 0$, is called:*

- *Mean square stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|z(t,\phi)|^2 < \varepsilon$, $t \ge 0$, provided that $\|\phi\|^2 = \sup_{s\le 0} \mathbf{E}|\phi(s)|^2 < \delta$;*
- *Asymptotically mean square stable if it is mean square stable and for each initial function ϕ the solution $z(t)$ of Equation* (7) *satisfies the condition $\lim_{t\to\infty} \mathbf{E}|z(t)|^2 = 0$.*

Remark 2. *The representation* (6) *in particular means that the level of nonlinearity of Equation* (5) *is more than one. In this case, it is known that sufficient conditions for the asymptotic mean square stability of the zero solution of the linear Equation* (7) *are also sufficient conditions for stability in probability of the zero solution of the nonlinear Equation* (5) *and therefore are sufficient conditions for stability in probability of the equilibrium x^* of Equation* (4) *[4].*

Let $z(t)$ be a value of Equation (7) solution in the time moment and t, $z_t = z(t+s)$, $s < 0$ be the trajectory of Equation (7) solution until the time moment t. Consider a functional $V(t,\varphi) : [0,\infty) \times H_2 \to \mathbf{R}_+$ that can be presented in the form $V(t,\varphi) = V(t,\varphi(0),\varphi(s))$, $s < 0$, and for $\varphi = z_t$ put
$$V_\varphi(t,z) = V(t,\varphi) = V(t,z_t) = V(t,z,z(t+s)), \quad z = \varphi(0) = z(t), \quad s < 0. \tag{11}$$

Denote by D the set of the functionals, for which the function $V_\varphi(t,z)$ defined in (11) has a continuous derivative with respect to t and two continuous derivatives with respect to z. Let $'$ be the sign of transpose, ∇ and ∇^2 be the first and the second derivatives, respectively, of the function $V_\varphi(t,z)$ with respect to z. For the functionals from D, the generator L of Equation (7) has the form [4,10]

$$
\begin{aligned}
LV(t,z_t) =& \frac{\partial V_\varphi(t,z(t))}{\partial t} + \nabla V'_\varphi(t,z(t))\left[\left(A+\sum_{i=1}^{m}F_iD_{1i}\right)z(t)\right. \\
&\left.+\sum_{i=1}^{k}B_iu_i(z_t)+\sum_{i=1}^{m}D_{2i}v_i(z_t)\right]+\frac{1}{2}\sum_{j=1}^{l}z'(t)C'_j\nabla^2V_\varphi(t,z(t))C_jz(t), \\
&u_i(z_t)=\int_0^\infty z(t-s)dK_i(s), \qquad v_i(z_t)=\int_0^\infty z(t-s)dF_i(s).
\end{aligned}
\tag{12}
$$

Theorem 1 ([4]). *Let there exist a functional $V(t,\varphi)\in D$ and positive constants c_1, c_2, c_3, such that the following conditions hold:*

$$
\mathbf{E}V(t,z_t)\geq\ c_1\mathbf{E}|z(t)|^2, \qquad \mathbf{E}V(0,\phi)\leq\ c_2\|\phi\|^2, \qquad \mathbf{E}LV(t,z_t)\leq\ -c_3\mathbf{E}|z(t)|^2.
$$

Then the zero solution of Equation (7) is asymptotically mean square stable.

Lemma 1 ([9]). *Let $R\in\mathbf{R}^{n\times n}$ be a positive definite matrix, $z=\int_Q y(s)\mu(\mathrm{d}s)$, where $z,y(s)\in\mathbf{R}^n$, $\mu(\mathrm{d}s)$ is some measure on Q such that $\mu(Q)<\infty$ and the integral is defined in the Lebesgue sense. Then*

$$
z'Rz\leq\mu(Q)\int_Q y'(s)Ry(s)\mu(\mathrm{d}s). \tag{13}
$$

Definition 3 ([4]). *The trace of the k-th order of a $n\times n$ matrix $A=\|a_{ij}\|$ is defined as follows:*

$$
S_k=\sum_{1\leq i_1<\ldots<i_k\leq n}\begin{vmatrix} a_{i_1i_1} & \ldots & a_{i_1i_k} \\ \ldots & \ldots & \ldots \\ a_{i_ki_1} & \ldots & a_{i_ki_k}\end{vmatrix}, \qquad k=1,\ldots,n.
$$

Here, in particular, $S_1=Tr(A)$, $S_n=\det(A)$, $S_{n-1}=\sum_{i=1}^{n}A_{ii}$, where A_{ii} is the algebraic complement of the diagonal element a_{ii} of the matrix A.

