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Lie Symmetry Analysis of the One-Dimensional Saint-Venant-Exner Model

Andronikos Paliathanasis [1,2]

[1] Institute of Systems Science, Durban University of Technology, P.O. Box 1334, Durban 4000, South Africa; anpaliat@phys.uoa.gr

[2] Instituto de Ciencias Físicas y Matemáticas, Universidad Austral de Chile, Valdivia 5090000, Chile

Abstract: We present the Lie symmetry analysis for a hyperbolic partial differential system known as the one-dimensional Saint-Venant-Exner model. The model describes shallow-water systems with bed evolution given by the Exner terms. The sediment flux is considered to be a power-law function of the velocity of the fluid. The admitted Lie symmetries are classified according to the power index of the sediment flux. Furthermore, the one-dimensional optimal system is determined in all cases. From the Lie symmetries we derive similarity transformations which are applied to reduce the hyperbolic system into a set of ordinary differential equations. Closed-form exact solutions, which have not been presented before in the literature, are presented. Finally, the initial value problem for the similarity solutions is discussed.

Keywords: lie symmetries; Saint-Venant-Exner model; shallow-water equations

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1. Introduction

The Saint-Venant system is a set of hyperbolic nonlinear partial differential equations known as the shallow-water equations. The Saint-Venant system describes the flow of a fluid under a pressure surface and covers many applications in real-world systems; see for instance [1–6] and references therein. There are various hyperbolic systems in the literature that describe physical phenomena and can be seen as extensions of the Saint-Venant system [7–11]. Because of the nonlinearity of the partial differential equations, various techniques of mathematical analysis have been applied to the construction of conserved quantities and for the determination of analytic and exact solutions [12,13].

A systematic mathematical approach which has been widely studied for the investigation of hyperbolic systems in fluid dynamics is Lie symmetry analysis. At the end of the 19th century, Sophus Lie published, in a series of books [14–16], the fundamental elements of the modern treatment of symmetries. The idea of Lie's approach is the invariance properties of differential equations which follow from the infinitesimal representations of the finite transformations of continuous groups. The generator of the invariant transformation which keeps a differential equation invariant is known as a Lie symmetry. The determination of the Lie symmetries for a system of differential equations is essential because they can be used to simplify the system by reducing the number of independent variables or by reducing the order of the differential equations. In addition, symmetries can be used for the construction of conservation laws. For more details on the modern treatment of Lie's approach to differential equations, we refer the reader to the following basic references of symmetry analysis [17–20].

The Lie point symmetries for rotating shallow-water without a gravitational field were studied in [21] while a nonzero constant gravitational field was introduced in [22]. It was found that the existence of the gravitational field affects the admitted Lie point symmetries. The Lie symmetries of shallow-water equations with a varying bottom were the subject of study in [23,24]. Moreover, the algebraic properties for a one-dimensional Saint-Venant system without a gravitational field and a constant bottom is presented in [25], while the analysis for a polytropic gas in Lorentz coordinates is given in [26]. For more studies, we refer the reader to [27–30] and references therein.

In this piece of work, we are interested in the investigation of the algebraic properties for the Saint-Venant-Exner system (SVE) [31]. The SVE system is an extension of the shallow-water system, where a new dynamical variable is introduced that describes the bed evolution of the bottom topography. The SVE system has been mainly considered for the description of model bedload sediment transport phenomena that occur in large time- and space-scale fluid dynamics systems [32–36]. In the following, we apply Lie's algorithm to determine the Lie point symmetries for the SVE system and then to determine the one-dimensional optimal system. The latter is essential in order to determine all the possible independent reductions, to perform a complete classification of the Lie symmetries and of the invariant transformations, and, consequently, to determine new analytic and exact solutions. The structure of the paper is as follows.

In Section 2, we present the basic properties and definitions for the Lie symmetry analysis as we determine the Lie symmetries for the SVE system. In Section 3, we calculate the invariants for the adjoint representation of the admitted Lie point symmetries and we derive the one-dimensional optimal system. The application of the Lie point symmetries for reductions is given in Section 4. Finally, in Section 5 we summarize the results and we discuss the families of initial and boundary conditions in which the similarity transformations are solutions to the initial value problem.

