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Law of Large Numbers, Central Limit Theorem, and Law of the Iterated Logarithm for Bernoulli Uncertain Sequence

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Abstract: In order to describe human uncertainty more precisely, Baoding Liu established uncertainty theory. Thus far, uncertainty theory has been successfully applied to uncertain finance, uncertain programming, uncertain control, etc. It is well known that the limit theorems represented by law of large numbers (LLN), central limit theorem (CLT), and law of the iterated logarithm (LIL) play a critical role in probability theory. For uncertain variables, basic and important research is also to obtain the relevant limit theorems. However, up to now, there has been no research on these limit theorems for uncertain variables. The main results to emerge from this paper are a strong law of large numbers (SLLN), a weak law of large numbers (WLLN), a CLT, and an LIL for Bernoulli uncertain sequence. For studying these theorems, we first propose an assumption, which can be regarded as a generalization of the duality axiom for uncertain measure in the case that the uncertainty space can be finitely partitioned. Additionally, several new notions such as weakly dependent, Bernoulli uncertain sequence, and continuity from below or continuity from above of uncertain measure are introduced. As far as we know, this is the first study of the LLN, the CLT, and the LIL for uncertain variables. All the theorems proved in this paper can be applied to uncertain variables with symmetric or asymmetric distributions. In particular, the limit of uncertain variables is symmetric in (c) of the third theorem, and the asymptotic distribution of uncertain variables in the fifth theorem is symmetrical.

Keywords: uncertain measure; Bernoulli uncertain sequence; weakly dependent; law of large numbers; central limit theorem; law of the iterated logarithm

Citation: Qu, Z.; Zong, Z.; Hu, F. Law of Large Numbers, Central Limit Theorem, and Law of the Iterated Logarithm for Bernoulli Uncertain Sequence. *Symmetry* **2022**, *14*, 1642. https://doi.org/10.3390/sym14081642

Academic Editors: Jian Zhou and José Carlos R. Alcantud

Received: 9 July 2022
Accepted: 7 August 2022
Published: 9 August 2022

1. Introduction

As the fundamental limit theorems in the theory of probability and statistics, the law of large numbers (LLN), central limit theorem (CLT), and law of the iterated logarithm (LIL) have made significant contributions to the development and application of probability and other theories. The first study of the LLN was reported by Cardano in the sixteenth century, and the LLN for a binary random variable was proved by Bernoulli [1] in 1713. With the further study and development of the LLN, two prominent forms of the LLN were discovered. In 1930, Kolmogorov [2] proposed the strong law of large numbers (SLLN) for independent and identically distributed Lebesgue integrable random variables. In the same period, Khinchin [3] established the weak law of large numbers (WLLN) for independent and identically distributed random variables with a finite expected value. The earliest version of the CLT was proposed by De Moivre [4] in 1738, and Lindeberg [5] gave the modern general form of the CLT in 1920. The original version of LIL for Bernoulli random variables was established by Khinchin [6] in 1924. Kolmogorov [7] generalized the applicable object of LIL from Bernoulli random variables to independent random variables in 1929. After the LLN, CLT, and LIL were established, many mathematicians contributed to the refinement of the limit theorems, including Poisson, Chebyshev, Markov, Lyapunov, Winter, Strassen, etc.

As a commonly used tool to handle the fuzzy phenomena, fuzzy theory was established by Zadeh [8] in 1965. In [8], a concept of fuzzy set was presented which can be characterized by a type of membership function that satisfies normality, nonnegativity, and maximality axioms. After that, Zadeh [9] further established a possibility theory. The Fuzzy measures, the Choquet integral, and the Sugeno integral were studied in [10–13]. Research on the theorems such as the LLN for fuzzy variables has also been ongoing. The LLN for fuzzy sets was first presented by Fullér [14] in 1992. Afterwards, Triesch [15] proposed the LLN for mutually T-related fuzzy numbers. As an extension of the early results of [14,15], Hong and Kim [16] discussed the LLN for fuzzy numbers in a Banach space. For more details, see [17–19].

In order to study human uncertainty, Baoding Liu [20] pioneered the uncertainty theory in 2007, and he further refined it [21] in 2009 based on normality, duality, subadditivity, and product axioms. Same as in probability theory, an uncertain variable was employed to model the uncertain quantity, an uncertain measure was used to denote the belief degree that an uncertain event may happen, and a concept of uncertainty distribution was adopted to describe uncertain variables. After that, many researchers have contributed a lot in this area. Since sequence convergence plays a very important role in probability theory, it has also been studied a lot in the field of uncertain measure. Baoding Liu [20] first introduced several convergence concepts such as convergence in measure, convergence in mean, convergence almost surely, and convergence in distribution. You [22] gave the concept of convergence uniformly almost surely. Guo and Xu [23] proposed the concept of convergence in mean square for uncertain sequence. Inspired by these, Chen, Ning, and Wang [24] first studied the convergence of complex uncertain sequences in 2016. Further studies on complex uncertain sequences have been done by many other researchers. For more details, we can refer to [25–29]. Up to now, uncertainty theory has been widely used in uncertain finance (see, e.g., Peng and Yao [30], Yu [31]), uncertain programming (see, e.g., Liu [32], Liu and Chen [33]), uncertain statistics (see, e.g., Tripathy and Nath [34]), uncertain differential equation (see, e.g., Liu [35], Chen and Liu [36]), and so on. However, the limit theorems for uncertain variables such as LLN, CLT, and LIL have not been studied.

Over the past decades, the LLN in uncertainty theory has only been discussed for uncertain random variables under chance space. For dealing with the complex phenomenon where uncertainty and randomness coexist, Yuhan Liu [37] established the chance theory on the basis of probability theory and uncertainty theory in 2013. In [37], several fundamental concepts were introduced. As an integration of probability measure and uncertain measure, a chance measure was employed to represent the possibility that an uncertain random event occurs. A concept of chance space was defined as the product space of probability space and uncertainty space, and the concept of uncertain random variable, chance distribution, etc., were further presented. The literature devoted to LLN in chance theory is very rich. For more details, we can refer to [38–43]. Yao and Gao [38] first proposed the LLN for uncertain random variables being functions of independent, identically distributed random variables and independent, identically distributed regular uncertain variables. As a generalization of [38], Gao and Sheng [39] weakened the conditions of the LLN in which random variables are independent, identically distributed and uncertain variables are independent but not identically distributed. Recently, Nowak and Hryniewicz [43] proved three types of laws of large numbers for uncertain random variables. First of all, the LLN proved in [38] was further extended to cases where random variables are pairwise independent, identically distributed and uncertain variables are regular, independent, and identically distributed. Then, the Marcinkiewicz–Zygmund-type LLN and the Chow-type LLN for sequences of uncertain random variables were also presented.

