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Intuitionistic Fuzzy Topology Based on Intuitionistic Fuzzy Logic

Osama R. Sayed [1,*,†], Ayman A. Aly [2,†] and Shaoyu Zhang [3]

1 Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt
2 Department of Mechanical Engineering, College of Engineering, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia
3 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 102488, China
* Correspondence: o_sayed@aun.edu.eg
† These authors contributed equally to this work.

Abstract: There are many symmetries in intuitionistic fuzzifying topology. In the present paper, the notion of intuitionistic fuzzifying topology as an extension of fuzzifying topology and a preliminary of the research on bi-intuitionistic fuzzy topology is introduced. A theory of intuitionistic fuzzy topology with the semantic method of intuitionistic fuzzy logic is established.

Keywords: intuitionistic fuzzy logic; topology; neighborhood structure; interior; closure; derived set; boundary

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1. Introduction

Symmetry takes place not only in geometry but also in other branches of mathematics. In particular, there are many symmetries in groups, rings, fields, lattices, fuzzy logic, topology, and fuzzy topology. In fuzzy topology, the intuitionistic fuzzifying topology is one of the most basic notions. Intuitionism is a very important mathematical property, and it exists in many research areas, such as vector space, metric space, and partially ordered sets. By abstracting the common properties of intuitionistic fuzzy sets, the intuitionistic fuzzy structures were proposed. We already see that serious research on this topic is absolutely necessary, even in hindsight.

Fuzzy sets were introduced by L. A. Zadeh [1] in 1965. In 1983, Atanassov proposed the concept of intuitionistic fuzzy set [2]. Some basic results on intuitionistic fuzzy sets were published in [3,4], and the book [5] offers a comprehensive study and applications of intuitionistic fuzzy sets. In fact, intuitionistic fuzzy sets are a topic of research by many scholars [6,7]. In particular, elements of intuitionistic fuzzy logic and, in the area of applications, were studied by Atanassov and co-workers (see [5,8,9]). According to Ref. [10], the kind of topologies defined by Chang [11] and Goguen [12] is called the topologies of fuzzy subset, and is called L-topological spaces if a lattice L of membership values has been chosen. From another side, Höhle in [13] proposed the term L-fuzzy topology to be a map from $\mathcal{P}(\mathcal{X})$ to $\mathcal{X}$. The authors in [14,15] defined an L-fuzzy topology to be a map from $L^{\mathcal{X}}$ of $\mathcal{X}$. Çoker [16,17] established the intuitionistic fuzzy topological space. The intuitionistic fuzzy topology is a family τ of intuitionistic fuzzy subsets of a non-empty set $\mathcal{X}$, and τ satisfies the basic conditions of classical topologies [18]. Ying, in 1991–1993 [19–21], used a semantical method of multi-valued logic to develop methodically fuzzifying topology. Now, we suggest the following problems: What are the details of the many-valued theories beyond the level of intuitionistic predicates calculus? To provide a partial answer to this issue in topology, we use a semantical method of intuitionistic fuzzy logic to develop methodically intuitionistic fuzzifying topology. Intuitionistic fuzzifying topology is an extension of fuzzifying topology and introductory research on bi-intuitionistic fuzzy topology. The rest of this paper is organized as follows. The next section contains necessary concepts and properties. Section 3 is exclusively devoted to intuitionistic fuzzy logics. In Section 4, the concept of bi-intuitionistic fuzzy topological spaces is defined and some examples are introduced. In Section 5, in the intuitionistic fuzzifying topology, we discuss the neighborhood system of a point. In Section 6, we introduce the concepts of intuitionistic fuzzifying closure, intuitionistic fuzzifying boundary, intuitionistic fuzzifying derived set, and intuitionistic fuzzifying interior. The goal of the last section is to conclude this paper with a succinct but precise recapitulation of our main findings, and to give some lines for future research.

2. Preliminaries

First, we recall some necessary notations that will be used throughout the whole paper.

Definition 1 ([2])**.** *Let $\mathcal{X}$ be a non-empty set. An intuitionistic fuzzy set ($\mathcal{IF}$ set for short) $\hat{\mathcal{A}}$, is an object having the form $(m\hat{\mathcal{A}}, n\hat{\mathcal{A}})$, where the functions $m\hat{\mathcal{A}} : \mathcal{X} \to [0,1]$ and $n\hat{\mathcal{A}} : \mathcal{X} \to [0,1]$ denote the degree of membership (namely $m\hat{\mathcal{A}}(x)$) and the degree of non-membership (namely $n\hat{\mathcal{A}}(x)$) of each element $x \in \mathcal{X}$ to the set $\hat{\mathcal{A}}$, respectively, and $0 \leq m\hat{\mathcal{A}}(x) + n\hat{\mathcal{A}}(x) \leq 1$ for each $x \in \mathcal{X}$. The set of all intuitionistic fuzzy set of $\mathcal{X}$ is denoted by $\mathcal{IF}(\mathcal{X})$. Note that an intuitionistic fuzzy set becomes a fuzzy set when we dispense with non-membership.*

The next definitions present some basic set-theoretic operations for intuitionistic fuzzy sets.

Definition 2 ([4])**.** *Let $\mathcal{X}$ be a non-empty set, and $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{X})$. Then*
(a) $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}$ if and only if $m\hat{\mathcal{A}}(x) \leq m\hat{\mathcal{B}}(x)$ and $n\hat{\mathcal{A}}(x) \geq n\hat{\mathcal{B}}(x)$ for all $x \in \mathcal{X}$;
(b) $\hat{\mathcal{A}} = \hat{\mathcal{B}}$ if and only if $\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}$ and $\hat{\mathcal{B}} \subseteq \hat{\mathcal{A}}$;
(c) $\neg\hat{\mathcal{A}} = (n\hat{\mathcal{A}}, m\hat{\mathcal{A}})$;
(d) $\hat{\mathcal{A}} \cap \hat{\mathcal{B}} = (m\hat{\mathcal{A}} \wedge m\hat{\mathcal{B}}, n\hat{\mathcal{A}} \vee n\hat{\mathcal{B}}))$;
(e) $\hat{\mathcal{A}} \cup \hat{\mathcal{B}} = (m\hat{\mathcal{A}} \vee m\hat{\mathcal{B}}, n\hat{\mathcal{A}} \wedge n\hat{\mathcal{B}})$.

Definition 3 ([16])**.** *Let $\{\hat{\mathcal{A}}_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{IF}(\mathcal{X})$. Then*
(a) $\bigcap_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda = \left(\bigwedge_{\lambda\in\Lambda} m\hat{\mathcal{A}}_\lambda(x), \bigvee_{\lambda\in\Lambda} n\hat{\mathcal{A}}_\lambda(x)\right)$;
(b) $\bigcup_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda = \left(\bigvee_{\lambda\in\Lambda} m\hat{\mathcal{A}}_\lambda(x), \bigwedge_{\lambda\in\Lambda} n\hat{\mathcal{A}}_\lambda(x)\right)$;
(c) $\hat{1} = (1,0)$ and $\hat{0} = (0,1)$.

Remark 1. *(1) If $\mathcal{A} \in \mathcal{P}(\mathcal{X})$, then $\mathcal{A}$ is identified with an $\mathcal{IF}$ set $\hat{\mathcal{A}}$ of $\mathcal{X}$ such that $m\hat{\mathcal{A}} = \mathcal{A}$ and $n\hat{\mathcal{A}} = \mathcal{X} - \mathcal{A}$;*
(2) If $\tilde{\mathcal{A}} \in \Im(\mathcal{X})$ ($\Im(\mathcal{X})$ is the set of all fuzzy subsets of $\mathcal{X}$), then $\tilde{\mathcal{A}}$ is identified with an $\mathcal{IF}$ set $\hat{\mathcal{A}}$ of $\mathcal{X}$ such that $m\hat{\mathcal{A}} = \tilde{\mathcal{A}}$ and $n\hat{\mathcal{A}} = 1 - n\tilde{\mathcal{A}}$ [2];
(3) $\hat{\mathcal{A}} \in \mathcal{IF}(\mathcal{X})$ is called normal if and only if there exists $x \in \mathcal{X}$ such that $\hat{\mathcal{A}}(x) = (1,0)$. The set of all normal $\mathcal{IF}$ sets of $\mathcal{X}$ will be denoted by $\mathcal{IF}^{\mathcal{N}}(\mathcal{X})$;
(4) The dual complement of $\hat{\mathcal{A}}$, denoted by $\neg\hat{\mathcal{A}}$, is defined as follows:

$$\neg\hat{\mathcal{A}}(x) = \neg(\hat{\mathcal{A}}(x)) = (n\hat{\mathcal{A}}(x), m\hat{\mathcal{A}}(x)).$$

3. Intuitionistic Fuzzy Logic

Now, we give the intuitionistic fuzzy logical and corresponding intuitionistic fuzzy set-theoretical notions. Let

$$\mathcal{Z} = \{(\alpha,\beta)|(\alpha,\beta) \in [0,1] \times [0,1], \alpha + \beta \leq 1\}.$$

By an intuitionistic fuzzy well formed formula φ we mean any proposition with respect to which we assign an element $(a,b) \in \mathcal{Z}$. We say that $a = m(\varphi)$ and $b = n(\varphi)$ and called them the truth degree and the falsity degree of φ, respectively. Let $\mathcal{M}$ denote the set of all intuitionistic fuzzy well formed formula and consider the function $\mathcal{V} : \mathcal{M} \longrightarrow \mathcal{Z}$. If $\varphi \in \mathcal{M}$ and $\mathcal{V}(\varphi) = (a, b)$ we write $[\varphi] = (a, b)$. We say that φ is valid (or a tautology) and we type $\models \varphi$ if $[\varphi] = (1, 0)$.

Definition 4. *(1) The binary relation "$=$", "$<$", "$\leq$", "$\wedge$" and "$\vee$" and the unary operation "$\neg$" on $\mathcal{Z}$ are defined as:*

(a) $(\alpha, \beta) = (\gamma, \delta)$ *if and only if* $\alpha = \gamma$ *and* $\beta = \delta$;
(b) $(\alpha, \beta) \leq (\gamma, \delta)$ *if and only if* $\alpha \leq \gamma$ *and* $\beta \geq \delta$;
(c) $(\alpha, \beta) < (\gamma, \delta)$ *if and only if* $\alpha < \gamma$ *and* $\beta > \delta$;
(d) $[(\alpha, \beta) \wedge (\gamma, \delta)] := (\min\{\alpha, \gamma\}, \max\{\beta, \delta\})$;
(e) $[(\alpha, \beta) \vee (\gamma, \delta)] := (\max\{\alpha, \gamma\}, \min\{\beta, \delta\})$;
(f) $[\neg(\alpha, \beta)] := (\beta, \alpha)$.
(2) (a) The intuitionistic fuzzy implications "$\rightarrow, \dot{\rightarrow}$" $\in \mathcal{Z}^{\mathcal{Z}\times\mathcal{Z}}$ are defined as follows:

$$[(\alpha, \beta) \rightarrow (\gamma, \delta)] = (\min(1, 1 - \alpha + \gamma, 1 - \delta + \beta), \max(0, \alpha - \gamma, \delta - \beta)),$$

$$[(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)] = (\min(1, \beta + \gamma), \max(0, \alpha + \delta - 1));$$

(b) For each $\alpha \in \mathcal{Z}^{\mathcal{X}}$ *(i.e.,* $\alpha : \mathcal{X} \rightarrow \mathcal{Z}$*), we have*

$$[\forall x \alpha(x)] := \bigwedge_{x \in \mathcal{X}} \alpha(x) := \left(\bigwedge_{x \in \mathcal{X}} m\alpha(x), \bigvee_{x \in \mathcal{X}} n\alpha(x) \right).$$

For example, if $\alpha : \mathcal{X} \rightarrow \mathcal{Z}$ *and* $\mathcal{X} = \{a, b, c\}$*, where* $\alpha(a) = (0.3, 0.6)$*,* $\alpha(b) = (0.4, 0.3)$ *and* $\alpha(c) = (0.5, 0.1)$*, then* $[\forall x \alpha(x)] = (0.3, 0.6)$*.*

(3) (a)

$$[(\alpha, \beta) \curlywedge (\gamma, \delta)] := [\neg((\alpha, \beta) \rightarrow \neg(\gamma, \delta))],$$

$$[(\alpha, \beta) \bar{\barwedge} (\gamma, \delta)] := [\neg((\alpha, \beta) \dot{\rightarrow} \neg(\gamma, \delta))];$$

(b)

$$[(\alpha, \beta) \curlyvee (\gamma, \delta)] := [\neg(\alpha, \beta) \rightarrow (\gamma, \delta)],$$

$$[(\alpha, \beta) \veebar (\gamma, \delta)] := [\neg(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)];$$

(4) (a)

$$[(\alpha, \beta) \longleftrightarrow (\gamma, \delta)] := [((\alpha, \beta) \rightarrow (\gamma, \delta)) \wedge ((\gamma, \delta) \rightarrow (\alpha, \beta))],$$

$$[(\alpha, \beta) \dot{\longleftrightarrow} (\gamma, \delta)] := [((\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)) \wedge ((\gamma, \delta) \dot{\rightarrow} (\alpha, \beta))].$$

For example, if $(\alpha, \beta) = (0.6, 0.3)$ *and* $(\gamma, \delta) = (0.5, 0.4)$*, then*
$[(\alpha, \beta) \rightarrow (\gamma, \delta)] = (0.9, 0.1)$ *and* $[(\gamma, \delta) \rightarrow (\alpha, \beta)] = (1, 0)$*.*
Therefore $[(\alpha, \beta) \longleftrightarrow (\gamma, \delta)] = (0.9, 0.1) \wedge (1, 0) = (0.9, 0.1)$*.*
Similarly, $[(\alpha, \beta) \dot{\longleftrightarrow} (\gamma, \delta)] = (0.9, 0)$*.*
(b) $[\exists x \alpha(x)] := [\neg(\forall x \neg(\alpha(x)))]$;
(5) Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{X})$*,*
(a) $[x \in \hat{\mathcal{A}}] := [\hat{\mathcal{A}}(x)]$;
The intuitionistic fuzzy inclusions "$\subseteq$ and $\Subset$", and the intuitionistic fuzzy equalities "$\equiv, \approx, \doteq$ and $\dot{\approx}$" between two intuitionistic fuzzy sets are defined as follows:
(b) $[\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}] := [\forall x (x \in \hat{\mathcal{A}} \rightarrow x \in \hat{\mathcal{B}})]$;
(c) $[\hat{\mathcal{A}} \equiv \hat{\mathcal{B}}] := [(\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}) \wedge (\hat{\mathcal{B}} \subseteq \hat{\mathcal{A}})]$;
(d) $[\hat{\mathcal{A}} \approx \hat{\mathcal{B}}] := [(\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}) \curlywedge (\hat{\mathcal{B}} \subseteq \hat{\mathcal{A}})]$;
(e) $[\hat{\mathcal{A}} \Subset \hat{\mathcal{B}}] := [\forall x (x \in \hat{\mathcal{A}} \dot{\rightarrow} x \in \hat{\mathcal{B}})]$;
(f) $[\hat{\mathcal{A}} \doteq \hat{\mathcal{B}}] := [(\hat{\mathcal{A}} \Subset \hat{\mathcal{B}}) \wedge (\hat{\mathcal{B}} \Subset \hat{\mathcal{A}})]$;
(g) $[\hat{\mathcal{A}} \dot{\approx} \hat{\mathcal{B}}] := [(\hat{\mathcal{A}} \Subset \hat{\mathcal{B}}) \bar{\barwedge} (\hat{\mathcal{B}} \Subset \hat{\mathcal{A}})]$.
(6) Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ *be a function,* $\hat{\mathcal{A}} \in \mathcal{IF}(\mathcal{X})$ *and* $\hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{Y})$*.*

The image $f(\hat{\mathcal{A}})$ of $\hat{\mathcal{A}}$ under f is an intuitionistic fuzzy set in $\mathcal{Y}$ defined by:

$$f(\hat{\mathcal{A}})(y) = (f(m\hat{\mathcal{A}})(y), 1 - f(1 - n\hat{\mathcal{A}})(y))\ \forall y \in \mathcal{Y}.$$

The inverse image $f^{-1}(\hat{\mathcal{B}})$ of $\hat{\mathcal{B}}$ under f is an intuitionistic fuzzy set in $\mathcal{X}$ defined by:

$$f^{-1}(\hat{\mathcal{B}}) = (f^{-1}(m\hat{\mathcal{B}})(x), f^{-1}(n\hat{\mathcal{B}})(x))\ \forall x \in \mathcal{X}.$$

Theorem 1. *$(\mathcal{Z}, \leq, \wedge, \vee, \neg)$ is a complete completely distributive lattice with least element $(0,1)$ and greatest element $(1,0)$, this is equipped with an order reversing involution "$\neg$".*

Remark 2. *(1) The completely distributive law in $\mathcal{Z}$ is of the form:*

$$\bigwedge_{\lambda \in \Lambda} \bigvee_{(\alpha,\beta) \in \wp_\lambda} (\alpha, \beta) = \bigvee_{f \in \underset{\lambda \in \Lambda}{\pi} \wp_\lambda} \bigwedge_{\lambda \in \Lambda} f(\lambda),$$

where $\wp_\lambda \subseteq \mathcal{Z}$ for all $\lambda \in \Lambda$.

