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Stability of Peakons and Periodic Peakons for the mCH–Novikov–CH Equation

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Abstract: Peakons and periodic peakons are two kinds of special symmetric traveling wave solutions, which have important applications in physics, optical fiber communication, and other fields. In this paper, we study the orbital stability of peakons and periodic peakons for a generalized Camassa–Holm equation with quadratic and cubic nonlinearities (mCH–Novikov–CH equation). It is a generalization of some classical equations, such as the Camassa–Holm (CH) equation, the modified Camassa–Holm (mCH) equation, and the Novikov equation. By constructing an inequality related to the maximum and minimum of solutions with the conservation laws, we prove that the peakons and periodic peakons are orbitally stable under small perturbations in the energy space.

Keywords: mCH–Novikov–CH equation; peakon; periodic peakon; orbital stability

MSC: 35B35; 37K05; 37K45

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1. Introduction

This paper is concerned with the following generalized Camassa–Holm equation with quadratic and cubic nonlinearities (mCH–Novikov–CH equation) [1]

$$\begin{cases} m_t + k_1[(u^2 - u_x^2)m]_x + k_2(u^2 m_x + 3uu_x m) + k_3(um_x + 2u_x m) = 0, \\ u(x,0) = u_0(x), \ t > 0, \end{cases} \tag{1}$$

where $m = u - u_{xx}$, $x \in \mathbb{X}$, $\mathbb{X} = \mathbb{R}$ or $\mathbb{S}$, $k_i = 0$, $(i = 1,2,3)$ are all real-valued parameters. Qin, Yan, and Guo [1] introduced Equation (1), and showed Equation (1) possesses symmetric peakons and periodic peakons. Equation (1) is a generalization of some classical equations, such as the Camassa–Holm (CH) equation, the modified Camassa–Holm (mCH) equation, and the Novikov equation.

When $k_1 = 0$, $k_2 = 0$ and $k_3 = 1$, Equation (1) reduces to the CH Equation [2]

$$m_t + 2u_x m + um_x = 0, \quad m = u - u_{xx}, \tag{2}$$

which was derived as a model for the unidirectional propagation of the shallow water waves over a flat bottom [3,4]. Using the method of recursive operators, Fokas and Fuchssteiner [5] found that Equation (2) has the bi-Hamiltonian structure with an infinite number of conserved quantities. Equation (2) has many important properties: existence of peaked solitons [2,6], complete integrability [2,5], and wave breaking phenomena [7–9]. It is remarkable that Equation (2) possesses the peakon in the form of

$$u(x,t) = c\varphi(x,t) = ce^{-|x-ct|}, \tag{3}$$

which was proved orbitally stable by Constantin and Strauss in [10]. Inspired by [10], Lenells [11] studied the stability of periodic peakons for Equation (2). Then, by means of variational methods, Constantin and Molinet proved the orbital stability of the peakon [12]. The orbital stability of multi-peakon solutions for Equation (2) was discussed in [13]. It is remarkable that Wang, Han, and Lu [14] showed that the b-family of the Novikov equation possessed some symmetric traveling wave solutions. In addition, Ray and Sahoo [15] constructed the analytical exact solutions of Riesz Time-Fractional Camassa–Holm Equation via modified homotopy analysis method (MHAM).

The following Degasperis–Procesi (DP) equation that is similar to Equation (2)

$$m_t + 3u_x m + u m_x = 0 \tag{4}$$

is also completely integrable. Lin and Liu [16] proved the orbital stability of the single peakons for the DP equation.

When $k_1 = 0$, $k_2 = 1$, and $k_3 = 0$, Equation (1) becomes the Novikov equation

$$m_t + u^2 m_x + 3uu_x m = 0, \tag{5}$$

which is an integrable Camassa–Holm type equation with cubic nonlinearity. Liu, Liu, and Qu [17] studied the orbital stability of the peaked solitons for Equation (5). In addition, Wang and Tian [18] extended Lenell's approach to discuss the orbital stability of periodic peakons for Equation (5). Moreover, Moon [19] proved the existence of peaked traveling wave solutions for the generalized μ-Novikov equation with nonlocal cubic and quadratic nonlinearities.

When $k_1 = 1$, $k_2 = 0$, and $k_3 = 0$, Equation (1) is transformed into the integral mCH equation with cubic nonlinearities

$$m_t + \left(\left(u^2 - u_x^2\right)m\right)_x = 0, \tag{6}$$

which was obtained by the tri-Hamiltonian duality approach [20]. In [21], the peakons and periodic peakons of Equation (6) were proved orbitally stable in the energy space. Liu, Liu, and Olver [22] proved that the peakons and periodic peakons for a generalization of the modified Camassa–Holm equation were obitally stable. In [23], Moon studied the dynamical stability of periodic peaked solitary waves for the generalized modified Camassa–Holm equation. Chen, Di, and Liu [24] showed the stability of peaked waves for the mCH–Novikov equation without restrictions on the sign of the momentum density. On the base of [24], Chen, Deng, and Qiao [25] first verify that the existence of global peakon and periodic peakon solutions and the orbital stability of peakons and periodic peakons for a nonlinear quartic Camassa–Holm equation.

It is not difficult to find that the special case of Equation (1) in this paper just corresponds to the above-mentioned special equations. In other words, it can be seen that studying the stability of the peakon of Equation (1) will further deepen the understanding of the stability of peakons of the CH-type equation if k_1, k_2, k_3 are non-zero. By [1], we know that Equation (1) has the following form of single peaked traveling wave solutions

$$\varphi(x,t) = a_1 e^{-|x-ct|}, \tag{7}$$

where

$$a_1 = \begin{cases} \dfrac{-3k_3 \pm \sqrt{9k_3^2 + 12c(2k_1 + 3k_2)}}{2(2k_1 + 3k_2)}, & 2k_1 + 3k_2 \neq 0, \\ c/k_3, & 2k_1 + 3k_2 = 0, \;\; k_1 k_2 k_3 \neq 0. \end{cases} \tag{8}$$

Equation (1) also has the following form of periodic peaked traveling wave solutions in [1]

$$\varphi(x,t) = a_2 \cosh\left(\frac{1}{2} - (x - ct) + \lfloor x - ct \rfloor\right), \tag{9}$$

where $\lfloor \cdot \rfloor$ means the floor function or the greatest integer function and

$$a_2 = \begin{cases} \dfrac{-3k_3 \cosh\left(\frac{1}{2}\right) \pm \sqrt{9k_3^2 \cosh^2\left(\frac{1}{2}\right) + 12cf(k_1,k_2)}}{2f(k_1,k_2)}, & f(k_1,k_2) \neq 0, \\ \dfrac{\operatorname{sech}\left(\frac{1}{2}\right)}{k_3}, & f(k_1,k_2) = 0, \ k_1k_2k_3 \neq 0. \end{cases} \tag{10}$$

Here, we denote $f(k_1,k_2) = (2 + \cosh(1))k_1 + 3\cosh^2\left(\frac{1}{2}\right)k_2$.

When $f \neq 0$, there is the above-mentioned Figure 1. In addition, when $f \neq 0$, $a_{2,1}$ represents the maximum value corresponding to the periodic peakon, and, similarly, $a_{2,2}$ represents the minimum value of the periodic peakon. The definition of Equations (7) and (9) will be described exhaustively in Section 2. Moreover, Equation (1) has multi-peaked traveling wave solutions in [1]. In [26], Hwang and Moon proved the existence of periodic peaked solitary waves to the equation of μ-Camassa–Holm–Novikov. Then, the exact solutions can be attained by eliminating logarithmic nonlinearity [27], symmetry analysis [28], and the modifed Khater method [29]

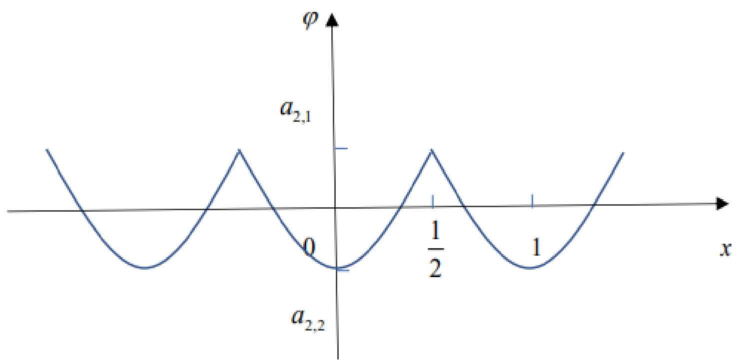

Figure 1. The profile of periodic peakon for $f \neq 0$.