Lemma 2 ([4]). *A 2×2 matrix A is the Hurwitz matrix if and only if $S_1<0$, $S_2>0$. A 3×3 matrix A is the Hurwitz matrix if and only if $S_1<0$, $S_1S_2<S_3<0$.*

2. Stability

In this section, we obtain sufficient conditions for the asymptotic mean square stability of the zero solutions of Equations (7) and (10), which, following Remark 2, are also sufficient conditions for stability in probability of the equilibrium x^* of Equation (4).

Note that the sign "*" inside of a matrix indicates a symmetric element of a symmetric matrix, and the matrix inequality $\Psi<0$ indicates that the symmetric matrix Ψ is a negative definite one.

Theorem 2. *Let there exist positive definite $n\times n$ matrices P, $Q_1,\ldots,Q_k$ and $R_1,\ldots,R_m$, such that the LMI is satisfied:*

$$
\begin{aligned}
&\Psi_0=\begin{bmatrix}
\Phi_0 & PB_1 & \ldots & PB_k & PD_{21} & \ldots & PD_{2m} \\
* & -Q_1 & \ldots & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
* & * & \ldots & -Q_k & 0 & \ldots & 0 \\
* & * & \ldots & * & -R_1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
* & * & \ldots & * & * & \ldots & -R_m
\end{bmatrix}<0, \\
&\Phi_0=P\left(A+\sum_{i=1}^{m}F_iD_{1i}\right)+\left(A+\sum_{i=1}^{m}F_iD_{1i}\right)'P+\sum_{j=1}^{l}C'_jPC_j+\sum_{i=1}^{k}K_i^2Q_i+\sum_{i=1}^{m}F_i^2R_i.
\end{aligned}
\tag{14}
$$

Then the equilibrium x^* of Equation (4) is stable in probability.

Proof. Following Remark 2, it is enough to prove that the zero solution of the linear Equation (7) is asymptotically mean square stable. Let L be the generator of Equation (7) [4,10]. Following the procedure of Lyapunov functional construction [4], we will construct the Lyapunov functional for Equation (7) in the form $V = V_1 + V_2$, where $V_1(z(t)) = z'(t)Pz(t)$, $P > 0$. Using (12) for $V_1(z(t))$, we have

$$\begin{aligned} LV_1(z(t)) =&2z'(t)P\left[\left(A+\sum_{i=1}^{m}F_iD_{1i}\right)z(t)+\sum_{i=1}^{k}B_iu_i(z_t)+\sum_{i=1}^{m}D_{2i}v_i(z_t)\right] \\ &+\sum_{j=1}^{l}z'(t)C_j'PC_jz(t) \\ =&z'(t)\left[P\left(A+\sum_{i=1}^{m}F_iD_{1i}\right)+\left(A+\sum_{i=1}^{m}F_iD_{1i}\right)'P+\sum_{j=1}^{l}C_j'PC_j\right]z(t) \\ &+2\sum_{i=1}^{k}z'(t)PB_iu_i(z_t)+2\sum_{i=1}^{m}z'(t)PD_{2i}v_i(z_t). \end{aligned} \tag{15}$$

Let us choose the additional functional V_2 in the form

$$\begin{aligned} V_2(t,z_t) =& \sum_{i=1}^{k}K_i\int_0^\infty\int_{t-s}^{t}z'(\tau)Q_iz(\tau)d\tau dK_i(s) \\ &+\sum_{i=1}^{m}F_i\int_0^\infty\int_{t-s}^{t}z'(\tau)R_iz(\tau)d\tau dF_i(s), \quad Q_i, R_i > 0. \end{aligned}$$

Using (2) and the inequality (13), we have

$$\begin{aligned} u_i'(t)Q_iu_i(t) &\le K_i\int_0^\infty z'(t-s)Q_iz(t-s)dK_i(s), \\ v_i'(t)R_iv_i(t) &\le F_i\int_0^\infty z'(t-s)R_iz(t-s)dF_i(s). \end{aligned}$$