(2) It is clear that $(\mathcal{Z}, \leq)$ is not totally ordered set (Indeed, $(\frac{1}{4}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}) \in \mathcal{Z}$, but $(\frac{1}{4}, \frac{1}{3})$ is not comparable with $(\frac{1}{2}, \frac{1}{2})$ by the relation "$\leq$".

Theorem 2. *Let $(\alpha, \beta), (\gamma, \delta) \in \mathcal{Z}$. Then*

(1) $[\neg(\alpha, \beta)] = [(\alpha, \beta) \dot{\rightarrow} (0,1)]$;

(2) $[(\alpha, \beta) \curlywedge (\gamma, \delta)] = (\max(0, \alpha - \delta, \gamma - \beta), \min(1, 1 - \alpha + \delta, 1 - \gamma + \beta))$

(3) $[(\alpha, \beta) \curlyvee (\gamma, \delta)] = (\min(1, 1 - \beta + \gamma, 1 - \delta + \alpha), \max(0, \beta - \gamma, \delta - \alpha))$

(4) $[(\alpha, \beta) \barwedge (\gamma, \delta)] = (\max(0, \alpha + \gamma - 1), \min(1, \beta + \delta))$;

(5) $[(\alpha, \beta) \veebar (\gamma, \delta)] = (\min(1, \alpha + \gamma), \max(0, \beta + \delta - 1))$;

(6) $[(\alpha, \beta) \rightarrow (\gamma, \delta)] = [\neg(\alpha, \beta) \curlyvee (\gamma, \delta)]$;

(7) $[(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)] = [\neg(\alpha, \beta) \veebar (\gamma, \delta)]$;

(8) $[\neg((\alpha, \beta) \wedge (\gamma, \delta))] = [\neg(\alpha, \beta) \vee \neg(\gamma, \delta)]$;

(9) $[\neg((\alpha, \beta) \vee (\gamma, \delta))] = [\neg(\alpha, \beta) \wedge \neg(\gamma, \delta)]$;

(10) $[(\alpha, \beta) \curlyvee (\gamma, \delta)] = [\neg(\neg(\alpha, \beta) \curlywedge \neg(\gamma, \delta))]$;

(11) $[(\alpha, \beta) \veebar (\gamma, \delta)] = [\neg(\neg(\alpha, \beta) \barwedge \neg(\gamma, \delta))]$.

Proof. (1) $[(\alpha, \beta) \dot{\longrightarrow} (0,1)] = (\min(1, \beta + 0), \max(0, \alpha + 1 - 1)) = (\beta, \alpha) = [\neg(\alpha, \beta)]$;

(2) $[(\alpha, \beta) \curlywedge (\gamma, \delta)] = [\neg((\alpha, \beta) \longrightarrow \neg(\gamma, \delta))] = [\neg((\alpha, \beta) \longrightarrow (\delta, \gamma))]$
$= \neg(\min(1, 1 - \alpha + \delta, 1 - \gamma + \beta), \max(0, \alpha - \delta, \gamma - \beta))$
$= (\max(0, \alpha - \delta, \gamma - \beta), \min(1, 1 - \alpha + \delta, 1 - \gamma + \beta))$;

(3) $[(\alpha, \beta) \curlyvee (\gamma, \delta)] = [\neg(\neg(\alpha, \beta) \curlywedge \neg(\gamma, \delta))] = [\neg((\beta, \alpha) \curlywedge (\delta, \gamma))]$
$= [\neg(\neg((\beta, \alpha) \longrightarrow \neg(\delta, \gamma)))] = [(\beta, \alpha) \longrightarrow (\gamma, \delta)]$
$= (\min(1, 1 - \beta + \gamma, 1 - \delta + \alpha), \max(0, \beta - \gamma, \delta - \alpha))$;

(4) $[(\alpha, \beta) \barwedge (\gamma, \delta)] = [\neg((\alpha, \beta) \dot{\longrightarrow} \neg(\gamma, \delta))] = [\neg((\alpha, \beta) \dot{\longrightarrow} (\delta, \gamma))]$
$= \neg(\min(1, \beta + \delta), \max(0, \alpha + \gamma - 1)) = (\max(0, \alpha + \gamma - 1), \min(1, \beta + \delta))$;

(5) $[(\alpha, \beta) \veebar (\gamma, \delta)] = [\neg(\alpha, \beta) \dot{\longrightarrow} (\gamma, \delta)] = [(\beta, \alpha) \dot{\longrightarrow} (\gamma, \delta)] = (\min(1, \alpha + \gamma), \max(0, \beta + \delta - 1))$;

(6) $[\neg(\alpha, \beta) \curlyvee (\gamma, \delta)] = [\neg(\neg(\alpha, \beta)) \longrightarrow (\gamma, \delta)] = [(\alpha, \beta) \longrightarrow (\gamma, \delta)]$;

(7) $[\neg(\alpha, \beta) \veebar (\gamma, \delta)] = [\neg(\neg(\alpha, \beta)) \dot{\longrightarrow} (\gamma, \delta)] = [(\alpha, \beta) \dot{\longrightarrow} (\gamma, \delta)]$;

(8) $[\neg((\alpha, \beta) \wedge (\gamma, \delta))] = \neg(\min(\alpha, \gamma), \max(\beta, \delta)) = (\max(\beta, \delta), \min(\alpha, \gamma)) = [(\beta, \alpha) \vee (\delta, \gamma)] = [\neg(\alpha, \beta) \vee \neg(\gamma, \delta)]$;

(9) The proof is similar to that of (8);

(10) $[\neg(\neg(\alpha, \beta) \curlywedge \neg(\gamma, \delta))] = [\neg((\beta, \alpha) \curlywedge (\delta, \gamma))] = \neg(\max(0, \beta - \gamma, \delta - \alpha), \min(1, 1 - \beta + \gamma, 1 - \delta + \alpha)) = (\min(1, 1 - \beta + \gamma, 1 - \delta + \alpha), \max(0, \beta - \gamma, \delta - \alpha)) = [(\alpha, \beta) \curlyvee (\gamma, \delta)]$;

(11) $[\neg(\neg(\alpha, \beta) \barwedge \neg(\gamma, \delta))] = [\neg((\beta, \alpha) \barwedge (\delta, \gamma))] = \neg(\max(0, \beta + \delta - 1), \min(1, \alpha + \gamma)) = (\min(1, \alpha + \gamma), \max(0, \beta + \delta - 1)) = [(\alpha, \beta) \veebar (\gamma, \delta)]$. □

Theorem 3. *(1) $\barwedge$ and $\curlywedge$ are isotone functions;*

(2) $(\mathcal{Z}, \barwedge, (1,0))$ is a commutative monoid;

(3) $(\mathcal{Z}, \curlywedge)$ is a commutative semi-group such that $[(\alpha,\beta) \curlywedge (1,0)] = [(1,0) \curlywedge (\alpha,\beta)] = (1-\beta,\beta)$;

(4) $\barwedge$ is distributive over arbitrary joins, i.e.,

$$(\alpha,\beta) \barwedge \bigvee_{j\in\Lambda} (\gamma_j,\delta_j) = \bigvee_{j\in\Lambda} ((\alpha,\beta) \barwedge (\gamma_j,\delta_j)),$$

for every $(\alpha,\beta) \in \mathcal{Z}, \{(\gamma_j,\delta_j) : j \in \Lambda\} \subseteq \mathcal{Z}$.

Proof. (1) To prove that $\barwedge$ is isotone function, suppose that $(\alpha,\beta) \leq (\alpha_1,\beta_1)$ and $(\gamma,\delta) \leq (\gamma_1,\delta_1)$. Then $\alpha \leq \alpha_1$, $\beta \geq \beta_1$, $\gamma \leq \gamma_1$ and $\delta \geq \delta_1$. Hence, $\alpha+\gamma-1 \leq \alpha_1+\gamma_1-1$ and $\beta+\delta \geq \beta_1+\delta_1$. Therefore,

$[(\alpha,\beta) \barwedge (\gamma,\delta)] = (\max(0,\alpha+\gamma-1), \min(1,\beta+\delta))$
$\leq (\max(0,\alpha_1+\gamma_1-1), \min(1,\beta_1+\delta_1)) = [(\alpha_1,\beta_1) \barwedge (\gamma_1,\delta_1)]$.

Similarly, from Theorem 2 (2) $\curlywedge$ is isotone function.

(2) To prove the commutative law, suppose that $(\alpha,\beta),(\gamma,\delta) \in \mathcal{Z}$.
$[(\alpha,\beta) \barwedge (\gamma,\delta)] = (\max(0,\alpha+\gamma-1), \min(1,\beta+\delta))$
$= (\max(0,\gamma+\alpha-1), \min(1,\delta+\beta)) = [(\gamma,\delta) \barwedge (\alpha,\beta)]$.

Now, we want to prove $\barwedge$ is associative. Therefore, we prove that $[((\alpha,\beta) \barwedge (\gamma,\delta)) \barwedge (\lambda,\xi)] = [(\alpha,\beta) \barwedge ((\gamma,\delta) \barwedge (\lambda,\xi))]$ i.e.,

$(\max(0,\max(0,\alpha+\gamma-1)+\lambda-1), \min(1,\min(1,\beta+\delta)+\xi)) = (\max(0,\alpha+\max(0,\gamma+\lambda-1)-1), \min(1,\beta+\min(1,\delta+\xi)))$. To prove that,

$L.H.S = \max(0,\max(0,\alpha+\gamma-1)+\lambda-1) = \max(0,\alpha+\max(0,\gamma+\lambda-1)-1) = R.H.S$.

First, we prove that $L.H.S \geq R.H.S$. If $R.H.S = 0$, then the result holds.

If $\max(0,\gamma+\lambda-1) = 0$, then $L.H.S \geq \max(0,\alpha-1) = R.H.S$. If $\max(0,\gamma+\lambda-1) = \gamma+\lambda-1$, then $R.H.S = \alpha+\gamma+\lambda-2$. Now, If $\alpha+\gamma \geq 1$, then $L.H.S = \max(0,\alpha+\gamma+\lambda-2) \geq R.H.S$. If $\alpha+\gamma \leq 1$, then $L.H.S = 0 = R.H.S$. Thus $L.H.S \geq R.H.S$.

Second, we prove that $L.H.S \leq R.H.S$. If $\gamma+\lambda-1 \geq 0$ and $\alpha+\gamma-1 \geq 0$, then $R.H.S = \max(0,\alpha+\gamma+\lambda-2) \geq \alpha+\gamma+\lambda-2 = L.H.S$. If $\gamma+\lambda-1 < 0$ and $\alpha+\gamma-1 \geq 0$, then $R.H.S = 0 = \max(0,\gamma+\lambda-1+\alpha-1) = L.H.S$. If $\gamma+\lambda-1 < 0$ and $\alpha+\gamma-1 < 0$, then $R.H.S = 0 = L.H.S$. Hence $L.H.S \leq R.H.S$. Therefore, $L.H.S = R.H.S$.

Similarly, $\min(1,\min(1,\beta+\delta)+\xi) = \min(1,\beta+\min(1,\delta+\xi))$. Therefore, the associative law holds.

Now, $[(\alpha,\beta) \barwedge (1,0)] = (\max(0,\alpha+1-1), \min(1,\beta+0)) = (\alpha,\beta) = [(1,0) \barwedge (\alpha,\beta)]$. Hence $(\mathcal{Z}, \barwedge, (1,0))$ is a commutative moniod.

(3) The proof is similar to the proof of (2);

(4) $[(\alpha,\beta) \barwedge \bigvee_{j\in\Lambda} (\gamma_j,\delta_j)] = [(\alpha,\beta) \barwedge (\bigvee_{j\in\Lambda} \gamma_j, \bigwedge_{j\in\Lambda} \delta_j)]$
$= (\max(0, \bigvee_{j\in\Lambda} \gamma_j+\alpha-1), \min(1, \bigwedge_{j\in\Lambda} \delta_j+\beta))$
$= (\bigvee_{j\in\Lambda} \max(0,\gamma_j+\alpha-1), \bigwedge_{j\in\Lambda} \min(1,\delta_j+\beta))$
$= \bigvee_{j\in\Lambda} (\max(0,\gamma_j+\alpha-1), \min(1,\delta_j+\beta))$
$= [\bigvee_{j\in\Lambda} ((\alpha,\beta) \barwedge (\gamma_j,\delta_j))]$. □

Theorem 4. *(1) If $(\alpha,\beta) \leq (\alpha_1,\beta_1)$, then $[(\alpha_1,\beta_1) \rightarrow (\gamma,\delta)] \leq [(\alpha,\beta) \rightarrow (\gamma,\delta)]$, and $[(\alpha_1,\beta_1) \dot{\rightarrow} (\gamma,\delta)] \leq [(\alpha,\beta) \dot{\rightarrow} (\gamma,\delta)]$.*

(2) If $(\gamma,\delta) \leq (\gamma_1,\delta_1)$, then $[(\alpha,\beta) \rightarrow (\gamma,\delta)] \leq [(\alpha,\beta) \rightarrow (\gamma_1,\delta_1)]$, and $[(\alpha,\beta) \dot{\rightarrow} (\gamma,\delta)] \leq [(\alpha,\beta) \dot{\rightarrow} (\gamma_1,\delta_1)]$;

(3) (a) $[(0,1) \rightarrow (\gamma,\delta)] = (1,0)$, $[(0,1) \dot{\rightarrow} (\gamma,\delta)] = (1,0)$, $[(1,0) \rightarrow (\gamma,\delta)] = (\gamma,1-\gamma)$, and $[(1,0) \dot{\rightarrow} (\gamma,\delta)] = (\gamma,\delta)$.

