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Article

On the Solutions for a Fifth Order Kudryashov–Sinelshchikov Type Equation

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Abstract: The fifth order Kudryashov–Sinelshchikov equation models the evolution of the nonlinear waves in a gas–liquid mixture, taking into account an interphase heat transfer, surface tension, and weak liquid compressibility simultaneously at the derivation of the equations for non-linear-waves. We prove the well-posedness of the solutions for the Cauchy problem associated with this equation for each choice of the terminal time T.

Keywords: existence; uniqueness; stability; the fifth order Kudryashov–Sinelshchikov type equation; Cauchy problem

MSC: 35G25; 35K55

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1. Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) + \nu\partial_x^2 u + \delta\partial_x^3 u + \alpha u\partial_x^3 u + \kappa\partial_x u\partial_x^2 u & \\ \qquad +q(\partial_x u)^2 + \gamma\partial_x^5 u + \beta^2\partial_x^4 u = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1}$$

with

$$\nu,\ \delta,\ \alpha,\ \kappa,\ q,\ \gamma,\ \beta \in \mathbb{R}, \quad \beta,\ \gamma \neq 0. \tag{2}$$

On the flux f, we assume

$$f(u) \in C^1(\mathbb{R}), \quad |f'(u)| \leq C_0\left(1 + |u|^2\right), \tag{3}$$

for some positive constant C_0.

On the initial datum, we assume

$$u_0 \in H^2(\mathbb{R}), \quad u_0 \neq 0. \tag{4}$$

Taking

$$f(u) = b_1 u^2 + b_2 u^3, \quad b_1,\ b_2 \in \mathbb{R}, \tag{5}$$

Equation (1) reads

$$\begin{aligned} &\partial_t u + b_1\partial_x u^2 + b_2\partial_x u^3 + \nu\partial_x^2 u + \delta\partial_x^3 u + \alpha u\partial_x^3 u \\ &\qquad + \kappa\partial_x u\partial_x^2 u + q(\partial_x u)^2 + \gamma\partial_x^5 u + \beta^2\partial_x^4 u = 0. \end{aligned} \tag{6}$$

Equation (6) is deduced in [1,2] to model the evolution of the nonlinear-waves in a gas–liquid mixture, taking into account an interphase heat transfer, a surface tension, and a weak liquid compressibility simultaneously at the derivation of the equations for non-linear-waves. In particular, in [1], the authors show that (6) is obtained by a perturbation of the Burgers–Korteweg-de Vries equation, in correspondence with main influence of dispersion nonlinear waves propagation. Finally, Equation (6) is deduced in [3] in the context of ray tracing through a crystalline lens and pressure waves in mixtures liquid–gas bubbles under the consideration of heat transfer and viscosity.

From a mathematical point of view, in [4], special solutions of (6) are studied. In particular, some elliptic and simple periodic traveling waves solution are constructed. In [5], the authors proved that (6) does not belong to the class of integrable equations. Moreover, they also proved that (6) admits classical and non-classical symmetries. In [6], approximate invariant solutions for (6) are analyzed.

Equation (6) is a generalization of the following equation:

$$\partial_t u + b_1 \partial_x u^2 - b_3^2 \partial_x^2 u + \delta \partial_x^3 u + \alpha u \partial_x^3 u + \kappa \partial_x u \partial_x^2 u = 0, \tag{7}$$

with is deduced in [7]. Equation (7) is also derived for water waves by Olver [8] (see also [9]) using Hamiltonian perturbation theory, with further generalization given by Craig and Groves [10].

Mathematical properties of (7) have been studied in significant detail, including the existence of the travelling wave solutions in [11–15], the solitary and periodic wave solutions [16,17], the periodic loop solutions [18], the soliton solutions [19], and the quasi-exact solutions [20]. Methods to find exact solutions are in [21–25], while, in [26], under appropriate assumptions of b_1, α, κ, and

$$u_0 \in H^{\ell}(\mathbb{R}), \quad \ell \in \{1, 2\}, \tag{8}$$

the existence of the solutions for (7) is proven. Finally, following [27–29], in [30], the convergence of the solution of (7) to the unique entropy in one of the Burgers equations is proven.