So, for the functional V_2, we obtain

$$\begin{aligned} LV_2(t,z_t) =& \sum_{i=1}^{k}K_i^2z'(t)Q_iz(t)-\sum_{i=1}^{k}K_i\int_0^\infty z'(t-s)Q_iz(t-s)dK_i(s) \\ &+\sum_{i=1}^{m}F_i^2z'(t)R_iz(t)-\sum_{i=1}^{m}F_i\int_0^\infty z'(t-s)R_iz(t-s)dF_i(s) \\ \le& z'(t)\left(\sum_{i=1}^{k}K_i^2Q_i+\sum_{i=1}^{m}F_i^2R_i\right)z(t)-\sum_{i=1}^{k}u_i'(t)Q_iu_i(t)-\sum_{i=1}^{m}v_i'(t)R_iv_i(t). \end{aligned} \tag{16}$$

From (15) and (16) for the functional $V = V_1 + V_2$, it follows that

$$\begin{aligned} LV(t,z_t) \le& z'(t)\Phi_0z(t)+2\sum_{i=1}^{k}z'(t)PB_iu_i(z_t)+2\sum_{i=1}^{m}z'(t)PD_{2i}v_i(z_t) \\ &-\sum_{i=1}^{k}u_i'(t)Q_iu_i(t)-\sum_{i=1}^{m}v_i'(t)R_iv_i(t) \\ =&\zeta'(t)\Psi_0\zeta(t), \end{aligned}$$

where the matrices Φ_0 and Ψ_0 are defined in (14) and $\zeta(t) = col(z(t), u_1(t), \dots, u_k(t), v_1(t), \dots, v_m(t))$.

So, the construction above the Lyapunov functional V satisfies Theorem 1. Thus, the zero solution of Equation (7) is asymptotically mean square stable, and, therefore, the equilibrium x^* of Equation (4) is stable in probability. The proof is completed. □

Theorem 3. *Let be*

$$\begin{aligned} \alpha &= \sum_{i=1}^{k} k_i \|B_i\| + \sum_{i=1}^{m} q_i \|D_{2i}\| < 1, \\ k_i &= \int_0^\infty s dK_i(s), \qquad q_i = \int_0^\infty s dF_i(s), \end{aligned} \tag{17}$$

where $\|.\|$ is the matrix norm, and there exist positive definite $n \times n$ matrices P, $Q_1, \dots, Q_k$, and $R_1, \dots, R_m$, such that the LMI is satisfied.

$$\Psi_1 = \begin{bmatrix} \Phi_1 & (A+H)'PB_1 & \dots & (A+H)'PB_k & (A+H)'PD_{21} & \dots & (A+H)'PD_{2m} \\ * & -Q_1 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & -Q_k & 0 & \dots & 0 \\ * & * & \dots & * & -R_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & * & * & \dots & -R_m \end{bmatrix} < 0, \tag{18}$$
$$\Phi_1 = P(A+H) + (A+H)'P + \sum_{j=1}^{l} C_j' P C_j + \sum_{i=1}^{k} k_i^2 Q_i + \sum_{i=1}^{m} q_i^2 R_i.$$

Then, the equilibrium x^* of Equation (4) is stable in probability.

Proof. Let L be the generator of Equation (10). Then, for the functional $V_1(z_t) = Z'(t)PZ(t)$, $P > 0$, via (10), we have

$$\begin{aligned} LV_1(z_t) =& 2(z(t) + G(t))'P(A+H)z(t) + \sum_{j=1}^{l} z'(t)C_j'PC_jz(t) \\ =& 2\left(z(t) + \sum_{i=1}^{k} B_i\xi_i(t) + \sum_{i=1}^{m} D_{2i}\eta_i(t) \right)' P(A+H)z(t) \\ &+ \sum_{j=1}^{l} z'(t)C_j'PC_jz(t) \\ =& 2z'(t)P(A+H)z(t) + 2\sum_{i=1}^{k} \xi_i'(t)B_i'P(A+H)z(t) \\ &+ 2\sum_{i=1}^{m} \eta_i'(t)D_{2i}'P(A+H)z(t) + \sum_{j=1}^{l} z'(t)C_j'PC_jz(t). \end{aligned} \tag{19}$$