Notice that the peaked function (3) is the traveling wave solutions to Equation (2) and travels with speed c and has a corner (that is, a finite jump in its derivative) at its peak of height c. Furthermore, the peaked function (3) is a soliton: two traveling waves reconstitute their shape and size after interacting with each other [30]. As for the periodic peakon, through [31], we know that periodic plane waves, termed swell, do not change along the wave crest, and move the same in any direction parallel to the crest line. Similarly, the interaction between periodic traveling waves does not change their shapes. Importantly, Equation (2) is the first such equation found that models the solitons' interaction of peaked traveling waves. Thus, we consider quantitative analysis of the stability of peakons and periodic peakons for Equation (1) in this paper.

The peakon can generate another solitary wave with different speed and phase translation under small perturbations. The stability mentioned in this paper refers to the orbital stability, that is, the wave with initial profile always maintains a similar distance from the peakon for all later time, and has the similar shape of wave all the time. Inspired by the work of [10,11], in this paper, we will prove that the symmetric peakons and periodic peakons of Equation (1) are orbitally stable under small perturbations in the energy space.

In order to facilitate the readers to understand the following Theorem 1, we briefly explain the several professional terms. Firstly, it is found that the $H^1(\mathbb{R})$-norm of the solitary wave u is equivalent to the conserved quantity $H_1[u]$. In other words, energy space refers to H^1 space. Secondly, the definition of strong solutions is not described in detail in Section 1; see Section 2 for details.

Theorem 1. *Let $\mathbb{X} = \mathbb{R}$ or $\mathbb{S}$, where $\mathbb{R}$ and $\mathbb{S}$ are referred to the real field and the unit circle. For every $\varepsilon > 0$, there is a $\delta > 0$ such that, if $u \in C([0,T); H^1(\mathbb{X}))$ is a solution to Equation (1) with*

$$\|u(\cdot,0) - \varphi\|_{H^1(\mathbb{X})} < \delta, \tag{11}$$

then

$$\|u(\cdot,t) - \varphi(\cdot - \xi(t))\|^2_{H^1(\mathbb{X})} < \varepsilon, \tag{12}$$

where $t \in [0, T)$ and $\xi \in \mathbb{X}$ is an extreme point where the function $u(\cdot, t)$ attains its maximum. Therefore, the peakons (or periodic peakons) are orbitally stable.

From Figure 2, we can intuitively see that the wave u with the initial profile is always close enough to the peakon φ in later times. The translation does not change the properties of the wave u, as long as u and φ satisfy the condition (11), they will reach orbital stability in the subsequent time.

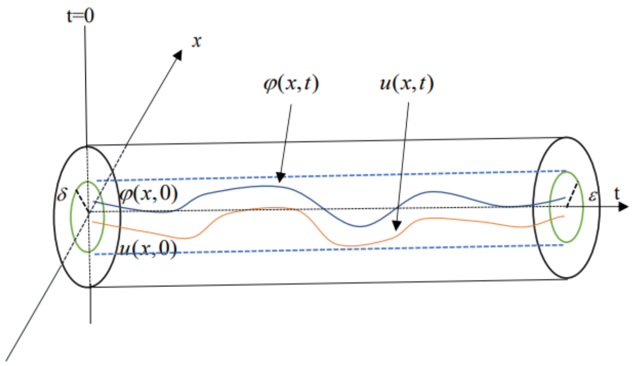

Figure 2. The graphical example of orbital stability.

The outline of this paper is as follows: In Section 2, we briefly recall the well-posedness result, three conservation laws, and the important definition for Equation (1). In Section 3, the orbital stability for peakons and periodic peakons are established in the energy space $H^1(\mathbb{R})$-norm. In Section 4, we give a brief conclusion.

2. Preliminary

In this section, the well-posedness result, two important definitions, and three conservation laws of Equation (1) are shown below.

Definition 1 ([1])**.** *If*

$$u \in C([0, T); H^s(\mathbb{X})) \cap C^1\Big([0, T); H^{s-1}(\mathbb{X})\Big)$$

with $s > \frac{1}{2}$ is a solution to Equation (1), then $u(x, t)$ is called strong solution to Equation (1).

Proposition 1 ([1])**.** *Let $u_0 \in H^s(\mathbb{X})$ with $s > \frac{1}{2}$. Then, there exists a time $T > 0$ such that the initial value problem of Equation (1) has a unique strong solution $u \in C([0, T); H^s(\mathbb{X})) \cap C^1([0, T); H^{s-1}(\mathbb{X}))$, and the map $m_0 \to m$ is continuous from a neighborhood of u_0 in $H^s(\mathbb{X})$ into $u \in C([0, T); H^s(\mathbb{X})) \cap C^1([0, T); H^{s-1}(\mathbb{X}))$.*

Definition 2 ([1])**.** *Given the initial data $u_0 \in W^{1,3}$, the function $u \in L^\infty_{loc}([0, T); W^{1,3}_{loc})$ is said to be a weak solution to the Cauchy problem (1) if it satisfies the following identity:*

$$\begin{aligned} \int_0^T \int_{\mathbb{X}} \Big\{ & u\psi_t + \frac{k_1 + k_2}{3} u^3 \psi_x + \frac{k_1}{3} u_x^3 \psi + \frac{k_3}{2} u^2 \psi_x - \psi p * \left[\left(\frac{k_1}{3} + \frac{k_2}{2} \right) u_x^3 \right] \\ & + \psi_x p * \left[\left(\frac{2k_1}{3} + k_2 \right) u^3 + \left(k_1 + \frac{3k_2}{2} \right) u u_x^2 + k_3 u^2 + \frac{k_3}{2} u_x^2 \right] \Big\} dxdt \\ & + \int_{\mathbb{X}} u(x,0) \psi(x,0) dx = 0, \end{aligned} \tag{13}$$

for any smooth test function $\psi(x, t) \in C_c^\infty(\mathbb{X} \times [0, T))$. If u is a weak solution on $[0, T)$ for every $T > 0$, then it is called a global weak solution. Note that u can be formulated by the Green function p in [1] as

$$u = (1 - \partial_x^2)^{-1} m = p * m, \tag{14}$$

where $*$ denotes the convolution product on $\mathbb{X}$, defined by

$$(f * g)(x) = \int_{\mathbb{X}} f(y) g(x - y) dy. \tag{15}$$

Notice that Equation (1) has the conservation laws as follows:

$$
\begin{aligned}
&H_0[u] = \int_{\mathbb{X}} \left[(k_1+k_3)u + k_2 m^{\frac{2}{3}}\right]dx, \quad H_1[u] = \int_{\mathbb{X}} \left(u^2+u_x^2\right)dx,\\
&H_2[u] = \int_{\mathbb{X}} \left[\left(\frac{k_1}{4}+k_2\right)\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)+k_3\left(u^3+uu_x^2\right)\right]dx.
\end{aligned} \tag{16}
$$

Those conservation laws will be helpful for our proof of the orbital stability. Furthermore, the conservation laws of time fractional coupled equations can be obtained in [32–34]. In addition, we can easily verify that $H_2[u]$ is the conservation law of the Equation (1) by integration by parts in Appendix A.

3. Stability

3.1. Stability of Peakons

In this subsection, we firstly prove the orbital stability of peakons for $\mathbb{X} = \mathbb{R}$ when $2k_1 + 3k_2 \neq 0$. In addition, the profile of peakon for $2k_1 + 3k_2 = 0$ has been shown in Figure 3. For the sake of simplicity, we consider proving the orbital stability of the following form of peakon:

$$
\varphi(x) = \frac{q}{2k_1+3k_2}e^{-|x|}, \tag{17}
$$

where $q \triangleq \frac{-3k_3+\sqrt{9k_3^2+12c(2k_1+3k_2)}}{2}$. Similarly, when $2k_1+3k_2 = 0,\ k_1k_2k_3 \neq 0$, the orbital stability of peakons can be proved in the same way. Without loss of generality, we assume that $k_i\ (i = 1,2,3)$ and c are positive constants.