Taking $b_2 = b_3 = \beta = q = 0$, (7) reads

$$\partial_t u + b_1 \partial_x u^2 + \delta \partial_x^3 u + \alpha u \partial_x^3 u + \kappa \partial_x u \partial_x^2 u + \gamma \partial_x^5 u = 0. \tag{9}$$

From a physical point of view, (7) was derived by Olver [8,31] in the context of water waves. In the case $b_1 = 0$, (9) was derived by Benney [32] as a model to describe the interaction effects between short and long waves.

From a mathematical point of view, under suitable assumptions on b_1, δ, α, κ, and γ, the existence of the travelling waves solutions for (9) is proven in [33,34], while a method to find exact solutions of (9) is given in [35]. In [36] the local well-posedness of the Cauchy problem of (9) is proven, while, in [37,38], under appropriate assumptions of b_1, α, κ, the global well-posedness is showed.

Taking $\alpha = \kappa = q = 0$, (1) reads

$$\partial_t u + \partial_x f(u) + \nu \partial_x^2 u + \delta \partial_x^3 u + \gamma \partial_x^5 u + \beta^2 \partial_x^4 u = 0. \tag{10}$$

Under Assumption (3), (10) was first introduced by Benney [39] and later by Lin [40] to describe the evolution of long waves in various problems in fluid dynamics (see also [41]).

In [42,43], under Assumptions $f(u) = b_1 u^2$ and appropriate assumptions on b_1, ν, δ, γ, and β using the energy space technique, the local and global well-posedness of weak solutions for (10) is proven, while, in [44], the well-posedness is proven using the Bourgain bilinear estimate technique. In [45], the well-posedness of (10) is proven under Assumption (4) for every choice of β and T.

Taking $b_2 = \alpha = \kappa = q = \gamma = 0$, (6) reads

$$\partial_t u + b_1 \partial_x u^2 + \nu \partial_x^2 u + \delta \partial_x^3 u + \beta^2 \partial_x^4 u = 0. \tag{11}$$

Equation (11) was derived independently by Kuramoto [46–48] as a model for phase turbulence in reaction–diffusion systems and by Sivashinsky [49] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front. It also describes incipient instabilities in a variety of physical and chemical systems [50–52]. It was derived by Kuramoto in the study of phase turbulence in the Belousov–Zhabotinsky reaction [53].

In [54–56] the well-posedness of the Cauchy problem for (11) is proven using the energy space technique the fixed point method, a priori estimates together with an application of the Cauchy–Kovalevskaya, and a priori estimates together with an application of the Aubin–Lions Lemma, respectively. In [57–59], the initial boundary value problem for (11) is studied using a priori estimates together with an application of the Cauchy–Kovalevskaya and the energy space technique. Finally, in [60], the convergence of the solution of (11) to the unique entropy in one of the Burgers equations is proven. Here, we extend some of those results considering the fifth order case.

Taking $b_2 = \alpha = \kappa = \gamma = 0$, in (6), we have the following equation:

$$\partial_t u + b_1 \partial_x u^2 + \nu \partial_x^2 u + \delta \partial_x^3 u + q(\partial_x u)^2 + \beta^2 \partial_x^4 u = 0. \tag{12}$$

Equation (12), known as the Kuramoto–Velarde equation, describes slow space-time variations of disturbances at interfaces, diffusion–reaction fronts, and plasma instability fronts [61–63]. It also describes Benard–Marangoni cells that occur when there is large surface tension on the interface [64–66] in a microgravity environment.

In [67], the exact solutions for (12) are studied, while in [68], the initial boundary problem is analyzed. In [61,69], the existence of the solitons is proven, while in [70], the existence of traveling wave solutions for (12) is analyzed. In [71], the author analyzes the existence of the periodic solution for (12) under appropriate assumptions on b_1, ν, δ, q, β. The local well-posedness of the Cauchy problem of (12) is proven in [72] using the energy space technique and assuming $b_1 = 0$, while in [73], the well-posedness of classical solutions is proven under Assumption (4) and suitable assumptions on β, T, and u_0. Finally, in [74], the authors prove the existence of appropriate rescalings, in which the well-posedness of the Cauchy problem of (12) holds for each choice of T, and under the assumption

$$u_0 \in H^1(\mathbb{R}), \quad u_0 \neq 0. \tag{13}$$

Taking $b_2 = \nu = \alpha = \kappa = q = \gamma = \beta = 0$, (6) reads

$$\partial_t u + b_1 \partial_x u^2 + \delta \partial_x^3 u = 0, \tag{14}$$

known as the Korteweg–de Vries equation [75]. It has a very wide range of applications, such as magnetic fluid waves, ion sound waves, and longitudinal astigmatic waves.