From (10), (13) and (17) it follows that

$$\begin{aligned} \xi_i'(t)Q_i\xi_i(t) &\le k_i \int_0^\infty \int_{t-s}^{t} z'(\tau)Q_iz(\tau)d\tau dK_i(s), \\ \eta_i'(t)R_i\eta_i(t) &\le q_i \int_0^\infty \int_{t-s}^{t} z'(\tau)R_iz(\tau)d\tau dF_i(s). \end{aligned} \tag{20}$$

So, for the functional

$$\begin{aligned} V_2(t, z_t) = &\sum_{i=1}^{k} k_i \int_0^\infty \int_{t-s}^{t} (\tau - t + s)z'(\tau)Q_iz(\tau)d\tau dK_i(s) \\ &+ \sum_{i=1}^{m} q_i \int_0^\infty \int_{t-s}^{t} (\tau - t + s)z'(\tau)R_iz(\tau)d\tau dF_i(s) \end{aligned}$$

via (20), we obtain

$$
\begin{aligned}
LV_2(t,z_t) =& \sum_{i=1}^{k} k_i^2 z'(t)Q_i z(t) - \sum_{i=1}^{k} k_i \int_0^\infty \int_{t-s}^{t} z'(\tau)Q_i z(\tau)d\tau dK_i(s) \\
&+ \sum_{i=1}^{m} q_i^2 z'(t)R_i z(t) - \sum_{i=1}^{m} q_i \int_0^\infty \int_{t-s}^{t} z'(\tau)R_i z(\tau)d\tau dF_i(s) \\
\le& z'(t)\left(\sum_{i=1}^{k} k_i^2 Q_i + \sum_{i=1}^{m} q_i^2 R_i\right)z(t) - \sum_{i=1}^{k} \xi_i'(t)Q_i\xi_i(t) - \sum_{i=1}^{m} \eta_i'(t)R_i\eta_i(t).
\end{aligned}
\tag{21}
$$

Using (19) and (21) for the functional $V = V_1 + V_2$, we have

$$
\begin{aligned}
LV(t,z_t) \le& z'(t)\Phi_1 z(t) + 2\sum_{i=1}^{k} \xi_i'(t)B_i'P(A+H)z(t) + 2\sum_{i=1}^{m} \eta_i'(t)D_{2i}'P(A+H)z(t) \\
&- \sum_{i=1}^{k} \xi_i'(t)Q_i\xi_i(t) - \sum_{i=1}^{m} \eta_i'(t)R_i\eta_i(t) \\
=& \zeta'(t)\Psi_1\zeta(t),
\end{aligned}
$$

where the matrices Φ_1 and Ψ_1 are defined in (18) and $\zeta(t) = col\{z(t), \xi_1(t), \dots, \xi_k(t), \eta_1(t), \dots, \eta_m(t)\}$.

This indicates that by using the conditions (17) and (18), the zero solution of Equation (10) is asymptotically mean square stable [4], and therefore the equilibrium x^* of Equation (4) is stable in probability. The proof is completed. □

Remark 3. *In the scalar case* $(n = 1)$ *without loss of generality in the LMIs* (14), (18) *one can use* $P = 1$. *In the general case, the LMIs of the type* (14) *and* (18) *are successfully investigated using MATLAB (see, for instance, [12–14]).*

Corollary 1. *In the case* $k = 0$, $n = m = 1$, *the LMIs* (14) *and* (18) *are equivalent to the conditions*

$$
A + F_1 D_{11} < 0, \qquad |A + F_1 D_{11}| > F_1|D_{21}| + \frac{1}{2}C_1^2, \tag{22}
$$

and

$$
A + F_1(D_{11} + D_{21}) < 0, \qquad |A + F_1(D_{11} + D_{21})|(1 - q_1|D_{21}|) > \frac{1}{2}C_1^2, \tag{23}
$$

respectively.

Proof. Note that via Remark 3, the matrices Ψ_0 and Ψ_1 by the given conditions are

$$
\Psi_0 = \begin{bmatrix} \Phi_0 & D_{21} \\ * & -R_1 \end{bmatrix}, \qquad \Phi_0 = 2(A + F_1 D_{11}) + C_1^2 + F_1^2 R_1 < 0,
$$

and

$$
\Psi_1 = \begin{bmatrix} \Phi_1 & (A+H)D_{21} \\ * & -R_1 \end{bmatrix}, \quad \Phi_1 = 2(A+H) + C_1^2 + q_1^2 R_1 < 0, \quad H = F_1(D_{11} + D_{21}),
$$

respectively.