(b) $[(\alpha,\beta) \rightarrow (1,0)] = (1,0)$, $[(\alpha,\beta) \dot{\rightarrow} (1,0)] = (1,0)$, $[(\alpha,\beta) \rightarrow (0,1)] = (\beta,1-\beta)$, and $[(\alpha,\beta) \dot{\rightarrow} (0,1)] = \neg(\alpha,\beta)$.

(c) $[(\alpha,\beta) \barwedge (1,0)] = (\alpha,\beta)$, and $[(\alpha,\beta) \barwedge (0,1)] = (0,1)$;

(4) (a) $(\alpha, \beta) \leq (\gamma, \delta)$ *if and only if* $[(\alpha, \beta) \rightarrow (\gamma, \delta)] = (1, 0)$
(b) $\beta + \gamma \geq 1$ *if and only if* $[(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)] = (1, 0)$;
(5) (a) $[(\alpha, \beta) \rightarrow (\gamma, \delta)] = [\neg(\gamma, \delta) \rightarrow \neg(\alpha, \beta)]$;
(b) $[(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)] = [\neg(\gamma, \delta) \dot{\rightarrow} \neg(\alpha, \beta)]$;
(6) $(\lambda, \xi) \leq [(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)]$ *if and only if* $[(\lambda, \xi) \,\bar{\wedge}\, (\alpha, \beta)] \leq (\gamma, \delta)$;
(7)

$$[(\gamma, \delta) \rightarrow (\lambda, \xi)] \leq [((\alpha, \beta) \curlywedge (\gamma, \delta)) \rightarrow ((\alpha, \beta) \curlywedge (\lambda, \xi))];$$

(8)

$$[(\alpha, \beta) \rightarrow (\gamma, \delta)] \leq [((\lambda, \xi) \rightarrow (\alpha, \beta)) \rightarrow ((\lambda, \xi) \rightarrow (\gamma, \delta))];$$

$$[(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)] \leq [((\lambda, \xi) \dot{\rightarrow} (\alpha, \beta)) \dot{\rightarrow} ((\lambda, \xi) \dot{\rightarrow} (\gamma, \delta))];$$

(9)

$$[(\alpha, \beta) \rightarrow (\gamma, \delta)] \leq [((\gamma, \delta) \rightarrow (\lambda, \xi)) \rightarrow ((\alpha, \beta) \rightarrow (\lambda, \xi))];$$

(10)

$$[(\alpha, \beta) \,\bar{\wedge}\, (\gamma, \delta) \dot{\rightarrow} (\lambda, \xi)] = [(\alpha, \beta) \dot{\rightarrow} ((\gamma, \delta) \dot{\rightarrow} (\lambda, \xi))];$$

(11) $(\gamma, \delta) \leq [(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)]$;
(12)

$$\bigwedge_{j \in \Lambda} [(\alpha_j, \beta_j) \rightarrow (\gamma_j, \delta_j)] \leq [\bigvee_{j \in \Lambda} (\alpha_j, \beta_j) \rightarrow \bigvee_{j \in \Lambda} (\gamma_j, \delta_j)];$$

$$\bigwedge_{j \in \Lambda} [(\alpha_j, \beta_j) \dot{\rightarrow} (\gamma_j, \delta_j)] \leq [\bigvee_{j \in \Lambda} (\alpha_j, \beta_j) \dot{\rightarrow} \bigvee_{j \in \Lambda} (\gamma_j, \delta_j)];$$

$$\bigwedge_{j \in \Lambda} [(\alpha_j, \beta_j) \rightarrow (\gamma_j, \delta_j)] \leq [\bigwedge_{j \in \Lambda} (\alpha_j, \beta_j) \rightarrow \bigwedge_{j \in \Lambda} (\gamma_j, \delta_j)];$$

$$\bigwedge_{j \in \Lambda} [(\alpha_j, \beta_j) \dot{\rightarrow} (\gamma_j, \delta_j)] \leq [\bigwedge_{j \in \Lambda} (\alpha_j, \beta_j) \dot{\rightarrow} \bigwedge_{j \in \Lambda} (\gamma_j, \delta_j)];$$

(13)

$$[(\alpha, \beta) \rightarrow ((\gamma, \delta) \wedge (\lambda, \xi))] = [(\alpha, \beta) \rightarrow (\gamma, \delta)] \wedge [(\alpha, \beta) \rightarrow (\lambda, \xi)];$$

$$[(\alpha, \beta) \dot{\rightarrow} ((\gamma, \delta) \wedge (\lambda, \xi))] = [(\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)] \wedge [(\alpha, \beta) \dot{\rightarrow} (\lambda, \xi)];$$

(14)

$$\models (\alpha, \beta) \,\bar{\wedge}\, (\gamma, \delta) \rightarrow (\alpha, \beta) \wedge (\gamma, \delta);$$

$$\models ((\alpha, \beta) \dot{\rightarrow} (\gamma, \delta)) \rightarrow ((\alpha, \beta) \rightarrow (\gamma, \delta)).$$

Proof. (1) $[(\alpha_1, \beta_1) \longrightarrow (\gamma, \delta)] = (\min(1, 1 - \alpha_1 + \gamma, 1 - \delta + \beta_1), \max(0, \alpha_1 - \gamma, \delta - \beta_1))$.

Since $(\alpha, \beta) \leq (\alpha_1, \beta_1)$, then $\alpha \leq \alpha_1$ and $\beta_1 \leq \beta$. Therefore,

$\min(1, 1 - \alpha_1 + \gamma, 1 - \delta + \beta_1) \leq \min(1, 1 - \alpha + \gamma, 1 - \delta + \beta)$ and $\max(0, \alpha - \gamma, \delta - \beta) \leq \max(0, \alpha_1 - \gamma, \delta - \beta_1)$. Hence

$[(\alpha_1, \beta_1) \longrightarrow (\gamma, \delta)] \leq (\min(1, 1 - \alpha + \gamma, 1 - \delta + \beta), \max(0, \alpha - \gamma, \delta - \beta)) = [(\alpha, \beta) \longrightarrow (\gamma, \delta)]$.

Similarly, $[(\alpha_1, \beta_1) \dot{\longrightarrow} (\gamma, \delta)] \leq [(\alpha, \beta) \dot{\longrightarrow} (\gamma, \delta)]$.

(2) The proof is similar to that of (1).

(3) Since $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$, the result holds.

(4) (a) Since $(\alpha, \beta) \leq (\gamma, \delta)$, then $\alpha \leq \gamma$ and $\delta \leq \beta$. Hence

$[(\alpha, \beta) \longrightarrow (\gamma, \delta)] = (\min(1, 1 - \alpha + \gamma, 1 - \delta + \beta), \max(0, \alpha - \gamma, \delta - \beta)) = (1, 0)$.

Conversely, if $[(\alpha, \beta) \longrightarrow (\gamma, \delta)] = (1, 0)$, then $\min(1, 1 - \alpha + \gamma, 1 - \delta + \beta) = 1$ or $1 - \alpha + \gamma \geq 1$ and $1 - \delta + \beta \geq 1$. Therefore, $\alpha \leq \gamma$ and $\delta \leq \beta$. Hence $(\alpha, \beta) \leq (\gamma, \delta)$. Additionally, the result holds if we choose $\max(0, \alpha - \gamma, \delta - \beta) = 0$.

(b) Since $\beta + \gamma \geq 1$, $\alpha + \beta \leq 1$ and $\gamma + \delta \leq 1$, then $\alpha + \delta \leq 1$. Therefore, the result holds.

(5) Follows from Definition 4 (2).

(6) $(\lambda,\xi) \leq [(\alpha,\beta)\dot{\longrightarrow}(\gamma,\delta)]$
$\Leftrightarrow (\lambda,\xi) \leq (\min(1,\beta+\gamma),\max(0,\alpha+\delta-1))$
$\Leftrightarrow \lambda \leq \min(1,\beta+\gamma)$ and $\xi \geq \max(0,\alpha+\delta-1)$
$\Leftrightarrow \lambda \leq \beta+\gamma$ and $\xi \geq \alpha+\delta-1$
$\overset{\alpha+\beta\leq 1}{\Leftrightarrow} \lambda+\alpha-1 \leq \gamma$ and $\xi+\beta \geq \delta$
$\Leftrightarrow \max(0,\lambda+\alpha-1) \leq \gamma$ and $\min(1,\xi+\beta) \geq \delta$
$\Leftrightarrow (\max(0,\lambda+\alpha-1),\min(1,\xi+\beta)) \leq (\gamma,\delta)$
$\Leftrightarrow [(\lambda,\xi)\,\overline{\wedge}\,(\alpha,\beta)] \leq (\gamma,\delta)$.

(7) $R.H.S = [((\alpha,\beta)\curlywedge(\gamma,\delta))\rightarrow((\alpha,\beta)\curlywedge(\lambda,\xi))]$
$= [(\max(0,\alpha-\delta,\gamma-\beta),\min(1,1-\alpha+\delta,1-\gamma+\beta)) \rightarrow (\max(0,\alpha-\xi,\lambda-\beta),\min(1,1 -\alpha+\xi,1-\lambda+\beta))]$
$= (\min(1,1-\max(0,\alpha-\delta,\gamma-\beta)+\max(0,\alpha-\xi,\lambda-\beta),1-\min(1,1-\alpha+\xi,1-\lambda+\beta)+\min(1,1-\alpha+\delta,1-\gamma+\beta)),\max(0,\max(0,\alpha-\delta,\gamma-\beta)-\max(0,\alpha-\xi,\lambda-\beta),\min(1,1-\alpha+\xi,1-\lambda+\beta)-\min(1,1-\alpha+\delta,1-\gamma+\beta)))$
$= (\min(1,\min(1,1-\alpha+\delta,1-\gamma+\beta)+\max(0,\alpha-\xi,\lambda-\beta)),\max(0,\max(0,\alpha-\delta,\gamma-\beta)-\max(0,\alpha-\xi,\lambda-\beta)))$
$\geq (\min(1,\min(1,1-\xi+\delta,1-\gamma+\lambda)),\max(0,\max(0,\gamma-\lambda,\xi-\delta)))$
$= (\min(1,1-\gamma+\lambda,1-\xi+\delta),\max(0,\gamma-\lambda,\xi-\delta)) = [(\gamma,\delta)\curlywedge(\lambda,\xi)] = L.H.S.$

(8) $[((\lambda,\xi)\rightarrow(\alpha,\beta))\rightarrow((\lambda,\xi)\rightarrow(\gamma,\delta))]$
$= [(\min(1,1-\lambda+\alpha,1-\beta+\xi),\max(0,\lambda-\alpha,\beta-\xi)) \rightarrow (\min(1,1-\lambda+\gamma,1-\delta+\xi),\max(0,\lambda-\gamma,\delta-\xi))]$
$= (\min(1,1-\min(1,1-\lambda+\alpha,1-\beta+\xi)+\min(1,1-\lambda+\gamma,1-\delta+\xi),1-\max(0,\lambda-\gamma,\delta-\xi)+\max(0,\lambda-\alpha,\beta-\xi)),\max(0,\min(1,1-\lambda+\alpha,1-\beta+\xi)-\min(1,1-\lambda+\gamma,1-\delta+\xi),\max(0,\lambda-\gamma,\delta-\xi)-\max(0,\lambda-\alpha,\beta-\xi)))$
$= (\min(1,1-\min(1,1+\alpha-\lambda,1-\beta+\xi)+\min(1,1-\lambda+\gamma,1-\delta+\xi)),\max(0,\max(0,\lambda-\gamma,\delta-\xi)-\max(0,\lambda-\alpha,\beta-\xi)))$
$\geq (\min(1,1-\alpha+\gamma,1-\delta+\beta),\max(0,\alpha-\gamma,\delta-\beta)) = [(\alpha,\beta)\rightarrow(\gamma,\delta)]$.
Similarly, $[(\alpha,\beta)\dot{\rightarrow}(\gamma,\delta)] \leq [((\lambda,\xi)\dot{\rightarrow}(\alpha,\beta))\dot{\rightarrow}((\lambda,\xi)\dot{\rightarrow}(\gamma,\delta))]$.

(9) The proof is similar to that of (8).

(10) $[(\alpha,\beta)\,\overline{\wedge}\,(\gamma,\delta)\dot{\rightarrow}(\lambda,\xi)] = (\max(0,\alpha+\gamma-1),\min(1,\beta+\delta))\dot{\rightarrow}(\lambda,\xi)$
$= (\min(1,\min(1,\beta+\delta)+\lambda),\max(0,\max(0,\alpha+\gamma-1)+\xi-1))$
$= (\min(1,\beta+\min(1,\delta+\lambda)),\max(0,\alpha+\max(0,\gamma+\xi-1)-1))$
$= [(\alpha,\beta)\dot{\rightarrow}(\min(1,\delta+\lambda),\max(0,\gamma+\xi-1))]$
$= [(\alpha,\beta)\dot{\rightarrow}((\gamma,\delta)\dot{\rightarrow}(\lambda,\xi))]$.

(11) Obvious.

(12) $\bigwedge_{j\in\Lambda}[(\alpha_j,\beta_j)\rightarrow(\gamma_j,\delta_j)]$
$= \bigwedge_{j\in\Lambda}(\min(1,1-\alpha_j+\gamma_j,1-\delta_j+\beta_j),\max(0,\alpha_j-\gamma_j,\delta_j-\beta_j))$
$= (\bigwedge_{j\in\Lambda}\min(1,1-\alpha_j+\gamma_j,1-\delta_j+\beta_j),\bigvee_{j\in\Lambda}\max(0,\alpha_j-\gamma_j,\delta_j-\beta_j))$
$= (\min(1,\bigwedge_{j\in\Lambda}(1-\alpha_j+\gamma_j),\bigwedge_{j\in\Lambda}(1-\delta_j+\beta_j)),\max(0,\bigvee_{j\in\Lambda}(\alpha_j-\gamma_j),\bigvee_{j\in\Lambda}(\delta_j-\beta_j)))$
$\leq (\min(1,1-\bigvee_{j\in\Lambda}\alpha_j+\bigvee_{j\in\Lambda}\gamma_j,1-\bigwedge_{j\in\Lambda}\delta_j+\bigwedge_{j\in\Lambda}\beta_j),\max(0,\bigvee_{j\in\Lambda}\alpha_j-\bigvee_{j\in\Lambda}\gamma_j,\bigwedge_{j\in\Lambda}\delta_j-\bigwedge_{j\in\Lambda}\beta_j))$
$= [(\bigvee_{j\in\Lambda}\alpha_j,\bigwedge_{j\in\Lambda}\beta_j) \longrightarrow (\bigvee_{j\in\Lambda}\gamma_j,\bigwedge_{j\in\Lambda}\delta_j)] = [\bigvee_{j\in\Lambda}(\alpha_j,\beta_j) \longrightarrow \bigvee_{j\in\Lambda}(\gamma_j,\delta_j)]$.
Also,
$\bigwedge_{j\in\Lambda}[(\alpha_j,\beta_j)\dot{\longrightarrow}(\gamma_j,\delta_j)] = \bigwedge_{j\in\Lambda}(\min(1,\beta_j+\gamma_j),\max(0,\alpha_j+\delta_j-1))$
$= (\bigwedge_{j\in\Lambda}\min(1,\beta_j+\gamma_j),\bigvee_{j\in\Lambda}\max(0,\alpha_j+\delta_j-1))$
$\leq (\min(1,\bigwedge_{j\in\Lambda}\beta_j+\bigvee_{j\in\Lambda}\gamma_j),\max(0,\bigvee_{j\in\Lambda}\alpha_j+\bigwedge_{j\in\Lambda}\delta_j-1))$
$= [(\bigvee_{j\in\Lambda}\alpha_j,\bigwedge_{j\in\Lambda}\beta_j)\dot{\longrightarrow}(\bigvee_{j\in\Lambda}\gamma_j,\bigwedge_{j\in\Lambda}\delta_j)] = [\bigvee_{j\in\Lambda}(\alpha_j,\beta_j)\dot{\longrightarrow}\bigvee_{j\in\Lambda}(\gamma_j,\beta_j)]$.
The proofs of the other statements are similar.