In this paper, our aim is to obtain an LLN, a CLT, and aLIL for Bernoulli uncertain sequence. To achieve our goal, we first propose several new notions such as weakly dependent, Bernoulli uncertain sequence and continuity from below or continuity from above of uncertain measure. Secondly, in order to illustrate the point of this paper, we propose Assumption 1. It is shown that, when the uncertainty space can be finitely partitioned, the duality of the uncertain measure defined on the uncertainty space can be generalized. After that, Theorems 1 and 2 are established to study the relationship between probability measure and uncertain measure on σ-algebra generated by Bernoulli uncertain sequence. Lastly, by applying Theorems 1 and 2, we successively obtain an SLLN, a WLLN, a CLT, and an LIL for Bernoulli uncertain sequence.

This paper is organized as follows. In Section 2, we give a brief exposition of notions, assumption, and lemma which will be used in this paper. Section 3 is dedicated to the main theorems of this paper. Theorems 1 and 2 are established as the fundamental theorems for deriving the main results of this paper. Then, LLN, CLT, and LIL for Bernoulli uncertain sequence are proved. A brief conclusion is presented in Section 4. Finally, the theorems mentioned in this paper are presented in Appendix A.

2. Preliminaries

In this section, several fundamental concepts concerning uncertainty theory will be reviewed first. Then, we will give other notions used in the article. Finally, we will make an assumption, which is the premise to illustrate the viewpoint of this paper.

Definition 1 (see [20])**.** *Let $\mathcal{L}$ be a σ-algebra on a non-empty set Γ. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:*
Axioms 1 (Normality Axiom): $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ;
Axioms 2 (Duality Axiom): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axioms 3 (Subadditivity Axiom): For every countable sequence of $\{\Lambda_j\} \subset \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [21] as follows:
Axioms 4 (Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2 (see [20])**.** *An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set of B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event. The notion $\sigma(\xi)$ stands for the smallest σ-algebra containing $\{\{\xi \leq x\}, \forall x \in \mathbb{R}\}$.

Definition 3 (see [20])**.** *The uncertainty distribution ϕ of an uncertain variable ξ is defined by*

$$\phi(x) = \mathcal{M}\{\xi \leq x\}, \forall x \in \mathbb{R}.$$

Definition 4 (see [20])**.** *Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} \mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\} \mathrm{d}x$$

provided that at least one of the two integrals is finite.

Definition 5 (see [20])**.** *Uncertain variables are said to be identically distributed if they have the same uncertainty distribution.*

Definition 6. *The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be weakly dependent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\xi_i \in B_i\}\right\} \in \left[\prod_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}, \bigwedge_{i=1}^{n}\mathcal{M}\{\xi_i \in B_i\}\right]$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Definition 7. *A sequence $\{\xi_k\}_{k=1}^{\infty}$ is called Bernoulli uncertain sequence if it satisfies*
(i) For each $n \in \mathbb{N}$, the uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are weakly dependent;
(ii) ξ_i and ξ_j are identically distributed uncertain variables for any $i \neq j$, $i, j \in \mathbb{N}$;
(iii) For each n, ξ_n takes on the values of $\{a_1, a_2, \cdots, a_p\}$, where $a_i \in \mathbb{R}$, $i = 1, \cdots, p$, and $p \in \mathbb{N}$.

Definition 8. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. An uncertain measure $\mathcal{M}$ is called continuity from below or continuity from above if it satisfies*
(i) Continuity from below: $\Lambda_n \in \mathcal{L}, n \in \mathbb{N}, \Lambda_n \uparrow$,

$$\lim_{n\to\infty}\mathcal{M}\{\Lambda_n\} = \mathcal{M}\left\{\bigcup_{n=1}^{\infty}\Lambda_n\right\}.$$

(ii) Continuity from above: $\Lambda_n \in \mathcal{L}, n \in \mathbb{N}, \Lambda_n \downarrow$,

$$\lim_{n\to\infty}\mathcal{M}\{\Lambda_n\} = \mathcal{M}\left\{\bigcap_{n=1}^{\infty}\Lambda_n\right\}.$$

Assumption 1. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space. For a given $n \in \mathbb{N}$, suppose that $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, such that $\Lambda_i \bigcap \Lambda_j = \varnothing$, $i \neq j$, $i, j = 1, \cdots, n$, and $\Lambda_1 \bigcup \cdots \bigcup \Lambda_n = \Gamma$. Then,*

$$\mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\} + \cdots + \mathcal{M}\{\Lambda_n\} = 1. \tag{1}$$

Remark 1. *The idea of making this assumption is quite natural. Next, we will explain its rationality. Suppose that $\Lambda_1, \Lambda_2, \Lambda_3 \in \mathcal{L}$, such that $\Lambda_i \bigcap \Lambda_j = \varnothing$, $i \neq j$, $i, j = 1, 2, 3$, $\Lambda_1 \bigcup \Lambda_2 \bigcup \Lambda_3 = \Gamma$, and $\mathcal{L}$ represents the smallest σ-algebra containing $\{\{\Lambda_1\}, \{\Lambda_2\}, \{\Lambda_3\}\}$. It is easily checked that*

$$\mathcal{L} = \{\varnothing, \{\Lambda_1\}, \{\Lambda_2\}, \{\Lambda_3\}, \{\Lambda_1, \Lambda_2\}, \{\Lambda_1, \Lambda_3\}, \{\Lambda_2, \Lambda_3\}, \Gamma\}.$$

Applying the duality of $\mathcal{M}$, we have,

$$\mathcal{M}\Big\{\{\Lambda_1\}\bigcup\{\Lambda_2\}\Big\} + \mathcal{M}\{\Lambda_3\} = 1.$$

Denote

$$\mathcal{M}\Big\{\{\Lambda_1\}\bigcup\{\Lambda_2\}\Big\} = p, \quad \mathcal{M}\{\Lambda_3\} = 1 - p.$$

It is easily seen that if $p = 0$, then

$$\mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\} + \mathcal{M}\{\Lambda_3\} = 1.$$

If $p \neq 0$, then we set

$$\Gamma_1 = \{\Lambda_1, \Lambda_2\}, \quad \mathcal{L}_1 = \{\varnothing, \{\Lambda_1\}, \{\Lambda_2\}, \Gamma_1\}, \quad \mathcal{M}_1 = \frac{1}{p}\mathcal{M},$$

where $\mathcal{L}_1$ is the smallest σ-algebra containing $\{\{\Lambda_1\}, \{\Lambda_2\}\}$. It is obvious that $\mathcal{M}_1$ satisfies the normality and subadditivity on $\mathcal{L}_1$. We hope that $\mathcal{M}_1$ is still an uncertain measure on $\mathcal{L}_1$, so we assume that

$$\mathcal{M}_1\{\Lambda_1\} + \mathcal{M}_1\{\Lambda_2\} = 1.$$

Then we have

$$\mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\} + \mathcal{M}\{\Lambda_3\} = 1.$$

Using the same method, for a given $n \in \mathbb{N}$, we can show that (1) holds. Therefore, our assumption is reasonable.