Putting R_1 for the matrices Φ_0 and Φ_1

$$
R_1 = \frac{2|A + F_1 D_{11}| - C_1^2}{2F_1^2} \qquad \text{and} \qquad R_1 = \frac{2|A + H| - C_1^2}{2q_1^2},
$$

respectively, we obtain $\Phi_0 = -F_1^2 R_1$, $\Phi_1 = -q_1^2 R_1$, and via Lemma 2, the matrices Ψ_0 and Ψ_1 are negative definite by the conditions $F_1 R_1 > |D_{21}|$ and $q_1 R_1 > |(A+H)D_{21}|$, respectively, which coincides with (22) and (23). The proof is completed. □

Remark 4. *Note that the condition* (17) *in the second inequality* (23) *takes the form* $q_1|D_{21}| < 1$ *and holds.*

Remark 5. *It is known that the condition* $\alpha < 1$ *in* (17) *provides exponential stability of the integral equation* $Z(t) = 0$, *i.e.,* $z(t) = -G(t)$ (10) *[4,15]. Sometimes this condition can be relaxed. For instance, for the simple integral equation*

$$z(t) = -\int_0^\infty \int_{t-s}^t Bz(\tau)d\tau dK(s) \tag{24}$$

similarly to [9,16], it can be shown that if there exists a positive definite matrix $S \in \mathbf{R}^{n\times n}$ *such that the LMI*

$$k_1 B'SB - k_1^{-1}S < 0, \qquad k_1 = \int_0^\infty s dK(s), \tag{25}$$

holds, and then the integral Equation (24) *is exponentially stable.*

The condition (17) *for the integral Equation* (24) *has the form* $k_1\|B\| < 1$ *and is simpler, but, generally speaking, rougher than* (25). *Note, however, that in the scalar case* $(n = 1)$, *both these conditions coincide.*

3. Application to Known Mathematical Models

In this section, several applications of the Theorems 2 and 3 for some known mathematical models are considered.

3.1. Glassy-Winged Sharpshooter Population

The nonlinear mathematical model of the glassy-winged sharpshooter under stochastic perturbations is described by the equation

$$\begin{aligned} dN(t) &= \left(I - cN(t) - rN(t)\ln\frac{N(t-\tau)}{K}\right)dt + \sigma(N(t) - N^*)dw(t), \\ N(s) &= N_0(s), \qquad s \in [-\tau, 0], \end{aligned} \tag{26}$$

where I, c, r, τ and K are positive parameters [17,18].

The equation for the equilibrium N^* of Equation (26) can be written in the form

$$r\ln\frac{N}{K} + c = \frac{I}{N}. \tag{27}$$

Note that for $N > 0$, the function from the left-hand part of this equation increases from $-\infty$ to $+\infty$, and the function from the right-hand part of this equation decreases from $+\infty$ to zero. So, it is clear that this equation has a unique positive solution, N^*.

Equation (26) is a particular case of Equation (4) with

$$\begin{gathered} n = m = l = 1, \qquad k = 0, \qquad a = I, \qquad A = -c, \\ f(N(t), N(t-\tau)) = -rN(t)\ln\frac{N(t-\tau)}{K}, \qquad dF_1(s) = \delta(s-\tau)ds, \end{gathered}$$

where $\delta(s)$ is the Dirac function.

Note that

$$\begin{aligned} &D_{11} = -r\ln\frac{N^*}{K}, \qquad D_{21} = -r, \qquad F_1 = 1, \qquad q_1 = \tau, \\ &A + F_1 D_{11} = -\frac{I}{N^*}, \qquad A + F_1(D_{11} + D_{21}) = -\frac{I}{N^*} - r. \end{aligned} \tag{28}$$

Using (28) and (27), the linear Equation (7) takes the form

$$\begin{aligned} dz(t) =& \left(-\frac{I}{N^*}z(t) - rz(t-\tau)\right)dt + \sigma z(t)dw(t), \\ z(s) =& \phi(s), \qquad s \in [-\tau, 0]. \end{aligned} \tag{29}$$

By the well-known [4] conditions

$$\frac{I}{N^*} > r + \frac{1}{2}\sigma^2 \qquad \text{or} \qquad \left(\frac{I}{N^*} + r\right)(1 - r\tau) > \frac{1}{2}\sigma^2 \tag{30}$$

the zero solution of Equation (29) is asymptotically mean square stable, and therefore (Remark 2) the equilibrium N^* of Equation (26) is stable in probability.