(13) $[(\alpha,\beta)\rightarrow((\gamma,\delta)\wedge(\lambda,\xi))] = [(\alpha,\beta) \longrightarrow (\gamma\wedge\lambda, \delta\vee\xi)]$
$= (\min(1, 1-\alpha+\gamma\wedge\lambda, 1-\delta\vee\xi+\beta), \max(0, \alpha-\gamma\wedge\lambda, \delta\vee\xi-\beta))$
$= (\min(1, (1-\alpha+\gamma)\wedge(1-\alpha+\lambda), (1-\delta+\beta)\wedge(1-\xi+\beta)), \max(0, (\alpha-\gamma)\vee(\alpha-\lambda), (\delta-\beta)\vee(\xi-\beta)))$
$= (\min(1, 1-\alpha+\gamma, 1-\alpha+\lambda, 1-\delta+\beta, 1-\xi+\beta), \max(0, \alpha-\gamma, \alpha-\lambda, \delta-\beta, \xi-\beta))$
$= (\min(1, 1-\alpha+\gamma, 1-\delta+\beta), \max(0, \alpha-\gamma, \delta-\beta))\wedge$
$(\min(1, 1-\alpha-\lambda, 1-\xi+\beta), \max(0, \alpha-\lambda, \xi-\beta))$
$= [(\alpha,\beta)\rightarrow(\gamma,\delta)]\wedge[(\alpha,\beta)\rightarrow(\lambda,\xi)]$.
Similarly, we can prove that
$[(\alpha,\beta)\dot{\rightarrow}((\gamma,\delta)\wedge(\lambda,\xi))] = [(\alpha,\beta)\dot{\rightarrow}(\gamma,\delta)]\wedge[(\alpha,\beta)\dot{\rightarrow}(\lambda,\xi)]$.
(14) $[(\alpha,\beta)\,\overline{\wedge}\,(\gamma,\delta) \rightarrow (\alpha,\beta)\wedge(\gamma,\delta)]$
$= [(\max(0, \alpha+\gamma-1), \min(1, \beta+\delta)) \rightarrow (\min(\alpha,\gamma), \max(\beta,\delta))]$
$= (\min(1, 1-\max(0,\alpha+\gamma-1)+\min(\alpha,\gamma), 1-\max(\beta,\delta)+\min(1,\beta+\delta)), \max(0, \max(0,\alpha+\gamma-1)-\min(\alpha,\gamma), \max(\beta,\delta)-\min(1,\beta+\delta))) = (1,0)$.
Similarly, $[((\alpha,\beta)\dot{\rightarrow}(\gamma,\delta)) \rightarrow ((\alpha,\beta) \rightarrow (\gamma,\delta))] = (1,0)$. □

Lemma 1. *Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in \mathcal{IF}(\mathcal{X})$. Then*
(1) $\models \hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \rightarrow (\hat{\mathcal{B}} \subseteq \hat{\mathcal{C}} \rightarrow \hat{\mathcal{A}} \subseteq \hat{\mathcal{C}})$;
(2) $\models \hat{\mathcal{A}} \equiv \hat{\mathcal{B}} \rightarrow (\hat{\mathcal{B}} \equiv \hat{\mathcal{C}} \rightarrow \hat{\mathcal{A}} \equiv \hat{\mathcal{C}})$;
(3) $\models \hat{\mathcal{A}} \approx \hat{\mathcal{B}} \rightarrow (\hat{\mathcal{B}} \approx \hat{\mathcal{C}} \rightarrow \hat{\mathcal{A}} \approx \hat{\mathcal{C}})$;
(4) $\models \hat{\mathcal{B}} \subseteq \hat{\mathcal{C}} \rightarrow (\hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \subseteq \hat{\mathcal{C}})$;
(5) $\models \hat{\mathcal{B}} \equiv \hat{\mathcal{C}} \rightarrow (\hat{\mathcal{A}} \equiv \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \equiv \hat{\mathcal{C}})$;
(6) $\models \hat{\mathcal{B}} \approx \hat{\mathcal{C}} \rightarrow (\hat{\mathcal{A}} \approx \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \approx \hat{\mathcal{C}})$;
(7) $\models \hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \rightarrow \neg(\hat{\mathcal{B}}) \subseteq \neg(\hat{\mathcal{A}})$;
(8) $\models \hat{\mathcal{A}} \equiv \hat{\mathcal{B}} \rightarrow \neg(\hat{\mathcal{B}}) \equiv \neg(\hat{\mathcal{A}})$;
(9) $\models \hat{\mathcal{A}} \approx \hat{\mathcal{B}} \rightarrow \neg(\hat{\mathcal{B}}) \approx \neg(\hat{\mathcal{A}})$.

Lemma 2. *Let $\{\hat{\mathcal{A}}_\lambda : \lambda \in \Lambda\}, \{\hat{\mathcal{B}}_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{IF}(\mathcal{X})$.*
(1) $\models \forall(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \subseteq \hat{\mathcal{B}}_\lambda) \rightarrow \bigcap\limits_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda \subseteq \bigcap\limits_{\lambda\in\Lambda} \hat{\mathcal{B}}_\lambda$;
(2) $\models \forall(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \equiv \hat{\mathcal{B}}_\lambda) \rightarrow \bigcap\limits_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda \equiv \bigcap\limits_{\lambda\in\Lambda} \hat{\mathcal{B}}_\lambda$;
(3) $\models \forall(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \approx \hat{\mathcal{B}}_\lambda) \rightarrow \bigcap\limits_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda \approx \bigcap\limits_{\lambda\in\Lambda} \hat{\mathcal{B}}_\lambda$;
(4) $\models \forall(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \subseteq \hat{\mathcal{B}}_\lambda) \rightarrow \bigcup\limits_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda \subseteq \bigcup\limits_{\lambda\in\Lambda} \hat{\mathcal{B}}_\lambda$;
(5) $\models \forall(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \equiv \hat{\mathcal{B}}_\lambda) \rightarrow \bigcup\limits_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda \equiv \bigcup\limits_{\lambda\in\Lambda} \hat{\mathcal{B}}_\lambda$;
(6) $\models \forall(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \approx \hat{\mathcal{B}}_\lambda) \rightarrow \bigcup\limits_{\lambda\in\Lambda} \hat{\mathcal{A}}_\lambda \approx \bigcup\limits_{\lambda\in\Lambda} \hat{\mathcal{B}}_\lambda$.

Lemma 3. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a function, $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{X})$ and $\hat{\mathcal{C}}, \hat{\mathcal{D}} \in \mathcal{IF}(\mathcal{Y})$. Then*
(1) $\models \hat{\mathcal{A}} \subseteq \hat{\mathcal{B}} \rightarrow f(\hat{\mathcal{A}}) \subseteq f(\hat{\mathcal{B}})$;
(2) $\models \hat{\mathcal{A}} \equiv \hat{\mathcal{B}} \rightarrow f(\hat{\mathcal{A}}) \equiv f(\hat{\mathcal{B}})$;
(3) $\models \hat{\mathcal{A}} \approx \hat{\mathcal{B}} \rightarrow f(\hat{\mathcal{A}}) \approx f(\hat{\mathcal{B}})$;
(4) $\models \hat{\mathcal{C}} \subseteq \hat{\mathcal{D}} \rightarrow f^{-1}(\hat{\mathcal{C}}) \subseteq f^{-1}(\hat{\mathcal{D}})$;
(5) $\models \hat{\mathcal{C}} \equiv \hat{\mathcal{D}} \rightarrow f^{-1}(\hat{\mathcal{C}}) \equiv f^{-1}(\hat{\mathcal{D}})$;
(6) $\models \hat{\mathcal{C}} \approx \hat{\mathcal{D}} \rightarrow f^{-1}(\hat{\mathcal{C}}) \approx f^{-1}(\hat{\mathcal{D}})$.

Remark 3. *Lemmas 1–3 are true when we replace "$\subseteq, \rightarrow, \equiv$ and $\approx$" by "$\Subset, \dot{\rightarrow}, \dot{\equiv}$ and $\dot{\approx}$", respectively.*

Lemma 4. *Let $\hat{\mathcal{A}}, \hat{\mathcal{B}}, \hat{\mathcal{C}} \in \mathcal{IF}(\mathcal{X})$. Then*
(1) $\models \hat{\mathcal{A}} \Subset \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \subseteq \hat{\mathcal{B}}$;
(2) $\models \hat{\mathcal{A}} \dot{\approx} \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \dot{\equiv} \hat{\mathcal{B}}$;
(3) $\models \hat{\mathcal{A}} \dot{\equiv} \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \equiv \hat{\mathcal{B}}$;
(4) $\models \hat{\mathcal{A}} \dot{\approx} \hat{\mathcal{B}} \rightarrow \hat{\mathcal{A}} \equiv \hat{\mathcal{B}}$;

(5) $\models \hat{\mathcal{A}} \Subset \hat{\mathcal{B}} \barwedge \hat{\mathcal{B}} \Subset \hat{\mathcal{C}} \rightarrow \hat{\mathcal{A}} \Subset \hat{\mathcal{C}}$;
(6) $\models \hat{\mathcal{A}} \doteqdot \hat{\mathcal{B}} \barwedge \hat{\mathcal{B}} \doteqdot \hat{\mathcal{C}} \rightarrow \hat{\mathcal{A}} \doteqdot \hat{\mathcal{C}}$.

Lemma 5. *Let $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{X})$. Then the following are equivalent:*
(1) $\hat{\mathcal{A}}(x) = \hat{\mathcal{B}}(x)$;
(2) $\models \hat{\mathcal{A}} \equiv \hat{\mathcal{B}}$;
(3) $\models \hat{\mathcal{A}} \doteqdot \hat{\mathcal{B}}$;
(4) $\models \hat{\mathcal{A}} \approx \hat{\mathcal{B}}$;
(5) $\models \hat{\mathcal{A}} \approxeq \hat{\mathcal{B}}$.

4. Bi-Intuitionistic Fuzzy Topological Space

Now, we introduce the definition of the bi-intuitionistic fuzzy topology and other related topologies. Additionally, we give some examples.

Definition 5. *Let $\mathcal{X}$ be a universe of discourse, $\tau \in \mathcal{IF}(\mathcal{IF}(\mathcal{X}))(\tau : \mathcal{IF}(\mathcal{X}) \rightarrow \mathcal{Z})$, satisfy the following axioms:*
(1) $\models \hat{1} \in \tau, \models \hat{0} \in \tau$;
(2) *For any* $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{X}), \models \hat{\mathcal{A}} \in \tau \wedge \hat{\mathcal{B}} \in \tau \rightarrow \hat{\mathcal{A}} \cap \hat{\mathcal{B}} \in \tau$;
(3) $\models$ *For any* $\{\hat{\mathcal{A}}_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{IF}(\mathcal{X})$,

$$\models \forall\lambda(\lambda \in \Lambda \dot{\rightarrow} \hat{\mathcal{A}}_\lambda \in \tau) \rightarrow \bigcup_{\lambda \in \Lambda} \hat{\mathcal{A}}_\lambda \in \tau.$$

Then, τ is called a bi-intuitionistic fuzzy topology (BIFT for short) and (X, τ) is a bi-intuitionistic fuzzy topological space. Specially, if $\tau \in \mathcal{P}(\mathcal{IF}(\mathcal{X}))(\tau : \mathcal{IF}(\mathcal{X}) \rightarrow \{0,1\})$, then τ is an intuitionistic fuzzy topology (IFT for short), if $\tau \in \mathcal{IF}(\mathcal{P}(\mathcal{X}))(\tau : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{Z})$, then τ is called an intuitionistic fuzzifying topology (IFuT for short), if $\tau \in IF(\mathcal{F}(\mathcal{X}))(\tau : \mathcal{F}(\mathcal{X}) \rightarrow \mathcal{Z})$, then τ is called an intuitionistic-fuzzy fuzzy topology (IFFT for short), if $\tau \in \mathcal{F}(\mathcal{IF}(\mathcal{X}))(\tau : \mathcal{IF}(\mathcal{X}) \rightarrow [0,1])$, then τ is a fuzzy intuitionistic fuzzy topology (FIFT for short), if $\tau \in \mathcal{F}(\mathcal{F}(\mathcal{X}))(\tau : \mathcal{F}(\mathcal{X}) \rightarrow [0,1])$, then τ is a bifuzzy topology (BFT for short), if $\tau \in \mathcal{F}(\mathcal{P}(\mathcal{X}))(\tau : \mathcal{P}(\mathcal{X}) \rightarrow [0,1])$, then τ is a fuzzifying topology (FuT for short), if $\tau \in \mathcal{P}(\mathcal{F}(\mathcal{X}))(\tau : \mathcal{F}(\mathcal{X}) \rightarrow \{0,1\})$, then τ is a fuzzy topology (FT for short), and if $\tau \in \mathcal{P}(\mathcal{P}(\mathcal{X}))(\tau : \mathcal{P}(\mathcal{X}) \rightarrow \{0,1\})$, then τ is a classical topology (T for short). For any $\hat{\mathcal{A}} \in \mathcal{IF}(\mathcal{X})$, we always suppose that $\tau(\hat{\mathcal{A}}) = (m\tau(\hat{\mathcal{A}}), n\tau(\hat{\mathcal{A}}))$; the number $m\tau(\hat{\mathcal{A}})$ is the openness degree of $\hat{\mathcal{A}}$, while the number $n\tau(\hat{\mathcal{A}})$ is the non-openness degree of $\hat{\mathcal{A}}$.

