

## Article

# Tripolar Picture Fuzzy Ideals of BCK-Algebras

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**Abstract:** In this paper, we acquaint new kinds of ideals of BCK-algebras built on tripolar picture fuzzy structures. In fact, the notions of tripolar picture fuzzy ideal, tripolar picture fuzzy implicative ideal (commutative ideal) of BCK-algebra are introduced, and related properties are studied. Also, a relation among tripolar picture fuzzy ideal, and tripolar picture fuzzy implicative ideal is well-known. Furthermore, it is shown that a tripolar picture fuzzy implicative ideal of BCK-algebra may be a tripolar picture fuzzy ideal, but the converse is not correct in common. Further, it is obtained that in an implicative BCK-algebra, the converse of aforementioned statement is true. Finally, the opinion of tripolar picture fuzzy commutative ideal is given, and some useful properties are explored. Many examples are constructed to sport our study.

**Keywords:** fuzzy BCK-algebra; tripolar picture fuzzy ideal; tripolar picture fuzzy implicative ideal; tripolar picture fuzzy commutative ideal

**MSC:** 06F35; 03G25; 03B52



**Citation:** Muhiuddin, G.; Abughazalah, N.; Aljuhani, A.; Balamurugan, M. Tripolar Picture Fuzzy Ideals of BCK-Algebras. *Symmetry* **2022**, *14*, 1562. <https://doi.org/10.3390/sym14081562>

Academic Editors: Sergei D. Odintsov, Jian-Qiang Wang and Evgeny Nikulchev

Received: 31 March 2022

Accepted: 13 July 2022

Published: 29 July 2022

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## 1. Introduction

The study of symmetry is one of the most important and beautiful themes uniting various areas of contemporary arithmetic. Algebraic structures are useful structures in pure mathematics for learning a geometrical object's symmetries. For example, the theory of groups is also used to provide a broad theory of symmetry. There are various sorts of symmetries that may be studied using the theory of groups, which is already widely utilized as an algebraic tool.

Fuzzy set (briefly, FS) was originated by Zadeh [1] in 1965 while Atanassov [2] in 1986 yielded the notion of the intuitionistic fuzzy (briefly, IFS). IFS include both the degree of belonging and therefore the degree of non-belonging, while the FS contains one the degree of belonging. The thought of "BCK/BCI-algebras" was existing through Iseki, and associates [3–5]. Xi [6] started the process of incorporating the theories of fuzzy BCK-algebra, FS, and BCK algebra. In [7], Ahmad gave the knowledge of fuzzy BCI-algebras. As a follow-up, plenty of effort on "BCK/BCI-algebras", and related topics in the FS atmosphere specified given by many authors [8,9]. IF subalgebra and IF ideal in BCK-algebras were offered by Jun and Kim [10] by way of a generality of the FS theory in BCK-algebra. The ideal structure in BCK/BCI-algebras has been the subject of many research papers; for example, The article was written by Meng et al. [11] fuzzy implicative ideals of BCK-algebras were developed. The p-ideals of BCI-algebras were introduced by Muhiuddin [12]. Furthermore, Muhiuddin et al. investigated various types of ideals are made known the following view: (i) The BCK-algebras are based on neutrosophic N-structures [13]; (ii) N-Soft p-ideal of BCI-algebras [14]; and (iii) In BCK/BCI-algebras, hesitant fuzzy translations, and extensions of ideals [15].

The notion of bipolar fuzzy set (briefly, BFS) was introduced by Zhang [16] which deals with the degree of positive, and negative belonging of an element. A positive effect may be classified as such, while a negative effect can be classified as such. An example of a bipolar fuzzy environment as it is developed. Serials on televisions have both positive and negative effects on the younger generation. In 2013, the perception as regards picture FS was originated by Cuong [17].

In [18,19], Lee suggested the notions of bipolar fuzzy sets (briefly, BFSs) as an expansion of fuzzy sets (briefly, FSs). An increasing number of researchers have devoted themselves to studying some BF algebraic structures in recent years in order to preserve the findings of BFSs. Lee [20] involved the BFS theory existing to BCK/BCI-algebras to investigate several properties. After the commencement of the picture fuzzy set, different types of research works within the framework of picture fuzzy set were done by some researchers [3,21,22]. Wei [23–25] developed cosine, and dice similarity events for picture FSs, and their applications. More associated concepts have been studied in [26–31].

The paper's structure is arranged as follows: Section 2 reviews certain theories and properties connecting to BCK-algebras, ideals, and fuzzy ideals that are essential to yield its crucial results. The idea of tripolar picture fuzzy ideal (briefly, TPPFI) of BCK-algebras is deliberated in Section 3. Section 4 is devoted to the analysis of tripolar picture fuzzy implicative ideal (briefly, TPPFII) while Section 5 deals with the study of a tripolar picture fuzzy commutative ideal of BCK-algebras (briefly, TPPFCI). Finally, a conclusion with some future prospects for potential work is given. The paper concluded with a bibliography.

## 2. Preliminaries

In this section, we label the well-known contents of BCK-algebra, which are aimed at the enhancement of this paper.

If a non-empty set  $F$  has a particular element  $0$ , and a binary operation  $\odot$  satisfies the following assets:

- (1)  $(x, q, \vartheta \in F) ((x \odot q) \odot (x \odot \vartheta)) \odot (\vartheta \odot q) = 0$ ;
- (2)  $(x, q \in F) (x \odot (x \odot q)) \odot q = 0$ ;
- (3)  $(x \in F) x \odot x = 0$ ;
- (4)  $(x \in F) 0 \odot x = 0$ ;
- (5)  $(x, q \in F) x \odot q = 0$  and  $q \odot x = 0 \Rightarrow x = q$ .

Then,  $F$  is a BCK-algebra.

**Proposition 1.** In a BCK-algebra  $F$ , the following hold:

- i.  $x \odot 0 = x$
- ii.  $x \odot (x \odot q) \leq q$
- iii.  $x \odot q \leq x$
- iv.  $(x \odot q) \odot \vartheta = (x \odot \vartheta) \odot q$
- v.  $(x \odot (x \odot (x \odot q))) = x \odot q$  for all  $x, q, \vartheta \in F$ .

A BCK algebra  $F$  is an implicative if  $x = x \odot (q \odot x)$  for all  $x, q \in F$ .

A BCK algebra  $F$  is commutative if  $q \odot (q \odot x) = x \odot (x \odot q)$  for all  $x, q \in F$ .

A non-empty subset  $\mathfrak{A}$  of  $F$  is an ideal of  $F$  if it fulfills

- (I<sub>1</sub>)  $0 \in \mathfrak{A}$ ,
- (I<sub>2</sub>)  $\forall x, q \in F, x \odot q \in \mathfrak{A}, q \in \mathfrak{A} \Rightarrow x \in \mathfrak{A}$ .

A nonempty subset  $\mathfrak{A}$  of  $F$  is an implicative ideal of  $F$  if it fulfills

- (I<sub>1</sub>) and (I<sub>3</sub>)  $\forall x, q, \vartheta \in F, (x \odot (q \odot x)) \odot \vartheta \in \mathfrak{A}, \vartheta \in \mathfrak{A} \Rightarrow x \in \mathfrak{A}$ .

A non-empty subset  $\mathfrak{A}$  is a positive implicative ideal of  $F$  if it fulfills

- (I<sub>1</sub>) and (I<sub>4</sub>)  $\forall x, q, \vartheta \in F, (x \odot q) \odot \vartheta \in \mathfrak{A}, q \odot \vartheta \in \mathfrak{A} \Rightarrow x \odot \vartheta \in \mathfrak{A}$ .

A non-empty subset  $\mathfrak{A}$  is a commutative ideal of  $F$  if it fulfills

- (I<sub>1</sub>) and (I<sub>5</sub>)  $\forall x, q, \vartheta \in F, (x \odot q) \odot \vartheta \in \mathfrak{A}, \vartheta \in \mathfrak{A} \Rightarrow x \odot (q \odot (q \odot x)) \in \mathfrak{A}$ .

A FS  $\omega$  is a fuzzy ideal [6] of  $F$  if it fulfills

- (F<sub>1</sub>)  $\omega(0) \geq \omega(x)$ ,  
 (F<sub>2</sub>)  $\omega(x) \geq \omega(x \odot q) \wedge \omega(q)$ .

A FS  $\omega$  is a fuzzy implicative ideal [11] of  $F$  if it fulfills

- (F<sub>1</sub>) and (F<sub>3</sub>)  $\omega(x) \geq \omega((x \odot (q \odot x)) \odot \theta) \wedge \omega(\theta)$ .

A FS  $\omega$  is a fuzzy positive implicative ideal of  $F$  if it fulfills

- (F<sub>1</sub>) and (F<sub>4</sub>)  $\omega(x \odot \theta) \geq \omega((x \odot q) \odot \theta) \wedge \omega(q \odot \theta)$ .

**Definition 1** ([16]). A bipolar fuzzy set (BFS)  $P = \{(x, \omega_P(x), \varsigma_P(x)) : x \in F\}$ , where  $\omega_P : F \rightarrow [0, 1]$ , and  $\varsigma_P : F \rightarrow [-1, 0]$  are any mappings.

Bipolar FSs are a FS extension with such a belonging grade range of  $[-1, 1]$ . In a bipolar FS, the belonging grade 0 means the element has no relevancy to the property, the belonging grade  $(0, 1]$  of an element designates that the element some extent fulfills the property, and the belonging grade  $[-1, 0)$  of an element designates that the element some extent fulfills the implied tackle-property.

**Definition 2** ([17]). Let  $F$  be the set of universe. Then, a picture fuzzy set  $P$  over  $F$  is defined as  $P = \{(x, \omega_P(x), \varsigma_P(x), \omega_P(x)) : x \in F\}$ , where  $\omega_P(x)$ ,  $\varsigma_P(x)$ ,  $\omega_P(x) \in [0, 1]$  are the grade of positive, neutral, and negative membership of  $x$  in  $P$  with the condition  $0 \leq \omega_P(x) + \varsigma_P(x) + \omega_P(x) \leq 1$  for all  $x \in F$ . For all  $x \in F$ ,  $1 - (\omega_P(x) + \varsigma_P(x) + \omega_P(x))$  is the degree of refusal membership  $x \in P$ . We call  $(\omega_P(x), \varsigma_P(x), \omega_P(x))$  the picture fuzzy value for  $x \in F$ .