It is easy to see that both inequalities (30) also follow from the conditions (22) and (23) of Corollary 1, and the condition (17) takes here the form $r\tau < 1$.

3.2. SIR Epidemic Model

Consider the very popular mathematical model of the spread of infectious diseases used in research, the so-called SIR epidemic model (see, for instance, [4,9,11,14,19–22] and references therein). The SIR epidemic model under stochastic perturbations can be described by the system of stochastic differential equations with distributed delay:

$$\begin{aligned} dS(t) &= \left(b - \beta S(t)\int_0^\infty I(t-s)dK(s) - \mu_1 S(t)\right)dt + \sigma_1(S(t) - S^*)dw_1(t), \\ dI(t) &= \left(\beta S(t)\int_0^\infty I(t-s)dK(s) - (\mu_2 + \lambda)I(t)\right)dt + \sigma_2(I(t) - I^*)dw_2(t), \\ dR(t) &= (\lambda I(t) - \mu_3 R(t))dt + \sigma_3(R(t) - R^*)dw_3(t). \end{aligned} \tag{31}$$

All parameters, b, β, λ, μ_1, μ_2 and μ_3, are positive constants, $K(s)$ is a nondecreasing function, such that $\int_0^\infty dK(s) = 1$, $w_i(t)$ and $i = 1,2,3$, are mutually independent standard Wiener processes. Equilibria of the system (31) are defined by the system of algebraic equations

$$b = \beta SI + \mu_1 S, \quad \beta SI = (\mu_2 + \lambda)I, \quad \lambda I = \mu_3 R, \tag{32}$$

with two solutions: $E_0^* = (b\mu_1^{-1}, 0, 0)$ and the positive equilibrium $E_1^* = (S^*, I^*, R^*)$, where

$$S^* = \frac{\mu_2 + \lambda}{\beta} < \frac{b}{\mu_1}, \quad I^* = \frac{b(S^*)^{-1} - \mu_1}{\beta}, \quad R^* = \frac{\lambda I^*}{\mu_3}.$$

The system (31) is a particular case of Equation (4) with

$$\begin{gathered} n = l = 3, \quad k = 0, \quad m = 1, \\ dF_1(s) = dK(s), \quad F_1 = 1, \quad q_1 = \int_0^\infty s dK(s), \\ x(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}, \quad a = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -\mu_1 & 0 & 0 \\ 0 & -(\mu_2 + \lambda) & 0 \\ 0 & \lambda & -\mu_3 \end{bmatrix}, \\ f_1(x(t), x(t-s)) = \begin{bmatrix} -\beta S(t)I(t-s) \\ \beta S(t)I(t-s) \\ 0 \end{bmatrix}, \end{gathered} \tag{33}$$

$$\begin{gathered} \text{the matrix} \quad C_j \quad \text{has all zeros elements instead of} \\ c_{jj} = \sigma_j, \quad j = 1,2,3. \end{gathered}$$

Note that

$$D_{11} = \begin{bmatrix} -\beta I^* & 0 & 0 \\ \beta I^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -\beta S^* & 0 \\ 0 & \beta S^* & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

So, for the equilibrium $E_0^* = (b\mu_1^{-1}, 0, 0)$, we have

$$D_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -\beta b\mu_1^{-1} & 0 \\ 0 & \beta b\mu_1^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A + D_{11} = A. \tag{35}$$

Similarly, for the equilibrium E_1^* via (33) and (32), we obtain

$$\begin{gathered} D_{11} = \begin{bmatrix} -\beta I^* & 0 & 0 \\ \beta I^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -\beta S^* & 0 \\ 0 & \beta S^* & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A + D_{11} = \begin{bmatrix} -b(S^*)^{-1} & 0 & 0 \\ \beta I^* & -\beta S^* & 0 \\ 0 & \lambda & -\mu_3 \end{bmatrix}. \end{gathered} \tag{36}$$