Lemma 1. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space satisfying Assumption 1. For a given $n \in \mathbb{N}$, suppose that $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, such that $\Lambda_i \bigcap \Lambda_j = \emptyset$, $i \neq j$, $i, j = 1, \cdots, n$, and $\Lambda_1 \bigcup \cdots \bigcup \Lambda_n = \Gamma$. Then,*

$$\mathcal{M}\left\{\bigcup_{j=i_1}^{i_k} \Lambda_j\right\} = \sum_{j=i_1}^{i_k} \mathcal{M}\{\Lambda_j\}, \tag{2}$$

where $\{i_1, \cdots, i_k\}$ is the top k items of any permutation of $\{1, \cdots, n\}$, $k = 1, 2, \cdots, n$.

Proof. Let

$$\Lambda = \bigcup_{j=i_1}^{i_k} \Lambda_j, \quad \Lambda^c = \bigcup_{k=i_{k+1}}^{i_n} \Lambda_k,$$

where $\{i_1, \cdots, i_n\}$ is any permutation of $\{1, \cdots, n\}$.

Applying the subadditivity of $\mathcal{M}$, we can obtain

$$\mathcal{M}\{\Lambda\} \leq \sum_{j=i_1}^{i_k} \mathcal{M}\{\Lambda_j\} \tag{3}$$

and

$$\mathcal{M}\{\Lambda^c\} \leq \sum_{k=i_{k+1}}^{i_n} \mathcal{M}\{\Lambda_k\}. \tag{4}$$

If either (3) or (4) is a strict inequality, then (3) plus (4) implies $1 < 1$, which contradicts the facts. Hence, (2) is proved. □

3. Main Results

Theorem 1. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space satisfying Assumption 1 and $\{\xi_k\}_{k=1}^{\infty}$ be a Bernoulli uncertain sequence relative to $\mathcal{M}$. Then there exists a probability measure $\mathbb{P}$ defined on $\sigma(\xi_1, \xi_2, \cdots, \xi_n, \cdots)$ such that*
(a) $\mathbb{P}\{\Lambda\} \leq \mathcal{M}\{\Lambda\}$, $\forall \Lambda \in \sigma(\xi_1, \xi_2, \cdots, \xi_n, \cdots)$.
(b) $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of independent random variables relative to $\mathbb{P}$.
(c) $E[\xi_k^n] = E_{\mathbb{P}}[\xi_k^n]$, for any $k, n \in \mathbb{N}$, where $E_{\mathbb{P}}[\cdot]$ denotes the expected value in the sense of probability measure $\mathbb{P}$.

Proof. (a) For each $k \in \mathbb{N}$, we have

$$\sigma(\xi_k) = \left\{\emptyset, \bigcup_{j=1}^{m} \{\xi_k = a_{i_j}\}\right\},$$

where $\{i_1, \cdots, i_m\}$ is the top m items of any permutation of $\{1, \cdots, p\}$, $m = 1, \cdots, p$.

By Lemma 1, for any $k \in \mathbb{N}$, we can define a finitely additive measure $\mathbb{P}$ on $\sigma(\xi_k)$ such that

$$\mathbb{P}\{\Lambda\} = \mathcal{M}\{\Lambda\}, \quad \forall \Lambda \in \sigma(\xi_k).$$

For any fixed n, we want to define a finitely additive measure $\mathbb{P}'$ on $\sigma(\xi_1, \xi_2, \cdots, \xi_n)$. For simplicity, we still denote $\mathbb{P}'$ as $\mathbb{P}$. If $\Lambda = \emptyset$, then we define $\mathbb{P}\{\emptyset\} = \mathcal{M}\{\emptyset\} = 0$. For any $\Lambda \in \sigma(\xi_1, \xi_2, \cdots, \xi_n)$ and $\Lambda \neq \emptyset$, it is easy to check that Λ has the form

$$\left\{\bigcup_{j_1=1}^{m_1}\left\{\xi_1 = a_{i_{j_1}^1}\right\}\right\} \bigcap \left\{\bigcup_{j_2=1}^{m_2}\left\{\xi_2 = a_{i_{j_2}^2}\right\}\right\} \bigcap \cdots \bigcap \left\{\bigcup_{j_n=1}^{m_n}\left\{\xi_n = a_{i_{j_n}^n}\right\}\right\},$$

where $\left\{i_1^q, \cdots, i_{m_q}^q\right\}$ is the top m_q items of any permutation of $\{1, \cdots, p\}$, $q = 1, \cdots, n$, and $m_1, \cdots, m_n = 1, \cdots, p$.

Thus, $\mathbb{P}$ on $\sigma(\xi_1, \xi_2, \cdots, \xi_n)$ is defined as

$$\begin{aligned}
\mathbb{P}\{\Lambda\} &= \sum_{j_n=1}^{m_n} \cdots \sum_{j_2=1}^{m_2} \sum_{j_1=1}^{m_1} \mathbb{P}\left\{\left\{\xi_1 = a_{i_{j_1}^1}\right\} \bigcap \left\{\xi_2 = a_{i_{j_2}^2}\right\} \bigcap \cdots \bigcap \left\{\xi_n = a_{i_{j_n}^n}\right\}\right\} \\
&= \sum_{j_n=1}^{m_n} \cdots \sum_{j_2=1}^{m_2} \sum_{j_1=1}^{m_1} \mathbb{P}\left\{\xi_1 = a_{i_{j_1}^1}\right\} \mathbb{P}\left\{\xi_2 = a_{i_{j_2}^2}\right\} \cdots \mathbb{P}\left\{\xi_n = a_{i_{j_n}^n}\right\} \\
&= \sum_{j_n=1}^{m_n} \cdots \sum_{j_2=1}^{m_2} \sum_{j_1=1}^{m_1} \mathcal{M}\left\{\xi_1 = a_{i_{j_1}^1}\right\} \mathcal{M}\left\{\xi_2 = a_{i_{j_2}^2}\right\} \cdots \mathcal{M}\left\{\xi_n = a_{i_{j_n}^n}\right\}. \qquad (5)
\end{aligned}$$