For a part of family  $\{x_i \mid i \in \Lambda\}$  of real numbers, we state

$$\begin{aligned} \vee \{x_i \mid i \in \Lambda\} &:= \begin{cases} \max \{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup \{x_i \mid i \in \Lambda\} & \text{otherwise,} \end{cases} \\ \wedge \{x_i \mid i \in \Lambda\} &:= \begin{cases} \min \{x_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf \{x_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases} \end{aligned}$$

Moreover, if  $\Lambda = \{1, 2, \dots, n\}$ , then  $\vee \{x_i \mid i \in \Lambda\}$  and  $\wedge \{x_i \mid i \in \Lambda\}$  are symbolized by  $x_1 \vee x_2 \vee \dots \vee x_n$ , and  $x_1 \wedge x_2 \wedge \dots \wedge x_n$ , respectively.

In what follows, let  $F$  specify a BCK-algebra unless otherwise indicated.

### 3. Tripolar Picture Fuzzy Ideal

**Definition 3.** Let  $P = \{(x, \omega_P(x)) : x \in F\}$  be a TPFS over the set of universe  $F$ , where  $\omega_P : F \rightarrow [0, 1]^3$ . In this case,  $[0, 1]^3$  is the poset with regard to partial order relation " $\leq$ " which is well-defined like:  $x \leq q$  iff  $\rho_i(x) \leq \rho_i(q)$  for  $i = 1, 2, 3$ , where  $\rho_i : [0, 1]^3 \rightarrow [0, 1]$  is called 3rd projection mapping. It is easy to understand that  $(0, 0, 0) = 0$ ,  $(1, 1, 1) = 1 \in [0, 1]^3$ .

**Definition 4.** Let  $F$  be the set of universe. Then, a TPPFS  $P$  over the universe  $F$  is demarcated by way of  $P = \{(x, \omega_P(x), \varsigma_P(x), \omega_P(x)) : x \in F\}$ , where  $\omega_P, \varsigma_P$ , and  $\omega_P : F \rightarrow [0, 1]^3$  with the condition  $0 \leq \rho_i \circ \omega_P(x) + \rho_i \circ \varsigma_P(x) + \rho_i \circ \omega_P(x) \leq 1$  for all  $x \in F$  and for  $i = 1, 2, 3$ . For all  $x \in F$ ,  $\omega_P(x)$ ,  $\varsigma_P(x)$  and  $\omega_P(x)$  is a 3-tuple fuzzy value. Here,  $\rho_i \circ \omega_P(x)$ ,  $\rho_i \circ \varsigma_P(x)$  and  $\rho_i \circ \omega_P(x)$  represent 3-components of  $\omega_P(x)$ ,  $\varsigma_P(x)$  and  $\omega_P(x)$  respectively for  $i = 1, 2, 3$ .

We will write  $P = (\omega_P, \varsigma_P, \omega_P)$  instead of  $P = \{(x, \omega_P(x), \varsigma_P(x), \omega_P(x)) : x \in F\}$  for shortness.

**Definition 5.** A TPPFS  $P = (\omega_P, \varsigma_P, \omega_P)$  in  $F$  is a TPPFI of  $F$  if it satisfies the following assertions:

- (i)  $\omega_P(0) \geq \omega_P(x)$ ,  $\varsigma_P(0) \geq \varsigma_P(x)$  and  $\omega_P(0) \leq \omega_P(x)$ ,  
 (ii)  $\omega_P(x) \geq \omega_P(x \odot q) \wedge \omega_P(q)$ ,  $\varsigma_P(x) \geq \varsigma_P(x \odot q) \wedge \varsigma_P(q)$  and  $\omega_P(x) \leq \omega_P(x \odot q) \vee \omega_P(q)$ .

That is,

- (i)  $\rho_i \circ \omega_P(0) \geq \rho_i \circ \omega_P(\varkappa), \rho_i \circ \zeta_P(0) \geq \rho_i \circ \zeta_P(\varkappa)$  and  $\rho_i \circ \omega_P(0) \leq \rho_i \circ \omega_P(\varkappa)$ ,  
(ii)  $\rho_i \circ \omega_P(\varkappa) \geq \rho_i \circ \omega_P(\varkappa \odot q) \wedge \rho_i \circ \omega_P(q), \rho_i \circ \zeta_P(\varkappa) \geq \rho_i \circ \zeta_P(\varkappa \odot q) \wedge \rho_i \circ \zeta_P(q)$  and  
 $\rho_i \circ \omega_P(\varkappa) \leq \rho_i \circ \omega_P(\varkappa \odot q) \vee \rho_i \circ \omega_P(q)$   
for all  $\varkappa, q \in F, i = 1, 2, 3$ .

**Example 1.** Let  $F = \{0, \varkappa, q, \vartheta, \kappa\}$  be a BCK-algebra with the succeeding Cayley table:

$\odot$	0	$\varkappa$	q	$\vartheta$	$\kappa$
0	0	0	0	0	0
$\varkappa$	$\varkappa$	0	$\varkappa$	0	0
q	q	q	0	0	0
$\vartheta$	$\vartheta$	$\vartheta$	$\vartheta$	0	0
$\kappa$	$\kappa$	$\kappa$	$\kappa$	$\vartheta$	0

Now, let us suppose a TPPFS  $P$  as follows:

$$\omega_P(\delta) = \begin{cases} (0.35, 0.36, 0.38), & \text{if } \delta = 0 \\ (0.25, 0.27, 0.28), & \text{if } \delta = \varkappa \\ (0.15, 0.19, 0.22), & \text{if } \delta = q, \vartheta, \kappa \end{cases}$$

$$\zeta_P(\delta) = \begin{cases} (0.38, 0.40, 0.42), & \text{if } \delta = 0 \\ (0.30, 0.31, 0.32), & \text{if } \delta = \varkappa \\ (0.15, 0.17, 0.18), & \text{if } \delta = q, \vartheta, \kappa \end{cases}$$

and

$$\omega_P(\delta) = \begin{cases} (0.03, 0.04, 0.05), & \text{if } \delta = 0 \\ (0.22, 0.23, 0.25), & \text{if } \delta = \varkappa \\ (0.40, 0.42, 0.45), & \text{if } \delta = q, \vartheta, \kappa \end{cases}$$

Clearly,  $P$  is a TPPFI of  $F$ .

**Theorem 1.** Let  $P = (\omega_P, \zeta_P, \omega_P)$  be a TPPFI of  $F$ . Then,  $\omega_P(\varkappa) \geq \omega_P(q)$ ,  $\zeta_P(\varkappa) \geq \zeta_P(q)$  and  $\omega_P(\varkappa) \leq \omega_P(q)$  for  $\varkappa, q \in F$  with  $\varkappa \leq q$ .

**Proof.** Let  $\varkappa, q \in F, \varkappa \leq q$ . Then,  $\varkappa \odot q = 0$ .

Now,

$$\begin{aligned} \omega_P(\varkappa) &\geq \omega_P(\varkappa \odot q) \wedge \omega_P(q) \text{ [as } P \text{ is a TPPFI of } F] \\ &= \omega_P(0) \wedge \omega_P(q) \\ &= \omega_P(q) \text{ [as } P \text{ is a TPPFI of } F], \end{aligned}$$

$$\begin{aligned} \zeta_P(\varkappa) &\geq \zeta_P(\varkappa \odot q) \wedge \zeta_P(q) \text{ [as } P \text{ is a TPPFI of } F] \\ &= \zeta_P(0) \wedge \zeta_P(q) \\ &= \zeta_P(q) \text{ [as } P \text{ is a TPPFI of } F] \end{aligned}$$

and

$$\begin{aligned} \omega_P(\varkappa) &\leq \omega_P(\varkappa \odot q) \vee \omega_P(q) \text{ [as } P \text{ is a TPPFI of } F] \\ &= \omega_P(0) \vee \omega_P(q) \\ &= \omega_P(q) \text{ [as } P \text{ is a TPPFI of } F]. \end{aligned}$$

Thus,  $\omega_P(\varkappa) \geq \omega_P(q)$ ,  $\zeta_P(\varkappa) \geq \zeta_P(q)$  and  $\omega_P(\varkappa) \leq \omega_P(q)$  for  $\varkappa \in F$  with  $\varkappa \leq q$ .  $\square$

**Theorem 2.** Let  $P = (\omega_P, \zeta_P, \varpi_P)$  be a TPPFI of  $F$ . Then,  $\varkappa \odot \varrho \leq \vartheta$  implies  $\omega_P(\varkappa) \geq \omega_P(\varrho) \wedge \omega_P(\vartheta)$ ,  $\zeta_P(\varkappa) \geq \zeta_P(\varrho) \wedge \zeta_P(\vartheta)$  and  $\varpi_P(\varkappa) \leq \varpi_P(\varrho) \vee \varpi_P(\vartheta)$ , for all  $\varkappa, \varrho, \vartheta \in F$ .

**Proof.** Let  $\varkappa, \varrho, \vartheta \in F$ ,  $\varkappa \odot \varrho \leq \vartheta$ . Then,  $(\varkappa \odot \varrho) \odot \vartheta = 0$ .

Now,

$$\begin{aligned}\omega_P(\varkappa) &\geq \omega_P(\varkappa \odot \varrho) \wedge \omega_P(\varrho) \\ &\geq \omega_P((\varkappa \odot \varrho) \odot \vartheta) \wedge \omega_P(\vartheta) \wedge \omega_P(\varrho) \text{ [as } P \text{ is a TPPFI of } F] \\ &= \omega_P(0) \wedge \omega_P(\vartheta) \wedge \omega_P(\varrho) \\ &= \omega_P(\varrho) \wedge \omega_P(\vartheta) \text{ [as } P \text{ is a TPPFI of } F],\end{aligned}$$

$$\begin{aligned}\zeta_P(\varkappa) &\geq \zeta_P(\varkappa \odot \varrho) \wedge \zeta_P(\varrho) \\ &\geq \zeta_P((\varkappa \odot \varrho) \odot \vartheta) \wedge \zeta_P(\vartheta) \wedge \zeta_P(\varrho) \text{ [as } P \text{ is a TPPFI of } F] \\ &= \zeta_P(0) \wedge \zeta_P(\vartheta) \wedge \zeta_P(\varrho) \\ &= \zeta_P(\varrho) \wedge \zeta_P(\vartheta) \text{ [as } P \text{ is a TPPFI of } F]\end{aligned}$$

and

$$\begin{aligned}\varpi_P(\varkappa) &\leq \varpi_P(\varkappa \odot \varrho) \vee \varpi_P(\varrho) \\ &\leq \varpi_P((\varkappa \odot \varrho) \odot \vartheta) \vee \varpi_P(\vartheta) \vee \varpi_P(\varrho) \text{ [as } P \text{ is a TPPFI of } F] \\ &= \varpi_P(0) \vee \varpi_P(\vartheta) \vee \varpi_P(\varrho) \\ &= \varpi_P(\varrho) \vee \varpi_P(\vartheta) \text{ [as } P \text{ is a TPPFI of } F].\end{aligned}$$

Thus, it is obtained that  $\omega_P(\varkappa) \geq \omega_P(\varrho) \wedge \omega_P(\vartheta)$ ,  $\zeta_P(\varkappa) \geq \zeta_P(\varrho) \wedge \zeta_P(\vartheta)$  and  $\varpi_P(\varkappa) \leq \varpi_P(\varrho) \vee \varpi_P(\vartheta)$ .  $\square$

The subsequent theorem is a generalization of Theorem 2.