Corollary 2. *Let there exist positive definite* 3×3 *matrices* P, R_1, *satisfying the LMI*

$$\begin{gathered} \Psi_0 = \begin{bmatrix} \Phi_0 & PD_{21} \\ * & -R_1 \end{bmatrix} < 0, \\ \Phi_0 = P(A + D_{11}) + (A + D_{11})'P + R_1 + \sum_{i=1}^{3} C_i' P C_i. \end{gathered} \tag{37}$$

Then, the equilibrium E_0^* *(in the case of* (35)*) and the equilibrium* E_1^* *(in the case of* (36)*) of the system* (31) *are stable in probability.*

Remark 6. *From Lemma 2, it follows that in the both cases* (35) *and* (36)*, the matrix* $A + D_{11}$ *is the Hurwitz matrix. So, for a small enough* σ_i, $i = 1, 2, 3$, *the matrix* Φ_0 *is a negative definite one.*

Note that stability conditions for both equilibria E_0^* *and* E_1^* *of the SIR epidemic model* (31) *were investigated in [4,11] and significantly improved in [14] by virtue of the method considered here, which included a detailed investigation of LMIs of the type* (14) *and* (18) *by virtue of MATLAB.*

3.3. Heroin Model

Consider the heroin model [23] with stochastic perturbations

$$\begin{aligned} dS(t) = & \left(\Lambda - \beta S(t) \int_0^{h_1} U_1(t-s)dF_1(s) - \mu S(t)\right)dt \\ & + \sigma_1(S(t) - S^*)dw_1(t), \\ dU_1(t) = & \left(\beta S(t) \int_0^{h_1} U_1(t-s)dF_1(s) - \nu U_1(t) + p\int_0^{h_2} U_1(t-s)dK_1(s)\right)dt \\ & + \sigma_2(U_1(t) - U_1^*)dw_2(t), \\ dU_2(t) = & \left(p\, U_1(t) - \gamma U_2(t) - p\int_0^{h_2} U_1(t-s)dK_1(s)\right)dt \\ & + \sigma_3(U_2(t) - U_2^*)dw_3(t), \end{aligned} \tag{38}$$

where

$$\begin{gathered} dF_1(s) = f(s)e^{-\nu s}ds, \quad f(s) > 0, \quad dK_1(s) = g(s)e^{-\gamma s}ds, \quad g(s) > 0, \\ \nu = \mu + \delta_1 + p, \quad \gamma = \mu + \delta_2, \quad a = \int_0^{h_1} dF_1(s) < 1, \quad b = \int_0^{h_2} dK_1(s) < 1. \end{gathered} \tag{39}$$

Equilibria of the system (38) are defined by the system of algebraic equations

$$(\mu + a\beta U_1)S = \Lambda, \quad (a\beta S + pb)U_1 = \nu U_1, \quad p(1-b)U_1 = \gamma U_2, \tag{40}$$

with two solutions: $E_0^* = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ and

$$E_1^* = (S^*, U_1^*, U_2^*), \quad S^* = \frac{\nu - pb}{a\beta}, \quad U_1^* = \frac{\Lambda(S^*)^{-1} - \mu}{a\beta}, \quad U_2^* = \frac{p(1-b)}{\gamma}U_1^*. \tag{41}$$

Note that the equilibrium E_1^* is a positive one by the condition

$$\Re_0 = \frac{a\beta\Lambda}{\mu(\nu - pb)} > 1. \tag{42}$$

The system (38) is a particular case of Equation (4) with

$$\begin{gathered} n = l = 3, \quad k = m = 1, \quad K_1 = b, \quad F_1 = a, \\ q_1 = \int_0^{h_1} s dF_1(s), \quad k_1 = \int_0^{h_2} s dK_1(s), \\ x(t) = \begin{bmatrix} S(t) \\ U_1(t) \\ U_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -\mu & 0 & 0 \\ 0 & -\nu & 0 \\ 0 & p & -\gamma \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & p & 0 \\ 0 & -p & 0 \end{bmatrix}, \\ f(x(t), x(t-s)) = \begin{bmatrix} -\beta S(t)U_1(t-s) \\ \beta S(t)U_1(t-s) \\ 0 \end{bmatrix}, \\ \text{the matrix} \quad C_j \quad \text{has all zeros elements instead of} \\ c_{jj} = \sigma_j, \quad j = 1,2,3. \end{gathered} \tag{43}$$