Remark 4. *The conditions in Definition 4 may be rewritten respectively as follows:*
(1)$'$ $\tau(\hat{1}) = (1,0) = \tau(\hat{0})$;
(2)$'$ *For any* $\hat{\mathcal{A}}, \hat{\mathcal{B}} \in \mathcal{IF}(\mathcal{X})$,

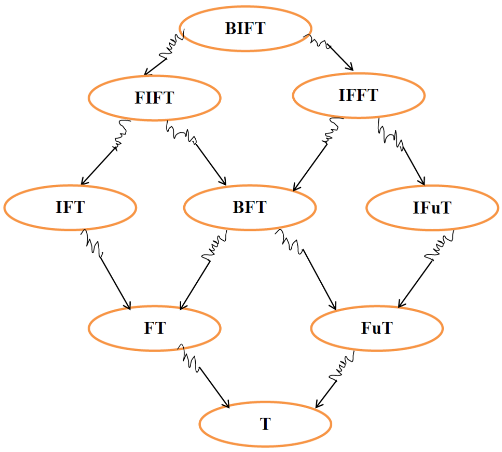
$$m\tau(\hat{\mathcal{A}}) \wedge m\tau(\hat{\mathcal{B}}) \leq m\tau(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}) \text{ and } n\tau(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}) \leq n\tau(\hat{\mathcal{A}}) \vee n\tau(\hat{\mathcal{B}});$$

(3)$'$ *For any* $\{\hat{\mathcal{A}}_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{IF}(\mathcal{X})$,

$$\bigwedge_{\lambda \in \Lambda} m\tau(\hat{\mathcal{A}}_\lambda) \leq m\tau\left(\bigcup_{\lambda \in \Lambda} \hat{\mathcal{A}}_\lambda\right) \text{ and } n\tau\left(\bigcup_{\lambda \in \Lambda} \hat{\mathcal{A}}_\lambda\right) \leq \bigvee_{\lambda \in \Lambda} n\tau(\hat{\mathcal{A}}_\lambda).$$

Remark 5. *In the above diagram, and in the sequel, the symbol $\mathcal{H} \rightsquigarrow \mathcal{K}$ means that the concept $\mathcal{K}$ is a special case of the concept $\mathcal{H}$:*

Theorem 5. *Let $\mathcal{X}$ be a non-empty set.*
(1) *If τ is a BIFT, then $\tau\big|_{\mathcal{F}(\mathcal{X})}$ is a IFFT;*
(2) *If τ is a BIFT, then $\tau\big|_{\mathcal{P}(\mathcal{X})}$ is a IFuT;*
(3) *If τ is a FIFT, then $\tau\big|_{\mathcal{F}(\mathcal{X})}$ is a BFT;*



(4) If τ is a FIFT, then $\tau\Big|_{\mathcal{P}(\mathcal{X})}$ is a FuT;

(5) If τ is a IFFT, then $\tau\Big|_{\mathcal{P}(\mathcal{X})}$ is a IFuT;

(6) If τ is a BFT, then $\tau\Big|_{\mathcal{P}(\mathcal{X})}$ is a FuT;

(7) If τ is a IFT, then $\tau\Big|_{\mathcal{F}(\mathcal{X})}$ is a FT;

(8) If τ is a IFT, then $\tau\Big|_{\mathcal{P}(\mathcal{X})}$ is a T;

(9) If τ is a FT, then $\tau\Big|_{\mathcal{P}(\mathcal{X})}$ is a T.

Theorem 6. *(1) τ is a IFuT if and only if for any $(\alpha,\beta)\in\mathcal{Z}$, $\tau_{(\alpha,\beta)}$ is a classical topology, where $\tau_{(\alpha,\beta)}=\{\mathcal{A}:\tau(\mathcal{A})\geq(\alpha,\beta)\}=\{\mathcal{A}:m\tau(\mathcal{A})\geq\alpha,\ \ n\tau(\mathcal{A})\leq\beta\}$ is the (α,β)-level set of τ.*

(2) τ is BIFT if and only if for any $(\alpha,\beta)\in\mathcal{Z}$, $\tau_{(\alpha,\beta)}$ is a IFT.

Proof. We only prove (1) as (2) is similar.

($\Longrightarrow$:) Suppose $(\alpha,\beta)\in\mathcal{Z}$. We prove that $\tau_{(\alpha,\beta)}$ is an ordinary topology.

(1) Since $\tau(\mathcal{X})=\tau(\phi)=(1,0)\geq(\alpha,\beta)$, then $\mathcal{X},\phi\in\tau_{(\alpha,\beta)}$.

(2) Suppose $\mathcal{A},\mathcal{B}\in\tau_{(\alpha,\beta)}$. Then $\tau(\mathcal{A})\geq(\alpha,\beta)$ and $\tau(\mathcal{B})\geq(\alpha,\beta)$. So, $\tau(\mathcal{A}\cap\mathcal{B})\geq\tau(\mathcal{A})\wedge\tau(\mathcal{B})\geq(\alpha,\beta)$. Therefore $\mathcal{A}\cap\mathcal{B}\in\tau_{(\alpha,\beta)}$.

(3) Suppose $\{\mathcal{A}_\lambda:\lambda\in\Lambda\}\subseteq\tau_{(\alpha,\beta)}$. Then for each $\lambda\in\Lambda$, we have $\tau(\mathcal{A}_\lambda)\geq(\alpha,\beta)$. So, $\bigwedge\limits_{\lambda\in\Lambda}\tau(\mathcal{A}_\lambda)\geq(\alpha,\beta)$. Hence $\tau\left(\bigcup\limits_{\lambda\in\Lambda}\mathcal{A}_\lambda\right)\geq\bigwedge\limits_{\lambda\in\Lambda}\tau(\mathcal{A}_\lambda)\geq(\alpha,\beta)$. Therefore $\bigcup\limits_{\lambda\in\Lambda}\mathcal{A}_\lambda\in\tau_{(\alpha,\beta)}$.

(:$\Longleftarrow$) We prove that τ is an intuitionistic fuzzifying topology.

(1) For any $(\alpha,\beta)\in\mathcal{Z}$, $\mathcal{X}\in\tau_{(\alpha,\beta)}$. Then, for any $(\alpha,\beta)\in\mathcal{Z}$, $\tau(\mathcal{X})\geq(\alpha,\beta)$. Therefore, $\tau(\mathcal{X})\geq\bigvee\limits_{(\alpha,\beta)\in\mathcal{Z}}(\alpha,\beta)=(1,0)$. Hence $\vDash\mathcal{X}\in\tau$. Similar $\vDash\phi\in\tau$.

(2) Suppose $\mathcal{A},\mathcal{B}\subseteq\mathcal{X}$ such that $\tau(\mathcal{A})\wedge\tau(\mathcal{B})\geq(\alpha,\beta)$. Then $\tau(\mathcal{A})\geq(\alpha,\beta)$ and $\tau(\mathcal{B})\geq(\alpha,\beta)$. Hence $\mathcal{A},\mathcal{B}\in\tau_{(\alpha,\beta)}$. Thus, $\mathcal{A}\cap\mathcal{B}\in\tau_{(\alpha,\beta)}$, i.e., $\tau(\mathcal{A}\cap\mathcal{B})\geq(\alpha,\beta)$. Therefore, $\tau(\mathcal{A}\cap\mathcal{B})\geq\tau(\mathcal{A})\wedge\tau(\mathcal{B})$.

(3) Suppose $\{\mathcal{A}_\lambda|\lambda\in\Lambda\}\subseteq\mathcal{P}(\mathcal{X})$ such that $\bigwedge\limits_{\lambda\in\Lambda}\tau(\mathcal{A}_\lambda)\geq(\alpha,\beta)$. Then $\{\mathcal{A}_\lambda|\lambda\in\Lambda\}\subseteq\tau_{(\alpha,\beta)}$ and so $\bigcup\limits_{\lambda\in\Lambda}\mathcal{A}_\lambda\in\tau_{(\alpha,\beta)}$. Hence $\tau\left(\bigcup\limits_{\lambda\in\Lambda}\mathcal{A}_\lambda\right)\geq(\alpha,\beta)$. Therefore $\tau\left(\bigcup\limits_{\lambda\in\Lambda}\mathcal{A}_\lambda\right)\geq\bigwedge\limits_{\lambda\in\Lambda}\tau(\mathcal{A}_\lambda)$. $\square$

Example 1. *(1) If τ is a IFuT and $\tau(\mathcal{P}(\mathcal{X}))\subseteq\{(0,1),(1,0)\}$, then τ is an ordinary topology.*

(2) If τ is a FIFT and for each $\tilde{\mathcal{A}}\in\Im(\mathcal{X})$, $m\tau(\tilde{\mathcal{A}})+n\tau(\tilde{\mathcal{A}})=1$, then τ is a fuzzifying topology.

Example 2. *Suppose that τ is an intuitionistic fuzzifying topology and $\check{\tau} \in \mathcal{IF}(\mathcal{IF}(\mathcal{X}))$ is defined as $\hat{\mathcal{A}} \in \check{\tau} := \exists \mathcal{A}((\mathcal{A} \in \tau) \wedge \mathcal{A} \equiv \hat{\mathcal{A}})$. Then $\check{\tau}$ is a bi-intuitionistic fuzzy topology such that $\check{\tau}|_{\mathcal{P}(\mathcal{X})} = \tau$. In fact, it suffices to check the three conditions in Definition 5.*

(1) $\check{\tau}(\tilde{1}) = \check{\tau}(\tilde{0}) = (1, 0)$.

(2)

$$\begin{aligned}
\check{\tau}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}) &= \bigvee_{\mathcal{C} \in \mathcal{P}(\mathcal{X})} (\tau(\mathcal{C}) \wedge [\mathcal{C} \equiv \hat{\mathcal{A}} \cap \hat{\mathcal{B}}]) \\
&= \bigvee_{\mathcal{A},\mathcal{B} \subseteq \mathcal{X}} (\tau(\mathcal{A} \cap \mathcal{B}) \wedge [\mathcal{A} \cap \mathcal{B} \equiv \hat{\mathcal{A}} \cap \hat{\mathcal{B}}]) \\
&\geq \bigvee_{\mathcal{A},\mathcal{B} \subseteq \mathcal{X}} (\tau(\mathcal{A}) \wedge \tau(\mathcal{B}) \wedge [\mathcal{A} \equiv \hat{\mathcal{A}}] \wedge [\mathcal{B} \equiv \hat{\mathcal{B}}]) \\
&= \bigvee_{\mathcal{A},\mathcal{B} \subseteq \mathcal{X}} (\tau(\mathcal{A}) \wedge [\mathcal{A} \equiv \hat{\mathcal{A}}]) \wedge \bigvee_{\mathcal{A},\mathcal{B} \subseteq \mathcal{X}} (\tau(\mathcal{B}) \wedge [\mathcal{B} \equiv \hat{\mathcal{B}}]) \\
&= \check{\tau}(\hat{\mathcal{A}}) \wedge \check{\tau}(\hat{\mathcal{B}}).
\end{aligned}$$

(3)

$$\begin{aligned}
\left[\bigcup_{\lambda \in \Lambda} \hat{\mathcal{A}}_\lambda \in \check{\tau}\right] &= \bigvee_{\mathcal{C} \subseteq \mathcal{X}} \left(\tau(\mathcal{C}) \wedge \left[\mathcal{C} \equiv \bigcup_{\lambda \in \Lambda} \hat{\mathcal{A}}_\lambda\right]\right) \\
&= \bigvee_{\mathcal{A}_\lambda \subseteq \mathcal{X}} \left(\tau\left(\bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda\right) \wedge \left[\bigcup_{\lambda \in \Lambda} \mathcal{A}_\lambda \equiv \bigcup_{\lambda \in \Lambda} \hat{\mathcal{A}}_\lambda\right]\right) \\
&\geq \bigvee_{\mathcal{A}_\lambda \subseteq \mathcal{X}} \left(\bigwedge_{\lambda \in \Lambda} \tau(\mathcal{A}_\lambda) \wedge \bigwedge_{\lambda \in \Lambda} [\mathcal{A}_\lambda \equiv \hat{\mathcal{A}}_\lambda]\right) \\
&= \bigwedge_{\lambda \in \Lambda} \bigvee_{\mathcal{A}_\lambda \subseteq \mathcal{X}} (\tau(\mathcal{A}_\lambda) \wedge [\mathcal{A}_\lambda \equiv \hat{\mathcal{A}}_\lambda]) \\
&= \bigwedge_{\lambda \in \Lambda} \check{\tau}(\hat{\mathcal{A}}_\lambda).
\end{aligned}$$

Example 3. *Suppose that τ is an inituitionistic fuzzifying topology and $\check{\tau} \in \mathcal{IF}(\mathcal{IF}(\mathcal{X}))$ is defined as*

$$\hat{\mathcal{A}} \in \check{\tau} := \forall (\alpha, \beta) \Big(\hat{\mathcal{A}}_{[(\alpha,\beta)]} \in \tau\Big),$$

where $\tau_{[(\alpha,\beta)]} = \{\hat{\mathcal{A}} : \tau(\hat{\mathcal{A}}) > (\alpha, \beta)\}$=$\{\hat{\mathcal{A}} : m\tau(\hat{\mathcal{A}}) > \alpha,\ \ n\tau(\hat{\mathcal{A}}) < \beta\}$ is the strong $[(\alpha, \beta)]$-level set of τ. Then $\check{\tau}$ is a bi-intuitionistic fuzzy topology such that $\check{\tau}\Big|_{\mathcal{P}(\mathcal{X})} = \tau$. To show this, it suffices to check all conditions in Definition 5.

(1) Since $\hat{1}(x) = (1, 0)$ and $(\alpha, \beta) \leq (1, 0)$, then $\hat{1}_{[(\alpha,\beta)]} = \mathcal{X}$ and $\tau(\hat{1}) = (1, 0)$. Hence $[\hat{1} \in \check{\tau}] = \check{\tau}(\hat{1}) = \bigwedge_{(\alpha,\beta) \in \mathcal{Z}} (1, 0) = (1, 0)$. Similarly, $[\hat{0} \in \check{\tau}] = (1, 0)$.

(2)

$$\begin{aligned}
\check{\tau}(\hat{\mathcal{A}} \cap \hat{\mathcal{B}}) &= \bigwedge_{(\alpha,\beta) \in \mathcal{Z}} \tau((\hat{\mathcal{A}} \cap \hat{\mathcal{B}})_{[(\alpha,\beta)]}) \\
&= \bigwedge_{(\alpha,\beta) \in \mathcal{Z}} \tau\Big(\hat{\mathcal{A}}_{[(\alpha,\beta)]} \cap \hat{\mathcal{B}}_{[(\alpha,\beta)]}\Big) \\
&\geq \bigwedge_{(\alpha,\beta) \in \mathcal{Z}} \Big(\tau(\hat{\mathcal{A}}_{[(\alpha,\beta)]}) \wedge \tau(\hat{\mathcal{B}}_{[(\alpha,\beta)]})\Big) \\
&= \bigwedge_{(\alpha,\beta) \in \mathcal{Z}} \tau(\hat{\mathcal{A}}_{[(\alpha,\beta)]}) \wedge \bigwedge_{(\alpha,\beta) \in \mathcal{Z}} \tau(\hat{\mathcal{B}}_{[(\alpha,\beta)]}) \\
&= \check{\tau}(\hat{\mathcal{A}}) \wedge \check{\tau}(\hat{\mathcal{B}}).
\end{aligned}$$

(3)

$$\check{\tau}\left(\bigcup_{\lambda\in\Lambda}\hat{\mathcal{A}}_\lambda\right) = \bigwedge_{(\alpha,\beta)\in\mathcal{Z}}\tau\left(\bigcup_{\lambda\in\Lambda}(\hat{\mathcal{A}}_\lambda)_{[(\alpha,\beta)]}\right)$$
$$\geq \bigwedge_{(\alpha,\beta)\in\mathcal{Z}}\bigwedge_{\lambda\in\Lambda}\tau\Big((\hat{\mathcal{A}}_\lambda)_{[(\alpha,\beta)]}\Big) = \bigwedge_{\lambda\in\Lambda}\check{\tau}(\hat{\mathcal{A}}_\lambda).$$

Definition 6. *The family of bi-intuitionistic fuzzy closed sets, denoted by $\mathcal{F}\in\mathcal{IF}(\mathcal{IF}(\mathcal{X}))$, is defined as $\hat{\mathcal{A}}\in\mathcal{F} := \neg(\hat{\mathcal{A}})\in\tau$, i.e., $\mathcal{F}(\hat{\mathcal{A}}) = \tau(\neg(\hat{\mathcal{A}}))$, or equivalently $m\mathcal{F}(\hat{\mathcal{A}}) = m\tau(\neg(\hat{\mathcal{A}}))$ and $n\mathcal{F}(\hat{\mathcal{A}}) = n\tau(\neg(\hat{\mathcal{A}}))$.*

Theorem 7. *(1) $\mathcal{F}(\hat{1}) = \mathcal{F}(\hat{0}) = (1,0)$;*
(2) for any $\hat{\mathcal{A}},\hat{\mathcal{B}}\in\mathcal{IF}(\mathcal{X})$, $\models \hat{\mathcal{A}}\in\mathcal{F}\wedge\hat{\mathcal{B}}\in\mathcal{F}\dot{\to}\hat{\mathcal{A}}\cup\hat{\mathcal{B}}\in\mathcal{F}$;
(3) for any $\{\hat{\mathcal{A}}_\lambda : \lambda\in\Lambda\}\subseteq\mathcal{IF}(\mathcal{X})$,

$$\models \forall\lambda\big(\lambda\in\Lambda\dot{\to}\hat{\mathcal{A}}_\lambda\in\mathcal{F}\big)\to\bigcup_{\lambda\in\Lambda}\hat{\mathcal{A}}_\lambda\in\mathcal{F}.$$

In the sequel, we discuss only intuitionistic fuzzifying topological spaces as a preliminary of the research on bi-intuitionistic fuzzy topology.