**Theorem 3.** If  $P = (\omega_P, \zeta_P, \varpi_P)$  is a TPPFI of  $F$ , then, for all  $\varkappa, \delta_1, \delta_2, \dots, \delta_n \in F$ ,

$$\prod_{i=1}^n \varkappa \odot \delta_i = 0 \Rightarrow \begin{pmatrix} \omega_P(\varkappa) \geq \omega_P(\delta_1) \wedge \omega_P(\delta_2) \wedge \dots \wedge \omega_P(\delta_n) \\ \zeta_P(\varkappa) \geq \zeta_P(\delta_1) \wedge \zeta_P(\delta_2) \wedge \dots \wedge \zeta_P(\delta_n) \\ \varpi_P(\varkappa) \leq \varpi_P(\delta_1) \vee \varpi_P(\delta_2) \vee \dots \vee \varpi_P(\delta_n) \end{pmatrix} \quad (1)$$

where  $\prod_{i=1}^n \varkappa \odot \delta_i = (\dots((\varkappa \odot \delta_1) \odot \delta_2) \odot \dots) \odot \delta_n$ .

**Proof.** The proof can be found on  $n$ . Let  $P = (\omega_P, \zeta_P, \varpi_P)$  be a TPPFI of  $F$ . Theorem 2 shows that the condition (1) is valid for  $n = 2$ . Assume that  $P = (\omega_P, \zeta_P, \varpi_P)$  satisfies the condition (1) for  $n = k$ , that is, for all  $\varkappa, \delta_1, \delta_2, \dots, \delta_k \in F$ ,  $\prod_{i=1}^k \varkappa \odot \delta_i = 0$  implies

$$\begin{aligned}\omega_P(\varkappa) &\geq \omega_P(\delta_1) \wedge \omega_P(\delta_2) \wedge \dots \wedge \omega_P(\delta_k), \\ \zeta_P(\varkappa) &\geq \zeta_P(\delta_1) \wedge \zeta_P(\delta_2) \wedge \dots \wedge \zeta_P(\delta_k)\end{aligned}$$

and

$$\varpi_P(\varkappa) \leq \varpi_P(\delta_1) \vee \varpi_P(\delta_2) \vee \dots \vee \varpi_P(\delta_k).$$

Let  $\varkappa, \delta_1, \delta_2, \dots, \delta_k, \delta_{k+1} \in F$  be such that  $\prod_{i=1}^{k+1} \varkappa \odot \delta_i = 0$ . Then,

$$\begin{aligned}\omega_P(\varkappa \odot \delta_1) &\geq \omega_P(\delta_2) \wedge \omega_P(\delta_3) \wedge \dots \wedge \omega_P(\delta_{k+1}), \\ \zeta_P(\varkappa \odot \delta_1) &\geq \zeta_P(\delta_2) \wedge \zeta_P(\delta_3) \wedge \dots \wedge \zeta_P(\delta_{k+1})\end{aligned}$$

and

$$\varpi_P(\varkappa \odot \delta_1) \leq \varpi_P(\delta_2) \vee \varpi_P(\delta_3) \vee \dots \vee \varpi_P(\delta_{k+1}).$$

Since  $P = (\omega_P, \varsigma_P, \varpi_P)$  is a TPPFI of  $F$ , it follows from Definition 5 (ii) that

$$\begin{aligned}\omega_P(\varkappa) &\geq \omega_P(\varkappa \odot \delta_1) \wedge \omega_P(\delta_1) \\ &\geq \omega_P(\delta_1) \wedge \omega_P(\delta_2) \wedge \omega_P(\delta_3) \wedge \dots \wedge \omega_P(\delta_{k+1}), \\ \varsigma_P(\varkappa) &\geq \varsigma_P(\varkappa \odot \delta_1) \wedge \varsigma_P(\delta_1) \\ &\geq \varsigma_P(\delta_1) \wedge \varsigma_P(\delta_2) \wedge \varsigma_P(\delta_3) \wedge \dots \wedge \varsigma_P(\delta_{k+1})\end{aligned}$$

and

$$\begin{aligned}\omega_P(\varkappa) &\leq \omega_P(\varkappa \odot \delta_1) \vee \omega_P(\delta_1) \\ &\leq \omega_P(\delta_1) \vee \omega_P(\delta_2) \vee \omega_P(\delta_3) \vee \dots \vee \omega_P(\delta_{k+1}).\end{aligned}$$

□

**Theorem 4.** Let  $P = (\omega_P, \varsigma_P, \varpi_P)$  be a TPPFI of  $F$ . Then,

$$\begin{aligned}\omega_P(\varkappa \odot \varrho) &\geq \omega_P(\varkappa \odot \vartheta) \wedge \omega_P(\vartheta \odot \varrho) \\ \varsigma_P(\varkappa \odot \varrho) &\geq \varsigma_P(\varkappa \odot \vartheta) \wedge \varsigma_P(\vartheta \odot \varrho),\end{aligned}$$

and

$$\varpi_P(\varkappa \odot \varrho) \leq \varpi_P(\varkappa \odot \vartheta) \vee \varpi_P(\vartheta \odot \varrho)$$

for all  $\varkappa, \varrho, \vartheta \in F$ .

**Proof.** It is worth noting that  $((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) \leq (\vartheta \odot \varrho)$ , for all  $\varkappa, \varrho, \vartheta \in F$ . It follows for Theorem 1 that

$$\begin{aligned}\omega_P((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) &\geq \omega_P(\vartheta \odot \varrho), \\ \varsigma_P((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) &\geq \varsigma_P(\vartheta \odot \varrho)\end{aligned}$$

and

$$\varpi_P((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) \leq \varpi_P(\vartheta \odot \varrho).$$

By Definition 5 (ii),

$$\begin{aligned}\omega_P(\varkappa \odot \varrho) &\geq \omega_P((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) \wedge \omega_P(\varkappa \odot \vartheta) \\ &\geq \omega_P(\vartheta \odot \varrho) \wedge \omega_P(\varkappa \odot \vartheta) \\ &= \omega_P(\varkappa \odot \vartheta) \wedge \omega_P(\vartheta \odot \varrho) \\ \varsigma_P(\varkappa \odot \varrho) &\geq \varsigma_P((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) \wedge \varsigma_P(\varkappa \odot \vartheta) \\ &\geq \varsigma_P(\vartheta \odot \varrho) \wedge \varsigma_P(\varkappa \odot \vartheta) \\ &= \varsigma_P(\varkappa \odot \vartheta) \wedge \varsigma_P(\vartheta \odot \varrho)\end{aligned}$$

and

$$\begin{aligned}\varpi_P(\varkappa \odot \varrho) &\leq \varpi_P((\varkappa \odot \varrho) \odot (\varkappa \odot \vartheta)) \vee \varpi_P(\varkappa \odot \vartheta) \\ &\leq \varpi_P(\vartheta \odot \varrho) \vee \varpi_P(\varkappa \odot \vartheta) \\ &= \varpi_P(\varkappa \odot \vartheta) \vee \varpi_P(\vartheta \odot \varrho)\end{aligned}$$

for all  $\varkappa, \varrho, \vartheta \in F$ . □

**Theorem 5.** Let  $P = (\omega_P, \varsigma_P, \varpi_P)$  be a TPPFI of  $F$ . Then,  $\omega_P(\varkappa \odot (\varkappa \odot \varrho)) \geq \omega_P(\varrho)$ ,  $\varsigma_P(\varkappa \odot (\varkappa \odot \varrho)) \geq \varsigma_P(\varrho)$  and  $\varpi_P(\varkappa \odot (\varkappa \odot \varrho)) \leq \varpi_P(\varrho)$  for all  $\varkappa, \varrho \in F$ .

**Proof.** Let  $P = (\omega_P, \zeta_P, \varpi_P)$  be a TPPFI of  $F$ . Then, for all  $\varkappa, \varrho \in F$ ,

$$\begin{aligned}\omega_P(\varkappa \odot (\varkappa \odot \varrho)) &\geq \omega_P((\varkappa \odot (\varkappa \odot \varrho)) \odot \varrho) \wedge \omega_P(\varrho) \\ &= \omega_P(0) \wedge \omega_P(\varrho) \\ &= \omega_P(\varrho), \\ \zeta_P(\varkappa \odot (\varkappa \odot \varrho)) &\geq \zeta_P((\varkappa \odot (\varkappa \odot \varrho)) \odot \varrho) \wedge \zeta_P(\varrho) \\ &= \zeta_P(0) \wedge \zeta_P(\varrho) \\ &= \zeta_P(\varrho)\end{aligned}$$

and

$$\begin{aligned}\varpi_P(\varkappa \odot (\varkappa \odot \varrho)) &\leq \varpi_P((\varkappa \odot (\varkappa \odot \varrho)) \odot \varrho) \vee \varpi_P(\varrho) \\ &= \varpi_P(0) \vee \varpi_P(\varrho) \\ &= \varpi_P(\varrho).\end{aligned}$$

□

**Theorem 6.** Let  $P = (\omega_P, \zeta_P, \varpi_P)$  be a TPPFS in  $F$  satisfying the condition (1). Then,  $P = (\omega_P, \zeta_P, \varpi_P)$  is a TPPFI of  $F$ .