2. Lie Symmetries for the Saint-Venant-Exner Model

The one-dimensional SVE model is defined by the following three nonlinear partial differential equations [37]

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(uh)=0\,, \tag{1}$$

$$\frac{\partial}{\partial t}(uh)+\frac{\partial}{\partial x}\left(hu^2+\frac{1}{2}gh^2\right)+gh\frac{\partial B}{\partial x}=0\,, \tag{2}$$

$$\frac{\partial B}{\partial t}+\xi\frac{\partial Q}{\partial x}=0. \tag{3}$$

Parameters $h(t,x)$, $u(t,x)$ represent the height above the bottom of the surface and the velocity in the $x-$direction, respectively; $B(t,x)$ is the height of the bed, i.e., the bed level, parameter ξ is defined as $\xi=\frac{1}{1-\varepsilon}$, in which ε is the porosity of the bed, and g is the gravitational constant. Finally, $Q(t,x)=Q(u(t,x),h(t,x))$ defines the volumetric bedload sediment flux. Recall that when there is not any bed evolution, SVE systems (1)–(3) reduces to the one-dimensional shallow-water system. In the following, for the bedload sediment flux $Q(t,x)$, we consider

$$Q(t,x)=A_g u^m, \tag{4}$$

where A_g, m are constants and $m\geq 1$ [38]. From experimental data, parameter m is constrained as $1\leq m\leq 4$ [37], while the case of $m=3$ is considered in [37]. Formula (4) for the description of the sediment flux is not the unique solution that has been proposed in the literature, see for instance [39,40] and references therein. However, the power-law expression (4) is the simplest that has been considered. Moreover, for special values of the free parameters of the power-law expression (4) an analytic solution determined for the first time in [41].

Consider now the infinitesimal point transformation

$$t' = t+\epsilon\xi^t(t,x,u,h,B)\,, \tag{5}$$

$$x' = x+\epsilon\xi^x(t,x,u,h,B)\,, \tag{6}$$

$$u' = u+\epsilon\eta^u(t,x,u,h,B)\,, \tag{7}$$

$$h' = h+\epsilon\eta^h(t,x,u,h,B)\,, \tag{8}$$

$$B' = B+\epsilon\eta^B(t,x,u,h,B)\,, \tag{9}$$

which form the infinitesimal generator defined as a vector field

$$X = \frac{\partial t'}{\partial \epsilon}\partial_t + \frac{\partial x'}{\partial \epsilon}\partial_x + \frac{\partial u'}{\partial \epsilon}\partial_u + \frac{\partial h'}{\partial \epsilon}\partial_h + \frac{\partial B'}{\partial \epsilon}\partial_B \,, \tag{10}$$

where $O\left(\epsilon^2\right) \simeq 0$.

By definition, the SVE system of nonlinear hyperbolic partial differential equations $\mathbf{H}\left(y^A, u, u_{,A}, h, h_{,A}, B, B_{,A}\right)$, $y^A = (t, x)$ is invariant under the action of the infinitesimal transformation if and only if [18],

$$\lim_{\varepsilon \to 0} \frac{\mathbf{H}\left(y'^A, u', u'_{,A}, h', h'_{,A}, B', B'_{,A}; \epsilon\right) - \mathbf{H}\left(y^A, u, u_{,A}, h, h_{,A}, B, B_{,A}\right)}{\epsilon} = 0 \,, \tag{11}$$

or equivalently for the functions ξ^t, ξ^x, η^u, η^h and η^B which solve the following system

$$\xi^t = \xi^t(t, x) \,,\; \xi^x = \xi^x(t, x),$$

$$0 = \zeta u^{m-1}\frac{\partial \eta^u}{\partial x} + \frac{\partial \eta^B}{\partial t} \,, \tag{12}$$

$$0 = u^{1-m} h \frac{\partial \eta^B}{\partial t} - \zeta\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\eta^h \,, \tag{13}$$

$$0 = u^{2-m} h \frac{\partial \eta^B}{\partial h} - u^{1-m} h \frac{\partial \xi^x}{\partial t} + \zeta\left(\frac{\partial}{\partial B} - \frac{\partial}{\partial h}\right)\eta^u \,, \tag{14}$$

$$0 = \zeta\left(\frac{\partial \eta^u}{\partial B} + g\frac{\partial \xi^t}{\partial x}\right) - u^{1-m}\left(g\frac{\partial \eta^B}{\partial u} - \frac{\partial \xi^x}{\partial t}\right), \tag{15}$$

$$0 = u^{1-m} h\left(\frac{\partial \eta^B}{\partial u} + \frac{\partial \xi^x}{\partial t}\right) - \zeta\left(g\frac{\partial}{\partial u} - u\frac{\partial}{\partial B}\right)\eta^h \,, \tag{16}$$

$$0 = \zeta\left(u\frac{\partial \eta^u}{\partial t} - g\frac{\partial \eta^h}{\partial t}\right) + \left(u^{1-m}\left(gh - u^2\right)\frac{\partial}{\partial t} + gu\frac{\partial}{\partial x}\right)\eta^B \,, \tag{17}$$

$$0 = \zeta\left(\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\xi^x - u\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\xi^t\right) + \zeta u\frac{\partial \eta^h}{\partial B} - \zeta\eta^u + u^{1-m} h\left(\frac{\partial \xi^x}{\partial t} - u\frac{\partial \eta^B}{\partial h}\right), \tag{18}$$