From weakly dependent of $\mathcal{M}$, we have,

$$\mathcal{M}\left\{\xi_1 = a_{i_{j_1}^1}\right\} \mathcal{M}\left\{\xi_2 = a_{i_{j_2}^2}\right\} \cdots \mathcal{M}\left\{\xi_n = a_{i_{j_n}^n}\right\}$$

$$\leq \mathcal{M}\left\{\left\{\xi_1 = a_{i_{j_1}^1}\right\} \bigcap \left\{\xi_2 = a_{i_{j_2}^2}\right\} \bigcap \cdots \bigcap \left\{\xi_n = a_{i_{j_n}^n}\right\}\right\}. \qquad (6)$$

Further, we obtain

$$\mathbb{P}\{\Lambda\} \leq \sum_{j_n=1}^{m_n} \cdots \sum_{j_2=1}^{m_2} \sum_{j_1=1}^{m_1} \mathcal{M}\left\{\left\{\xi_1 = a_{i_{j_1}^1}\right\} \bigcap \left\{\xi_2 = a_{i_{j_2}^2}\right\} \bigcap \cdots \bigcap \left\{\xi_n = a_{i_{j_n}^n}\right\}\right\} \qquad (7)$$

by (5) and (6). Note that Λ on $\sigma(\xi_1, \xi_2, \cdots, \xi_n)$ can be represented as the union of finite sets, and the sets are pairwise disjoint. From Lemma 1, it follows that

$$\mathcal{M}\{\Lambda\} = \sum_{j_n=1}^{m_n} \cdots \sum_{j_2=1}^{m_2} \sum_{j_1=1}^{m_1} \mathcal{M}\left\{\left\{\xi_1 = a_{i_{j_1}^1}\right\} \bigcap \left\{\xi_2 = a_{i_{j_2}^2}\right\} \bigcap \cdots \bigcap \left\{\xi_n = a_{i_{j_n}^n}\right\}\right\}. \qquad (8)$$

Hence, from (7) and (8), we obtain $\mathbb{P}\{\Lambda\} \leq \mathcal{M}\{\Lambda\}, \forall \Lambda \in \sigma(\xi_1, \xi_2, \cdots, \xi_n)$.

It is easily seen that $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$ is an algebra, and for any $\Lambda \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$, there exists $n_1 \in \mathbb{N}$, such that $\Lambda \in \sigma(\xi_1, \xi_2, \cdots, \xi_{n_1})$. So, we define

$$\mathbb{P}''\{\Lambda\} = \mathbb{P}\{\Lambda\}, \quad \forall \Lambda \in \bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n).$$

For simplicity, we still denote $\mathbb{P}''$ as $\mathbb{P}$. By (5), we know that $\mathbb{P}$ satisfies finite additivity on $\sigma(\xi_1, \xi_2, \cdots, \xi_n)$. Thus, $\mathbb{P}$ is a finitely additive measure defined on $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$.

We can endow the space Γ with an auxiliary compact topology. This topology has as basis the algebra $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$ itself (see Lemma 9 [44] p. 155). From the definition of regular (see Definition A1 in Appendix A), we can verify that $\mathbb{P}$ is regular on $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$. By Theorem A1 in Appendix A, it follows that $\mathbb{P}$ is a countably additive measure on $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$. Therefore, $\mathbb{P}$ is a probability measure on $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$.

Since, $\sigma(\xi_1, \xi_2, \cdots, \xi_n, \cdots)$ is the σ-algebra generated by $\bigcup_{n=1}^{\infty} \sigma(\xi_1, \xi_2, \cdots, \xi_n)$, then a standard application of Caratheodory's extension theorem (see Theorem A2 in Appendix A) ensures the existence of a unique probability measure on $\sigma(\xi_1, \xi_2, \cdots, \xi_n, \cdots)$ that extends $\mathbb{P}$. We still denote it as $\mathbb{P}$. This is the probability measure on $\sigma(\xi_1, \xi_2, \cdots, \xi_n, \cdots)$ we are looking for. Thus, (a) is proved.

(b) By (5), we know that $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of independent random variables relative to $\mathbb{P}$.

(c) Without loss of generality, we only prove that ξ_k is a positive uncertain variable and the value of ξ_k is either a_1 or a_2. The proof of other cases is similar.

Let us set $0 < a_1 < a_2$. By Definition 4, we have,

$$
\begin{aligned}
E[\xi_k^n] &= \int_0^{\infty} \mathcal{M}\{\xi_k^n \geq x\} \mathrm{d}x \\
&= \int_0^{a_1^n} 1 \mathrm{d}x + \int_{a_1^n}^{a_2^n} \mathcal{M}\{\xi_k^n = a_2^n\} \mathrm{d}x \\
&= a_1^n + (a_2^n - a_1^n) \mathcal{M}\{\xi_k^n = a_2^n\} \\
&= a_1^n \mathcal{M}\{\xi_k^n = a_1^n\} + a_2^n \mathcal{M}\{\xi_k^n = a_2^n\} \\
&= a_1^n \mathbb{P}\{\xi_k^n = a_1^n\} + a_2^n \mathbb{P}\{\xi_k^n = a_2^n\} \\
&= E_{\mathbb{P}}[\xi_k^n].
\end{aligned}
$$

Thus, (c) is proved. □

Theorem 2. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space satisfying Assumption 1 and $\{\xi_k\}_{k=1}^{\infty}$ be a Bernoulli uncertain sequence relative to $\mathcal{M}$. For fixed n, uncertain variable Y_n is a measurable function on $\sigma(\xi_1, \xi_2, \cdots, \xi_n)$. Suppose that $\mathbb{P}$ is the probability measure provided by Theorem 1. Then,*

(a)

$$
\lim_{p\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \lim_{k\to\infty} \mathcal{M}\left\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\}
$$
$$
\geq \mathbb{P}\left\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\right\}. \tag{9}
$$

(b)

$$
\lim_{m\to\infty} \lim_{n\to\infty} \lim_{p\to\infty} \lim_{k\to\infty} \mathcal{M}\left\{\gamma : Y_n \leq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \leq \alpha + \frac{1}{m} - \frac{1}{p}\right\}
$$
$$
\geq \mathbb{P}\left\{\gamma : \limsup_{n\to\infty} Y_n \leq \alpha\right\}. \tag{10}
$$

(c)

$$
\lim_{n\to\infty} \lim_{p\to\infty} \lim_{k\to\infty} \mathcal{M}\left\{\left\{\gamma : Y_n > \alpha - \frac{1}{p}\right\} \bigcup \cdots \bigcup \left\{\gamma : Y_{n+k} > \alpha - \frac{1}{p}\right\}\right\}
$$
$$
\geq \mathbb{P}\left\{\gamma : \limsup_{n\to\infty} Y_n \geq \alpha\right\}. \tag{11}
$$