Replacing u by $\varphi(x) = \frac{q}{2k_1+3k_2}e^{-|x|}$, we find that

$$
\begin{aligned}
H_1[\varphi] &= \int_{\mathbb{R}} \left(\varphi^2+\varphi_x^2\right)dx = \frac{2q^2}{(2k_1+3k_2)^2},\\
H_2[\varphi] &= \int_{\mathbb{R}} \left[\left(\frac{k_1}{4}+k_2\right)\left(\varphi^4+2\varphi^2\varphi_x^2-\frac{1}{3}\varphi_x^4\right)+k_3\left(\varphi^3+\varphi\varphi_x^2\right)\right]dx\\
&= \frac{(k_1+4k_2)q^4}{3(2k_1+3k_2)^4} + \frac{4k_3q^3}{3(2k_1+3k_2)^3}.
\end{aligned} \tag{18}
$$

Then, we consider the expansion of the conservation law H_1 around the peakon φ in the $H^1(\mathbb{R})$-norm.

Lemma 1. *For any $u \in H^1(\mathbb{R})$ and $\xi \in \mathbb{R}$,*

$$
H_1[u] - H_1[\varphi] = \|u-\varphi(\cdot-\xi)\|^2_{H^1(\mathbb{R})} + \frac{4q}{2k_1+3k_2}\left(u(\xi) - \frac{q}{2k_1+3k_2}\right). \tag{19}
$$

Proof. It follows from integration by parts that

$$
\begin{aligned}
&\|u-\varphi(\cdot-\xi)\|^2_{H^1(\mathbb{R})}\\
=&H_1[u]+H_1[\varphi] - 2\int_{\mathbb{R}} u(x)\varphi(x-\xi)dx - 2\int_{\mathbb{R}} u_x(x)\varphi_x(x-\xi)dx\\
=&H_1[u]+H_1[\varphi] - 2\int_{\mathbb{R}} u(x)\varphi(x-\xi)dx - 2\int_{-\infty}^{\xi} u_x(x)\varphi(x-\xi)dx\\
&+2\int_{\xi}^{+\infty} u_x(x)\varphi(x-\xi)dx\\
=&H_1[u]+H_1[\varphi] - \frac{4q}{2k_1+3k_2}u(\xi).
\end{aligned}
$$

Thus, we have

$$
\|u-\varphi(\cdot-\xi)\|^2_{H^1(\mathbb{R})} = H_1[u]-H_1[\varphi] + \frac{4q}{2k_1+3k_2}\left(\frac{q}{2k_1+3k_2} - u(\xi)\right),
$$

which completes the proof of Lemma 1. □

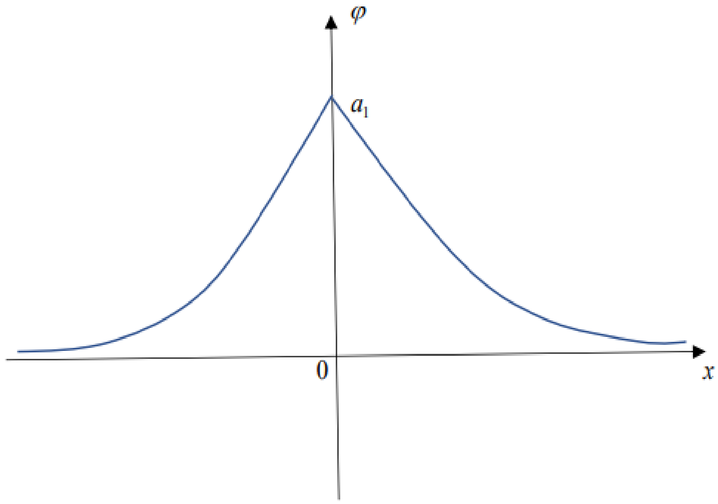

Figure 3. The profile of peakon for $2k_1 + 3k_2 = 0$.

Lemma 1 sets up a global identity about conserved quantities. We will determine the disturbance term of maximum height of u and φ through the following lemmas, so as to quantitatively estimate the global disturbance between the maximum value of the solitary wave u by disturbance near the peakon φ and the peakon φ.

Lemma 2. *For $0 < u(x) \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, let $M = max_{x\in\mathbb{R}}\{u(x)\}$, then*

$$H_2[u] \leq \left(\frac{k_1 + 4k_2}{3}M^2 + k_3 M\right)H_1[u] - \frac{k_1 + 4k_2}{3}M^4 - \frac{2k_3}{3}M^3. \tag{20}$$

Proof. Assume that $u(x)$ attains the maximum at $\xi \in \mathbb{R}$, then $M = u(\xi)$, and define

$$g(x) = \begin{cases} u(x) - u_x(x), & x < \xi, \\ u(x) + u_x(x), & x > \xi. \end{cases} \tag{21}$$

It is easy to show that

$$\begin{aligned} \int_{\mathbb{R}} g^2(x)dx &= \int_{-\infty}^{\xi} [u(x) - u_x(x)]^2 dx + \int_{\xi}^{+\infty} [u(x) + u_x(x)]^2 dx \\ &= H_1[u] - u^2(x)\Big|_{-\infty}^{\xi} + u^2(x)\Big|_{\xi}^{+\infty} \\ &= H_1[u] - 2M^2. \end{aligned} \tag{22}$$

On the other hand, we define $h(x)$ by

$$h(x) = \begin{cases} \left(\frac{k_1}{4} + k_2\right)\left(u^2 - \frac{2}{3}uu_x - \frac{1}{3}u_x^2\right) + k_3 u, & x < \xi, \\ \left(\frac{k_1}{4} + k_2\right)\left(u^2 + \frac{2}{3}uu_x - \frac{1}{3}u_x^2\right) + k_3 u, & x > \xi. \end{cases} \tag{23}$$

Then, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} h(x)g^2(x)dx \\
=&\int_{-\infty}^{\xi}\left[\left(\frac{k_1}{4}+k_2\right)\left(u^2-\frac{2}{3}uu_x-\frac{1}{3}u_x^2\right)+k_3u\right](u-u_x)^2dx \\
&+\int_{\xi}^{\infty}\left[\left(\frac{k_1}{4}+k_2\right)\left(u^2+\frac{2}{3}uu_x-\frac{1}{3}u_x^2\right)+k_3u\right](u+u_x)^2dx \\
=&\int_{\mathbb{R}}\left[\left(\frac{k_1}{4}+k_2\right)\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)+k_3\left(u^2+u_x^2\right)\right]dx \\
&-\frac{2}{3}(k_1+4k_2)\int_{-\infty}^{\xi}u^3u_xdx+\frac{2}{3}(k_1+4k_2)\int_{\xi}^{+\infty}u^3u_xdx \\
&-2k_3\int_{-\infty}^{\xi}u^2u_xdx+2k_3\int_{\xi}^{+\infty}u^2u_xdx \\
=&H_2[u]-\frac{k_1+4k_2}{3}M^4-\frac{4k_3}{3}M^3.
\end{aligned}
\tag{24}
$$

By the Young's inequality,

$$
h(x)=\left(\frac{k_1}{4}+k_2\right)\left(u^2\pm\frac{2}{3}uu_x-\frac{1}{3}u_x^2\right)+k_3u\le\frac{k_1+4k_2}{3}M^2+k_3M. \tag{25}
$$

Moreover, combining the above three Relations (22), (24), and (25), we obtain

$$
H_2[u]\le\left(\frac{k_1+4k_2}{3}M^2+k_3M\right)H_1[u]-\frac{k_1+4k_2}{3}M^4-\frac{2k_3}{3}M^3.
$$

Therefore, we have finished the proof of Lemma 2. □

Lemma 3. *For all $u\in H^s(\mathbb{R})$, $s>\frac{1}{2}$, if $\|u(\cdot,0)-\varphi(\cdot,0)\|_{H^1(\mathbb{R})}<\delta$ with $\delta\in(0,1)$, then*

$$
|H_1[u]-H_1[\varphi]|\le\delta\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right), \tag{26}
$$

and

$$
|H_2[u]-H_2[\varphi]|\le\delta Q(\delta), \tag{27}
$$

where

$$
\begin{aligned}
Q(\delta)=&\frac{\sqrt{2}}{2}\left[\sqrt{2}\left(\frac{k_1}{4}+k_2\right)\delta+\frac{(k_1+4k_2)q}{2k_1+3k_2}+k_3\right]\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2 \\
&+\frac{q}{2k_1+3k_2}\left(\frac{k_1}{2}q+k_3\right)\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right)+\frac{\sqrt{2}}{6}(k_1+4k_2)s'.
\end{aligned}
$$