$$\begin{aligned} 0 \;=\; & \zeta\left(\left(gh - u^2\right)\frac{\partial}{\partial B} - g\left(h\frac{\partial}{\partial h} - u\frac{\partial}{\partial u} + (m-1)\right)\right)\eta^u - \\ & \zeta gu\frac{\partial \eta^B}{\partial B} - \left(u^{1-m}\left(gh - u^2\right)\frac{\partial}{\partial t} + \zeta gu\frac{\partial}{\partial x}\right)\xi^x + \zeta gu\frac{\partial \xi^t}{\partial t} \,, \end{aligned} \tag{19}$$

$$\begin{aligned} 0 \;=\; & g\left(\left(u^{3-m} - ghu^{1-m}\right)\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial B}\right)\eta^B + \zeta g^2\frac{\partial \eta^h}{\partial u} + \zeta gu\frac{\partial \xi^t}{\partial t} \\ & + \left(\left(u^{3-m} + ghu^{1-m}\right)\frac{\partial}{\partial t} + \zeta gu\frac{\partial}{\partial x}\right)\xi^x \,, \end{aligned} \tag{20}$$

$$\begin{aligned} 0 \;=\; & \left(\left(\zeta u + u^{4-m} - gu^{2-m}h\right)\frac{\partial}{\partial h} - \zeta gu\frac{\partial}{\partial B}\right)\eta^B + \zeta gu\frac{\partial \eta^h}{\partial h} - \zeta\left(g + u^2\frac{\partial}{\partial h}\right)\eta^u \\ & + \left(\left(\zeta g + ghu^{1-m} - u^{3-m}\right)\frac{\partial}{\partial t} + \zeta gu\frac{\partial}{\partial x}\right)\xi^x - \zeta g\frac{\partial \xi^t}{\partial t} \,, \end{aligned} \tag{21}$$

$$
\begin{aligned}
0 \;=\;& \zeta\left(\left(\zeta g - ghu^{1-m} + u^{3-m}\right)\frac{\partial}{\partial B} + gu^{1-m}\left(h\frac{\partial}{\partial h} - 1\right) - gu^{2-m}\frac{\partial}{\partial u}\right)\eta^h + \zeta mghu^{-m}\eta^u \\
& -\zeta ghu^{1-m}\frac{\partial \eta^B}{\partial B} - h\left(\left(\zeta gu^{-m} + hu^{1-2m} - u^{3-2m}\right)\frac{\partial}{\partial t} + \zeta g\frac{\partial}{\partial x}\right)\xi^x - \zeta ghu^{1-m}\frac{\partial \xi^t}{\partial t}\,,
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
0 \;=\;& \zeta g^2\left(\zeta\tfrac{\partial}{\partial B} - u^{1-m}\right)\eta^h + \zeta g\left(\left((2-m)u^{2-m} + mghu^{-m}\right) - \zeta\tfrac{\partial}{\partial B}\right)\eta^u + \zeta g^2u^{2-m}\tfrac{\partial \eta^u}{\partial u} \\
& -\left(u^{-m}\left(\zeta g(gh+u^2) + u^{5-m} + ghu^{1-m}(g-2u^2)\right)\tfrac{\partial}{\partial t} + \zeta gu^{1-m}(gh-u^2)\tfrac{\partial}{\partial x}\right)\xi^x \\
& \left(\zeta gu^{1-m}(gh-u^2)\tfrac{\partial}{\partial t} + g^2\zeta^2\tfrac{\partial}{\partial x}\right)\xi^t\,,
\end{aligned} \tag{23}
$$

where the new parameter ζ is defined as $\zeta = m\xi A_g$. The latter system of partial differential equations is also known as the determining system, or as the Lie symmetry conditions.

We observe that the resulting solution of the Lie symmetry conditions depends on the value of the free parameter m. In the following proposition, the Lie symmetry classification is presented, where the classification of Patera et al. [42] is used.

Proposition 1. *The Lie point symmetries for the SVE systems (1)–(3) with sediment flux $Q(t,x) = A_g u^m$ are the vector fields*

$$X_1 = \partial_t\,,\ X_2 = \partial_x\,,\ X_3 = t\partial_t + x\partial_x\,,\ X_4 = \partial_B, \tag{24}$$

for arbitrary value of parameter m. The nonzero commutators for vector fields $\{X_1, X_2, X_3, X_4\}$ are $[X_1, X_3] = X_1$, $[X_1, X_3] = X_2$. Therefore, the vector fields $\{X_1, X_2, X_3\}$ form the $A_{3,3}$ Lie algebra. Consequently, the four Lie symmetry vectors $\{X_1, X_2, X_3, X_4\}$ form $A_{3,3} \otimes A_1$ Lie algebra.