5. Intuitionistic Fuzzifying Neighborhood Structure of a Point

In this section, we introduce the definition of the intuitionistic fuzzifying neighborhood structure of a point and study its properties.

Definition 7. *Let $x\in\mathcal{X}$. The intuitionistic fuzzifying neighborhood system of x, denoted by $\hat{\mathcal{N}}_x\in\mathcal{IF}(\mathcal{P}(\mathcal{X}))$, is defined as follows:*

$$\mathcal{A}\in\hat{\mathcal{N}}_x := \exists\mathcal{B}(\mathcal{B}\in\tau\wedge\mathcal{B}\in\mathcal{M}(x,\mathcal{A})),$$

where $\mathcal{M}(x,\mathcal{A}) = \{\mathcal{B} : \mathcal{B}\subseteq\mathcal{X}\ ,\ x\in\mathcal{B}\subseteq\mathcal{A}\}$.

Lemma 6. $\bigwedge_{x\in\mathcal{A}}\bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B}) = \tau(\mathcal{A})$.

Proof. First, for each $x\in\mathcal{A}$, we have $\bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B})\geq\tau(\mathcal{A})$. So,

$$\bigwedge_{x\in\mathcal{A}}\bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B})\geq\tau(\mathcal{A}).$$

Second, for any $f\in\prod_{x\in\mathcal{A}}\mathcal{M}(x,\mathcal{A})$ we have $\bigcup_{x\in\mathcal{A}}f(x) = \mathcal{A}$. Therefore

$$\tau(\mathcal{A}) = \tau\left(\bigcup_{x\in\mathcal{A}}f(x)\right)\geq\bigwedge_{x\in\mathcal{A}}\tau(f(x)).$$

So,

$$\tau(\mathcal{A})\geq\bigvee_{f\in\prod_{x\in\mathcal{A}}\mathcal{M}(x,\mathcal{A})}\bigwedge_{x\in\mathcal{A}}\tau(f(x)) = \bigwedge_{x\in\mathcal{A}}\bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B}).$$

□

Corollary 1. $\tau(\mathcal{A}) = \bigwedge_{x\in\mathcal{A}}\hat{\mathcal{N}}_x(\mathcal{A})$.

Theorem 8. *For any $x \in \mathcal{X}$ and $\mathcal{A} \in \mathcal{P}(\mathcal{X})$,*

$$\models \mathcal{A} \in \tau \longleftrightarrow \forall x\big(x \in \mathcal{A} \dot{\to} \exists \mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}_x} \wedge \mathcal{B} \subseteq \mathcal{A}\big)\big).$$

Proof.

$$\begin{aligned}
\big[\forall x\big(x \in \mathcal{A} \dot{\to} \exists \mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}_x} \wedge \mathcal{B} \subseteq \mathcal{A}\big)\big)\big] &= \bigwedge_{x\in\mathcal{A}} \bigvee_{\mathcal{B}\subseteq\mathcal{A}} \hat{\mathcal{N}_x}(\mathcal{B}) \\
&= \bigwedge_{x\in\mathcal{A}} \bigvee_{\mathcal{B}\subseteq\mathcal{A}} \bigvee_{\mathcal{C}\in\mathcal{M}(x,\mathcal{B})} \tau(\mathcal{C}) \\
&= \bigwedge_{x\in\mathcal{A}} \bigvee_{\mathcal{C}\in\mathcal{M}(x,\mathcal{A})} \tau(\mathcal{C}) \\
&= \bigwedge_{x\in\mathcal{A}} \hat{\mathcal{N}_x}(\mathcal{A}) = \tau(\mathcal{A}).
\end{aligned}$$

□

Theorem 9. *The mapping $\hat{\mathcal{N}} : \mathcal{X} \to \mathcal{IF}^{\mathcal{N}}(\mathcal{X})$, $x \mapsto \hat{\mathcal{N}_x}$, has the following properties:*

(1) For any $x \in \mathcal{A}$, $\models \mathcal{A} \in \hat{\mathcal{N}_x} \to x \in \mathcal{A}$;

(2) For any $x \in \mathcal{A}, \mathcal{B}$, $\models \mathcal{A} \in \hat{\mathcal{N}_x} \wedge \mathcal{B} \in \hat{\mathcal{N}_x} \to \mathcal{A} \cap \mathcal{B} \in \hat{\mathcal{N}_x}$;

(3) For any $x \in \mathcal{A}, \mathcal{B}$, $\models \mathcal{A} \subseteq \mathcal{B} \dot{\to} \big(\mathcal{A} \in \hat{\mathcal{N}_x} \to \mathcal{B} \in \hat{\mathcal{N}_x}\big)$;

(4) For any $x, \mathcal{A}$, $\models \mathcal{A} \in \hat{\mathcal{N}_x} \to \exists \mathcal{C}\big((\mathcal{C} \in \hat{\mathcal{N}_x}) \wedge (\mathcal{C} \subseteq \mathcal{A}) \wedge \forall(y \in \mathcal{C} \dot{\to} \mathcal{C} \in \hat{\mathcal{N}_y})\big)$.

Conversely, if a mapping $\hat{\mathcal{N}}$ satisfies (2) and (3), then τ is an intuitionistic fuzzifying topology which is defined $\mathcal{A} \in \tau := \forall x\big(x \in \mathcal{A} \dot{\to} \mathcal{A} \in \hat{\mathcal{N}_x}\big)$.

Specially, if it satisfies (1) and (4) also, then for any $x \in \mathcal{X}$, $\hat{\mathcal{N}_x}$ is the neighborhood system of x with respect to τ.

Proof. (A) Since $\hat{\mathcal{N}_x}(\mathcal{X}) = \bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{X})} \tau(\mathcal{B}) \geq \tau(\mathcal{X}) = (1,0)$, then $\hat{\mathcal{N}_x}$ is normal.

(1) If $\mathcal{A}(x) = (1,0)$, then $\mathcal{A}(x) \geq \hat{\mathcal{N}_x}(\mathcal{A})$. Now, suppose $\mathcal{A}(x) = (0,1)$. We need to prove that $\hat{\mathcal{N}_x}(\mathcal{A}) = (0,1)$. Indeed,

$$\begin{aligned}
\hat{\mathcal{N}_x}(\mathcal{A}) &= \bigvee_{\mathcal{B}\in\mathcal{P}(\mathcal{X})} (\tau(\mathcal{B}) \wedge x \in \mathcal{B} \wedge \mathcal{B} \subseteq \mathcal{A}) \\
&= \bigvee_{x\in\mathcal{B}} (\tau(\mathcal{B}) \wedge \mathcal{B} \subseteq \mathcal{A}) = \bigvee_{x\in\mathcal{B}} (\tau(\mathcal{B}) \wedge (0,1)) \\
&= \bigvee_{x\in\mathcal{B}} (m\tau(\mathcal{B}) \wedge 0, n\tau(\mathcal{B}) \vee 1) \\
&= \bigvee_{x\in\mathcal{B}} (0,1) = (0,1).
\end{aligned}$$

(2)

$$\begin{aligned}
\hat{\mathcal{N}_x}(\mathcal{A} \cap \mathcal{B}) &= \bigvee_{x\in\mathcal{C}\subseteq\mathcal{A}\cap\mathcal{B}} \tau(\mathcal{C}) \\
&= \bigvee_{x\in\mathcal{C}_1\subseteq\mathcal{A},\ x\in\mathcal{C}_2\subseteq\mathcal{B}} \tau(\mathcal{C}_1 \cap \mathcal{C}_2) \\
&\geq \bigvee_{x\in\mathcal{C}_1\subseteq\mathcal{A},\ x\in\mathcal{C}_2\subseteq\mathcal{B}} (\tau(\mathcal{C}_1) \wedge \tau(\mathcal{C}_2)) \\
&= \bigvee_{x\in\mathcal{C}_1\subseteq\mathcal{A}} \tau(\mathcal{C}_1) \wedge \bigvee_{x\in\mathcal{C}_2\subseteq\mathcal{B}} \tau(\mathcal{C}_2) \\
&= \bigvee_{\mathcal{C}_1\in\mathcal{M}(x,\mathcal{A})} \tau(\mathcal{C}_1) \wedge \bigvee_{\mathcal{C}_2\in\mathcal{M}(x,\mathcal{B})} \tau(\mathcal{C}_2) = \hat{\mathcal{N}_x}(\mathcal{A}) \wedge \hat{\mathcal{N}_x}(\mathcal{B}).
\end{aligned}$$

(3) If $[\mathcal{A} \subseteq \mathcal{B}] = (0,1)$, then the result holds. Suppose $[\mathcal{A} \subseteq \mathcal{B}] = (1,0)$ and so to complete the proof we need to prove that $\hat{\mathcal{N}_x}(\mathcal{A}) \leq \hat{\mathcal{N}_x}(\mathcal{B})$. Now, $\mathcal{M}(x,\mathcal{A}) \subseteq \mathcal{M}(x,\mathcal{B})$.
Then

$$\hat{\mathcal{N}_x}(\mathcal{A}) = \bigvee_{\mathcal{C}\in\mathcal{M}(x,\mathcal{A})} \tau(\mathcal{C}) \leq \bigvee_{\mathcal{C}\in\mathcal{M}(x,\mathcal{B})} \tau(\mathcal{C}) = \hat{\mathcal{N}_x}(\mathcal{B}).$$

Thus $[\mathcal{A} \in \hat{\mathcal{N}_x} \dot{\to} \mathcal{B} \in \hat{\mathcal{N}_x}] = (1,0)$.
(4)

$$\begin{aligned}
&\left[\exists \mathcal{C}\left((\mathcal{C} \in \hat{\mathcal{N}_x}) \wedge (\mathcal{C} \subseteq \mathcal{A}) \wedge \forall (y \in \mathcal{C} \dot{\to} \mathcal{C} \in \hat{\mathcal{N}_y})\right)\right] \\
&\quad = \bigvee_{\mathcal{C}\subseteq\mathcal{A}} \left(\hat{\mathcal{N}_x}(\mathcal{C}) \wedge \bigwedge_{y\in\mathcal{C}} \hat{\mathcal{N}_y}(\mathcal{C})\right) \\
&\quad = \bigvee_{\mathcal{C}\subseteq\mathcal{A}} \left(\hat{\mathcal{N}_x}(\mathcal{C}) \wedge \tau(\mathcal{C})\right) \\
&\quad = \bigvee_{\mathcal{C}\subseteq\mathcal{A}} \tau(\mathcal{C}) \geq \bigvee_{x\in\mathcal{C}\subseteq\mathcal{A}} \tau(\mathcal{C}) \\
&\quad = \bigvee_{\mathcal{C}\in\mathcal{M}(x,\mathcal{A})} \tau(\mathcal{C}) = \hat{\mathcal{N}_x}(\mathcal{A}).
\end{aligned}$$

(B) Conversely, we prove that $\tau(\mathcal{A}) = \bigwedge_{x\in\mathcal{A}} \hat{\mathcal{N}_x}(\mathcal{A})$ is an intuitionistic fuzzifying topology.

(1) (a) $\tau(\phi) = \bigwedge_{x\in\phi} \hat{\mathcal{N}_x}(\phi) = (1,0)$.

(b) For any $x \in \mathcal{X}$ and since $\mathcal{N}_x$ is normal, then there exists $\mathcal{A} \in \mathcal{P}(\mathcal{X})$ such that $\hat{\mathcal{N}_x}(\mathcal{A}) = (1,0)$. Thus, from Condition (3) we have $\hat{\mathcal{N}_x}(\mathcal{X}) \geq \hat{\mathcal{N}_x}(\mathcal{A}) = (1,0)$. Therefore, $\tau(\mathcal{X}) = \bigwedge_{x\in\mathcal{X}} \hat{\mathcal{N}_x}(\mathcal{X}) = (1,0)$.

(2) Applying Condition (2) we have

$$\begin{aligned}
\tau(\mathcal{A}\cap\mathcal{B}) &= \bigwedge_{x\in\mathcal{A}\cap\mathcal{B}} \hat{\mathcal{N}_x}(\mathcal{A}\cap\mathcal{B}) \\
&\geq \bigwedge_{x\in\mathcal{A}\cap\mathcal{B}} \left(\hat{\mathcal{N}_x}(\mathcal{A}) \wedge \hat{\mathcal{N}_x}(\mathcal{B})\right) \\
&= \bigwedge_{x\in\mathcal{A}\cap\mathcal{B}} \hat{\mathcal{N}_x}(\mathcal{A}) \wedge \bigwedge_{x\in\mathcal{A}\cap\mathcal{B}} \hat{\mathcal{N}_x}(\mathcal{B}) \\
&\geq \bigwedge_{x\in\mathcal{A}} \hat{\mathcal{N}_x}(\mathcal{A}) \wedge \bigwedge_{x\in\mathcal{B}} \hat{\mathcal{N}_x}(\mathcal{B}) = \tau(\mathcal{A}) \wedge \tau(\mathcal{B}).
\end{aligned}$$

(3) Applying Condition (3) we have

$$\begin{aligned}
\tau\left(\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda\right) &= \bigwedge_{x\in \bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda} \hat{\mathcal{N}_x}\left(\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda\right) \\
&= \bigwedge_{\lambda\in\Lambda} \bigwedge_{x\in\mathcal{A}_\lambda} \hat{\mathcal{N}_x}\left(\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda\right) \\
&\geq \bigwedge_{\lambda\in\Lambda} \bigwedge_{x\in\mathcal{A}_\lambda} \hat{\mathcal{N}_x}(\mathcal{A}_\lambda) = \bigwedge_{\lambda\in\Lambda} \tau(\mathcal{A}_\lambda).
\end{aligned}$$

(4) From Condition (4), $\hat{\mathcal{N}}_x(\mathcal{A}) \leq \bigvee_{\mathcal{B}\subseteq\mathcal{A}}\left(\hat{\mathcal{N}}_x(\mathcal{B}) \wedge \bigwedge_{y\in\mathcal{B}}\hat{\mathcal{N}}_y(\mathcal{B})\right)$ and from Condition (1), if $[x \in \mathcal{B}] = (0,1)$, then $\hat{\mathcal{N}}_x(\mathcal{B}) = (0,1)$. So,

$$\hat{\mathcal{N}}_x(\mathcal{A}) \leq \bigvee_{x\in\mathcal{B}\subseteq\mathcal{A}}\left(\hat{\mathcal{N}}_x(\mathcal{B}) \wedge \bigwedge_{y\in\mathcal{B}}\hat{\mathcal{N}}_y(\mathcal{B})\right) = \bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\bigwedge_{y\in\mathcal{B}}\hat{\mathcal{N}}_y(\mathcal{B}) = \bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B}).$$

On the other hand, for each $\mathcal{B} \in \mathcal{M}(x,\mathcal{A})$, from Condition (3) we obtain $\bigwedge_{y\in\mathcal{B}}\hat{\mathcal{N}}_y(\mathcal{B}) \leq \hat{\mathcal{N}}_x(\mathcal{B}) \leq \hat{\mathcal{N}}_x(\mathcal{A})$. Therefore,

$$\hat{\mathcal{N}}_x(\mathcal{A}) \geq \bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\bigwedge_{y\in\mathcal{B}}\hat{\mathcal{N}}_y(\mathcal{B}) = \bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B}).$$

Therefore, we obtain $\hat{\mathcal{N}}_x(\mathcal{A}) = \bigvee_{\mathcal{B}\in\mathcal{M}(x,\mathcal{A})}\tau(\mathcal{B})$. □

6. Intuitionistic Fuzzifying Fundamental Concepts

In this section, the concepts of intuitionistic fuzzifying closure, intuitionistic fuzzifying boundary, intuitionistic fuzzifying derived set, and intuitionistic fuzzifying interior are investigated. Additionally, the relations among these concepts are derived.