Proof. We observe that

$$
\sup_{x\in\mathbb{R}}|v(x)|\le\sqrt{\frac{1}{2}H_1[v]}=\frac{\sqrt{2}}{2}\|v\|_{H^1(\mathbb{R})}. \tag{28}
$$

The equality holds if and only if v is proportional to a translate of φ. Note that

$$
\begin{aligned}
|H_1[u]-H_1[\varphi]|&=(\|u\|_{H^1}+\|\varphi\|_{H^1})(\|u\|_{H^1}-\|\varphi\|_{H^1}) \\
&\le(\|u-\varphi\|_{H^1}+2\|\varphi\|_{H^1})\|u-\varphi\|_{H^1} \\
&=\delta\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right).
\end{aligned}
\tag{29}
$$

Similarly,

$$
\begin{aligned}
&|H_2[u]-H_2[\varphi]| \\
=&\left|\int_{\mathbb{R}}\left[\left(\frac{k_1}{4}+k_2\right)\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)+k_3\left(u^3+uu_x^2\right)\right]dx\right. \\
&\left.-\int_{\mathbb{R}}\left[\left(\frac{k_1}{4}+k_2\right)\left(\varphi^4+2\varphi^2\varphi_x^2-\frac{1}{3}\varphi_x^4\right)+k_3\left(\varphi^3+\varphi\varphi_x^2\right)\right]dx\right| \\
\leq&\left|\int_{\mathbb{R}}\left[\left(\frac{k_1}{4}+k_2\right)\left(u^2-\varphi^2\right)\left(u^2+2u_x^2\right)+k_3(u-\varphi)\left(u^2+u_x^2\right)\right]dx\right| \\
&+\left(\frac{k_1}{4}+k_2\right)\left|\int_{\mathbb{R}}\varphi^2\left(u^2+2u_x^2-\varphi^2-2\varphi_x^2\right)dx\right| \\
&+k_3\left|\int_{\mathbb{R}}\varphi\left(u^2+u_x^2-\varphi^2-\varphi_x^2\right)dx\right|+\frac{k_1+4k_2}{12}\left|\int_{\mathbb{R}}\left(u_x^4-\varphi_x^4\right)dx\right| \\
\stackrel{\Delta}{=}&I_3+I_4+I_5.
\end{aligned}
\tag{30}
$$

Now, we first estimate I_3 and I_4.

$$
\begin{aligned}
I_3 \leq& 2\int_{\mathbb{R}}\left|\left(\frac{k_1}{4}+k_2\right)(u+\varphi)(u-\varphi)\left(u^2+u_x^2\right)\right|dx+k_3\int_{\mathbb{R}}|u-\varphi|\left(u^2+u_x^2\right)dx \\
\leq&\left[\left(\frac{k_1}{2}+2k_2\right)(\|u-\varphi\|_{L^\infty}+2\|\varphi\|_{L^\infty})+k_3\right]\|u-\varphi\|_{L^\infty}H_1[u] \\
\leq&\frac{\sqrt{2}}{2}\left[\left(\frac{k_1}{2}+2k_2\right)\left(\frac{\sqrt{2}}{2}\|u-\varphi\|_{H^1}+\frac{2q}{2k_1+3k_2}\right)+k_3\right]\|u-\varphi\|_{H^1}H_1[u] \\
\leq&\frac{\sqrt{2}}{2}\delta\left[\sqrt{2}\left(\frac{k_1}{4}+k_2\right)\delta+\frac{(k_1+4k_2)q}{2k_1+3k_2}+k_3\right]\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2.
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
I_4 \leq&\left(\frac{k_1}{4}+k_2\right)\int_{\mathbb{R}}\varphi^2\left|(u-\varphi)^2+2\varphi(u-\varphi)+2(u_x-\varphi_x)^2+4\varphi_x(u_x-\varphi_x)\right|dx \\
&+k_3\int_{\mathbb{R}}\left|\varphi\left[(u-\varphi)^2+(u_x-\varphi_x)^2+2\varphi(u-\varphi)+2\varphi_x(u_x-\varphi_x)\right]\right|dx \\
\leq&\frac{(k_1+4k_2)q^2}{(4k_1+6k_2)^2}\left(2\|u-\varphi\|_{H^1}^2+4\|\varphi\|_{H^1}\|u-\varphi\|_{H^1}\right) \\
&+\frac{k_3q}{2k_1+3k_2}\left(\|u-\varphi\|_{H^1}^2+2\|\varphi\|_{H^1}\|u-\varphi\|_{H^1}\right) \\
\leq&\frac{\delta q}{2k_1+3k_2}\left(\frac{k_1}{2}q+k_3\right)\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right).
\end{aligned}
\tag{32}
$$

By means of Hölder's inequality, one finds

$$
\begin{aligned}
I_5 =& \frac{k_1+4k_2}{12}\left|\int_{\mathbb{R}}\left(u_x^2+\varphi_x^2\right)(u_x+\varphi_x)(u_x-\varphi_x)dx\right| \\
\leq& \frac{k_1+4k_2}{12}\left|\left(\int_{\mathbb{R}}\left(u_x^2+\varphi_x^2\right)^2(u_x+\varphi_x)^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}(u_x-\varphi_x)^2dx\right)^{\frac{1}{2}}\right| \\
\leq& \frac{\delta(k_1+4k_2)}{12}I_5{}',
\end{aligned}
\tag{33}
$$

where $I_5{}'=\left[\int_{\mathbb{R}}\left(u_x^6+2u_x^5\varphi_x+3u_x^4\varphi_x^2+4u_x^3\varphi_x^3+3u_x^2\varphi_x^4+2u_x\varphi_x^5+\varphi_x^6\right)dx\right]^{\frac{1}{2}}$.

According to Young's inequality, we have

$$
I_5 \leq \frac{\delta(k_1+4k_2)}{12}I_5{}' \leq \frac{\sqrt{2}}{6}\delta(k_1+4k_2)\left[\int_R\left(u_x^6+\varphi_x^6\right)dx\right]^{\frac{1}{2}}. \tag{34}
$$

Due to the following Gagliardo–Nirenberg–Sobolev inequality,

$$
\|u_x\|_{L^6(\mathbb{R})} \leq C'\|u_x\|_{L^2(\mathbb{R})}^{\frac{2}{3}}\|u_x\|_{W^{1,2}(\mathbb{R})}^{\frac{1}{3}}, \tag{35}
$$

where C' is an undermined constant. Using Equation (35), we obtain

$$\int_{\mathbb{R}} u_x^6 dx \leq C\|u_x\|^4_{L^2(\mathbb{R})}\|u_x\|^2_{W^{1,2}_{loc}(\mathbb{R})} \leq C\|u_x\|^6_{H^1(\mathbb{R})}, \tag{36}$$

where C is independent of u_x. Due to $\|\varphi_x\|^6_{L^6(\mathbb{R})} = \frac{q^6}{3(2k_1+3k_2)^6}$, we thus obtain

$$I_5 \leq \frac{\sqrt{2}}{6}(k_1+4k_2)s'\delta, \tag{37}$$

where the constant $s' > 0$ depends only the norm $\|u_x\|_{H^1(\mathbb{R})}$. Then, from the above three estimations, one deduces

$$|H_2[u] - H_2[\varphi]| \leq \delta Q(\delta).$$

Thus, we complete the proof of Lemma 3. □

Lemma 4. *For all $u \in H^s(\mathbb{R})$, $s > \frac{1}{2}$, let $M = max_{x\in\mathbb{R}}u(x)$. If*

$$|H_1[u] - H_1[\varphi]| \leq \delta\left(\delta + \frac{2\sqrt{2}q}{2k_1+3k_2}\right),$$

and

$$|H_2[u] - H_2[\varphi]| \leq \delta Q(\delta),$$

for some $\delta \in (0,1)$, then

$$\begin{aligned}&\left|M - \frac{q}{2k_1+3k_2}\right| \\ \leq&\delta^{\frac{1}{2}}\left\{\frac{3(2k_1+3k_2)^2}{(k_1+4k_2)q^2+4k_3q(2k_1+3k_2)}\left[\left(\frac{k_1+4k_2}{6}\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2\right.\right.\right. \\ &\left.\left.\left.+\frac{\sqrt{2}k_3}{2}\sqrt{\delta^2+\frac{2\sqrt{2}q}{2k_1+3k_2}\delta+\frac{q^2}{(2k_1+3k_2)^2}}\right)\cdot\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right)+Q(\delta)\right]\right\}^{\frac{1}{2}}.\end{aligned} \tag{38}$$