In the special case where $m = 3$, the SVE system admits the additional Lie point symmetry

$$X_5 = t\partial_t - 2u\partial_u - h\partial_h - 2B\partial_B. \tag{25}$$

The additional nonzero commutators are $[X_1, X_5] = X_1$, $[X_4, X_5] = -2X_4$, that is, the Lie point symmetries form the $A_{3,3} \otimes_s A_{2,1}$ Lie algebra.

3. One-Dimensional Optimal System

Lie symmetries are mainly applied for the construction of similarity transformations and the definition of invariant functions. Therefore, the SVE systems (1)–(3) can be written in a simpler form with the use of the Lie invariants. In order to calculate all the unique similarity transformations, we calculate the adjoint representation of the admitted Lie algebra. The later is essential in order to write down the one-dimensional optimal system.

Consider the two generic Lie symmetry vector fields $\mathbf{Z}$, $\mathbf{W}$ defined as

$$\mathbf{Z} = \sum_{A=1}^{n} a_A X_A\,,\ \mathbf{W} = \sum_{i=1}^{n} b_A X_A\,,\ \ a_A,\ b_A \text{ are constants.} \tag{26}$$

We shall say that $\mathbf{Z}$, $\mathbf{W}$ are equivalent if an only if $\mathbf{W} = \prod Ad(\exp(\varepsilon_A X_A))\mathbf{Z}$, or $a_A = cb_A$, $c = const$. $Ad(\exp(\varepsilon_A X_A))$ is the adjoint operator defined as [20]

$$Ad(\exp(\varepsilon X_A))X_B = X_A - \varepsilon[X_A, X_B] + \frac{1}{2}\varepsilon^2[X_A,[X_A,X_B]] + \dots\,, \tag{27}$$

where $[X_A, X_B]$ is the commutator.

As a first step the invariants $\phi(a_A)$ of the adjoint action should be calculated. They invariants given by the following system

$$\Delta_A(\phi) = C^C_{AB} a^B \frac{\partial}{\partial a^C} \phi \equiv 0 \,, \tag{28}$$

where C^C_{AB} are the structure constants of the Lie algebra $[X_A, X_B] = C^C_{AB} X_C$. The commutators for the admitted Lie symmetries are given in Table 1.

Table 1. Commutators of the admitted Lie point symmetries for the SVE system.

$[\,,\,]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$
$\mathbf{X}_1$	0	0	X_1	0	X_1
$\mathbf{X}_2$	0	0	X_2	0	0
$\mathbf{X}_3$	$-X_1$	$-X_2$	0	0	0
$\mathbf{X}_4$	0	0	0	0	$-2X_4$
$\mathbf{X}_5$	0	0	0	$2X_4$	0

Hence, from Table 1 and (28) we define the system

$$a_3 \frac{\partial \phi}{\partial a_1} = 0 \,, \; a_3 \frac{\partial \phi}{\partial a_2} = 0 \,, \; a_5 \frac{\partial \phi}{\partial a_1} = 0 \,, \; -2a_5 \frac{\partial \phi}{\partial a_4} = 0. \tag{29}$$

Therefore, for $m \neq 3$, we find $\phi = \phi(a_3, a_4)$, that is, that the adjoint invariants are a_3 and a_4. On the other hand, for $m = 3$, it follows $\phi = \phi(a_3, a_5)$ from where we infer that the adjoint invariants are a_3 and a_5.

With the use of the adjoint representation given in Table 2 and the invariants we determine the one-dimensional optimal system. The results are presented in the following proposition.

Table 2. Adjoint representation for the admitted Lie point symmetries of the SVE system.

$Ad(\exp(\varepsilon X_A))X_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$
$\mathbf{X}_1$	X_1	X_2	$X_3 - \varepsilon X_1$	X_4	$X_5 - \varepsilon X_1$
$\mathbf{X}_2$	X_1	X_2	$X_3 - \varepsilon X_2$	X_4	X_5
$\mathbf{X}_3$	$e^\varepsilon X_1$	$e^\varepsilon X_2$	X_3	X_4	X_5
$\mathbf{X}_4$	X_1	X_2	X_3	X_4	$X_5 + 2\varepsilon X_4$
$\mathbf{X}_5$	$e^\varepsilon X_1$	X_2	X_3	$e^{-2\varepsilon} X_4$	X_5

Proposition 2. *The one-dimensional optimal system for the SVE systems (1)–(3) with sediment flux $Q(t,x) = A_g u^m$ consists of the one-dimensional Lie algebras*

$$\{X_1\} \,,\; \{X_2\} \,,\; \{X_3\} \,,\; \{X_4\} \,,\; \{X_1 + \alpha X_2\} \,,\; \{X_1 + \alpha X_4\} \,,$$

$$\{X_2 + \alpha X_4\},\; \{X_3 + \alpha X_4\} \,,\; \{X_1 + \alpha X_2 + \beta X_4\}.$$

Furthermore, when $m = 3$, the extra one-dimensional Lie algebras exist

$$\{X_5\},\; \{X_2 + \alpha X_5\},\; \{X_3 + \alpha X_5\}.$$

In the following Section we continue our analysis with the application of the Lie symmetries to define similarity transformations in order to reduce the SVE system and to determine similarity solutions.