**Proof.** It is worth noting that  $(\dots ((0 \odot \varkappa) \odot \varkappa) \odot \dots) \odot \varkappa = 0$  for all  $\varkappa \in F$ . According to (1),  $\omega_P(0) \geq \omega_P(\varkappa)$ ,  $\zeta_P(0) \geq \zeta_P(\varkappa)$  and  $\varpi_P(0) \leq \varpi_P(\varkappa)$  for all  $\varkappa \in F$ . Let  $\varkappa, \varrho, \vartheta \in F$ ,  $\varkappa \odot \varrho \leq \vartheta$ . Then,  $0 = (\varkappa \odot \varrho) \odot \vartheta = (\dots ((\varkappa \odot \varrho) \odot \vartheta) \odot \dots) \odot 0$ , and so

$$\begin{aligned}\omega_P(\varkappa) &\geq \omega_P(\varrho) \wedge \omega_P(\vartheta) \wedge \omega_P(0) \\ &= \omega_P(\varrho) \wedge \omega_P(\vartheta), \\ \zeta_P(\varkappa) &\geq \zeta_P(\varrho) \wedge \zeta_P(\vartheta) \wedge \zeta_P(0) \\ &= \zeta_P(\varrho) \wedge \zeta_P(\vartheta)\end{aligned}$$

and

$$\begin{aligned}\varpi_P(\varkappa) &\leq \varpi_P(\varrho) \vee \varpi_P(\vartheta) \vee \varpi_P(0) \\ &= \varpi_P(\varrho) \vee \varpi_P(\vartheta).\end{aligned}$$

Hence, by Theorem 2, we conclude that  $P = (\omega_P, \zeta_P, \varpi_P)$  is a TPPFI of  $F$ . □

#### 4. Tripolar Picture Fuzzy Implicative Ideal

**Definition 6.** A TPPFS  $P = (\omega_P, \zeta_P, \varpi_P)$  in  $F$  is called TPPFII of  $F$  if it satisfies the following assertions:

- (i)  $\omega_P(0) \geq \omega_P(\varkappa)$ ,  $\zeta_P(0) \geq \zeta_P(\varkappa)$  and  $\varpi_P(0) \leq \varpi_P(\varkappa)$ ,
- (ii)  $\omega_P(\varkappa) \geq \omega_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \wedge \omega_P(\vartheta)$ ,  $\zeta_P(\varkappa) \geq \zeta_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \wedge \zeta_P(\vartheta)$  and  $\varpi_P(\varkappa) \leq \varpi_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \vee \varpi_P(\vartheta)$ .

That is,

- (i)  $\rho_i \circ \omega_P(0) \geq \rho_i \circ \omega_P(\varkappa)$ ,  $\rho_i \circ \zeta_P(0) \geq \rho_i \circ \zeta_P(\varkappa)$  and  $\rho_i \circ \varpi_P(0) \leq \rho_i \circ \varpi_P(\varkappa)$ ,
- (ii)  $\rho_i \circ \omega_P(\varkappa) \geq \rho_i \circ \omega_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \wedge \rho_i \circ \omega_P(\vartheta)$ ,  $\rho_i \circ \zeta_P(\varkappa) \geq \rho_i \circ \zeta_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \wedge \rho_i \circ \zeta_P(\vartheta)$  and  $\rho_i \circ \varpi_P(\varkappa) \leq \rho_i \circ \varpi_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \vee \rho_i \circ \varpi_P(\vartheta)$

for all  $\varkappa, \varrho, \vartheta \in F$ ,  $i = 1, 2, 3$ .

**Example 2.** Take a look at the BCK-algebra  $(F; \odot, 0)$  as follows:

$\odot$	0	$\varkappa$	$\varrho$	$\vartheta$	$\kappa$
0	0	0	0	0	0
$\varkappa$	$\varkappa$	0	$\varkappa$	0	0
$\varrho$	$\varrho$	$\varrho$	0	0	0
$\vartheta$	$\vartheta$	$\vartheta$	$\vartheta$	0	0
$\kappa$	$\kappa$	$\vartheta$	$\kappa$	$\varkappa$	0

Now consider the following TPPFS  $P$ :

$$\omega_P(\delta) = \begin{cases} (0.29, 0.31, 0.32), & \text{if } \delta = 0, \varkappa, \varrho \\ (0.15, 0.17, 0.2), & \text{if } \delta = \vartheta, \kappa \end{cases}$$

$$\varsigma_P(\delta) = \begin{cases} (0.27, 0.29, 0.3), & \text{if } \delta = 0, \varkappa, \varrho \\ (0.19, 0.23, 0.25), & \text{if } \delta = \vartheta, \kappa \end{cases}$$

and

$$\varpi_P(\delta) = \begin{cases} (0.04, 0.07, 0.08), & \text{if } \delta = 0, \varkappa, \varrho \\ (0.2, 0.22, 0.25), & \text{if } \delta = \vartheta, \kappa \end{cases}$$

It can be easily shown that  $P$  is TPPFII of  $F$ .

**Lemma 1.** Every TPPFII of  $F$  is order preserving.

**Proof.** Let  $P$  be a TPPFII of  $F$ . Take  $\varkappa, \varrho, \vartheta \in F$ ,  $\varkappa \leq \varrho$ . Then,

$$\begin{aligned} \omega_P(\varkappa) &\geq \omega_P((\varkappa \odot (\vartheta \odot \varkappa)) \odot \varrho) \wedge \omega_P(\varrho) \\ &= \omega_P((\varkappa \odot \varrho) \odot (\vartheta \odot \varkappa)) \wedge \omega_P(\varrho) \\ &= \omega_P(0 \odot (\vartheta \odot \varkappa)) \wedge \omega_P(\varrho) \\ &= \omega_P(0) \wedge \omega_P(\varrho) \\ &= \omega_P(\varrho), \end{aligned}$$

$$\begin{aligned} \varsigma_P(\varkappa) &\geq \varsigma_P((\varkappa \odot (\vartheta \odot \varkappa)) \odot \varrho) \wedge \varsigma_P(\varrho) \\ &= \varsigma_P((\varkappa \odot \varrho) \odot (\vartheta \odot \varkappa)) \wedge \varsigma_P(\varrho) \\ &= \varsigma_P(0 \odot (\vartheta \odot \varkappa)) \wedge \varsigma_P(\varrho) \\ &= \varsigma_P(0) \wedge \varsigma_P(\varrho) \\ &= \varsigma_P(\varrho) \end{aligned}$$

and

$$\begin{aligned} \varpi_P(\varkappa) &\leq \varpi_P((\varkappa \odot (\vartheta \odot \varkappa)) \odot \varrho) \vee \varpi_P(\varrho) \\ &= \varpi_P((\varkappa \odot \varrho) \odot (\vartheta \odot \varkappa)) \vee \varpi_P(\varrho) \\ &= \varpi_P(0 \odot (\vartheta \odot \varkappa)) \vee \varpi_P(\varrho) \\ &= \varpi_P(0) \vee \varpi_P(\varrho) \\ &= \varpi_P(\varrho). \end{aligned}$$

□

**Theorem 7.** Any TPPFII of  $F$  must be a TPPFI of  $F$ .

**Proof.** Let  $P = (\omega_P, \varsigma_P, \varpi_P)$  be a TPPFII of  $F$ . Then, for all  $\varkappa, \varrho, \vartheta \in F$ ,

$$\begin{aligned} \omega_P(\varkappa) &\geq \omega_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \wedge \omega_P(\vartheta), \\ \varsigma_P(\varkappa) &\geq \varsigma_P((\varkappa \odot (\varrho \odot \varkappa)) \odot \vartheta) \wedge \varsigma_P(\vartheta) \end{aligned}$$



and

$$\omega_P(\kappa) \leq \omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \vee \omega_P(\vartheta).$$

Putting  $q = \kappa$  and  $\vartheta = q$ ,

$$\begin{aligned}\omega_P(\kappa) &\geq \omega_P((\kappa \odot (\kappa \odot \kappa)) \odot q) \wedge \omega_P(q) \\ &= \omega_P((\kappa \odot 0) \odot q) \wedge \omega_P(q) \\ &= \omega_P(\kappa \odot q) \wedge \omega_P(q), \\ \zeta_P(\kappa) &\geq \zeta_P((\kappa \odot (\kappa \odot \kappa)) \odot q) \wedge \zeta_P(q) \\ &= \zeta_P((\kappa \odot 0) \odot q) \wedge \zeta_P(q) \\ &= \zeta_P(\kappa \odot q) \wedge \zeta_P(q)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa) &\leq \omega_P((\kappa \odot (\kappa \odot \kappa)) \odot q) \vee \omega_P(q) \\ &= \omega_P((\kappa \odot 0) \odot q) \vee \omega_P(q) \\ &= \omega_P(\kappa \odot q) \vee \omega_P(q).\end{aligned}$$

This shows that  $P = (\omega_P, \zeta_P, \omega_P)$  satisfies Definition 5 (ii). Hence,  $P = (\omega_P, \zeta_P, \omega_P)$  is a TPPFI of  $F$ .  $\square$

**Theorem 8.** If  $F$  is an implicative BCK-algebra, then every TPPFI of  $F$  is a TPPFII.

**Proof.** Because  $F$  is an implicative BCK-algebra,  $\kappa = \kappa \odot (q \odot \kappa)$  for all  $\kappa, q \in F$ . Let  $P = (\omega_P, \zeta_P, \omega_P)$  be a TPPFI of  $F$ . Then, by Definition 5 (ii),

$$\begin{aligned}\omega_P(\kappa) &\geq \omega_P(\kappa \odot \vartheta) \wedge \omega_P(\vartheta) \\ &= \omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \wedge \omega_P(\vartheta), \\ \zeta_P(\kappa) &\geq \zeta_P(\kappa \odot \vartheta) \wedge \zeta_P(\vartheta) \\ &= \zeta_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \wedge \zeta_P(\vartheta)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa) &\leq \omega_P(\kappa \odot \vartheta) \vee \omega_P(\vartheta) \\ &= \omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \vee \omega_P(\vartheta)\end{aligned}$$

for all  $\kappa, q, \vartheta \in F$ . Hence,  $P = (\omega_P, \zeta_P, \omega_P)$  is a TPPFII of  $F$ .  $\square$

**Theorem 9.** If  $F$  is an implicative BCK-algebra, then a TPPFS  $P = (\omega_P, \zeta_P, \omega_P)$  of  $F$  is a TPPFI of  $F$  if and only if it is a TPPFII of  $F$ .

**Proposition 2.** A TPPFI  $P = (\omega_P, \zeta_P, \omega_P)$  is a TPPFII of  $F$  iff  $P^C$  satisfies the following:

- (i)  $\omega_P^C(0) \leq \omega_P^C(\kappa), \zeta_P^C(0) \leq \zeta_P^C(\kappa)$  and  $\omega_P^C(0) \geq \omega_P^C(\kappa)$ ,
- (ii)  $\omega_P^C(\kappa) \leq \omega_P^C((\kappa \odot (q \odot \kappa)) \odot \vartheta) \vee \omega_P^C(\vartheta), \zeta_P^C(\kappa) \leq \zeta_P^C((\kappa \odot (q \odot \kappa)) \odot \vartheta) \vee \zeta_P^C(\vartheta)$  and  $\omega_P^C(\kappa) \geq \omega_P^C((\kappa \odot (q \odot \kappa)) \odot \vartheta) \wedge \omega_P^C(\vartheta)$ .