(d) Furthermore, if $\mathcal{M}$ satisfies (i) and (ii) in Definition 8, then

$$\mathcal{M}\left\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\right\} \geq \mathbb{P}\left\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\right\}. \tag{12}$$

$$\mathcal{M}\left\{\gamma : \limsup_{n\to\infty} Y_n \leq \alpha\right\} \geq \mathbb{P}\left\{\gamma : \limsup_{n\to\infty} Y_n \leq \alpha\right\}. \tag{13}$$

$$\mathcal{M}\left\{\gamma : \limsup_{n\to\infty} Y_n \geq \alpha\right\} \geq \mathbb{P}\left\{\gamma : \limsup_{n\to\infty} Y_n \geq \alpha\right\}. \tag{14}$$

Proof. (a) By Theorem 1 (a), we have

$$\mathcal{M}\left\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\}$$
$$\geq \mathbb{P}\left\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\},$$

since

$$\bigcap_{k=n}^{\infty}\left\{\gamma : Y_k \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\} = \left\{\gamma : \inf_{k\geq n} Y_k \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\}, \tag{15}$$

then

$$\lim_{k\to\infty} \mathcal{M}\left\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\}$$

$$\geq \mathbb{P}\left\{\gamma : \inf_{k\geq n} Y_k \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\},$$

by (15) and the continuity from above of $\mathbb{P}$.

Note

$$\bigcup_{m=1}^{\infty}\left\{\gamma : \inf_{k\geq n} Y_k \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\} = \left\{\gamma : \inf_{k\geq n} Y_k > \alpha - \frac{1}{p}\right\}. \tag{16}$$

From (16) and the continuity from below of $\mathbb{P}$, it follows that

$$\lim_{m\to\infty}\lim_{k\to\infty} \mathcal{M}\left\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\}$$

$$\geq \mathbb{P}\left\{\gamma : \inf_{k\geq n} Y_k > \alpha - \frac{1}{p}\right\}.$$

Next, it is easily checked that

$$\bigcup_{n=1}^{\infty}\left\{\gamma : \inf_{k\geq n} Y_k > \alpha - \frac{1}{p}\right\} = \left\{\gamma : \liminf_{n\to\infty} Y_n > \alpha - \frac{1}{p}\right\}. \tag{17}$$

Thus, by (17) and the continuity from below of $\mathbb{P}$, we have,

$$\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty} \mathcal{M}\left\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\right\}$$

$$\geq \mathbb{P}\Big\{\gamma : \liminf_{n\to\infty} Y_n > \alpha - \frac{1}{p}\Big\}.$$

Finally, it is obvious that

$$\bigcap_{p=1}^{\infty}\Big\{\gamma : \liminf_{n\to\infty} Y_n > \alpha - \frac{1}{p}\Big\} = \Big\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\Big\}, \tag{18}$$

we obtain

$$\lim_{p\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\mathcal{M}\Big\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\Big\}$$

$$\geq \mathbb{P}\Big\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\Big\},$$

by applying (18) and the continuity from above of $\mathbb{P}$. Hence, the proof of (a) is completed.

The proof of Theorem 2 (b) and (c) can be established using the technique of that of Theorem 2 (a), so we omit it.

(d) In the following proof, we only prove (12) holds, and (13) and (14) can be proved by the same method.

Note that $\mathcal{M}$ satisfies (i) and (ii) in Definition 8. We can conclude that

$$\lim_{p\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\mathcal{M}\Big\{\gamma : Y_n \geq \alpha + \frac{1}{m} - \frac{1}{p}, \cdots, Y_{n+k} \geq \alpha + \frac{1}{m} - \frac{1}{p}\Big\}$$

$$= \mathcal{M}\Big\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\Big\}.$$

By applying (9), it follows that

$$\mathcal{M}\Big\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\Big\} \geq \mathbb{P}\Big\{\gamma : \liminf_{n\to\infty} Y_n \geq \alpha\Big\}.$$

Hence, (12) is proved. □

Theorem 3 (SLLN). *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space satisfying Assumption 1 and* $\{\xi_k\}_{k=1}^{\infty}$ *be a Bernoulli uncertain sequence relative to* $\mathcal{M}$. *Set* $\mu = E[\xi_k]$, $S_n = \sum_{i=1}^{n}\xi_i$. *Then,*

(a)

$$\lim_{p\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\mathcal{M}\Big\{\gamma : \frac{S_n}{n} \geq \mu + \frac{1}{m} - \frac{1}{p}, \cdots, \frac{S_{n+k}}{n+k} \geq \mu + \frac{1}{m} - \frac{1}{p}\Big\} = 1. \tag{19}$$

(b)

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{p\to\infty}\lim_{k\to\infty}\mathcal{M}\Big\{\gamma : \frac{S_n}{n} \leq \mu + \frac{1}{m} - \frac{1}{p}, \cdots, \frac{S_{n+k}}{n+k} \leq \mu + \frac{1}{m} - \frac{1}{p}\Big\} = 1. \tag{20}$$

(c) Furthermore, if $\mathcal{M}$ *satisfies (i) and (ii) in Definition 8, then*

$$\mathcal{M}\Big\{\gamma : \lim_{n\to\infty}\frac{S_n}{n} = \mu\Big\} = 1. \tag{21}$$

Proof. (a) By Theorem 2 (a), we have

$$\lim_{p\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\mathcal{M}\left\{\gamma:\frac{S_n}{n}\geq\mu+\frac{1}{m}-\frac{1}{p},\cdots,\frac{S_{n+k}}{n+k}\geq\mu+\frac{1}{m}-\frac{1}{p}\right\}$$
$$\geq\mathbb{P}\left\{\gamma:\liminf_{n\to\infty}\frac{S_n}{n}\geq\mu\right\}.$$

From Theorem 1 (b) and (c), it can be shown that $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of independent random variables relative to $\mathbb{P}$, $\mu=E[\xi_k]=E_{\mathbb{P}}[\xi_k]$, for any $k\in\mathbb{N}$. Hence, by applying Kolmogorov's strong law of large numbers (see Theorem A3 in Appendix A), it follows that

$$\mathbb{P}\left\{\gamma:\liminf_{n\to\infty}\frac{S_n}{n}\geq\mu\right\}=1,$$

which implies,

$$\lim_{p\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty}\mathcal{M}\left\{\gamma:\frac{S_n}{n}\geq\mu+\frac{1}{m}-\frac{1}{p},\cdots,\frac{S_{n+k}}{n+k}\geq\mu+\frac{1}{m}-\frac{1}{p}\right\}=1.$$

Hence, the proof of (a) is completed.