Definition 8. *The intuitionistic fuzzifying derived (resp. closure, interior, boundary) operation will be denoted by $\hat{\mathcal{D}}$ (resp. $\widehat{Cl}$, $\widehat{Int}$, $\hat{b}$) $\in (\mathcal{IF}(\mathcal{X}))^{\mathcal{P}(\mathcal{X})}$ and defined as follows:*
$x \in \hat{\mathcal{D}}(\mathcal{A}) := \forall\mathcal{B}(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} \mathcal{B} \cap (\mathcal{A} - \{x\}) \neq \phi)$;
$x \in \widehat{Cl}(\mathcal{A}) := \forall\mathcal{B}((\mathcal{B} \supseteq \mathcal{A}) \wedge \mathcal{B} \in \mathcal{F} \dot{\to} x \in \mathcal{B})$;
$x \in \widehat{Int}(\mathcal{A}) := \mathcal{A} \in \widehat{\mathcal{N}_x}$;
$x \in \hat{b}(\mathcal{A}) := x \in \widehat{Cl}(\mathcal{A}) \wedge \widehat{Cl}(\mathcal{X} - \mathcal{A})$.

Theorem 10. *For any $x, \mathcal{A}$ (1) $\hat{\mathcal{D}}(\mathcal{A})(x) = \neg(\hat{\mathcal{N}}_x((\mathcal{X} - \mathcal{A}) \cup \{x\})$;*
(2) $\models \mathcal{A} \in \mathcal{F} \longleftrightarrow \hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}$.

Proof. (1)

$$\begin{aligned}
\hat{\mathcal{D}}(\mathcal{A})(x) &= \bigwedge_{\mathcal{B}\cap(\mathcal{A}-\{x\})=\phi} \neg(\hat{\mathcal{N}}_x(\mathcal{B})) \\
&= \neg(\bigvee_{\mathcal{B}\subseteq(\mathcal{X}-\mathcal{A})\cup\{x\}} \hat{\mathcal{N}}_x(\mathcal{B})) \\
&= \neg(\hat{\mathcal{N}}_x((\mathcal{X} - \mathcal{A}) \cup \{x\})).
\end{aligned}$$

(2)

$$\begin{aligned}
[\hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}] &= \left[\forall x\left(x \in \hat{\mathcal{D}}(\mathcal{A}) \dot{\to} x \in \mathcal{A}\right)\right] \\
&= \bigwedge_{x\in\mathcal{X}-\mathcal{A}} \neg(\hat{\mathcal{D}}(\mathcal{A}))(x) \\
&= \bigwedge_{x\in\mathcal{X}-\mathcal{A}} \hat{\mathcal{N}}_x((\mathcal{X} - \mathcal{A}) \cup \{x\}) \\
&= \bigwedge_{x\in\mathcal{X}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) = \tau(\mathcal{X} - \mathcal{A}) = \mathcal{F}(\mathcal{A}).
\end{aligned}$$

□

Lemma 7. $\models \widehat{Cl}(\mathcal{A}) \equiv \neg((\mathcal{X} - \mathcal{A}) \in \widehat{\mathcal{N}_x})$.

Proof.

$$\begin{aligned}\hat{C}l(\mathcal{A})(x) &= \bigwedge_{x\in\mathcal{X}-\mathcal{B}\subseteq\mathcal{X}-\mathcal{A}} \neg(\mathcal{F}(\mathcal{B})) \\ &= \neg\left(\bigvee_{x\in\mathcal{X}-\mathcal{B}\subseteq\mathcal{X}-\mathcal{A}} \mathcal{F}(\mathcal{B})\right) \\ &= \neg\left(\bigvee_{x\in\mathcal{X}-\mathcal{B}\subseteq\mathcal{X}-\mathcal{A}} \tau(\mathcal{X}-\mathcal{B})\right) \\ &= \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})).\end{aligned}$$

□

Corollary 2. *(1)* $\models \widehat{Cl}(\mathcal{A}) \doteq \neg((\mathcal{X}-\mathcal{A}) \in \widehat{\mathcal{N}_x})$;
(2) $\models \widehat{Cl}(\mathcal{A}) \approx \neg((\mathcal{X}-\mathcal{A}) \in \widehat{\mathcal{N}_x})$

Theorem 11. *(1)* $\widehat{Cl}(\phi) = \phi$;
(2) For any $\mathcal{A}, \models \mathcal{A} \Subset \widehat{Cl}(\mathcal{A})$;
(3) For any $\mathcal{A}, \mathcal{B}$, *if* $[\mathcal{A} \subseteq \mathcal{B}] = (1,0)$, *then* $\models \widehat{Cl}(\mathcal{A}) \subseteq \widehat{Cl}(\mathcal{B})$;
(4) For any, $\models \widehat{Cl}(\mathcal{A}\cup\mathcal{B}) \equiv \widehat{Cl}(\mathcal{A}) \cup \widehat{Cl}(\mathcal{B})$.

Proof. (1) $\widehat{Cl}(\phi)(x) = \neg(\hat{\mathcal{N}}_x(\mathcal{X})) = \neg(1,0) = (0,1) = \phi(x)$.
(2)

$$\begin{aligned}[\mathcal{A} \Subset \widehat{Cl}(\mathcal{A})] &= \left[\forall x\left(x \in \mathcal{A} \dot{\rightarrow} x \in \widehat{Cl}(\mathcal{A})\right)\right] \\ &= \bigwedge_{x\in\mathcal{A}} \widehat{Cl}(\mathcal{A})(x) = \bigwedge_{x\in\mathcal{A}} \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})) \\ &= \neg\left(\bigvee_{x\in\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})\right) = \neg(0,1) = (1,0).\end{aligned}$$

(3) Suppose that $[\mathcal{A} \subseteq \mathcal{B}] = (1,0)$. Then, from Theorem 9 (3), we have $\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{B}) \leq \hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})$. Hence $\neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})) \leq \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{B}))$. Therefore, $\widehat{Cl}(\mathcal{A})(x) \leq \widehat{Cl}(\mathcal{B})(x)$. Therefore $\models \widehat{Cl}(\mathcal{A}) \subseteq \widehat{Cl}(\mathcal{B})$.

(4) Applying (3) above, we obtain $\models \widehat{Cl}(\mathcal{A}) \cup \widehat{Cl}(\mathcal{B}) \subseteq \widehat{Cl}(\mathcal{A}\cup\mathcal{B})$. Now, we need to prove that $\models \widehat{Cl}(\mathcal{A}\cup\mathcal{B}) \subseteq \widehat{Cl}(\mathcal{A}) \cup \widehat{Cl}(\mathcal{B})$. For any $x \in \mathcal{X}$,

$$\begin{aligned}\widehat{Cl}(\mathcal{A}\cup\mathcal{B})(x) &= \neg(\hat{\mathcal{N}}_x(\mathcal{X}-(\mathcal{A}\cup\mathcal{B}))) \\ &= \neg(\hat{\mathcal{N}}_x((\mathcal{X}-\mathcal{A})\cap(\mathcal{X}-\mathcal{B}))) \\ &\leq \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A}) \wedge \hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{B})) \\ &= \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})) \vee \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{B})) \\ &= \widehat{Cl}(\mathcal{A})(x) \vee \widehat{Cl}(\mathcal{B})(x).\end{aligned}$$

□

Theorem 12. *For any* $x, \mathcal{A}$,
(1) $\models \widehat{Cl}(\mathcal{A}) \approx \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})$, $\models \widehat{Cl}(\mathcal{A}) \doteq \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})$ *and* $\models \widehat{Cl}(\mathcal{A}) \equiv \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})$;
(2) $\models x \in \widehat{Cl}(\mathcal{A}) \longleftrightarrow \forall\mathcal{B}(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\rightarrow} \mathcal{A}\cap\mathcal{B} \neq \phi)$;
(3) $\models \mathcal{A} \in \mathcal{F} \rightarrow \widehat{Cl}(\mathcal{A}) \doteq \mathcal{A}$;
(4) $\models \mathcal{A} \in \mathcal{F} \rightarrow \widehat{Cl}(\mathcal{A}) \equiv \mathcal{A}$.

Proof. (1) If $\mathcal{A}(x) = (1,0)$, then $\widehat{Cl}(\mathcal{A})(x) = \neg(\hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})) = \neg(0,1) = (1,0)$, and so $\widehat{Cl}(\mathcal{A})(x) = \mathcal{A}(x) = \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})(x)$. If $\mathcal{A}(x) = (0,1)$, then $\widehat{Cl}(\mathcal{A})(x) = \neg(\hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})) = \neg(\hat{\mathcal{N}}_x((\mathcal{X} - \mathcal{A}) \cup \{x\})) = \hat{\mathcal{D}}(\mathcal{A})(x) = (\mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}))(x)$. Therefore, by Lemma 5, the results hold.

(2)

$$\begin{aligned}
\big[\forall\mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} \mathcal{A} \cap \mathcal{B} \neq \phi\big)\big] &= \bigwedge_{\mathcal{A}\cap\mathcal{B}=\phi} \neg(\hat{\mathcal{N}}_x(\mathcal{B})) \\
&= \neg\left(\bigvee_{\mathcal{A}\cap\mathcal{B}=\phi} (\hat{\mathcal{N}}_x(\mathcal{B}))\right) \\
&= \neg\left(\bigvee_{\mathcal{B}\subseteq\mathcal{X}-\mathcal{A}} (\hat{\mathcal{N}}_x(\mathcal{B}))\right) \\
&= \neg\big(\hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\big) = \widehat{Cl}(\mathcal{A})(x).
\end{aligned}$$

(3)

$$\begin{aligned}
\mathcal{F}(\mathcal{A}) = [\hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}] = [\hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}] \wedge [\mathcal{A} \Subset \mathcal{A}] &\leq [\mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}] \\
&= [\mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}] \wedge [\mathcal{A} \Subset \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})] \leq [\mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \doteq \mathcal{A}] \\
&= [\widehat{Cl}(\mathcal{A}) \doteq \mathcal{A}].
\end{aligned}$$

(4) $\models \mathcal{A} \in \mathcal{F} \to \widehat{Cl}(\mathcal{A}) \doteq \mathcal{A} \to \widehat{Cl}(\mathcal{A}) \equiv \mathcal{A}$. □

Theorem 13. *For any* $\mathcal{A}, \mathcal{B}, \models \mathcal{B} \approx \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \to \mathcal{B} \in \mathcal{F}$.

Proof. If $[\mathcal{A} \subseteq \mathcal{B}] = (0,1)$, then $[\mathcal{B} \approx \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})] = (0,1)$. Suppose $[\mathcal{A} \subseteq \mathcal{B}] = (1,0)$. Then

$$\begin{aligned}
[\mathcal{B} \Subset \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})] &= \big[\forall x\big(x \in \mathcal{B} \dot{\to} x \in \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})\big)\big] \\
&= \bigwedge_{x\in\mathcal{B}} [x \in \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})] \\
&= \bigwedge_{x\in\mathcal{B}} \big(m\mathcal{A}(x) \vee m\hat{\mathcal{D}}(\mathcal{A})(x), n\mathcal{A}(x) \wedge n\hat{\mathcal{D}}(\mathcal{A})(x)\big) \\
&= \bigwedge_{x\in\mathcal{B}-\mathcal{A}} (m\hat{\mathcal{D}}(\mathcal{A})(x), n\hat{\mathcal{D}}(\mathcal{A})(x)) = \bigwedge_{x\in\mathcal{B}-\mathcal{A}} \hat{\mathcal{D}}(\mathcal{A})(x) \\
&= \bigwedge_{x\in\mathcal{B}-\mathcal{A}} \neg\big(\hat{\mathcal{N}}_x((\mathcal{X} - \mathcal{A}) \cup \{x\})\big) \\
&= \bigwedge_{x\in\mathcal{B}-\mathcal{A}} \neg\big(\hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\big) \\
&= \neg\left(\bigvee_{x\in\mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\right).
\end{aligned}$$

$$
\begin{aligned}
[\mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{B}] &= \big[\forall x\big(x \in \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \dot{\rightarrow} x \in \mathcal{B}\big)\big] \\
&= \big[\forall x\big(x \notin \mathcal{B} \dot{\rightarrow} \neg\big(x \in \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})\big)\big)\big] \\
&= \bigwedge_{x \notin \mathcal{B}} \neg\big(x \in \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})\big) \\
&= \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \neg\big(m\mathcal{A}(x) \vee m\hat{\mathcal{D}}(\mathcal{A})(x), n\mathcal{A}(x) \wedge n\hat{\mathcal{D}}(\mathcal{A})(x)\big) \\
&= \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \neg\big(m\hat{\mathcal{D}}(\mathcal{A})(x), n\hat{\mathcal{D}}(\mathcal{A})(x)\big) \\
&= \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \neg(\hat{\mathcal{D}}(\mathcal{A})(x)) \\
&= \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}).
\end{aligned}
$$

Suppose $\mathcal{B} \dot{\approx} \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) > (t, t_o)$. Then

$$
\Bigg(\max\Bigg(0, m\neg\big(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\big) + m \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) - 1\Bigg),
$$
$$
\min\Bigg(1, n\Bigg(\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg)\Bigg) + n \bigwedge_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg)\Bigg) > (t, t_o).
$$

First,

$$
m\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg) + m \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) - 1 > t.
$$

Hence

$$
m \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) > 1 - m\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg) + t.
$$

For any $x \in \mathcal{X} - \mathcal{B}$,

$$
m\hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) > 1 - m\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg) + t,
$$

then

$$
m\Bigg(\bigvee_{x \in \mathcal{C} \subseteq \mathcal{X}-\mathcal{A}} \tau(\mathcal{C})\Bigg) > 1 - m\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg) + t,
$$

i.e., there exists $\mathcal{C}_x$ such that $x \in \mathcal{C}_x \subseteq \mathcal{X} - \mathcal{A}$ and

$$
m\tau(\mathcal{C}_x) > 1 - m\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg) + t > t.
$$

Therefore,

$$
m\mathcal{F}(\mathcal{B}) = m\tau(\mathcal{X} - \mathcal{B}) = m \bigwedge_{x \in \mathcal{X}-\mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{B}) \geq m\Bigg(\bigwedge_{x \in \mathcal{X}-\mathcal{B}} \tau(\mathcal{C}_x)\Bigg) > t.
$$