In [76–78], the Cauchy problem for (14) is studied. In particular, in [76,77], the well-posedness of the Cauchy problem of (14) is proven under Assumption (4) and for each choice of T. In [79], the author reviewed the travelling wave solutions for (14), while in [28,29,80], the convergence of the solution of (14) to the unique entropy in one of the Burgers equations is proven.

Taking $b_2 = \nu = \alpha = \kappa = q = \beta = 0$, we have the following equation:

$$\partial_t u + b_1 \partial_x u^2 + \delta \partial_x^3 u + \gamma \partial_x^5 u = 0. \tag{15}$$

This was derived by Kawahara [81] to describe small-amplitude gravity capillary waves on water of a finite depth when the Weber number is close to $1/3$ (see [82]). Moreover, in [81], the author deduced (15) to describe one-dimensional propagation of small-amplitude long waves in various problems of fluid dynamics and plasma physics.

In [83–87], the local and global well-posedness in Bourgain space for (15) is proven, while in [88–93], the local and global well-posedness in energy space for (15) is studied. In [94–96], the well-posedness of the initial boundary value problem on a bounded domain is analyzed, while in [97] (see also [98]), the well-posedness of the classical solution of the Cauchy problem of (15) is proven for each choice of T. In [99], the authors prove that the solution of (15) converges to the solution of (14), while, in [100,101] (see also [102]), the convergence of the solution of (15) to the unique entropy in one of the Burgers equations is proven.

The main result of this paper is the following theorem.

Theorem 1. *Assume* (2)–(4). *Given* ν, α, κ, q, T, *there exists a unique solution* u *of* (1), *such that*

$$\begin{aligned} &u \in L^{\infty}(0,T;H^2(\mathbb{R})) \cap L^4(0,\,T;W^{2,4}(\mathbb{R})) \cap L^6(0,\,T;W^{2,6}(\mathbb{R})),\\ &\partial_x^4 u \in L^2((0,T)\times\mathbb{R}). \end{aligned} \tag{16}$$

Moreover, if u_1 *and* u_2 *are two solutions of* (1) *in correspondence with the initial data* $u_{1,0}$ *and* $u_{2,0}$, *the following stability estimate holds:*

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t}\|u_{1,0}-u_{2,0}\|_{L^2(\mathbb{R})}, \tag{17}$$

for some suitable $C(T) > 0$ *and every,* $0 \le t \le T$.

The proof of Theorem 1 is based on the Aubin–Lions Lemma due to the functional setting [103–106]. Observe that using the Aubin–Lions Lemma under Assumption (4), Refs. [45,76,77] give the well-posedness of (10) and (14) for each choice of T. Therefore, thanks to Theorem 1, we find that the solution of (6) converges to the unique solutions of (10) and (14) under assumptions $\alpha = \kappa = q = \gamma = 0$ and $\alpha = \kappa = q = \gamma = \beta = 0$, respectively.

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (1). Those play a key role in the proof of our main result, which is given in Section 3.

2. Vanishing Viscosity Approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t,x)$ be the unique classical solution of the following problem [73]:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) + \nu\partial_x^2 u_\varepsilon + \delta\partial_x^3 u_\varepsilon + \alpha u_\varepsilon \partial_x^3 u_\varepsilon \\ \qquad\qquad +\kappa\partial_x u_\varepsilon \partial_x^2 u_\varepsilon + q(\partial_x u_\varepsilon)^2 + \gamma\partial_x^5 u_\varepsilon \\ \qquad\qquad +\beta^2\partial_x^4 u_\varepsilon = \varepsilon\partial_x^6 u_\varepsilon, & 0 < t < T, \quad x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{18}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0, such that

$$\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \le \|u_0\|_{H^2(\mathbb{R})}, \quad u_{\varepsilon,0} \neq 0. \tag{19}$$

Let us prove some a priori estimates on u_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$ the constants which depend also on T.