4. Similarity Transformations

Consider a function $F(\mathbf{y}^A, \mathbf{u}) = 0$ which is invariant under the action of a one-parameter point transformation with generator X. By definition, it follows $X(F) = 0$, that is,

$$\xi^{A}\frac{\partial F}{\partial \mathbf{y}^{A}}+\eta^{\mathbf{u}}\frac{\partial F}{\partial \mathbf{u}}=0. \tag{30}$$

Thus, the solution of the later system determines all the functions which are invariant under the infinitesimal generator X.

From condition (30) we can define the associated Lagrange system

$$\frac{dy^{A}}{\xi^{A}(\mathbf{y}^{A},\mathbf{u})}=\frac{d\mathbf{u}}{\eta^{\mathbf{u}}(\mathbf{y}^{A},\mathbf{u})},$$

from where we define the invariant functions $d\mathbf{W}^{A}=\frac{dy^{A}}{\xi^{A}(\mathbf{y}^{A},\mathbf{u})}-\frac{d\mathbf{u}}{\eta^{\mathbf{u}}(\mathbf{y}^{A},\mathbf{u})}$. For the invariant functions $\mathbf{W}^{A}$ holds $X(\mathbf{W}^{A})=0$. The invariant functions are used to define the similarity transformations which are used to simplify the system of partial differential equations.

4.1. Reduction with X_1

From the symmetry vector X_1 we determine the invariant functions x,u,h,B. Thus, by considering x to be the new independent variable, then $u=u(x),\ h=h(x)$ and $B=B(x)$. We replace in the SVE system and we find

$$\frac{\partial}{\partial x}(uh)=0\,, \tag{31}$$

$$\frac{\partial}{\partial x}\left(hu^{2}+\frac{1}{2}gh^{2}\right)+gh\frac{\partial B}{\partial x}=0\,, \tag{32}$$

$$\zeta u^{m-1}\frac{\partial u}{\partial x}=0. \tag{33}$$

Consequently, we determine the similarity solution $u=u_0\,,h=h_0\,,B=0$ with $u_0\neq 0$, or $u=0\,,\ B=-\frac{\partial h}{\partial x}$.

4.2. Reduction with X_2

From the symmetry vector X_2 we find the invariant functions t,u,h,B. Thus, $u=u(t)$, $h=h(t)$ and $B(t)$. Hence, from the SVE system we find the reduced equations

$$\frac{\partial h}{\partial t}=0\,,\ \frac{\partial B}{\partial t}=0\,, \tag{34}$$

$$\frac{\partial}{\partial t}(uh)=0\,, \tag{35}$$

from where we infer the similarity solutions

$$u=u_0\,,\ h=h_0,\ B=B_0. \tag{36}$$

4.3. Reduction with X_3

The Lie symmetry vector X_3 provides $u=u(\chi)$, $h=h(\chi)$ and $B=B(\chi)$ with $\chi=xt^{-1}$. In the new coordinates, the SVE system reads

$$-\chi\frac{dh}{d\chi}+\frac{d}{d\chi}(uh)=0\,, \tag{37}$$

$$-\chi\frac{d}{d\chi}(uh)+\frac{d}{d\chi}\left(hu^{2}+\frac{1}{2}gh^{2}\right)+gh\frac{\partial B}{\partial \chi}=0\,, \tag{38}$$

$$-\chi\frac{dB}{d\chi}+\zeta u^{m-1}\frac{du}{\partial \chi}=0. \tag{39}$$

Thus we end up with the reduced differential equation

$$\chi\frac{du}{d\chi} = \frac{u^2(2\chi^2 u - 2\chi^3 + \zeta g u^m)}{3\chi u^2(u-\chi) + \zeta g u^m((1-m)\chi + (1+m)u)}\ , \tag{40}$$

with

$$h(\chi) = \frac{\chi^2 u(2u-\chi) - \zeta g u^m(u-\chi) - \chi u^3}{g\chi u}\ , \tag{41}$$

$$B(\chi) = \frac{\zeta}{m}\int \frac{1}{\chi}\frac{d}{d\chi}(u^m)d\chi. \tag{42}$$

4.4. Reduction with X_4

Lie symmetry X_4 does not apply a similarity transformation which reduces the number of independent variables for the SVE system.