Proof. In view of Equation (20) in Lemma 2,

$$H_2[u] \leq \left(\frac{k_1+4k_2}{3}M^2 + k_3M\right)H_1[u] - \frac{k_1+4k_2}{3}M^4 - \frac{2k_3}{3}M^3.$$

Define the polynomial P by

$$P(y) = H_2[u] - \left(\frac{k_1+4k_2}{3}y^2 + k_3y\right)H_1[u] + \frac{k_1+4k_2}{3}y^4 + \frac{2k_3}{3}y^3. \tag{39}$$

When $H_1[u] = H_1[\varphi] = \frac{2q^2}{(2k_1+3k_2)^2}$ and $H_2[u] = H_2[\varphi] = \frac{(k_1+4k_2)q^4}{3(2k_1+3k_2)^4} + \frac{4k_3q^3}{3(2k_1+3k_2)^3}$, Equation (39) can be rewritten as

$$\begin{aligned}P_0(y) =&H_2[\varphi] - \left(\frac{k_1+4k_2}{3}y^2 + k_3y\right)H_1[\varphi] + \frac{k_1+4k_2}{3}y^4 + \frac{2k_3}{3}y^3 \\ =&\frac{1}{3}\left(y - \frac{q}{2k_1+3k_2}\right)^2\left[(k_1+4k_2)y^2 + \frac{q(2k_1+8k_2)}{2k_1+3k_2}y + 2k_3y\right. \\ &\left.+\frac{(k_1+4k_2)q^2}{(2k_1+3k_2)^2} + \frac{4k_3q}{2k_1+3k_2}\right].\end{aligned} \tag{40}$$

It follows that

$$P_0(M) = P(M) + \left(\frac{k_1+4k_2}{3}M^2 + k_3M\right)(H_1[u] - H_1[\varphi]) - (H_2[u] - H_2[\varphi]). \tag{41}$$

Then,

$$\begin{aligned} &\frac{q}{3(2k_1+3k_2)^2}[(k_1+4k_2)q+4k_3(2k_1+3k_2)]\left(M-\frac{q}{2k_1+3k_2}\right)^2 \\ \le&\left(\frac{k_1+4k_2}{3}M^2+k_3M\right)(H_1[u]-H_1[\varphi])-(H_2[u]-H_2[\varphi]), \end{aligned} \tag{42}$$

which along with the relation

$$0\le M^2\le\frac{H_1[u]}{2}\le\frac{1}{2}\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2, \tag{43}$$

yields

$$\begin{aligned} &\left|M-\frac{q}{2k_1+3k_2}\right| \\ \le&\left\{\frac{3(2k_1+3k_2)^2}{(k_1+4k_2)q^2+4k_3q(2k_1+3k_2)}\left[\left(\frac{k_1+4k_2}{3}M^2+k_3M\right)|H_1[u]-H_1[\varphi]|\right.\right. \\ &\left.\left.+|H_2[u]-H_2[\varphi]|\right]\right\}^{\frac{1}{2}} \\ \le&\delta^{\frac{1}{2}}\left\{\frac{3(2k_1+3k_2)^2}{(k_1+4k_2)q^2+4k_3q(2k_1+3k_2)}\left[\left(\frac{k_1+4k_2}{6}\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2\right.\right.\right. \\ &\left.\left.\left.+\frac{\sqrt{2}k_3}{2}\sqrt{\delta^2+\frac{2\sqrt{2}q}{2k_1+3k_2}\delta+\frac{q^2}{(2k_1+3k_2)^2}}\right)\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right)+Q(\delta)\right]\right\}^{\frac{1}{2}}. \end{aligned}$$

Therefore, we have completed the proof of Lemma 4. □

Now, we start to prove Theorem 1 for the case of $\mathbb{X}=\mathbb{R}$.

Proof. Since $H_1[u]$ and $H_2[u]$ are both conserved by Equation (1), we obtain $H_1[u(\cdot,t)]=H_1[u_0]$, $H_2[u(\cdot,t)]=H_2[u_0]$, $t\in(0,T)$. We apply Lemma 3 to u_0 and δ. In addition, the hypotheses of Lemma 4 are satisfied for $u(\cdot,t)$. Thus,

$$\begin{aligned} &\left|u(\xi(t),t)-\frac{q}{2k_1+3k_2}\right| \\ \le&\delta^{\frac{1}{2}}\left\{\frac{3(2k_1+3k_2)^2}{(k_1+4k_2)q^2+4k_3q(2k_1+3k_2)}\left[\left(\frac{k_1+4k_2}{6}\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2\right.\right.\right. \\ &\left.\left.\left.+\frac{\sqrt{2}k_3}{2}\sqrt{\delta^2+\frac{2\sqrt{2}q}{2k_1+3k_2}\delta+\frac{q^2}{(2k_1+3k_2)^2}}\right)\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right)+Q(\delta)\right]\right\}^{\frac{1}{2}}. \end{aligned}$$

Using Equation (19) of Lemma 1, we obtain

$$\begin{aligned} &\|u(\cdot,t)-\varphi(\cdot-\xi(t))\|^2_{H^1(\mathbb{R})} \\ =&H_1[u]-H_1[\varphi]-\frac{4q}{2k_1+3k_2}\left(u(\xi)-\frac{q}{2k_1+3k_2}\right) \\ \le&|H_1[u]-H_1[\varphi]|+\frac{4q}{2k_1+3k_2}\left|u(\xi)-\frac{q}{2k_1+3k_2}\right| \\ \le&\frac{4q\delta^{\frac{1}{2}}}{2k_1+3k_2}\left\{\frac{3(2k_1+3k_2)^2}{(k_1+4k_2)q^2+4k_3q(2k_1+3k_2)}\left[\left(\frac{k_1+4k_2}{6}\left(\delta+\frac{\sqrt{2}q}{2k_1+3k_2}\right)^2\right.\right.\right. \\ &\left.\left.\left.+\frac{\sqrt{2}k_3}{2}\sqrt{\delta^2+\frac{2\sqrt{2}q}{2k_1+3k_2}\delta+\frac{q^2}{(2k_1+3k_2)^2}}\right)\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right)+Q(\delta)\right]\right\}^{\frac{1}{2}} \\ &+\delta\left(\delta+\frac{2\sqrt{2}q}{2k_1+3k_2}\right). \end{aligned}$$

Hence, for any $\varepsilon>0$, we can take a $\delta(\varepsilon)$ such that

$$\|u(\cdot,t)-\varphi(\cdot-\xi(t))\|^2_{H^1(\mathbb{R})}<\varepsilon.$$

That is, we have finished the proof of Theorem 1 for $\mathbb{X} = \mathbb{R}$. $\square$

3.2. Stability of Periodic Peakons

In this subsection, we will prove the stability of periodic peakon of Equation (1). In particular, we have shown the profile of periodic peakon by Figure 1. When $f(x_1, x_2) \neq 0$, it is obvious that the periodic peaked function for $x \in [0, 1]$,

$$\varphi(x) = a\cosh\left(\frac{1}{2} - x\right), \quad a = \frac{-3k_3\cosh\left(\frac{1}{2}\right) \pm \sqrt{9k_3^2\cosh^2\left(\frac{1}{2}\right) + 12cf(k_1,k_2)}}{2f(k_1,k_2)}, \tag{44}$$

which can be extended to the whole line. Here, we still use $\mathbb{S}$ with the interval $[0, T)$ and treat all functions on $\mathbb{S}$ as periodic functions with the period T on the entire line. For the convenience of calculation, we set $a = \frac{-3k_3\cosh\left(\frac{1}{2}\right) + \sqrt{9k_3^2\cosh^2\left(\frac{1}{2}\right) + 12cf(k_1,k_2)}}{2f(k_1,k_2)} > 0$.

Equation (1) has the following three conservation laws:

$$\begin{aligned} H_0[u] &= \int_{\mathbb{S}} \left[(k_1 + k_3)u + k_2 m^{\frac{2}{3}}\right] dx, \quad H_1[u] = \int_{\mathbb{S}} \left(u^2 + u_x^2\right) dx, \\ H_2[u] &= \int_{\mathbb{S}} \left[\left(\frac{k_1}{4} + k_2\right)\left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) + k_3\left(u^3 + uu_x^2\right)\right] dx, \end{aligned} \tag{45}$$

where the functionals $H_i[u]$ $(i = 0, 1, 2)$ defined in Equation (45) are independent of $t \in [0, T)$.