Following [73], Lemma 1 and [107], Lemma 2.2, we prove the following result.

Lemma 1. *The following inequalities hold*

$$\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})},\ \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T), \tag{20}$$

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T), \tag{21}$$

$$\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{22}$$

$$\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{23}$$

$$\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{24}$$

$$\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \le C(T), \tag{25}$$

for every $0 \le t \le T$.

Proof. We begin by proving that

$$\begin{aligned}
&\frac{e^{\frac{2(2A^2C_0+a_5^2A+a_3^2)t}{A\beta^2}}}{\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)^2} \\
&\qquad + \frac{24C_0A^2\beta^4 + a_2^2A\beta^4 + a_3^2\beta^2 + a_4^2A}{A\beta^4(2C_0A^2 + a_5^2A + a_3^2)}\left(e^{\frac{2(2A^2C_0+a_5^2A+a_3^2)t}{A\beta^2}} - 1\right) \\
&\quad \ge \frac{1}{\left(\|u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})}\right)^2},
\end{aligned} \tag{26}$$

for every $0 \le t \le T$, where A is a generic positive constant, and

$$\begin{aligned}
&a_1^2 = 12\nu^2 + 1 \neq 0, \quad a_2^2 = 3(C_0 + q^2), \quad a_3^2 := 12\kappa^2 + 6\alpha^2, \\
&a_4^2 := 3(24\alpha^2 + 7\kappa^2)^2 + 1 \neq 0, \quad a_5^2 := a_1^2 + a_2.
\end{aligned} \tag{27}$$

Consider A a positive constant. Multiplying (18) by $2u_\varepsilon - 2A\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right) \\
&\quad = 2\int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon dx - 2A\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon\, dx \\
&\quad = \underbrace{-2\int_{\mathbb{R}} uf'(u)\partial_x u dx}_{=0} + 2A\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\nu\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon\, dx \\
&\qquad + 2A\nu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\delta\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx + 2A\delta\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon\, dx \\
&\qquad - 2\alpha\int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon dx + 2A\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx - 2\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon\, dx \\
&\qquad + 2A\kappa\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2q\int_{\mathbb{R}} u_\varepsilon (\partial_x u_\varepsilon)^2 dx + 2Aq\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^2 u_\varepsilon\, dx \\
&\qquad - 2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^5 u_\varepsilon dx + 2A\gamma\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon dx - 2\beta^2\int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon\, dx \\
&\qquad + 2A\beta^2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^6 u_\varepsilon dx - 2A\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon\, dx \\
&\quad = 2A\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\nu\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx + 2A\nu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\qquad + 2\delta\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\alpha\int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon dx + 2A\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon\, dx \\
&\qquad - 2\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2A\kappa\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2q\int_{\mathbb{R}} u_\varepsilon (\partial_x u_\varepsilon)^2\, dx \\
&\qquad + 2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx - 2A\gamma\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\beta^2\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon\, dx
\end{aligned}$$

$$
\begin{aligned}
&-2A\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5 u_\varepsilon dx+2A\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^5 u_\varepsilon\, dx\\
=&\,2A\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx-2\nu\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx+2A\nu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&-2\alpha\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon dx+2A\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx-2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx\\
&+2A\kappa\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx-2q\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx-2\gamma\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon\, dx\\
&-2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2A\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\, dx\\
&-2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
=&\,2A\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx-2\nu\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx+2A\nu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&-2\alpha\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon dx+2A\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx-2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\, dx\\
&+2A\kappa\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx-2q\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx-2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&-2A\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&=2A\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx-2\nu\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx+2A\nu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad-2\alpha\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon dx+2A\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx-2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\quad+2A\kappa\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx-2q\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx.
\end{aligned}
\tag{28}
$$

Observe that

$$
\begin{aligned}
\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx&=\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^2 u_\varepsilon dx=-\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x\left(\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\right)dx\\
&=-\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx-\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon dx.
\end{aligned}
$$

Consequently, we have that

$$
2\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx=-\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon dx. \tag{29}
$$

Moreover,

$$
\begin{aligned}
\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx&=\int_{\mathbb{R}} u_\varepsilon\partial_x\partial_x u_\varepsilon dx=-\int_{\mathbb{R}} u_\varepsilon\partial_x(u_\varepsilon\partial_x u_\varepsilon)\\
&=-\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2-\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon dx,
\end{aligned}
$$

which gives

$$
2\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx=-\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon dx. \tag{30}
$$