4.5. Reduction with $X_1 + \alpha X_2$ Solution

Application of the vector field $X_1 + \alpha X_2$ gives the travelling wave solution $u = u(\sigma)$, $h = h(\sigma)$, $B = B(\sigma)$ with $\sigma = x - \alpha t$. The reduced SVE system is

$$\frac{d}{d\sigma}((\alpha - u)h) = 0 \tag{43}$$

$$\frac{d}{d\sigma}\left(uh(u-\alpha) + \frac{1}{2}gh^2\right) + gh\frac{dB}{d\sigma} = 0$$

$$\frac{d}{d\sigma}\left(B - \frac{\zeta}{m}u^m\right) = 0 \tag{44}$$

Hence, we calculate the similarity solutions $u = u_0$, $h = h_0$ and $B = B_0$. In the special case in which $h = 0$, the exact solution is $B - \frac{\zeta}{m}u^m = B_0$.

4.6. Reduction with $X_1 + \alpha X_4$

Reduction with respect to the vector field $X_1 + \alpha X_4$ gives the similarity transformation $u = u(x)$, $h = h(x)$ and $B = at + b(x)$. By replacing in the SVE system we find the closed-form solution

$$u(x) = \zeta^{\frac{1}{m}} u_0 (x_0 - \alpha m x)^{-\frac{1}{m}}\ , \tag{45}$$

$$h(x) = \zeta^{-\frac{1}{m}}(x_0 - \alpha m x)^{-\frac{1}{m}}\ ,$$

and

$$b(x) = b_0 - u_0\zeta^{\frac{1}{m}} u_0 (x_0 - \alpha m x)^{-\frac{1}{m}} - \frac{1}{2g}\zeta^{-\frac{2}{m}}(x_0 - \alpha m x)^{-\frac{2}{m}}. \tag{46}$$

4.7. Reduction with $X_2 + \alpha X_4$

From the symmetry vector $X_2 + \alpha X_4$ it follows the similarity transformation $u = u(t)$, $h = h(t)$ and $B = \alpha x + b(t)$. Hence, the similarity solution is written

$$u(t) = u_0\ , \ h(t) = h_0\ , \ b(t) = b_0 - \alpha g t\ , \tag{47}$$

or

$$u(t) = u(t)\ , \ h(t) = h_0\ , \ b(t) = b_0\ . \tag{48}$$

4.8. Reduction with $X_3 + \alpha X_4$

Reduction with respect to the invariant functions provided by the Lie symmetry vector field $X_3 + \alpha X_4$ provides

$$K(\chi)\frac{du}{d\chi} = g\alpha(u - \chi), \tag{49}$$

$$K(\chi)\frac{dh}{d\chi}=g\alpha h, \tag{50}$$

$$K(\chi)\frac{db}{d\chi}=a\Big((\chi-u)^2-gh\Big), \tag{51}$$

in which

$$u=u(\chi)\ ,\ h=h(\chi)\ ,\ B=\alpha\ln t+b(\chi)\ ,\ \chi=\frac{x}{t}, \tag{52}$$

and function $K(\chi)$ is

$$K(\chi)=\chi\Big((\chi-u)^2-gh\Big)-\zeta g u^{m-1}(\chi-u). \tag{53}$$

4.9. Reduction with $X_1+\alpha X_2+\beta X_4$

We proceed by considering the vector field $X_1+\alpha X_2+\beta X_4$. We determine the invariant functions $u=u(\sigma)\ ,\ h=h(\sigma)$, $B=\beta t+b(\sigma)$ where $\sigma=x-\alpha t$.

Hence, the SVE system under the application of the invariants reads

$$\bar{K}(\sigma)\frac{du}{d\sigma}=\beta g(u-\alpha) \tag{54}$$

$$\bar{K}(\sigma)\frac{dh}{d\sigma}=\beta gh \tag{55}$$

$$\bar{K}(\sigma)\frac{db}{d\sigma}=\beta\Big((a-u)^2-gh\Big) \tag{56}$$

where now

$$\bar{K}(\sigma)=\alpha\Big((\alpha-u)^2-gu\Big)-\zeta g u^{m-1}(\alpha-u). \tag{57}$$

We can define the new independent variable Σ, $d\sigma=\bar{K}(\Sigma)d\Sigma$, such that the reduced SVE system to be

$$\frac{du}{d\Sigma}=\beta g(u-\alpha) \tag{58}$$

$$\frac{dh}{d\Sigma}=\beta gh \tag{59}$$

$$\frac{db}{d\Sigma}=\beta\Big((a-u)^2-gh\Big) \tag{60}$$

from where we infer the closed-form solution

$$u(\Sigma)=\alpha+u_0e^{\beta g\Sigma}\ ,\ h(\Sigma)=h_0e^{\beta g\Sigma}\ , \tag{61}$$

and

$$b(\Sigma)=\frac{(u_0)^2}{2g}e^{2\beta g\Sigma}-h_0e^{\beta g\Sigma}+b_0\ . \tag{62}$$

4.10. Reduction with X_5

For $m=3$ and the symmetry vector X_5 provides the similarity transformation $u=U(x)t^{-1}$, $h=H(x)t^{-2}$ and $B=b(x)t^{-2}$.