For an integer $n \geq 1$, let $H^n(\mathbb{S})$ be the Sobolev space of all square integrable functions $f \in L^2(\mathbb{S})$ with distributional derivatives $\partial_x^i f \in L^2(\mathbb{S})$ for $i = 1, \cdots, n$. These Hilbert spaces are endowed with the following inner product:

$$\langle f, g \rangle_{H^n(\mathbb{S})} = \sum_{i=0}^{n} \int_{\mathbb{S}} \left(\partial_x^i f\right)(x)\left(\partial_x^i g\right)(x) dx. \tag{46}$$

A function $u \in C([0, T); H^1(\mathbb{S}))$ is said to be a solution to Equation (1) on $[0, T)$ with the period $T > 0$ if the equation holds in the distribution sense. Clearly, φ is continuous on $\mathbb{S}$ with a peak at $x = 0$. Therefore, we calculate that

$$M_\varphi = \varphi(0) = a\cosh\left(\frac{1}{2}\right), \quad m_\varphi = a. \tag{47}$$

Thus, by integration by parts, we obtain

$$\begin{aligned} H_0[\varphi] =& (k_1 + k_3)\int_0^1 a\cosh\left(\frac{1}{2} - x\right) dx \\ =& 2a(k_1 + k_3)\sinh\left(\frac{1}{2}\right), \end{aligned} \tag{48}$$

$$\begin{aligned} H_1[\varphi] &= \int_{\mathbb{S}} \left(\varphi^2 + \varphi_x^2\right) dx = \int_{\mathbb{S}} \left(\varphi^2 - \varphi\varphi_{xx}\right) dx \\ &= \int_{\mathbb{S}} \left[\varphi^2 - \varphi\left(\varphi - 2a\sinh\left(\frac{1}{2}\right)\delta\right)\right] dx \\ &= 2a^2\sinh\left(\frac{1}{2}\right)\cosh\left(\frac{1}{2}\right), \end{aligned} \tag{49}$$

and

$$
\begin{aligned}
H_2[\varphi] &= \int_{\mathbb{S}} \left[\left(\frac{k_1}{4} + k_2 \right) \left(\varphi^4 + 2\varphi^2\varphi_x^2 - \frac{1}{3}\varphi_x^4 \right) + k_3 \left(\varphi^3 + \varphi\varphi_x^2 \right) \right] dx \\
&= a^4 \left(\frac{k_1}{4} + k_2 \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cosh^4(x) + 2\cosh^2(x)\sinh^2(x) - \frac{1}{3}\sinh^4(x) \right) dx \\
&\quad + a^3 k_3 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\cosh^3(x) + \cosh(x)\sinh^2(x) \right) dx \\
&= a^4 \left(\frac{k_1}{4} + k_2 \right) \left(\frac{2}{3}\sinh(1) + \frac{1}{6}\sinh(2) \right) \\
&\quad + a^3 k_3 \sinh\left(\frac{1}{2}\right) \left(\frac{4}{3}\sinh^2\left(\frac{1}{2}\right) + 2 \right).
\end{aligned}
\tag{50}
$$

Lemma 5. *For all $u \in H^1(\mathbb{S})$ and $\xi \in \mathbb{S}$,*

$$
H_1[u] - H_1[\varphi] = \|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 + 4a\sinh\left(\frac{1}{2}\right)(u(\xi) - M_\varphi). \tag{51}
$$

Proof. We calculate

$$
\begin{aligned}
&\|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 \\
=& H_1[u] + H_1[\varphi] - 2\int_{\mathbb{S}} u(x)\varphi(x-\xi)dx - 2\int_{\mathbb{S}} u_x(x)\varphi_x(x-\xi)dx \\
=& H_1[u] + H_1[\varphi] - 2\int_{\mathbb{S}} u(x)\varphi(x-\xi)dx + 2\int_{\mathbb{S}} u(x)\varphi_{xx}(x-\xi)dx.
\end{aligned}
\tag{52}
$$

Due to $\varphi_{xx}(x) = \varphi(x) - 2a\sinh\left(\frac{1}{2}\right)\delta(x)$, we obtain

$$
\begin{aligned}
\|u - \varphi(\cdot - \xi)\|_{H^1(\mathbb{S})}^2 &= H_1[u] + H_1[\varphi] - 4a\sinh\left(\frac{1}{2}\right)u(\xi) \\
&= H_1[u] - H_1[\varphi] + 4a\sinh\left(\frac{1}{2}\right)(M_\varphi - u(\xi)).
\end{aligned}
\tag{53}
$$

Thus, we complete the proof of Lemma 5. □

Similar to Lemma 1, Lemma 5 also constructs the global identity related to the conserved quantity, and combined with the following several lemmas, the proof of the orbital stability of periodic peakons can be completed.

Lemma 6. *For any positive $u \in H^1(\mathbb{S})$, let*

$$
F_u : \left\{ (M, m) \in \mathbb{R}^2 : M \geq m > 0 \right\} \to \mathbb{R} \tag{54}
$$

be the function defined by

$$
\begin{aligned}
&F_u(M, m) \\
=& \left[H_1[u] + 2m^2 \ln\left(\frac{M + \sqrt{M^2 - m^2}}{m} \right) - 2M\sqrt{M^2 - m^2} - m^2 \right] \\
&\cdot \left(\frac{k_1 + 4k_2}{3} M^2 + k_3 M - \frac{k_1 + 4k_2}{12} m^2 \right) + \frac{k_1 + 4k_2}{3} M \left(M^2 - m^2 \right)^{\frac{3}{2}} \\
&+ \frac{4}{3} k_3 \left(M^2 - m^2 \right)^{\frac{3}{2}} + \left(\frac{k_1}{4} + k_2 \right) m^2 H_1[u] + k_3 m^2 H_0[u] - H_2[u].
\end{aligned}
\tag{55}
$$

Then, we have

$$
F_u(M_u, m_u) \geq 0, \tag{56}
$$

where $M_u = \max_{x \in \mathbb{S}}\{u(x)\}$ and $m_u = \min_{x \in \mathbb{S}}\{u(x)\}$.

Proof. Noted that the periodic peakon φ satisfies the following equation:

$$\varphi_x = \begin{cases} -\sqrt{\varphi^2 - m_\varphi^2}, & 0 < x \le \dfrac{1}{2}, \\ \sqrt{\varphi^2 - m_\varphi^2}, & \dfrac{1}{2} \le x < 1. \end{cases} \tag{57}$$

Let $u \in H^1(\mathbb{S}) \subset C(\mathbb{S})$ be a positive function and write $M = M_u = \max_{x\in\mathbb{S}}\{u(x)\}$ and $m = m_u = \min_{x\in\mathbb{S}}\{u(x)\}$. Let ξ and η satisfy $u(\xi) = M$, $u(\eta) = m$. Similarly, we need to define the real function $g(x)$ as follows:

$$g(x) = \begin{cases} u_x + \sqrt{u^2 - m^2}, & \xi < x \le \eta, \\ u_x - \sqrt{u^2 - m^2}, & \eta \le x < \xi + 1, \end{cases} \tag{58}$$

and extend it periodically to the real line. Then,

$$\begin{aligned} \int_{\mathbb{S}} g^2(x)dx &= \int_\xi^\eta \left(u_x + \sqrt{u^2 - m^2}\right)^2 dx + \int_\eta^{\xi+1} \left(u_x - \sqrt{u^2 - m^2}\right)^2 dx \\ &= 2m^2 \ln\left(\frac{M + \sqrt{M^2 - m^2}}{m}\right) - 2M\sqrt{M^2 - m^2} - m^2 + H_1[u]. \end{aligned} \tag{59}$$

Next, we define

$$\mathrm{h}(x) = \begin{cases} \left(\frac{k_1}{4} + k_2\right)\left(u^2 + \frac{2}{3}u_x\sqrt{u^2 - m^2} - \frac{1}{3}u_x^2 - m^2\right) + k_3 u, & \xi < x \le \eta, \\ \left(\frac{k_1}{4} + k_2\right)\left(u^2 - \frac{2}{3}u_x\sqrt{u^2 - m^2} - \frac{1}{3}u_x^2 - m^2\right) + k_3 u, & \eta \le x < \xi + 1. \end{cases} \tag{60}$$