From Theorem 2 (b) and using the similar method of the proof of Theorem 3 (a), we can prove Theorem 3 (b). So it is omitted.

(c) Applying (12), (13) and Kolmogorov's strong law of large numbers, it yields that

$$\mathcal{M}\left\{\gamma:\liminf_{n\to\infty}\frac{S_n}{n}\geq\mu\right\}=1\quad\text{and}\quad\mathcal{M}\left\{\gamma:\limsup_{n\to\infty}\frac{S_n}{n}\leq\mu\right\}=1.\tag{22}$$

Now we show that (21) $\Leftrightarrow$ (22). Since, (21) $\Rightarrow$ (22) is obvious, we only need to prove that (22) $\Rightarrow$ (21). Let

$$\Lambda_1=\left\{\gamma:\liminf_{n\to\infty}\frac{S_n}{n}\geq\mu\right\},\ \Lambda_2=\left\{\gamma:\limsup_{n\to\infty}\frac{S_n}{n}\leq\mu\right\}.$$

Then,

$$\Lambda_1\bigcap\Lambda_2=\left\{\gamma:\lim_{n\to\infty}\frac{S_n}{n}=\mu\right\}.$$

Due to $\Gamma=\Lambda_1^c\bigcup\{\Lambda_1\bigcap\Lambda_2\}\bigcup\Lambda_2^c$, we obtain

$$\mathcal{M}\{\Gamma\}\leq\mathcal{M}\{\Lambda_1^c\}+\mathcal{M}\left\{\Lambda_1\bigcap\Lambda_2\right\}+\mathcal{M}\{\Lambda_2^c\},\tag{23}$$

by the subadditivity of $\mathcal{M}$. Note $\mathcal{M}\{\Lambda_1^c\}=0$, $\mathcal{M}\{\Lambda_2^c\}=0$. From (23), it follows that $\mathcal{M}\{\Lambda_1\bigcap\Lambda_2\}=1$, i.e.,

$$\mathcal{M}\left\{\gamma:\lim_{n\to\infty}\frac{S_n}{n}=\mu\right\}=1.$$

Hence, (c) is proved. □

Remark 2. *For (c), denote* $\xi=\lim_{n\to\infty}\frac{S_n-n\mu}{n}$*, then* $\mathcal{M}\{\gamma:\xi=0\}=1$*. Furthermore,*

$$\mathcal{M}\{\gamma:\xi\leq x\}+\mathcal{M}\{\gamma:\xi\leq -x\}=1,\ \forall x\in\mathbb{R}\backslash\{0\}.$$

Thus, ξ *is symmetrical (see, e.g., [45]).*

Theorem 4 (WLLN)**.** *Let* $(\Gamma,\mathcal{L},\mathcal{M})$ *be an uncertainty space satisfying Assumption 1 and* $\{\xi_k\}_{k=1}^{\infty}$ *be a Bernoulli uncertain sequence relative to* $\mathcal{M}$*. Set* $\mu=E[\xi_k]$*,* $S_n=\sum_{i=1}^{n}\xi_i$*. Then, for any* $\varepsilon>0$*, we have,*

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \left|\frac{S_n}{n} - \mu\right| < \epsilon\right\} = 1. \tag{24}$$

Proof. By Theorem 1 (a), we have,

$$\mathcal{M}\left\{\gamma : \mu - \varepsilon < \frac{S_n}{n} < \mu + \varepsilon\right\} \geq \mathbb{P}\left\{\gamma : \mu - \varepsilon < \frac{S_n}{n} < \mu + \varepsilon\right\}.$$

Note Theorem 1 (b) and (c). From Khinchin's weak law of large numbers (see Theorem A4 in Appendix A), for any $\varepsilon > 0$, it follows that

$$\lim_{n\to\infty} \mathbb{P}\left\{\gamma : \left|\frac{S_n}{n} - \mu\right| < \epsilon\right\} = 1,$$

which implies,

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \left|\frac{S_n}{n} - \mu\right| < \epsilon\right\} = 1.$$

□

Theorem 5 (CLT). *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space satisfying Assumption 1 and $\{\xi_k\}_{k=1}^{\infty}$ be a Bernoulli uncertain sequence relative to $\mathcal{M}$. Set $\mu = E[\xi_k]$, $\sigma^2 = E[\xi_k{}^2] - \mu^2 > 0$, $S_n = \sum_{i=1}^{n} \xi_i$. Then,*

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t \approx \phi(x),\ \forall x \in R, \tag{25}$$

where $\phi(x) = \left(1 + exp\left(\frac{-\pi x}{\sqrt{3}}\right)\right)^{-1}$.

Proof. Set $Y_n = \frac{S_n - n\mu}{\sqrt{n}\sigma}$. Then, by Theorem 1 (a), we have

$$\mathbb{P}\{\gamma : Y_n \leq x\} \leq \mathcal{M}\{\gamma : Y_n \leq x\}, \tag{26}$$

$$\mathbb{P}\{\gamma : Y_n > x\} \leq \mathcal{M}\{\gamma : Y_n > x\}. \tag{27}$$

If either (26) or (27) is a strict inequality, then (26) plus (27) implies $1 < 1$, which contradicts the facts. Therefore,

$$\mathbb{P}\{\gamma : Y_n \leq x\} = \mathcal{M}\{\gamma : Y_n \leq x\}.$$

From Theorem 1 (b) and (c), it follows that $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of independent random variables relative to $\mathbb{P}$, $\mu = E[\xi_k] = E_{\mathbb{P}}[\xi_k]$, and $\sigma^2 = E[\xi_k{}^2] - \mu^2 = E_{\mathbb{P}}[\xi_k^2] - \mu^2$, for any $k \in \mathbb{N}$.

Applying Lindeberg–Lévy's central limit theorem (see Theorem A5 in Appendix A), we have

$$\lim_{n\to\infty} \mathbb{P}\left\{\gamma : \frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t,$$

which implies,

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t.$$

□

Remark 3. *Note that*

$$\lim_{n\to\infty}\mathcal{M}\left\{\gamma:\frac{S_n-n\mu}{\sqrt{n}\sigma}\le x\right\}=\lim_{n\to\infty}\mathcal{M}\left\{\gamma:\frac{S_n-n\mu}{\sqrt{n}\sigma}\ge -x\right\},\ \forall x\in\mathbb{R}.$$

Hence, the asymptotic distribution of $\left\{\frac{S_n-n\mu}{\sqrt{n}\sigma}\right\}_{n=1}^{\infty}$ *is symmetrical (see, e.g., [45]).*