Thus, we replace it in the SVE system and we find the following set of ordinary differential equations

$$\frac{d}{dx}(uh)-2h=0, \tag{63}$$

$$\frac{d}{dx}\left(HU^2+\frac{1}{2}gH^2\right)+gH\frac{db}{dx}-3UH=0\ , \tag{64}$$

$$\zeta U^2\frac{dU}{dx}-2b=0. \tag{65}$$

or equivalently

$$\frac{dU}{dX} = \frac{2}{\zeta}\,, \tag{66}$$

$$bU\frac{dH}{dX} = -\frac{2}{\zeta}H\left(b-\zeta U^2\right), \tag{67}$$

$$bU\frac{db}{dX} = \frac{1}{g\zeta}\left(\zeta U^4 - 2\left(bU^2 - 2gH\left(b-\zeta U^2\right)\right)\right). \tag{68}$$

where $dX = bU^{-2}dx$. The latter system can be reduced to the following Abel equation

$$\begin{aligned} 0 \;=\;& \frac{d}{d\mathcal{X}}\mathcal{H} + \frac{3}{8\zeta g}\mathcal{X}\left(3g\zeta^2\mathcal{X} + 4g\zeta - 3\right)\mathcal{H}^3 - \frac{1}{8\zeta g}\left(8\zeta g + 21\zeta g^2\mathcal{X} - 30\right)\mathcal{H}^2 \\ & -\frac{1}{2\zeta g\mathcal{X}}\left(4\zeta^2 g\mathcal{X}^2 + 2\zeta g - 7\right)\mathcal{H} - \frac{\zeta^2 g\mathcal{X} - 2}{2\zeta g\mathcal{X}}, \end{aligned} \tag{69}$$

where now

$$\mathcal{X} = \frac{\mathcal{H}}{X^2},\; \mathcal{H} = \frac{2X^2}{\frac{dH}{dX}X - 2H} \tag{70}$$

and $U(X) = \frac{2}{\zeta}X$ and $b(X) = \frac{8HX^2}{\zeta\left(\frac{dH}{dX}X + 2H\right)}$.

A closed form solution of the later system

$$U(X) = \frac{2}{\zeta}X\,,\; b(X) = \frac{4}{3\zeta}X^2\,,\; H(X) = -\frac{4g\zeta - 3}{3g\zeta^2}X^2. \tag{71}$$

4.11. Reduction with $X_2 + \alpha X_5$

From the vector field $X_2 + \alpha X_5$ we derive the invariant functions $u = U(\omega)t^{-1}$, $h = H(\omega)t^{-2}$ and $B = B(\omega)t^{-2}$ with $\omega = x - \frac{1}{\alpha}\ln t$.

Thus, the reduced system reads

$$\hat{K}(\omega)\frac{dU}{d\omega} = \alpha(U(\alpha U - 1) + 2\alpha g(\alpha bU - H - b))\,, \tag{72}$$

$$\hat{K}(\omega)\frac{dH}{d\omega} = \alpha H\left(2\left(1 + g\alpha^2 b\right) - \alpha U(2\alpha g\zeta U + 1)\right), \tag{73}$$

$$\begin{aligned} -\frac{1}{\alpha}\hat{K}(\omega)\frac{db}{d\omega} \;=\;& 2\alpha bU(\alpha U - 2) - 2\alpha^2 gbH + 2b \\ & +\alpha\zeta(U(\alpha U - 1) - 2\alpha gH)\,, \end{aligned} \tag{74}$$

with

$$\hat{K}(\omega) = \alpha U(\alpha U - 2) - \alpha^2 gH + 1 + \alpha^2\zeta gU^2(\alpha U - 1)\,.$$

4.12. Reduction with $X_3 + \alpha X_5$

Finally, from the symmetry vector $X_3 + \alpha X_5$ we find $u = U(\lambda)t^{-\frac{\alpha}{1+\alpha}}$, $h = H(\lambda)t^{-\frac{2\alpha}{1+\alpha}}$ and $B = b(\lambda)t^{-\frac{2\alpha}{1+\alpha}}$, with $\lambda = xt^{-\frac{\alpha}{1+\alpha}}$.