From Equations (59) and (60), we have

$$\begin{aligned} &\int_{\mathbb{S}} h(x)g^2(x)dx \\ =&\left(\frac{k_1}{4} + k_2\right)\left[\int_\xi^\eta \left(u^2 + \frac{2}{3}u_x\sqrt{u^2 - m^2} - \frac{1}{3}u_x^2 - m^2\right)\left(u_x + \sqrt{u^2 - m^2}\right)^2 dx \right.\\ &\left. + \int_\eta^{\xi+1} \left(u^2 - \frac{2}{3}u_x\sqrt{u^2 - m^2} - \frac{1}{3}u_x^2 - m^2\right)\left(u_x - \sqrt{u^2 - m^2}\right)^2 dx\right] \\ &+ k_3 \int_\xi^\eta u\left(u_x + \sqrt{u^2 - m^2}\right)^2 dx + k_3 \int_\eta^{\xi+1} u\left(u_x - \sqrt{u^2 - m^2}\right)^2 dx. \end{aligned} \tag{61}$$

A direct calculation leads to

$$\begin{aligned} &\left(\frac{k_1}{4} + k_2\right) \int_\xi^\eta \left(u^2 + \frac{2}{3}u_x\sqrt{u^2 - m^2} - \frac{1}{3}u_x^2 - m^2\right)\left(u_x + \sqrt{u^2 - m^2}\right)^2 dx \\ &+ k_3 \int_\xi^\eta u\left(u_x + \sqrt{u^2 - m^2}\right)^2 dx \\ =&\int_\xi^\eta \left[\left(\frac{k_1}{4} + k_2\right)\left(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4\right) + k_3 u\left(u^2 + u_x^2\right)\right] dx \\ &+ \frac{2}{3}(k_1 + 4k_2) \int_\xi^\eta u^2 u_x\sqrt{u^2 - m^2}dx - \frac{k_1 + 4k_2}{6}m^2 \int_\xi^\eta u_x\sqrt{u^2 - m^2}dx \\ &- \left(\frac{k_1}{4} + k_2\right)m^2 \int_\xi^\eta \left(u^2 + u_x^2\right)dx - \left(\frac{k_1}{4} + k_2\right)m^2 \int_\xi^\eta g^2(x)dx \\ &- k_3 m^2 \int_\xi^\eta u dx + 2k_3 \int_\xi^\eta u u_x\sqrt{u^2 - m^2}dx, \end{aligned} \tag{62}$$

and

$$\begin{aligned}
&\left(\frac{k_1}{4}+k_2\right)\int_\eta^{\xi+1}\left(u^2-\frac{2}{3}u_x\sqrt{u^2-m^2}-\frac{1}{3}u_x^2-m^2\right)\left(u_x-\sqrt{u^2-m^2}\right)^2dx\\
&+k_3\int_\eta^{\xi+1}u\left(u_x-\sqrt{u^2-m^2}\right)^2dx\\
=&\int_\eta^{\xi+1}\left[\left(\frac{k_1}{4}+k_2\right)\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)+k_3u\left(u^2+u_x^2\right)\right]dx\\
&-\frac{2}{3}(k_1+4k_2)\int_\eta^{\xi+1}u^2u_x\sqrt{u^2-m^2}dx+\frac{k_1+4k_2}{6}m^2\int_\eta^{\xi+1}u_x\sqrt{u^2-m^2}dx\\
&-\left(\frac{k_1}{4}+k_2\right)m^2\int_\eta^{\xi+1}\left(u^2+u_x^2\right)dx-\left(\frac{k_1}{4}+k_2\right)m^2\int_\eta^{\xi+1}g^2(x)dx\\
&-k_3m^2\int_\eta^{\xi+1}udx-2k_3\int_\eta^{\xi+1}uu_x\sqrt{u^2-m^2}dx.
\end{aligned}\tag{63}$$

Since

$$\frac{1}{8}\frac{d}{dx}\left[m^4\ln\left(\frac{u+\sqrt{u^2-m^2}}{m}\right)+u\sqrt{u^2-m^2}\left(m^2-2u^2\right)\right]=u^2u_x\sqrt{u^2-m^2},\tag{64}$$

$$\frac{d}{dx}\left(-\frac{1}{3}\left(u^2-m^2\right)^{\frac{3}{2}}\right)=uu_x\sqrt{u^2-m^2},\tag{65}$$

and

$$\frac{d}{dx}\left[\frac{1}{2}\left(m^2\ln\left(\frac{u+\sqrt{u^2-m^2}}{m}\right)-u\sqrt{u^2-m^2}\right)\right]=u_x\sqrt{u^2-m^2},\tag{66}$$

we have

$$\begin{aligned}
\int_{\mathbb{S}}h(x)g^2(x)dx=&-\frac{(k_1+4k_2)}{3}M\left(M^2-m^2\right)^{\frac{3}{2}}-\left(\frac{k_1}{4}+k_2\right)m^2H_1[u]+H_2[u]\\
&-\left(\frac{k_1}{4}+k_2\right)m^2\int_{\mathbb{S}}g^2(x)dx-\frac{4}{3}k_3\left(M^2-m^2\right)^{\frac{3}{2}}-k_3m^2\int_{\mathbb{S}}udx.
\end{aligned}\tag{67}$$

It follows from Young's inequality that

$$\int_{\mathbb{S}}h(x)g^2(x)dx\le\left[\frac{k_1+4k_2}{3}\left(M^2-m^2\right)+k_3M\right]\int_{\mathbb{S}}g^2(x)dx.\tag{68}$$

Combining with Equation (67), we see that

$$\begin{aligned}
0<&\left[H_1[u]+2m^2\ln\left(\frac{M+\sqrt{M^2-m^2}}{m}\right)-2M\sqrt{M^2-m^2}-m^2\right]\\
&\cdot\left(\frac{k_1+4k_2}{3}M^2+k_3M-\frac{k_1+4k_2}{12}m^2\right)+\frac{k_1+4k_2}{3}M\left(M^2-m^2\right)^{\frac{3}{2}}\\
&+\frac{4}{3}k_3\left(M^2-m^2\right)^{\frac{3}{2}}+\left(\frac{k_1}{4}+k_2\right)m^2H_1[u]+k_3m^2\int_{\mathbb{S}}udx-H_2[u],
\end{aligned}\tag{69}$$

which completes the proof of Lemma 6. □

Similar to Lemma 2.5 in [11], we obtain the following properties of $F_\varphi(M,m)$.

Lemma 7. *The peaked function φ satisfies the following relations:*

$$\begin{aligned}
&F_\varphi(M_\varphi,m_\varphi)=0,\quad \frac{\partial F_\varphi}{\partial M}(M_\varphi,m_\varphi)=0,\\
&\frac{\partial F_\varphi}{\partial m}(M_\varphi,m_\varphi)=0,\quad \frac{\partial^2 F_\varphi}{\partial M\partial m}(M_\varphi,m_\varphi)=0,\\
&\frac{\partial^2 F_\varphi}{\partial M^2}(M_\varphi,m_\varphi)=-\frac{a}{3}\sinh\left(\frac{1}{2}\right)\left(8a(k_1+4k_2)\cosh\left(\frac{1}{2}\right)+12k_3\right),\\
&\frac{\partial^2 F_\varphi}{\partial m^2}(M_\varphi,m_\varphi)=-\frac{4}{3}(k_1+4k_2)a^2\sinh\left(\frac{1}{2}\right)\cosh\left(\frac{1}{2}\right).
\end{aligned}\tag{70}$$

Lemma 8. *Suppose $f \in H^1(\mathbb{S})$, then*

$$\max_{x\in\mathbb{S}}|f(x)| \le \sqrt{\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)}}\|f\|_{H^1(\mathbb{S})}. \tag{71}$$

Here, "equal" holds if and only if $f = \varphi(\cdot - \xi)$ for some $\xi \in \mathbb{R}$, that is, f is a peakon.