Second, $n\Big(\neg\Big(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Big)\Big) + n \bigwedge_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) < t_o$. Hence,

$$
n \bigwedge_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A}) < t_o - n\Bigg(\neg\Bigg(\bigvee_{x \in \mathcal{B}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\Bigg)\Bigg).
$$

There exists $x' \in \mathcal{X} - \mathcal{B}$ such that

$$n\big(\hat{\mathcal{N}}_{x'}(\mathcal{X} - \mathcal{A})\big) < t_o - n\left(\neg\left(\bigvee_{x \in \mathcal{B} - \mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\right)\right).$$

Thus

$$n\left(\bigvee_{x' \in \mathcal{C} \subseteq \mathcal{X} - \mathcal{A}} \tau(\mathcal{C})\right) < t_o - n\left(\neg\left(\bigvee_{x \in \mathcal{B} - \mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{A})\right)\right).$$

So, $n(\tau(\mathcal{C})) < t_o$. Therefore,

$$\begin{aligned} n\mathcal{F}(\mathcal{B}) &= n\tau(\mathcal{X} - \mathcal{B}) \\ &= n\left(\bigwedge_{x \in \mathcal{X} - \mathcal{B}} \mathcal{N}_x(\mathcal{X} - \mathcal{B})\right) \le n\big(\hat{\mathcal{N}}_x(\mathcal{X} - \mathcal{B})\big) \\ &\le n\left(\bigvee_{x \in \mathcal{C} \subseteq \mathcal{X} - \mathcal{A}} \tau(\mathcal{C})\right) \le n\tau(\mathcal{C}) < t_o. \end{aligned}$$

Hence $\mathcal{F}(\mathcal{B}) = (m\mathcal{F}(\mathcal{B}),\ n\mathcal{F}(\mathcal{B})) > (t, t_o)$. Therefore, $\mathcal{F}(\mathcal{B}) \ge [\mathcal{B} \approx \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A})]$. We obtain $\models \mathcal{B} \approx \mathcal{A} \cup \hat{\mathcal{D}}(\mathcal{A}) \rightarrow \mathcal{B} \in \mathcal{F}$. $\square$

Theorem 14. *For any $x, \mathcal{A}, \mathcal{B}$,*

(1) $\models (\mathcal{B} \in \tau) \wedge (\mathcal{B} \subseteq \mathcal{A}) \rightarrow \mathcal{B} \Subset \widehat{Int}(\mathcal{A})$;

(2) $\models \mathcal{A} \doteqdot \widehat{Int}(\mathcal{A}) \longleftrightarrow \mathcal{A} \in \tau$, and the statement is true when we replace "$\doteqdot$" by "$\approx$";

(3) $\models x \in \widehat{Int}(\mathcal{A}) \longleftrightarrow (x \in \mathcal{A}) \wedge (x \in \neg(\hat{\mathcal{D}}(\mathcal{X} - \mathcal{A})))$, and the statement is true when we replace "$\wedge$" by "$\barwedge$";

(4) $\models \widehat{Int}(\mathcal{A}) \equiv \neg(\widehat{Cl}(\mathcal{X} - \mathcal{A}))$, and the statement is true when we replace "$\equiv$" by "$\doteqdot$" or by "$\approx$";

(5) $\models \mathcal{B} \doteqdot \widehat{Int}(\mathcal{A}) \rightarrow \mathcal{B} \in \tau$, and the statement is true when we replace "$\doteqdot$" by "$\approx$".

Proof. (1) If $[\mathcal{B} \subseteq \mathcal{A}] = (0,1)$, then

$$[(\mathcal{B} \in \tau) \wedge (\mathcal{B} \subseteq \mathcal{A})] = \tau(\mathcal{B}) \wedge (0,1) = (0,1).$$

If $[\mathcal{B} \subseteq \mathcal{A}] = (1,0)$, then

$$\begin{aligned} [\mathcal{B} \Subset \widehat{Int}(\mathcal{A})] &= \Big[\forall x\Big(x \in \mathcal{B} \dot{\rightarrow} x \in \widehat{Int}(\mathcal{A})\Big)\Big] \\ &\le \bigwedge_{x \in \mathcal{B}} [(1,0) \dot{\rightarrow} \hat{\mathcal{N}}_x(\mathcal{A})] \\ &= \bigwedge_{x \in \mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{A}) \ge \bigwedge_{x \in \mathcal{B}} \hat{\mathcal{N}}_x(\mathcal{B}) \\ &= \tau(\mathcal{B}) = [\mathcal{B} \in \tau] = [(\mathcal{B} \in \tau) \wedge (1,0)] \\ &= [(\mathcal{B} \in \tau) \wedge (\mathcal{B} \subseteq \mathcal{A})]. \end{aligned}$$

(2) $[\mathcal{A} \doteq \widehat{Int}(\mathcal{A})] = [\mathcal{A} \Subset \widehat{Int}(\mathcal{A})] \wedge [\widehat{Int}(\mathcal{A}) \Subset \mathcal{A}]$. First,

$$\begin{aligned}
[\widehat{Int}(\mathcal{A}) \Subset \mathcal{A}] &= \left[\forall x\left(x \in \widehat{Int}(\mathcal{A}) \dot{\to} x \in \mathcal{A}\right)\right] \\
&= \bigwedge_{x \in \mathcal{X}-\mathcal{A}} [(1,0) \dot{\to} \neg(\hat{\mathcal{N}}_x(\mathcal{A}))] \\
&= \bigwedge_{x \in \mathcal{X}-\mathcal{A}} \neg(\hat{\mathcal{N}}_x(\mathcal{A})) \\
&= \neg\left(\bigvee_{x \in \mathcal{X}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{A})\right) = \neg(0,1) = (1,0).
\end{aligned}$$

Second,

$$\begin{aligned}
[\mathcal{A} \Subset \widehat{Int}(\mathcal{A})] &= \left[\forall x\left(x \in \mathcal{A} \dot{\to} x \in \widehat{Int}(\mathcal{A})\right)\right] \\
&= \bigwedge_{x \in \mathcal{A}} [(1,0) \dot{\to} \hat{\mathcal{N}}_x(\mathcal{A})] = \bigwedge_{x \in \mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{A}) = \tau(\mathcal{A}) = [\mathcal{A} \in \tau].
\end{aligned}$$

(3) If $\mathcal{A}(x) = (0,1)$, then $\widehat{Int}(\mathcal{A})(x) = (0,1)$ and the result holds. If $\mathcal{A}(x) = (1,0)$, then

$$[x \in \hat{\mathcal{D}}(\mathcal{X}-\mathcal{A})] = \hat{\mathcal{D}}(\mathcal{X}-\mathcal{A})(x) = \neg(\hat{\mathcal{N}}_x(\mathcal{A} \cup \{x\})) = \neg(\hat{\mathcal{N}}_x(\mathcal{A})).$$

Hence

$$[x \in \neg(\hat{\mathcal{D}}(\mathcal{X}-\mathcal{A}))] = \neg(\hat{\mathcal{D}}(\mathcal{X}-\mathcal{A}))(x) = \hat{\mathcal{N}}_x(\mathcal{A}) = [x \in \widehat{Int}(\mathcal{A})].$$

Therefore,

$$[(x \in \mathcal{A}) \wedge x \in \neg(\hat{\mathcal{D}}(\mathcal{X}-\mathcal{A}))] = [x \in \widehat{Int}(\mathcal{A})].$$

(4) Follows from Lemma 7.

(5) From (4) above, Theorems 12 and 13 (1) we have

$$[\mathcal{B} \doteq \widehat{Int}(\mathcal{A})] \equiv [\neg(\mathcal{B}) \doteq \neg(\widehat{Int}(\mathcal{A}))] \equiv [\mathcal{X}-\mathcal{B} \doteq \widehat{Cl}(\mathcal{X}-\mathcal{A})] \le [\mathcal{X}-\mathcal{B} \in \mathcal{F}] = [\mathcal{B} \in \tau].$$

□

Lemma 8.

$$\vDash x \in \hat{b}(\mathcal{A}) \longleftrightarrow \forall \mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} ((\mathcal{B} \cap \mathcal{A} \neq \phi) \wedge (\mathcal{B} \cap (\mathcal{X}-\mathcal{A}) \neq \phi))\big).$$

Proof.

$$\begin{aligned}
&\left[\forall \mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} ((\mathcal{B} \cap \mathcal{A} \neq \phi) \wedge (\mathcal{B} \cap (\mathcal{X}-\mathcal{A}) \neq \phi))\big)\right] \\
=\ &\left[\forall \mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} (\mathcal{B} \cap \mathcal{A} \neq \phi)\big)\right] \wedge \left[\forall \mathcal{B}\big(\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} (\mathcal{B} \cap (\mathcal{X}-\mathcal{A}) \neq \phi)\big)\right] \\
=\ &\bigwedge_{\mathcal{B} \subseteq \mathcal{X}-\mathcal{A}} [\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} (0,1)] \wedge \bigwedge_{\mathcal{B} \subseteq \mathcal{A}} [\mathcal{B} \in \hat{\mathcal{N}}_x \dot{\to} (0,1)] \\
=\ &\bigwedge_{\mathcal{B} \subseteq \mathcal{A}} \neg(\hat{\mathcal{N}}_x(\mathcal{B})) \wedge \bigwedge_{\mathcal{B} \subseteq \mathcal{X}-\mathcal{A}} \neg(\hat{\mathcal{N}}_x(\mathcal{B})) \\
=\ &\neg\left(\bigvee_{\mathcal{B} \subseteq \mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{B})\right) \wedge \neg\left(\bigvee_{\mathcal{B} \subseteq \mathcal{X}-\mathcal{A}} \hat{\mathcal{N}}_x(\mathcal{B})\right) \\
=\ &\neg(\hat{\mathcal{N}}_x(\mathcal{A})) \wedge \neg(\hat{\mathcal{N}}_x(\mathcal{X}-\mathcal{A})) \\
=\ &\neg(\widehat{Int}(\mathcal{A})(x)) \wedge \neg(\widehat{Int}(\mathcal{X}-\mathcal{A})(x)) \\
=\ &\widehat{Cl}(\mathcal{X}-\mathcal{A})(x) \wedge \widehat{Cl}(\mathcal{A})(x) \\
=\ &\hat{b}(\mathcal{A})(x).
\end{aligned}$$

□

Theorem 15. *For any $\mathcal{A}$,*
(1) $\models \hat{b}(\mathcal{A}) \equiv \hat{b}(\mathcal{X} - \mathcal{A})$;
(2) $\neg(\hat{b}(\mathcal{A})) \equiv \widehat{Int}(\mathcal{A}) \cup \widehat{Int}(\mathcal{X} - \mathcal{A})$;
(3) $\models \widehat{Cl}(\mathcal{A}) \equiv \mathcal{A} \cup \hat{b}(\mathcal{A})$, *and so* $\models \hat{b}(\mathcal{A}) \Subset \mathcal{A} \longleftrightarrow \mathcal{A} \in \mathcal{F}$.
(4) $\models \widehat{Int}(\mathcal{A}) \equiv \mathcal{A} \cap \neg(\hat{b}(\mathcal{A}))$, *and so* $\models (\hat{b}(\mathcal{A}) \cap \mathcal{A} \doteq \hat{0}) \longleftrightarrow \mathcal{A} \in \tau$.

Proof. (1) obvious.

(2) From Theorem 14 (4), we obtain

$$\widehat{Int}(\mathcal{A})(x) \vee \widehat{Int}(\mathcal{X} - \mathcal{A})(x) = \neg(\widehat{Cl}(\mathcal{X} - \mathcal{A})(x)) \vee \neg(\widehat{Int}(\mathcal{A})(x)) = \neg(\hat{b}(\mathcal{A})(x)).$$

(3) If $\mathcal{A}(x) = (1,0)$, then $\widehat{Cl}(\mathcal{A})(x) = (\mathcal{A} \cup \hat{b}(\mathcal{A}))(x) = (1,0)$. If $\mathcal{A}(x) = (0,1)$, then $(\mathcal{A} \cup \hat{b}(\mathcal{A}))(x) = \hat{b}(\mathcal{A})(x) = \neg(\widehat{Int}(\mathcal{A})(x)) \wedge \neg(\widehat{Int}(\mathcal{X} - \mathcal{A})(x)) = (1,0) \wedge \neg(\widehat{Int}(\mathcal{X} - \mathcal{A})(x)) = \neg(\widehat{Int}(\mathcal{X} - \mathcal{A})(x)) = \widehat{Cl}(\mathcal{A})(x)$. From Theorem 10 (2), we have $\models \mathcal{A} \in \mathcal{F} \longleftrightarrow \hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A} \longleftrightarrow (\hat{\mathcal{D}}(\mathcal{A}) \Subset \mathcal{A}) \wedge (\mathcal{A} \Subset \mathcal{A}) \longleftrightarrow (\hat{\mathcal{D}}(\mathcal{A}) \cup \mathcal{A} \Subset \mathcal{A}) \longleftrightarrow \widehat{Cl}(\mathcal{A}) \Subset \mathcal{A} \longleftrightarrow \mathcal{A} \cup \hat{b}(\mathcal{A}) \Subset \mathcal{A} \longleftrightarrow (\mathcal{A} \Subset \mathcal{A}) \wedge (\hat{b}(\mathcal{A}) \Subset \mathcal{A}) \longleftrightarrow (1,0) \wedge (\hat{b}(\mathcal{A}) \Subset \mathcal{A}) \longleftrightarrow \hat{b}(\mathcal{A}) \Subset \mathcal{A}$.

(4) $\widehat{Int}(\mathcal{A})(\mathcal{A}) \equiv \neg(\widehat{Cl}(\mathcal{X} - \mathcal{A})) \equiv \neg((\mathcal{X} - \mathcal{A}) \cup \hat{b}(\mathcal{X} - \mathcal{A})) \equiv \mathcal{A} \cap \neg(\hat{b}(\mathcal{X} - \mathcal{A})) = \mathcal{A} \cap \neg(\hat{b}(\mathcal{A}))$. From Theorem 14 (2), we obtain $\models \hat{b}(\mathcal{A}) \cap \mathcal{A} \doteq \hat{0} \longleftrightarrow \neg(\hat{b}(\mathcal{A})) \cup (\mathcal{X} - \mathcal{A}) \doteq \hat{1} \longleftrightarrow \mathcal{A} \Subset \neg(\hat{b}(\mathcal{A})) \longleftrightarrow \mathcal{A} \cap \neg(\hat{b}(\mathcal{A})) \doteq \mathcal{A} \longleftrightarrow \widehat{Int}(\mathcal{A}) \doteq \mathcal{A} \longleftrightarrow \mathcal{A} \in \tau$. □

Remark 6. *All statements in Theorem 15 are true when we replace "$\equiv$" by "$\doteq$" or by "$\dot{\approx}$".*

7. Conclusions

As an extension of fuzzifying topology, we discussed intuitionistic fuzzifying neighborhood system of a point. Additionally, we introduced the concepts of intuitionistic fuzzifying closure, intuitionistic fuzzifying boundary, intuitionistic fuzzifying derived set and intuitionistic fuzzifying interior. These concepts provide a theoretical basis for the further study of intuitionistic fuzzifying topology. In this regard, it is interesting to develop a mathematical framework that contains continuity, speration axioms, compactness, and vector spaces. Additionally, we believe that it would be interesting to apply the intuitionistic fuzzy logic to other structures such as proximity, uniformity, topogenous, syntopogenous, homotopy, etc. We intend to investigate some of these issues in future research works. We believe that it is very important to apply both fuzzy logic and intuitionistic fuzzy logic to convex spaces.

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