**Proof.** Let  $P = (\omega_P, \zeta_P, \omega_P)$  be a TPPFII of  $F$  and let  $\kappa, q, \vartheta \in F$ . Then,

$$\begin{aligned}\omega_P^C(0) &= 1 - \omega_P(0) \leq 1 - \omega_P(\kappa) = \omega_P^C(\kappa), \\ \zeta_P^C(0) &= 1 - \zeta_P(0) \leq 1 - \zeta_P(\kappa) = \zeta_P^C(\kappa), \\ \omega_P^C(0) &= 1 - \omega_P(0) \geq 1 - \omega_P(\kappa) = \omega_P^C(\kappa)\end{aligned}$$

and

$$\begin{aligned}
 \omega_P^C(\kappa) &= 1 - \omega_P(\kappa) \\
 &\leq 1 - \{\omega_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \omega_P(\vartheta)\} \\
 &= 1 - \{(1 - \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta)) \vee (1 - \omega_P^C(\vartheta))\} \\
 &= \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \omega_P^C(\vartheta), \\
 \varsigma_P^C(\kappa) &= 1 - \varsigma_P(\kappa) \\
 &\leq 1 - \{\varsigma_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \varsigma_P(\vartheta)\} \\
 &= 1 - \{(1 - \varsigma_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta)) \vee (1 - \varsigma_P^C(\vartheta))\} \\
 &= \varsigma_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \varsigma_P^C(\vartheta), \\
 \omega_P^C(\kappa) &= 1 - \omega_P(\kappa) \\
 &\geq 1 - \{\omega_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \omega_P(\vartheta)\} \\
 &= 1 - \{(1 - \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta)) \wedge (1 - \omega_P^C(\vartheta))\} \\
 &= \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \omega_P^C(\vartheta).
 \end{aligned}$$

Conversely, let  $P^C$  satisfy the following:

- (i)  $\omega_P^C(0) \leq \omega_P^C(\kappa), \varsigma_P^C(0) \leq \varsigma_P^C(\kappa)$  and  $\omega_P^C(0) \geq \omega_P^C(\kappa),$
- (ii)  $\omega_P^C(\kappa) \leq \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \omega_P^C(\vartheta), \varsigma_P^C(\kappa) \leq \varsigma_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \varsigma_P^C(\vartheta)$  and  $\omega_P^C(\kappa) \geq \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \omega_P^C(\vartheta).$

Then,

$$\begin{aligned}
 \omega_P(0) &= 1 - \omega_P^C(0) \geq 1 - \omega_P^C(\kappa) = \omega_P(\kappa), \\
 \varsigma_P(0) &= 1 - \varsigma_P^C(0) \geq 1 - \varsigma_P^C(\kappa) = \varsigma_P(\kappa), \\
 \omega_P(0) &= 1 - \omega_P^C(0) \leq 1 - \omega_P^C(\kappa) = \omega_P(\kappa)
 \end{aligned}$$

and

$$\begin{aligned}
 \omega_P(\kappa) &= 1 - \omega_P^C(\kappa) \\
 &\geq 1 - \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \omega_P^C(\vartheta) \\
 &= 1 - \{(1 - \omega_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta)) \wedge (1 - \omega_P(\vartheta))\} \\
 &= \omega_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \omega_P(\vartheta), \\
 \varsigma_P(\kappa) &= 1 - \varsigma_P^C(\kappa) \\
 &\geq 1 - \varsigma_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \varsigma_P^C(\vartheta) \\
 &= 1 - \{(1 - \varsigma_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta)) \wedge (1 - \varsigma_P(\vartheta))\} \\
 &= \varsigma_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \varsigma_P(\vartheta), \\
 \omega_P(\kappa) &= 1 - \omega_P^C(\kappa) \\
 &\leq 1 - \omega_P^C((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \wedge \omega_P^C(\vartheta) \\
 &= 1 - \{(1 - \omega_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta)) \vee (1 - \omega_P(\vartheta))\} \\
 &= \omega_P((\kappa \odot (\varrho \odot \kappa)) \odot \vartheta) \vee \omega_P(\vartheta).
 \end{aligned}$$

Thus,  $P = (\omega_P, \varsigma_P, \omega_P)$  is a TPPFII of  $F$ .  $\square$

**Theorem 10.** Let  $P = (\omega_P, \varsigma_P, \omega_P)$  be a TPPFII of  $F$ . Then, the set  $F_P = \{\kappa \in F : \omega_P(\kappa) = \omega_P(0), \varsigma_P(\kappa) = \varsigma_P(0) \text{ and } \omega_P(\kappa) = \omega_P(0)\}$  is an implicative ideal of  $F$ .

**Proof.** Obviously,  $0 \in F_P$ . Let  $\kappa, q, \vartheta \in F_P$ ,  $(\kappa \odot (q \odot \kappa)) \odot \vartheta \in F_P$  and  $\vartheta \in F_P$ . Then,

$$\begin{aligned}\omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) &= \omega_P(\vartheta) = \omega_P(0), \\ \zeta_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) &= \zeta_P(\vartheta) = \zeta_P(0)\end{aligned}$$

and

$$\omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) = \omega_P(\vartheta) = \omega_P(0).$$

It follows that

$$\begin{aligned}\omega_P(\kappa) &\geq \omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \wedge \omega_P(\vartheta) \\ &= \omega_P(0) \wedge \omega_P(0) \\ &= \omega_P(0), \\ \zeta_P(\kappa) &\geq \zeta_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \wedge \zeta_P(\vartheta) \\ &= \zeta_P(0) \wedge \zeta_P(0) \\ &= \zeta_P(0)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa) &\leq \omega_P((\kappa \odot (q \odot \kappa)) \odot \vartheta) \vee \omega_P(\vartheta) \\ &= \omega_P(0) \vee \omega_P(0) \\ &= \omega_P(0).\end{aligned}$$

Combining with Definition 6 (i), we obtain

$$\omega_P(\kappa) = \omega_P(0), \zeta_P(\kappa) = \zeta_P(0) \text{ and } \omega_P(\kappa) = \omega_P(0) \text{ and so } \kappa \in F_P.$$

Hence,  $F_P$  is an implicative ideal of  $F$ .  $\square$

**Proposition 3.** Let  $P = (\omega_P, \zeta_P, \omega_P)$  is a TPPFI of  $F$ . Then, the below stated statements are equivalent.

- (i)  $P$  is a TPPFII of  $F$ .
- (ii)  $\omega_P(\kappa) \geq \omega_P(\kappa \odot (q \odot \kappa)), \zeta_P(\kappa) \geq \zeta_P(\kappa \odot (q \odot \kappa))$  and  $\omega_P(\kappa) \leq \omega_P(\kappa \odot (q \odot \kappa))$  for all  $\kappa, q \in F$ .
- (iii)  $\omega_P(\kappa) = \omega_P(\kappa \odot (q \odot \kappa)), \zeta_P(\kappa) = \zeta_P(\kappa \odot (q \odot \kappa))$  and  $\omega_P(\kappa) = \omega_P(\kappa \odot (q \odot \kappa))$  for all  $\kappa, q \in F$ .

**Proof.**

(i)  $\Rightarrow$  (ii): Since  $P$  is a TPPFII of  $F$ , we have

$$\begin{aligned}\omega_P(\kappa) &\geq \omega_P((\kappa \odot (q \odot \kappa)) \odot 0) \wedge \omega_P(0) \\ &= \omega_P(\kappa \odot (q \odot \kappa)) \wedge \omega_P(0) \\ &= \omega_P(\kappa \odot (q \odot \kappa)), \\ \zeta_P(\kappa) &\geq \zeta_P((\kappa \odot (q \odot \kappa)) \odot 0) \wedge \zeta_P(0) \\ &= \zeta_P(\kappa \odot (q \odot \kappa)) \wedge \zeta_P(0) \\ &= \zeta_P(\kappa \odot (q \odot \kappa))\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa) &\leq \omega_P((\kappa \odot (q \odot \kappa)) \odot 0) \vee \omega_P(0) \\ &= \omega_P(\kappa \odot (q \odot \kappa)) \vee \omega_P(0) \\ &= \omega_P(\kappa \odot (q \odot \kappa))\end{aligned}$$

for all  $\kappa, q \in F$ .

(ii)  $\Rightarrow$  (iii): It is well-known by Proposition 1 that  $\kappa \odot (q \odot \kappa) \leq \kappa$ . Then, by Theorem 1,  $\omega_P(\kappa) \leq \omega_P(\kappa \odot (q \odot \kappa))$ ,  $\zeta_P(\kappa) \leq \zeta_P(\kappa \odot (q \odot \kappa))$  and  $\omega_P(\kappa) \geq \omega_P(\kappa \odot (q \odot \kappa))$  for all  $\kappa, q \in F$ . By (ii),  $\omega_P(\kappa) \geq \omega_P(\kappa \odot (q \odot \kappa))$ ,  $\zeta_P(\kappa) \geq \zeta_P(\kappa \odot (q \odot \kappa))$  and  $\omega_P(\kappa) \leq \omega_P(\kappa \odot (q \odot \kappa))$  for all  $\kappa, q \in F$ . As a result,  $\omega_P(\kappa) = \omega_P(\kappa \odot (q \odot \kappa))$ ,  $\zeta_P(\kappa) = \zeta_P(\kappa \odot (q \odot \kappa))$  and  $\omega_P(\kappa) = \omega_P(\kappa \odot (q \odot \kappa))$  for all  $\kappa, q \in F$ .