Theorem 6 (LIL)**.** *Let* $(\Gamma,\mathcal{L},\mathcal{M})$ *be an uncertainty space satisfying Assumption 1 and* $\{\xi_k\}_{k=1}^{\infty}$ *be a Bernoulli uncertain sequence relative to* $\mathcal{M}$*. Set* $E[\xi_k]=0$, $\sigma^2=E[\xi_k{}^2]>0$, $S_n=\sum_{i=1}^{n}\xi_i$, $\overline{S}_n=\frac{S_n}{\sqrt{2nloglogn}}$*. Then,*

(a)

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{p\to\infty}\lim_{k\to\infty}\mathcal{M}\left\{\gamma:\overline{S}_n\le\sigma+\frac{1}{m}-\frac{1}{p},\cdots,\overline{S}_{n+k}\le\sigma+\frac{1}{m}-\frac{1}{p}\right\}=1. \tag{28}$$

(b)

$$\lim_{n\to\infty}\lim_{p\to\infty}\lim_{k\to\infty}\mathcal{M}\left\{\left\{\gamma:\overline{S}_n>\sigma-\frac{1}{p}\right\}\bigcup\cdots\bigcup\left\{\gamma:\overline{S}_{n+k}>\sigma-\frac{1}{p}\right\}\right\}=1. \tag{29}$$

(c) Furthermore, if $\mathcal{M}$ *satisfies (i) and (ii) in Definition 8, then*

$$\mathcal{M}\left\{\gamma:\limsup_{n\to\infty}\overline{S}_n=\sigma\right\}=1. \tag{30}$$

Proof. (a) By Theorem 2 (b), we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{p\to\infty}\lim_{k\to\infty}\mathcal{M}\left\{\gamma:\overline{S}_n\le\sigma+\frac{1}{m}-\frac{1}{p},\cdots,\overline{S}_{n+k}\le\sigma+\frac{1}{m}-\frac{1}{p}\right\}$$
$$\ge\mathbb{P}\left\{\gamma:\limsup_{n\to\infty}\overline{S}_n\le\sigma\right\}.$$

According to Theorem 1 (b) and (c), we know that $\{\xi_k\}_{k=1}^{\infty}$ is a sequence of independent random variables relative to $\mathbb{P}$, $E[\xi_k]=E_{\mathbb{P}}[\xi_k]=0$, and $\sigma^2=E[\xi_k{}^2]=E_{\mathbb{P}}[\xi_k{}^2]$, for any $k\in\mathbb{N}$.

Applying Kolmogorov's law of the iterated logarithm (see Theorem A6 in Appendix A), we obtain

$$\mathbb{P}\left\{\gamma:\limsup_{n\to\infty}\overline{S}_n\le\sigma\right\}=1,$$

which implies,

$$\lim_{m\to\infty}\lim_{n\to\infty}\lim_{p\to\infty}\lim_{k\to\infty}\mathcal{M}\left\{\gamma:\overline{S}_n\le\sigma+\frac{1}{m}-\frac{1}{p},\cdots,\overline{S}_{n+k}\le\sigma+\frac{1}{m}-\frac{1}{p}\right\}=1.$$

Hence, the proof of (a) is completed.

From Theorem 2 (c) and using the similar method of the proof of Theorem 6 (a), we can prove Theorem 6 (b). So it is omitted.

(c) Combining (13), (14), and Kolmogorov's law of the iterated logarithm, it yields that

$$\mathcal{M}\left\{\gamma:\limsup_{n\to\infty}\overline{S}_n\le\sigma\right\}=1\quad\text{and}\quad\mathcal{M}\left\{\gamma:\limsup_{n\to\infty}\overline{S}_n\ge\sigma\right\}=1. \tag{31}$$

Now, we show that (30) $\Leftrightarrow$ (31). Since (30) $\Rightarrow$ (31) is obvious, we only need to prove that (31) $\Rightarrow$ (30). Let

$$\Lambda_1 = \left\{\gamma : \limsup_{n\to\infty} \overline{S}_n \le \sigma\right\},\ \Lambda_2 = \left\{\gamma : \limsup_{n\to\infty} \overline{S}_n \ge \sigma\right\}.$$

Then,

$$\Lambda_1 \bigcap \Lambda_2 = \left\{\gamma : \limsup_{n\to\infty} \overline{S}_n = \sigma\right\}.$$

With the similar proof of Theorem 3 (c), we have,

$$\mathcal{M}\left\{\gamma : \limsup_{n\to\infty} \overline{S}_n = \sigma\right\} = 1.$$

Hence, (c) is proved. □

Now, to better explain our main results, we give the following special case.

Example 1. *Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space satisfying Assumption 1 and $\mathcal{M}$ satisfies (i) and (ii) in Definition 8. Let $\{\xi_k\}_{k=1}^{\infty}$ be a Bernoulli uncertain sequence that takes values of 0 and 1. Suppose that $\mathcal{M}\{\xi_k = 0\} = p$, $\mathcal{M}\{\xi_k = 1\} = 1 - p = q$, $S_n = \sum_{i=1}^{n} \xi_i$. Then $\mu = q$, $\sigma^2 = q(1-q) > 0$, by Theorems 3–6, it follows that*
(a)

$$\mathcal{M}\left\{\gamma : \lim_{n\to\infty} \frac{S_n}{n} = q\right\} = 1. \tag{32}$$

(b) For any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \left|\frac{S_n}{n} - q\right| < \epsilon\right\} = 1. \tag{33}$$

(c)

$$\lim_{n\to\infty} \mathcal{M}\left\{\gamma : \frac{S_n - nq}{\sqrt{nq(1-q)}} \le x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} \mathrm{d}t \approx \phi(x),\ \forall x \in R. \tag{34}$$

(d) Set $\overline{S}_n = \dfrac{\sum_{i=1}^{n} (\xi_i - q)}{\sqrt{2nloglogn}}$. Then,

$$\mathcal{M}\left\{\gamma : \limsup_{n\to\infty} \overline{S}_n = \sqrt{q(1-q)}\right\} = 1. \tag{35}$$