The reduced SVE system reads

$$-\left(\lambda\frac{dH}{d\lambda} + 2\alpha H\right) + (1+\alpha)\frac{d}{d\lambda}(UH) = 0 \tag{75}$$

$$-\left(\lambda\frac{d}{d\lambda}(UH) + 2\alpha(UH)\right) + (1+\alpha)\frac{d}{dx}\left(HU^2 + \frac{1}{2}gH^2\right) + gH\frac{db}{d\lambda} = 0\,, \tag{76}$$

$$-\left(\lambda\frac{db}{d\lambda} + 2\alpha b\right) + \zeta(1+\alpha)U^2\frac{dU}{d\lambda} = 0 \tag{77}$$

For $\alpha = -1$ we end up with the similarity transformation $u = xU(t)$, $h = x^2 H(t)$ and $B = x^2 B(t)$ where the reduced SVE system admits the closed-form solutions

$$U(\lambda) = \frac{2}{3} t^{-1} \, , \; H(\lambda) = \frac{4}{27} \zeta t^{-2} \, , \; b = \frac{3 - 4g\zeta}{27g} t^{-2} \, , \tag{78}$$

and

$$U(\lambda) = U_0 t^{-1} \, , \; H(\lambda) = \frac{\zeta}{2} (U_0)^3 t^{-2} \, , \; b = 0 \, . \tag{79}$$

5. Conclusions

In this work we investigated the algebraic properties of the SVE system with a power-law sediment flux $Q(t,x) \simeq u^m$, $m \geq 1$ [38]. Specifically we solved the Lie symmetry conditions for the SVE model where we found that the SVE system admits four symmetry vectors for an arbitrary value of the power index m, which forms the $A_{3,3} \otimes A_1$ Lie algebra. However, in the special case where $m = 3$, the SVE system admits a fifth Lie point symmetry. In this case, the admitted symmetry vectors form the $A_{3,3} \otimes_s A_{2,1}$ Lie algebra.

Furthermore, for the admitted Lie symmetries, we calculated the adjoint representation and its invariants. We used these results in order to determine the one-dimensional optimal system. The latter is necessary in order to perform a complete classification of the independent similarity transformations. Indeed, for all the elements of the one-dimensional system we determined the invariant functions which are used to reduce the SVE model from a system of hyperbolic partial differential equations into a system of ordinary differential equations. New exact closed-form solutions derived which have not found before in the literature. These solutions can be related with numerical results presented in previous studies.

However, what we have not discussed is the problem of the initial and boundary conditions for the SVE model. It is true that for a given initial and value problem not all the similarity transformations can be applied. Thus, if we assume the initial and boundary problem with initial conditions $s_\alpha(y^A) = 0$ and boundary constraints $C_\alpha(y^A, u, h, B) = 0$, then the application of similarity transformation given by symmetry vector X for the differential equation $\mathbf{H}(y^A, u, u_{,A}, h, h_{,A}, B, B_{,A}) = 0$, provides a reduced system, a similarity solution which solves the initial value problem if and only if $s_\alpha(y^A)$ and $C_\alpha(y^A, u, h, B)$ are also invariant under the action of the symmetry vector X, that is, $X(s_\alpha(y^A)) = 0$ and $X(C_\alpha(y^A, u, h, B)) = 0$ [43].

With the application of these two conditions, we are able to determine the initial and boundary value problem for each element of the one-dimensional optimal system. We present some of the initial and boundary conditions. For arbitrary value of the power index m, and for the symmetry vector X_1, the initial and boundary conditions which provide a static solution are $s_a(x) = 0$, $C_\alpha(x, u, h, B) = 0$. Vector field X_2 gives $s_a(t) = 0$, $C_\alpha(t, u, h, B) = 0$. For the Lie symmetry vector X_3, we calculate $s_\alpha\left(\frac{x}{t}\right) = 0$, $C_\alpha\left(\frac{x}{t}, u, h, B\right) = 0$. Similarly for the $X_1 + \alpha X_2$ it follows $s_\alpha(x - \alpha t) = 0$, $C_\alpha(x - \alpha t, u, h, B) = 0$. In a similar way the initial value problem for the rest of the elements of the one-dimensional optimal system can be constructed with the use of the Lie invariant functions.

This work contributes to the application of the Lie symmetries in fluid dynamics and specifically in shallow-water systems. Inspired by the Ovsiannikov classification scheme [44], in a future work we plan to perform a complete classification for the functional form of the sediment flux $Q(t,x) = Q(u(t,x), h(t,x))$ where the SVE models admit Lie symmetries as able to construct new conservation laws for the SVE models which will be useful for the numerical analysis of the models.

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