(iii)  $\Rightarrow$  (i): Since  $P$  is a TPPFI of  $F$ ,

$$\begin{aligned}\omega_P(\kappa \odot (q \odot \kappa)) &\geq \omega_P((\kappa \odot (q \odot \kappa)) \odot \theta) \wedge \omega_P(\theta), \\ \zeta_P(\kappa \odot (q \odot \kappa)) &\geq \zeta_P((\kappa \odot (q \odot \kappa)) \odot \theta) \wedge \zeta_P(\theta)\end{aligned}$$

and

$$\omega_P(\kappa \odot (q \odot \kappa)) \leq \omega_P((\kappa \odot (q \odot \kappa)) \odot \theta) \vee \omega_P(\theta),$$

for all  $\kappa, q, \theta \in F$ . By (iii), we have

$$\begin{aligned}\omega_P(\kappa) &\geq \omega_P((\kappa \odot (q \odot \kappa)) \odot \theta) \wedge \omega_P(\theta), \\ \zeta_P(\kappa) &\geq \zeta_P((\kappa \odot (q \odot \kappa)) \odot \theta) \wedge \zeta_P(\theta)\end{aligned}$$

and

$$\omega_P(\kappa) \leq \omega_P((\kappa \odot (q \odot \kappa)) \odot \theta) \vee \omega_P(\theta)$$

for all  $\kappa, q, \theta \in F$ . Thus,  $P$  is a TPPFII of  $F$ .  $\square$

**Definition 7.** A TPPFS  $P = (\omega_P, \zeta_P, \omega_P)$  in a BCK-algebra  $F$  is called TPPFPII of  $F$  if it fulfills the succeeding assertions:

- (i)  $\omega_P(0) \geq \omega_P(\kappa)$ ,  $\zeta_P(0) \geq \zeta_P(\kappa)$  and  $\omega_P(0) \leq \omega_P(\kappa)$ ,
- (ii)  $\omega_P(\kappa \odot \theta) \geq \omega_P((\kappa \odot q) \odot \theta) \wedge \omega_P(q \odot \theta)$ ,  $\zeta_P(\kappa \odot \theta) \geq \zeta_P((\kappa \odot q) \odot \theta) \wedge \zeta_P(q \odot \theta)$  and  $\omega_P(\kappa \odot \theta) \leq \omega_P((\kappa \odot q) \odot \theta) \vee \omega_P(q \odot \theta)$ .

That is,

- (i)  $\rho_i \circ \omega_P(0) \geq \rho_i \circ \omega_P(\kappa)$ ,  $\rho_i \circ \zeta_P(0) \geq \rho_i \circ \zeta_P(\kappa)$  and  $\rho_i \circ \omega_P(0) \leq \rho_i \circ \omega_P(\kappa)$
- (ii)  $\rho_i \circ \omega_P(\kappa \odot \theta) \geq \rho_i \circ \omega_P((\kappa \odot q) \odot \theta) \wedge \rho_i \circ \omega_P(q \odot \theta)$ ,  $\rho_i \circ \zeta_P(\kappa \odot \theta) \geq \rho_i \circ \zeta_P((\kappa \odot q) \odot \theta) \wedge \rho_i \circ \zeta_P(q \odot \theta)$  and  $\rho_i \circ \omega_P(\kappa \odot \theta) \leq \rho_i \circ \omega_P((\kappa \odot q) \odot \theta) \vee \rho_i \circ \omega_P(q \odot \theta)$

for all  $\kappa, q, \theta \in F$ ,  $i = 1, 2, 3$ .

**Proposition 4.** A TPPFI  $P = (\omega_P, \zeta_P, \omega_P)$  of  $F$  is a TPPFPII iff  $\omega_P(\kappa \odot q) \geq \omega_P((\kappa \odot q) \odot q)$ ,  $\zeta_P(\kappa \odot q) \geq \zeta_P((\kappa \odot q) \odot q)$ , and  $\omega_P(\kappa \odot q) \leq \omega_P((\kappa \odot q) \odot q)$  for all  $\kappa, q \in F$ ,  $i = 1, 2, 3$ .

**Proof.** Suppose that TPPFI  $P = (\omega_P, \zeta_P, \omega_P)$  is a TPPFPII of  $F$ . Then,

$$\begin{aligned}\omega_P(\kappa \odot \theta) &\geq \omega_P((\kappa \odot q) \odot \theta) \wedge \omega_P(q \odot \theta), \\ \zeta_P(\kappa \odot \theta) &\geq \zeta_P((\kappa \odot q) \odot \theta) \wedge \zeta_P(q \odot \theta)\end{aligned}$$

and

$$\omega_P(\kappa \odot \theta) \leq \omega_P((\kappa \odot q) \odot \theta) \vee \omega_P(q \odot \theta).$$

Substituting  $\vartheta = \varrho$ , we have

$$\begin{aligned}\omega_P(\varkappa \odot \varrho) &\geq \omega_P((\varkappa \odot \varrho) \odot \varrho) \wedge \omega_P(\varrho \odot \varrho) \\ &= \omega_P((\varkappa \odot \varrho) \odot \varrho) \wedge \omega_P(0) \\ &= \omega_P((\varkappa \odot \varrho) \odot \varrho), \\ \varsigma_P(\varkappa \odot \varrho) &\geq \varsigma_P((\varkappa \odot \varrho) \odot \varrho) \wedge \varsigma_P(\varrho \odot \varrho) \\ &= \varsigma_P((\varkappa \odot \varrho) \odot \varrho) \wedge \varsigma_P(0) \\ &= \varsigma_P((\varkappa \odot \varrho) \odot \varrho)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\varkappa \odot \varrho) &\leq \omega_P((\varkappa \odot \varrho) \odot \varrho) \vee \omega_P(\varrho \odot \varrho) \\ &= \omega_P((\varkappa \odot \varrho) \odot \varrho) \vee \omega_P(0) \\ &= \omega_P((\varkappa \odot \varrho) \odot \varrho).\end{aligned}$$

On the other hand, suppose that  $P$  is a *TPPFI* of  $F$  satisfying the subsequent inequalities:  $\omega_P(\varkappa \odot \varrho) \geq \omega_P((\varkappa \odot \varrho) \odot \varrho)$ ,  $\varsigma_P(\varkappa \odot \varrho) \geq \varsigma_P((\varkappa \odot \varrho) \odot \varrho)$ ,  $\omega_P(\varkappa \odot \varrho) \leq \omega_P((\varkappa \odot \varrho) \odot \varrho)$  for all  $\varkappa, \varrho \in F$ ,  $i = 1, 2, 3$ ,

Now, we prove that

$$\begin{aligned}\omega_P(\varkappa \odot \vartheta) &\geq \omega_P((\varkappa \odot \varrho) \odot \vartheta) \wedge \omega_P(\varrho \odot \vartheta), \\ \varsigma_P(\varkappa \odot \vartheta) &\geq \varsigma_P((\varkappa \odot \varrho) \odot \vartheta) \wedge \varsigma_P(\varrho \odot \vartheta)\end{aligned}$$

and

$$\omega_P(\varkappa \odot \vartheta) \leq \omega_P((\varkappa \odot \varrho) \odot \vartheta) \vee \omega_P(\varrho \odot \vartheta).$$

Suppose, in contrast, that there exist  $\varkappa_0, \varrho_0 \in F$  such that

$$\begin{aligned}\omega_P(\varkappa_0 \odot \varrho_0) &\leq \omega_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0) \wedge \omega_P(\varrho_0 \odot \varrho_0) \\ &= \omega_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0) \wedge \omega_P(0) \\ &= \omega_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0), \\ \varsigma_P(\varkappa_0 \odot \varrho_0) &\leq \varsigma_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0) \wedge \varsigma_P(\varrho_0 \odot \varrho_0) \\ &= \varsigma_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0) \wedge \varsigma_P(0) \\ &= \varsigma_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\varkappa_0 \odot \varrho_0) &\geq \omega_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0) \vee \omega_P(\varrho_0 \odot \varrho_0) \\ &= \omega_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0) \vee \omega_P(0) \\ &= \omega_P((\varkappa_0 \odot \varrho_0) \odot \varrho_0),\end{aligned}$$

which is a contradiction.

Therefore,

$$\begin{aligned}\omega_P(\varkappa \odot \vartheta) &\geq \omega_P((\varkappa \odot \varrho) \odot \vartheta) \wedge \omega_P(\varrho \odot \vartheta), \\ \varsigma_P(\varkappa \odot \vartheta) &\geq \varsigma_P((\varkappa \odot \varrho) \odot \vartheta) \wedge \varsigma_P(\varrho \odot \vartheta)\end{aligned}$$

and

$$\omega_P(\varkappa \odot \vartheta) \leq \omega_P((\varkappa \odot \varrho) \odot \vartheta) \vee \omega_P(\varrho \odot \vartheta).$$

Thus,  $P$  is a *TPPFPII* of  $F$ .  $\square$

Since  $(\varkappa \odot q) \odot q \leq \varkappa \odot q$ , it follows from Theorem 1 that  $\omega_P(\varkappa \odot q) \leq \omega_P((\varkappa \odot q) \odot q)$ ,  $\zeta_P(\varkappa \odot q) \leq \zeta_P((\varkappa \odot q) \odot q)$ ,  $\varpi_P(\varkappa \odot q) \geq \varpi_P((\varkappa \odot q) \odot q)$  for all  $\varkappa, q \in F, i = 1, 2, 3$ . Thus, Proposition 4 above can be reformed in the subsequent way:

**Proposition 5.** A TPPFI  $P = (\omega_P, \zeta_P, \varpi_P)$  of  $F$  is a TPPFPII iff  $\omega_P(\varkappa \odot q) = \omega_P((\varkappa \odot q) \odot q)$ ,  $\zeta_P(\varkappa \odot q) = \zeta_P((\varkappa \odot q) \odot q)$ ,  $\varpi_P(\varkappa \odot q) = \varpi_P((\varkappa \odot q) \odot q)$  for all  $\varkappa, q \in F, i = 1, 2, 3$ .

### 5. Tripolar Picture Fuzzy Commutative Ideal

**Definition 8.** A TPPFS  $P = (\omega_P, \zeta_P, \varpi_P)$  is TPPFCI of  $F$  if it satisfies the following assertions:

- (i)  $\omega_P(0) \geq \omega_P(\varkappa), \zeta_P(0) \geq \zeta_P(\varkappa)$  and  $\varpi_P(0) \leq \varpi_P(\varkappa)$ ,
- (ii)  $\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \omega_P((\varkappa \odot q) \odot \vartheta) \wedge \omega_P(\vartheta), \zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \zeta_P((\varkappa \odot q) \odot \vartheta) \wedge \zeta_P(\vartheta)$  and  $\varpi_P(\varkappa \odot (q \odot (q \odot \varkappa))) \leq \varpi_P((\varkappa \odot q) \odot \vartheta) \vee \varpi_P(\vartheta)$ .