4. Conclusions

Nowadays, uncertainty theory has developed rapidly in uncertain finance, uncertain statistics, uncertain calculus, uncertain risk analysis, and other fields. However, so far, very little attention has been paid to the limit theorems such as LLN, CLT, and LIL for uncertain variables. This paper has been the first attempt to establish an SLLN, a WLLN, a CLT, and an LIL for Bernoulli uncertain sequence. We have known that these limit theorems are well developed in probability theory, so we have naturally thought of obtaining those for uncertain variables by exploring the relationship between probability measure and uncertain measure. In this paper, we have proposed a new definition called weakly dependent, which can be regarded as a generalization of the independence of uncertain variables. Based on weakly dependent, a Bernoulli uncertain sequence has been introduced where the uncertain variables take a finite number of values. Besides this, we have introduced a new way to define the continuity of uncertain measure. After that, in explaining our main idea, Assumption 1 has been put forward, which is the premise of all theorems in this paper, and it has stated that, when the uncertainty space can be finitely partitioned, we can generalize the duality of the uncertain measure defined on this uncertainty space. In Theorem 1, we have discussed the relationship between probability measure and uncertain measure on σ-algebra generated by Bernoulli uncertain sequence. Then, the Bernoulli uncertain sequence has been proved to be a sequence of independent random variables under probability measure. Lastly, we have shown that the expected value of the Bernoulli uncertain sequence in the sense of uncertain measure is equal to its expected value in the sense of probability measure. Theorem 2, as an application of Theorem 1 (a), combined with the continuity of the probability measure, has yielded more specific results. It is worth noting that both Theorems 1 and 2 are essential tools to prove the main results (Theorems 3–6) of this paper. Theorem 3 has been established as an SLLN for Bernoulli uncertain sequence. In a special case where the uncertain measure satisfies continuity from below and continuity from above, the SLLN for Bernoulli uncertain sequence becomes Kolmogorov's SLLN. In addition, Theorems 4 and 5 have been presented as a WLLN and a CLT for Bernoulli uncertain sequence, respectively. Theorem 4 has the form corresponding to Khinchin's WLLN. Theorem 5 has the form corresponding to Lindeberg–Lévy's CLT. Finally, we have illustrated an LIL for Bernoulli uncertain sequence by Theorem 6. Particularly, when the uncertain measure satisfies continuity from below and continuity from above, the LIL for Bernoulli uncertain sequence becomes Kolmogorov's LIL. Future research may consider generalizing the limit theorems for Bernoulli uncertain sequence proved in this paper to those for general uncertain sequence. Although the relationship between probability measure and uncertain measure based on general uncertain sequence is difficult to handle, we will actively explore more and better ways to solve this problem.

Author Contributions: All authors have contributed their efforts jointly to this manuscript. Z.Q. drafted the manuscript and performed the proofs in this research. F.H. proposed the main idea and methodology of this paper. F.H. also supervised and guided the entire work. Z.Z. carried out a theoretical analysis of the proposed idea, reviewed the manuscript, and improved the final version. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported partly by the National Natural Science Foundation of China (11801307) and the Natural Science Foundation of Shandong Province of China (ZR2017MA012, ZR2021MA009).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

LLN	Law of Large Numbers
SLLN	Strong Law of Large Numbers
WLLN	Weak Law of Large Numbers
CLT	Central Limit Theorem
LIL	Law of the Iterated Logarithm

Appendix A

Definition A1 (see [46]). *Let (Γ, ρ) is a metric space, and $\mathcal{O}$ and $\mathcal{C}$ are the classes of all open and closed sets in (Γ, ρ), respectively, and $\mathcal{L}$ is the Borel σ-algebra on Γ, i.e., it is the smallest σ-algebra containing $\mathcal{O}$. A measure μ on (Γ, ρ) is called regular, if for any $\Lambda \in \mathcal{L}$ and $\delta > 0$, there exists a closed set F_δ and an open set G_δ of Γ, such that $F_\delta \subset \Lambda \subset G_\delta$ and $\mu\{G_\delta - F_\delta\} < \delta$.*

Theorem A1 (Alexandroff Theorem, see [47]). *Let μ be a regular finitely additive measure defined on algebra Σ of subsets of a compact topological space Γ. Thus, μ is countably additive.*

Theorem A2 (Caratheodory's Extension Theorem, see [48]). *Let $\mathcal{A} \subset 2^\Omega$ be an algebra and $\mathbb{P}$ be a probability measure on $\mathcal{A}$. There exists a unique measure $\mathbb{P}_1$ on $\sigma(\mathcal{A})$ such that $\mathbb{P}_1\{\Lambda\} = \mathbb{P}\{\Lambda\}$ for all $\Lambda \in \mathcal{A}$.*

Theorem A3 (Kolmogorov's SLLN, see [49]). *Let $(\Gamma, \mathcal{L}, \mathbb{P})$ be a probability space and $\xi_1, \cdots, \xi_n, \cdots$ be i.i.d.random variables with a finite expected value $E_{\mathbb{P}}[\xi_1] = \mu \in \mathbb{R}$. Set $S_n = \sum_{i=1}^{n} \xi_i$. Then, $\mathbb{P}\left\{\lim_{n\to\infty} \frac{S_n}{n} = \mu\right\} = 1$.*

Theorem A4 (Khinchin's WLLN, see [49]). *Let $(\Gamma, \mathcal{L}, \mathbb{P})$ be a probability space and $\xi_1, \cdots, \xi_n, \cdots$ be i.i.d.random variables with a finite expected value $E_{\mathbb{P}}[\xi_1] = \mu \in \mathbb{R}$. Set $S_n = \sum_{i=1}^{n} \xi_i$. Then, for any $\varepsilon > 0$, $\lim_{n\to\infty} \mathbb{P}\left\{\mid \frac{S_n}{n} - \mu \mid < \epsilon\right\} = 1$.*

Theorem A5 (Lindeberg–Lévy's CLT, see [49]). *Let $(\Gamma, \mathcal{L}, \mathbb{P})$ be a probability space and $\xi_1, \cdots, \xi_n, \cdots$ be i.i.d.random variables with a finite expected value and variance. Set $E_{\mathbb{P}}[\xi_1] = \mu \in \mathbb{R}$, $\mathcal{D}_{\mathbb{P}}[\xi_1] = E_{\mathbb{P}}[{\xi_1}^2] - \mu^2 = \sigma^2 > 0$, $S_n = \sum_{i=1}^{n} \xi_i$. Then,*

$$\lim_{n\to\infty} \mathbb{P}\{\frac{S_n - n\mu}{\sqrt{n}\sigma} \le x\} = \phi(x),$$

where $\phi(x)$ is the distribution function of the standard normal distribution.

Theorem A6 (Kolmogorov's LIL, see [49]). *Let $(\Gamma, \mathcal{L}, \mathbb{P})$ be a probability space and $\xi_1, \cdots, \xi_n, \cdots$ be i.i.d.random variables with mean zero and finite variance. Set $\mathcal{D}_{\mathbb{P}}[\xi_1] = E_{\mathbb{P}}[{\xi_1}^2] = \sigma^2 > 0$, $S_n = \sum_{i=1}^{n} \xi_i$, $\overline{S}_n = \frac{S_n}{\sqrt{2n \log\log n}}$. Then,*

$$\mathbb{P}\left\{\limsup_{n\to\infty} \overline{S}_n = \sigma\right\} = 1,$$

where "log" is the natural logarithm.

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