Proof. For $x \in \mathbb{S}$, we have

$$\begin{aligned}\langle\varphi(\cdot - x), f\rangle_{H^1(\mathbb{S})} &= \int_{\mathbb{S}} (\varphi(y-x)f(y) + \varphi'(y-x)f'(y))dy \\ &= \int_{\mathbb{S}} (\varphi - \varphi'')(y-x)f(y)dy \\ &= \int_{\mathbb{S}} 2a\sinh\left(\frac{1}{2}\right)\delta(y-x)f(y)dy \\ &= 2a\sinh\left(\frac{1}{2}\right)f(x).\end{aligned} \tag{72}$$

Since

$$H_1[\varphi] = \|\varphi\|^2_{H^1(\mathbb{S})} = 2a^2\sinh\left(\frac{1}{2}\right)\cosh\left(\frac{1}{2}\right),$$

we obtain

$$\begin{aligned}f(x) &= \frac{1}{2a\sinh\left(\frac{1}{2}\right)}\langle\varphi(\cdot - x), f\rangle_{H^1(\mathbb{S})} \\ &\le \frac{1}{2a\sinh\left(\frac{1}{2}\right)}\|\varphi\|_{H^1(\mathbb{S})}\|f\|_{H^1(\mathbb{S})} = \sqrt{\frac{\cosh\left(\frac{1}{2}\right)}{2\sinh\left(\frac{1}{2}\right)}}\|f\|_{H^1(\mathbb{S})},\end{aligned} \tag{73}$$

where "equal" holds true if and only if f and $\varphi(\cdot - x)$ are proportional. Taking the maximum of Equation (73) over $\mathbb{S}$ completes the proof of Lemma 8. □

Lemma 9. *If $u \in C([0,T); H^1(\mathbb{S}))$, then $M_{u(t)} = \max_{x\in\mathbb{S}} u(x,t)$ and $m_{u(t)} = \min_{x\in\mathbb{S}} u(x,t)$ are continuous functions of $t \in [0,T)$.*

Proof. By Lemma 8, for $t, s \in [0,T)$, we have

$$\begin{aligned}\left|M_{u(t)} - M_{u(s)}\right| &= |\max_{x\in\mathbb{S}}u(x,t) - \max_{x\in\mathbb{S}}u(x,s)| \\ &\le \max_{x\in\mathbb{S}}|u(x,t) - u(x,s)| \\ &\le \sqrt{\frac{\cosh\left(\frac{1}{2}\right)}{2\sinh\left(\frac{1}{2}\right)}}\|u(x,t) - u(x,s)\|_{H^1(\mathbb{S})},\end{aligned} \tag{74}$$

which implies that $M_{u(t)}$ is continuous. The continuity of $m_{u(t)}$ is evident since $m_{u(t)} = -M_{-u(t)}$, which finish the proof of Lemma 9. □

Suppose $H_i[u] = H_i[\varphi] + \varepsilon_i$, $i = 0, 1, 2$. Then, we obtain

$$\begin{aligned}F_u(M,m) =& F_\varphi(M,m) + \left(\frac{k_1 + 4k_2}{3}M^2 + k_3M + \frac{k_1 + 4k_2}{6}m^2\right)\varepsilon_1 \\ &+ k_3m^2\varepsilon_0 - \varepsilon_2.\end{aligned} \tag{75}$$

Following the work from Lemma 2.9 in [11], we obtain the following lemma:

Lemma 10. *Let $u \in C([0,T); H^1(\mathbb{S}))$ be a solution of Equation (1). Given a small neighborhood $\mathcal{U}$ of (M_φ, m_φ) in $\mathbb{R}^2$, there is a $\delta > 0$ such that*

$$\left(M_{u(t)}, m_{u(t)}\right) \in \mathcal{U} \ \ for \ \ t \in [0,T) \ \ if \ \ \|u(\cdot,0) - \varphi\|_{H^1(\mathbb{S})} < \delta. \tag{76}$$

Finally, we prove Theorem 1 for the case of $\mathbb{X} = \mathbb{S}$.

Proof. Let $u \in C\big([0,T); H^1(\mathbb{S})\big)$ be a solution of Equation (1) and assume that we are given an $\varepsilon > 0$. Then, we choose a neighborhood $\mathcal{U}$ of (M_φ, m_φ) small enough that $\left|M - M_\varphi\right| < \frac{1}{8a\sinh\left(\frac{1}{2}\right)}\varepsilon$ if $(M, m) \in \mathcal{U}$. We can find a $\delta > 0$ in Lemma 10 such that Equation (76) holds. Then, choosing a smaller δ if necessary, we may suppose

$$|H_1[u] - H_1[\varphi]| < \frac{\varepsilon}{2} \;\; if \;\; \|u(\cdot,0) - \varphi\|_{H^1(\mathbb{S})} < \delta.$$

Therefore, we use Lemma 7 to deduce that

$$\|u - \varphi(\cdot - \xi)\|^2_{H^1(\mathbb{S})} = H_1[u] - H_1[\varphi] + 4a\sinh\left(\frac{1}{2}\right)\left(M_\varphi - M_{u(t)}\right) \le \varepsilon,$$

where $\xi(t) \in \mathbb{R}$ is any point satisfying $u(\xi(t), t) = M_{u(t)}$.

Therefore, we have proved Theorem 1 for the case of $\mathbb{X} = \mathbb{S}$. $\square$

4. Conclusions

In this paper, we obtain the orbital stability of symmetric peakons and periodic peakons for the mCH–Novikov–CH equation. The mCH–Novikov–CH equation is a generalization of some classical equations, such as the Camassa–Holm (CH) equation, the modified Camassa–Holm (mCH) equation, and the Novikov equation. The proof is inspired by [10,11]. In particular, we construct a polynomial inequality related to the maximum and the minimum with the conservation laws, which plays an important role in our proof of the orbital stability of peakons and periodic peakons for Equation (1). From the perspective of energy space $H^1(\mathbb{R})$, Theorem 1 shows that the shape of the (periodic) wave generated near the (periodic) peakon remains unchanged under the slight perturbation.

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Appendix A

In Appendix A, the proof of conserved quantities are as follows. In order to prove the qualities $H_1[u]$, $H_2[u]$ is independent of t, we set $v(x,t) = \int_{-\infty}^{x} u_t(z,t)dz$. Thus, we have

$$\begin{aligned}
\frac{dH_1[u]}{dt} &= \int_{\mathbb{X}} (2uv_x + 2u_x v_{xx})dx \\
&= \int_{\mathbb{X}} (2uv_x - 2uv_x)dx = 0. \\
\frac{dH_2[u]}{dt} &= \int_{\mathbb{X}} \left(\frac{k_1}{4} + k_2\right)\left(u^3 v_x + uu_x^2 v_x + u^2 u_x v_{xx} - \frac{1}{3}u_x^3 v_{xx}\right)dx \\
&\quad + \int_{\mathbb{X}} k_3\left(3u^2 v_x + u_x^2 v_x + 2uu_x v_{xx}\right)dx \\
&\triangleq I_1 + I_2.
\end{aligned}$$

By using integration by parts, one finds

$$
\begin{aligned}
I_1 &= \int_{\mathbb{X}} \left(\frac{k_1}{4}+k_2\right)\left(-12u^2u_x-4u_x^3-8uu_xv_{xx}\right)vdx \\
&\quad -\int_{\mathbb{X}} \left(\frac{k_1}{4}+k_2\right)\left(8uu_x^2+4u^2u_{xx}-4u_x^2u_{xx}\right)v_xdx \\
&= \int_{\mathbb{X}} \left(\frac{k_1}{4}+k_2\right)\left(-12u^2u_x+4u_x^3+16uu_xu_{xx}+4u^2u_{xxx}\right. \\
&\quad \left. -8u_xu_{xx}^2-4u_x^2u_{xxx}\right)vdx \\
&= \int_{\mathbb{X}} \left(\frac{k_1}{4}+k_2\right)\left(-4u^2m_x-8uu_xm+4u_x^2m_x-8u_xu_{xx}m\right)vdx \\
&= \int_{\mathbb{X}} \left(\frac{k_1}{4}+k_2\right)\left[m\left(u_x^2-u^2\right)\right]_x vdx, \\
I_2 &= \int_{\mathbb{X}} k_3\left[(-6uu_x-2u_xu_{xx})v-\left(2u_x^2+2uu_{xx}\right)v_x\right]dx \\
&= \int_{\mathbb{X}} k_3(-6uu_x+4u_xu_{xx}+2uu_{xxx})vdx \\
&= \int_{\mathbb{X}} 2k_3(-2u_xm-um_x)vdx.
\end{aligned}
$$

From the above calculations, we can see that

$$\frac{dH_2[u]}{dt} = I_1 + I_2 = 0,$$

that is, $H_2[u]$ is independent of t and is the conservation law of Equation (1).

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