That is,

- (i)  $\rho_i \circ \omega_P(0) \geq \rho_i \circ \omega_P(\varkappa), \zeta_P(0) \geq \zeta_P(\varkappa)$  and  $\varpi_P(0) \leq \varpi_P(\varkappa)$ ,
- (ii)  $\rho_i \circ \omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \rho_i \circ \omega_P((\varkappa \odot q) \odot \vartheta) \wedge \rho_i \circ \omega_P(\vartheta), \rho_i \circ \zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \rho_i \circ \zeta_P((\varkappa \odot q) \odot \vartheta) \wedge \rho_i \circ \zeta_P(\vartheta)$  and  $\rho_i \circ \varpi_P(\varkappa \odot (q \odot (q \odot \varkappa))) \leq \rho_i \circ \varpi_P((\varkappa \odot q) \odot \vartheta) \vee \rho_i \circ \varpi_P(\vartheta)$

for all  $\varkappa, q, \vartheta \in F, i = 1, 2, 3$ .

**Example 3.** Let us consider the BCK-algebra  $(F; \odot, 0)$  as follows:

$\odot$	0	$\varkappa$	q	$\vartheta$
0	0	0	0	0
$\varkappa$	$\varkappa$	0	0	$\varkappa$
q	q	$\varkappa$	0	q
$\vartheta$	$\vartheta$	$\vartheta$	$\vartheta$	0

Now, let us construct a TPPFS  $P$  as follows:

$$\omega_P(\delta) = \begin{cases} (0.29, 0.31, 0.32), & \text{if } \delta = 0 \\ (0.23, 0.25, 0.27), & \text{if } \delta = \varkappa \\ (0.12, 0.13, 0.13), & \text{if } \delta = q, \vartheta \end{cases}$$

$$\zeta_P(\delta) = \begin{cases} (0.3, 0.31, 0.34), & \text{if } \delta = 0 \\ (0.20, 0.22, 0.25), & \text{if } \delta = \varkappa \\ (0.15, 0.18, 0.22), & \text{if } \delta = q, \vartheta \end{cases}$$

and

$$\varpi_P(\delta) = \begin{cases} (0.05, 0.1, 0.12), & \text{if } \delta = 0 \\ (0.15, 0.22, 0.26), & \text{if } \delta = \varkappa \\ (0.50, 0.52, 0.53), & \text{if } \delta = q, \vartheta \end{cases}$$

Clearly,  $P$  is a TPPFCI of  $F$ .

**Proposition 6.** Every TPPFCI of a BCK-algebra is a TPPFI.

**Proof.** Let  $P = (\omega_P, \zeta_P, \varpi_P)$  be a TPPFCI of  $F$ . We have

$$\begin{aligned} (\varkappa \odot (0 \odot (0 \odot \varkappa))) &= (\varkappa \odot 0) \text{ [by Proposition 1]} \\ &= \varkappa \text{ [by Proposition 1]} \end{aligned}$$

Then,

$$\begin{aligned}\omega_P(\kappa) &= \omega_P(\kappa \odot (0 \odot (0 \odot \kappa))) \\ &\geq \omega_P((\kappa \odot 0) \odot \vartheta) \wedge \omega_P(\vartheta) \\ &= \omega_P(\kappa \odot \vartheta) \wedge \omega_P(\vartheta) \\ \zeta_P(\kappa) &= \zeta_P(\kappa \odot (0 \odot (0 \odot \kappa))) \\ &\geq \zeta_P((\kappa \odot 0) \odot \vartheta) \wedge \zeta_P(\vartheta) \\ &= \zeta_P(\kappa \odot \vartheta) \wedge \zeta_P(\vartheta)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa) &= \omega_P(\kappa \odot (0 \odot (0 \odot \kappa))) \\ &\leq \omega_P((\kappa \odot 0) \odot \vartheta) \vee \omega_P(\vartheta) \\ &= \omega_P(\kappa \odot \vartheta) \vee \omega_P(\vartheta)\end{aligned}$$

for all  $\kappa, \vartheta \in F$ . Consequently,  $P$  is a TPPFI of  $F$ .  $\square$

Proposition 6 is not true in the reverse direction, which we show by the following example:

**Example 4.** Let us consider the BCK-algebra given in Example 1. Observe that

$$\begin{aligned}\omega_P((q \odot (\vartheta \odot (\vartheta \odot q)))) &= \omega_P(q) = (0.15, 0.19, 0.22), \\ \omega_P((q \odot \vartheta) \odot 0) \wedge \omega_P(0) &= (0.35, 0.36, 0.38), \\ \zeta_P((q \odot (\vartheta \odot (\vartheta \odot q)))) &= \zeta_P(q) = (0.16, 0.17, 0.18), \\ \zeta_P((q \odot \vartheta) \odot 0) \wedge \zeta_P(0) &= (0.38, 0.40, 0.42).\end{aligned}$$

and

$$\begin{aligned}\omega_P((q \odot (\vartheta \odot (\vartheta \odot q)))) &= \omega_P(q) = (0.40, 0.42, 0.45), \\ \omega_P((q \odot \vartheta) \odot 0) \vee \omega_P(0) &= (0.03, 0.04, 0.05).\end{aligned}$$

Hence,

$$\begin{aligned}\omega_P((q \odot (\vartheta \odot (\vartheta \odot q)))) &\not\geq \omega_P((q \odot \vartheta) \odot 0) \vee \omega_P(0) \\ \zeta_P((q \odot (\vartheta \odot (\vartheta \odot q)))) &\not\geq \zeta_P((q \odot \vartheta) \odot 0) \wedge \zeta_P(0)\end{aligned}$$

and

$$\omega_P((q \odot (\vartheta \odot (\vartheta \odot q)))) \not\leq \omega_P((q \odot \vartheta) \odot 0) \wedge \omega_P(0).$$

Clearly,  $P$  is not a TPPFCI of  $F$ .

The converse of Proposition 6 above holds in commutative BCK-algebra, which is underlined in the following proposition.

**Proposition 7.** Every TPPFI in a commutative BCK-algebra is a TPPFCI.

**Proof.** Let  $P = (\omega_P, \zeta_P, \omega_P)$  be a TPPFI of a commutative BCK-algebra  $F$ . Then,

$$\begin{aligned}[(\kappa \odot (q \odot (q \odot \kappa))) \odot ((\kappa \odot q) \odot \vartheta)] \odot \vartheta &= ((\kappa \odot (q \odot (q \odot \kappa))) \odot \vartheta) \odot ((\kappa \odot q) \odot \vartheta) \text{ [by Proposition 1 (iv)]} \\ &\leq (\kappa \odot (q \odot (q \odot \kappa))) \odot (\kappa \odot q) \text{ [by Proposition 1 (iii)]} \\ &= (\kappa \odot (\kappa \odot (\kappa \odot q))) \odot (\kappa \odot q) \text{ [as } F \text{ is commutative]} \\ &= (\kappa \odot q) \odot (\kappa \odot q) \text{ [by Proposition 1 (v)]} \\ &= 0\end{aligned}$$

i.e.,  $(\varkappa \odot (q \odot (q \odot \varkappa))) \odot ((\varkappa \odot q) \odot \vartheta) \leq \vartheta$ .

Thus, by Theorem 2, it is obtained that

$$\begin{aligned}\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\geq \omega_P((\varkappa \odot q) \odot \vartheta) \wedge \omega_P(\vartheta), \\ \zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\geq \zeta_P((\varkappa \odot q) \odot \vartheta) \wedge \zeta_P(\vartheta)\end{aligned}$$

and

$$\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \leq \omega_P((\varkappa \odot q) \odot \vartheta) \vee \omega_P(\vartheta).$$

for all  $\varkappa, q, \vartheta \in F$ . Consequently,  $P$  is a  $TPPFCI$  of  $F$ .  $\square$

Now, we are interested in developing a relationship between  $TPPFII$  and  $TPPFCI$ . Before we get there, there are a few things we need to talk about. The following proposition was made by Meng et al. [11]

**Proposition 8.** *The followings hold in a BCK-algebra  $F$ .*

- i.  $((\varkappa \odot \vartheta) \odot \vartheta) \odot (q \odot \vartheta) \leq (\varkappa \odot t) \odot u$ .
- ii.  $(\varkappa \odot \vartheta) \odot (\varkappa \odot (\varkappa \odot \vartheta)) = (\varkappa \odot \vartheta) \odot \vartheta$ .
- iii.  $(\varkappa \odot (q \odot (q \odot \varkappa))) \odot (q \odot (\varkappa \odot (q \odot (q \odot \varkappa)))) \leq \varkappa \odot q$ .

**Proposition 9.** *A  $TPPFI$   $P = (\omega_P, \zeta_P, \omega_P)$  of  $(F, \odot, 0)$  is a  $TPPFCI$  iff  $\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \omega_P(\varkappa \odot q)$ ,  $\zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \zeta_P(\varkappa \odot q)$  and  $\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \leq \omega_P(\varkappa \odot q)$  for all  $\varkappa, q \in F$ .*

**Proof.** Suppose that  $P$  is a  $TPPFCI$  of  $F$ . From Definition 8,

$$\begin{aligned}\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\geq \omega_P((\varkappa \odot q) \odot \vartheta) \wedge \omega_P(\vartheta), \\ \zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\geq \zeta_P((\varkappa \odot q) \odot \vartheta) \wedge \zeta_P(\vartheta)\end{aligned}$$

and

$$\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \leq \omega_P((\varkappa \odot q) \odot \vartheta) \vee \omega_P(\vartheta)$$

Taking  $\vartheta = 0$ ,

$$\begin{aligned}\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\geq \omega_P((\varkappa \odot q) \odot 0) \wedge \omega_P(0) \\ &= \omega_P(\varkappa \odot q), \\ \zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\geq \zeta_P((\varkappa \odot q) \odot 0) \wedge \zeta_P(0) \\ &= \zeta_P(\varkappa \odot q)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) &\leq \omega_P((\varkappa \odot q) \odot 0) \vee \omega_P(0) \\ &= \omega_P(\varkappa \odot q).\end{aligned}$$

Conversely, assume that  $P$  satisfies  $\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \omega_P(\varkappa \odot q)$ ,  $\zeta_P(\varkappa \odot (q \odot (q \odot \varkappa))) \geq \zeta_P(\varkappa \odot q)$  and  $\omega_P(\varkappa \odot (q \odot (q \odot \varkappa))) \leq \omega_P(\varkappa \odot q)$  for all  $\varkappa, q \in F$ .