Note that

$$D_{11} = \begin{bmatrix} -\beta U_1^* & 0 & 0 \\ \beta U_1^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & -\beta S^* & 0 \\ 0 & \beta S^* & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{44}$$

So, for the equilibrium $E_0^* = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ via (43) and (44), we have $D_{11} = 0$, $H = bB_1 + aD_{21}$,

$$D_{21} = \begin{bmatrix} 0 & -\beta\Lambda\mu^{-1} & 0 \\ 0 & \beta\Lambda\mu^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A + H = \begin{bmatrix} -\mu & -a\beta\Lambda\mu^{-1} & 0 \\ 0 & -(\nu - pb - a\beta\Lambda\mu^{-1}) & 0 \\ 0 & p(1-b) & -\gamma \end{bmatrix}. \tag{45}$$

Similarly, for the equilibrium E_1^* via (43), (44) and (40)

$$A + aD_{11} = \begin{bmatrix} -\Lambda(S^*)^{-1} & 0 & 0 \\ a\beta U_1^* & -\nu & 0 \\ 0 & p & -\gamma \end{bmatrix}. \tag{46}$$

Corollary 3. *Let the condition (42) hold, and there exist positive definite 3×3 matrices P, Q_1, R_1, satisfying the LMI*

$$\Psi_0 = \begin{bmatrix} \Phi_0 & PB_1 & PD_{21} \\ * & -Q_1 & 0 \\ * & * & -R_1 \end{bmatrix} < 0,$$

$$\Phi_0 = P(A + aD_{11}) + (A + aD_{11})'P + b^2Q_1 + a^2R_1 + \sum_{i=1}^{3} C_i'PC_i.$$

Then, the equilibrium $E_1^* = (S^*, U_1^*, U_2^*)$ *of the system* (38) *is stable in probability.*

Remark 7. *From Lemma 2, it follows that the matrix* $A + aD_{11}$ (46) *is the Hurwitz matrix. So, for a small enough* σ_i, $i = 1, 2, 3$ *and* $Q_1 = R_1 = 0$, *the matrix* Φ_0 *is a negative definite one.*

Corollary 4. *Let the conditions* $q_1\|D_{21}\| < 1$, $\mathfrak{R}_0 < 1$ *hold, and there exist positive definite* 3×3 *matrices* P, Q_1, R_1, *satisfying the LMI*

$$\Psi_1 = \begin{bmatrix} \Phi_1 & (A+H)'PB_1 & (A+H)'PD_{21} \\ * & -Q_1 & 0 \\ * & * & -R_1 \end{bmatrix} < 0,$$

$$\Phi_1 = P(A+H) + (A+H)'P + b^2 Q_1 + a^2 R_1 + \sum_{i=1}^{3} C_i' P C_i.$$

Then, the equilibrium $E_0^* = \left(\frac{\Lambda}{\mu}, 0, 0\right)$ *of the system* (38) *is stable in probability.*

Remark 8. *Note that via* (42), *the condition* $\mathfrak{R}_0 < 1$ *indicates that the positive equilibrium* E_1^* *does not exist and* $\nu > pb + a\beta\Lambda\mu^{-1}$. *From Lemma 2, it follows that by this condition the matrix* $A + H$ (45) *is the Hurwitz matrix. So, for a small enough* σ_i, $i = 1, 2, 3$ *and* $Q_1 = R_1 = 0$, *the matrix* Φ_1 *is negative definite.*

4. Conclusions

A method of investigation to determine equilibria stability for nonlinear delay differential equations under stochastic perturbations and a high level of nonlinearity was described in [9]. As was noted there, in future research we plan to apply the proposed method to more complex nonlinear models. This paper devoted namely to extension of possible applications of the proposed research method to nonlinear stochastic delay differential equations of a much more general form. In addition, it is shown that the combination of the method of Lyapunov functionals with the method of Linear Matrix Inequalities (LMIs) gives very useful and productive results, allowing for this research method to be used in a lot of different applications. The author continues this work and hopes to involve all other interested researchers in it.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflicts of interest.

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