Since  $P$  is a  $TPPFI$ , we know that

$$\begin{aligned}\omega_P(\varkappa \odot q) &\geq \omega_P((\varkappa \odot q) \odot \vartheta) \wedge \omega_P(\vartheta), \\ \zeta_P(\varkappa \odot q) &\geq \zeta_P((\varkappa \odot q) \odot \vartheta) \wedge \zeta_P(\vartheta)\end{aligned}$$



and

$$\omega_P(\kappa \odot q) \leq \omega_P((\kappa \odot q) \odot \vartheta) \vee \omega_P(\vartheta).$$

Therefore,

$$\begin{aligned}\omega_P(\kappa \odot (q \odot (q \odot \kappa))) &\geq \omega_P((\kappa \odot q) \odot \vartheta) \wedge \omega_P(\vartheta), \\ \zeta_P(\kappa \odot (q \odot (q \odot \kappa))) &\geq \zeta_P((\kappa \odot q) \odot \vartheta) \wedge \zeta_P(\vartheta)\end{aligned}$$

and

$$\omega_P(\kappa \odot (q \odot (q \odot \kappa))) \leq \omega_P((\kappa \odot q) \odot \vartheta) \vee \omega_P(\vartheta).$$

Hence,  $P$  is  $TPPFCI$  of  $F$ .  $\square$

It is observed that  $(\kappa \odot q) \odot (\kappa \odot (q \odot (q \odot \kappa))) = 0$  and, using Theorem 1, we obtain  $\omega_P(\kappa \odot (q \odot (q \odot \kappa))) \leq \omega_P(\kappa \odot q), \zeta_P(\kappa \odot (q \odot (q \odot \kappa))) \leq \zeta_P(\kappa \odot q)$  and  $\omega_P(\kappa \odot (q \odot (q \odot \kappa))) \geq \omega_P(\kappa \odot q)$  for all  $\kappa, q \in F$ . Thus, Proposition 9 above can be modified as follows:

**Proposition 10.** A  $TPPFI$   $P = (\omega_P, \zeta_P, \omega_P)$  of  $(F, \odot, 0)$  is a  $TPPFCI$  iff  $\omega_P(\kappa \odot (q \odot (q \odot \kappa))) = \omega_P(\kappa \odot q), \zeta_P(\kappa \odot (q \odot (q \odot \kappa))) = \zeta_P(\kappa \odot q)$  and  $\omega_P(\kappa \odot (q \odot (q \odot \kappa))) = \omega_P(\kappa \odot q)$  for all  $\kappa, q \in F$ .

**Proposition 11.** A  $TPPFI$   $P = (\omega_P, \zeta_P, \omega_P)$  is a  $TPPFII$  iff  $P$  is both  $TPPFCI$  and  $TPPFPII$ .

**Proof.** Suppose that  $P$  is a  $TPPFII$ . Then, by Proposition 8 (i), and Theorem 2,

$$\begin{aligned}\omega_P((\kappa \odot q) \odot \vartheta) \wedge \omega_P(q \odot \vartheta) &\leq \omega_P((\kappa \odot \vartheta) \odot \vartheta) \\ &= \omega_P((\kappa \odot \vartheta) \odot (\kappa \odot (\kappa \odot \vartheta))) \text{ [by Proposition 8 (ii)]} \\ &= \omega_P(\kappa \odot \vartheta) \text{ [by Proposition 3 (iii)]}, \\ \zeta_P((\kappa \odot q) \odot \vartheta) \wedge \zeta_P(q \odot \vartheta) &\leq \zeta_P((\kappa \odot \vartheta) \odot \vartheta) \\ &= \zeta_P((\kappa \odot \vartheta) \odot (\kappa \odot (\kappa \odot \vartheta))) \text{ [by Proposition 8 (ii)]} \\ &= \zeta_P(\kappa \odot \vartheta) \text{ [by Proposition 3 (iii)]}\end{aligned}$$

and

$$\begin{aligned}\omega_P((\kappa \odot q) \odot \vartheta) \vee \omega_P(q \odot \vartheta) &\geq \omega_P((\kappa \odot \vartheta) \odot \vartheta) \\ &= \omega_P((\kappa \odot \vartheta) \odot (\kappa \odot (\kappa \odot \vartheta))) \text{ [by Proposition 8 (ii)]} \\ &= \omega_P(\kappa \odot \vartheta) \text{ [by Proposition 3 (iii)]}.\end{aligned}$$

Therefore, by Definition 7,  $P$  is a  $TPPFPII$ .

By Proposition 8 (iii) and Theorem 1 we obtain

$$\begin{aligned}\omega_P(\kappa \odot q) &\leq \omega_P((\kappa \odot (q \odot (q \odot \kappa))) \odot (q \odot (\kappa \odot (q \odot (q \odot \kappa))))) \\ &= \omega_P(\kappa \odot (q \odot (q \odot \kappa))) \text{ [by Proposition 3 (iii)]}, \\ \zeta_P(\kappa \odot q) &\leq \zeta_P((\kappa \odot (q \odot (q \odot \kappa))) \odot (q \odot (\kappa \odot (q \odot (q \odot \kappa))))) \\ &= \zeta_P(\kappa \odot (q \odot (q \odot \kappa))) \text{ [by Proposition 3 (iii)]}\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa \odot q) &\geq \omega_P((\kappa \odot (q \odot (q \odot \kappa))) \odot (q \odot (\kappa \odot (q \odot (q \odot \kappa))))) \\ &= \omega_P(\kappa \odot (q \odot (q \odot \kappa))) \text{ [by Proposition 3 (iii)]}.\end{aligned}$$

Therefore, by Proposition 9,  $P$  is a  $TPPFCI$  of  $F$ .

Conversely, let  $P$  be both  $TPPFPII$  and  $TPPFCI$  of  $F$ . Since  $(\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa) \leq \varkappa \odot (\varrho \odot \varkappa)$ , by Theorem 1,

$$\begin{aligned}\omega_P(\varkappa \odot (\varrho \odot \varkappa)) &\leq \omega_P((\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa)), \\ \varsigma_P(\varkappa \odot (\varrho \odot \varkappa)) &\leq \varsigma_P((\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa))\end{aligned}$$

and

$$\omega_P(\varkappa \odot (\varrho \odot \varkappa)) \geq \omega_P((\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa)).$$

By Proposition 5,

$$\begin{aligned}\omega_P((\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa)) &= \omega_P(\varrho \odot (\varrho \odot \varkappa)), \\ \varsigma_P((\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa)) &= \varsigma_P(\varrho \odot (\varrho \odot \varkappa))\end{aligned}$$

and

$$\omega_P((\varrho \odot (\varrho \odot \varkappa)) \odot (\varrho \odot \varkappa)) = \omega_P(\varrho \odot (\varrho \odot \varkappa)).$$

Therefore, it is obtained that

$$\omega_P(\varkappa \odot (\varrho \odot \varkappa)) \leq \omega_P(\varrho \odot (\varrho \odot \varkappa)), \quad (2)$$

$$\varsigma_P(\varkappa \odot (\varrho \odot \varkappa)) \leq \varsigma_P(\varrho \odot (\varrho \odot \varkappa)) \quad (3)$$

and

$$\omega_P(\varkappa \odot (\varrho \odot \varkappa)) \geq \omega_P(\varrho \odot (\varrho \odot \varkappa)). \quad (4)$$

In addition, by Proposition 1,  $\varkappa \odot \varrho \leq \varkappa \odot (\varrho \odot \varkappa)$ . Therefore, by Theorem 1,

$$\begin{aligned}\omega_P(\varkappa \odot (\varrho \odot \varkappa)) &\leq \omega_P(\varkappa \odot \varrho), \\ \varsigma_P(\varkappa \odot (\varrho \odot \varkappa)) &\leq \varsigma_P(\varkappa \odot \varrho)\end{aligned}$$

and

$$\omega_P(\varkappa \odot (\varrho \odot \varkappa)) \geq \omega_P(\varkappa \odot \varrho).$$

Since  $P$  is a  $TPPFCI$  therefore by Proposition 10, it is obtained that

$$\omega_P(\varkappa \odot (\varrho \odot \varkappa)) \leq \omega_P(\varkappa \odot (\varrho \odot (\varrho \odot \varkappa))), \quad (5)$$

$$\varsigma_P(\varkappa \odot (\varrho \odot \varkappa)) \leq \varsigma_P(\varkappa \odot (\varrho \odot (\varrho \odot \varkappa))) \quad (6)$$

and

$$\omega_P(\varkappa \odot (\varrho \odot \varkappa)) \geq \omega_P(\varkappa \odot (\varrho \odot (\varrho \odot \varkappa))). \quad (7)$$

Combining (2) and (5), (3) and (6), (4) and (7), it is obtained that

$$\begin{aligned}\omega_P(\varkappa \odot (\varrho \odot \varkappa)) &\leq \omega_P(\varkappa \odot (\varrho \odot (\varrho \odot \varkappa))) \wedge \omega_P(\varrho \odot (\varrho \odot \varkappa)) \\ &\leq \omega_P(\varkappa), \\ \varsigma_P(\varkappa \odot (\varrho \odot \varkappa)) &\leq \varsigma_P(\varkappa \odot (\varrho \odot (\varrho \odot \varkappa))) \wedge \varsigma_P(\varrho \odot (\varrho \odot \varkappa)) \\ &\leq \varsigma_P(\varkappa)\end{aligned}$$

and

$$\begin{aligned}\omega_P(\kappa \odot (q \odot \kappa)) &\geq \omega_P(\kappa \odot (q \odot (q \odot \kappa))) \vee \omega_P(q \odot (q \odot \kappa)) \\ &\geq \omega_P(\kappa).\end{aligned}$$

Thus, by Proposition 3,  $P$  is a  $TPPFII$  of  $F$ .  $\square$

## 6. Conclusions

In this paper, we popularized the view of tripolar picture fuzzy ideal (implicative ideal) in BCK-algebras. We obtained many related results associated with these notions. Moreover, we considered a relationship between tripolar picture fuzzy ideal (commutative ideal) in BCK-algebras. We are confident that the analysis of specific other types of algebraic structures from the perspective of the tripolar picture fuzzy set will be aided by our work. The findings of this work can be applied to a variety of algebras—for instance, BG-algebras, B-algebras, BCH-algebras, UP-algebras and MV-algebras. Future research should focus on a few key areas, including (1) developing methods for obtaining more useful results, (2) using these ideas and findings to study related ideas in other (hyper/soft) algebraic structures, and (3) examining the ideas of uni-soft filters based on picture fuzzy set theory.

**Author Contributions:** Conceptualization, G.M., N.A. and M.B.; methodology, G.M.; validation, N.A. and M.B.; formal analysis, N.A. and A.A.; investigation, M.B.; writing—original draft preparation, G.M.; writing—review and editing, A.A. and M.B. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research was funded by Princess Nourah bint Abdulrahman University Researchers Supporting Project Number (PNURSP2022R87), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors extend their appreciations to Princess Nourah bint Abdulrahman University Researchers Supporting Project Number (PNURSP2022R87), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

**Conflicts of Interest:** The authors declare no conflict of interest.

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