Hence, by (28)–(30), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\qquad+2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=2A\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx-2\nu\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx+2A\nu\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad-2\alpha\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon dx+2A\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx-2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\qquad-A\kappa\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon dx+q\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon dx.
\end{aligned} \tag{31}$$

Due to (3) and the Young inequality,

$$\begin{aligned}
&2A\int_{\mathbb{R}}|f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|dx\le 2AC_0\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|dx+2AC_0\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|\,dx\\
&\qquad+2AC_0\int_{\mathbb{R}} u_\varepsilon^2|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|\,dx\\
&\quad=2\int_{\mathbb{R}}\left|\frac{AC_0\partial_x u_\varepsilon}{\beta\sqrt{D_1}}\right|\left|\sqrt{D_1}\beta\partial_x^2 u_\varepsilon\right|dx+2\int_{\mathbb{R}}\left|\frac{AC_0 u_\varepsilon\partial_x u_\varepsilon}{\beta\sqrt{D_1}}\right|\left|\sqrt{D_1}\beta\partial_x^2 u_\varepsilon\right|dx\\
&\qquad+2\int_{\mathbb{R}}\left|\frac{AC_0 u_\varepsilon^2\partial_x u_\varepsilon}{\beta\sqrt{D_1}}\right|dx\\
&\quad\le\frac{A^2C_0}{D_1\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A^2C_0}{D_1\beta^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\,dx\\
&\qquad+\frac{A^2C_0}{D_1\beta^2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx+3D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{A^2C_0}{D_1\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A^2C_0}{D_1\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad+\frac{A^2C_0}{D_1\beta^2}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+3D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2|\nu|\int_{\mathbb{R}}|u_\varepsilon||\partial_x^2 u_\varepsilon|dx=2\int_{\mathbb{R}}\left|\frac{\nu u_\varepsilon}{\beta\sqrt{D_1}}\right|\left|\sqrt{D_1}\beta\partial_x^2 u_\varepsilon\right|dx\\
&\quad\le\frac{\nu^2}{\beta^2 D_1}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2|\alpha|\int_{\mathbb{R}} u_\varepsilon^2|\partial_x^3 u_\varepsilon|dx=2\int_{\mathbb{R}}\left|\frac{\alpha u_\varepsilon^2}{\sqrt{AD_2}\beta}\right|\left|\sqrt{AD_2}\beta\partial_x^3 u_\varepsilon\right|dx\\
&\quad\le\frac{\alpha^2}{\beta^2 AD_2}\int_{\mathbb{R}} u_\varepsilon^4 dx+AD_2\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{\alpha^2}{\beta^2 AD_2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+AD_2\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2A|\alpha|\int_{\mathbb{R}}|u_\varepsilon||\partial_x^2 u_\varepsilon||\partial_x^3 u_\varepsilon|dx=2A\int_{\mathbb{R}}\left|\frac{\alpha u_\varepsilon\partial_x^2 u_\varepsilon}{\beta\sqrt{D_2}}\right|\left|\sqrt{D_2}\beta\partial_x^3 u_\varepsilon\right|dx\\
&\quad\le\frac{\alpha^2 A}{D_2\beta^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x^2 u_\varepsilon)^2 dx+AD_2\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{\alpha^2 A}{D_2\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+AD_2\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\kappa\int_{\mathbb{R}}|u_\varepsilon\partial_x u_\varepsilon|\partial_x^2 u_\varepsilon|dx=2\int_{\mathbb{R}}\left|\frac{\kappa u_\varepsilon\partial_x u_\varepsilon}{\beta\sqrt{D_1}}\right|\left|\sqrt{D_1}\beta\partial_x^2 u_\varepsilon\right|dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\kappa^2}{D_1\beta^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx + D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\kappa^2}{D_1\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= \frac{\kappa^2 A}{AD_1\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
A|\kappa|\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^3 u_\varepsilon| dx &= 2A\int_{\mathbb{R}}\left|\frac{\kappa(\partial_x u_\varepsilon)^2}{2\beta\sqrt{D_2}}\right|\left|\sqrt{D_2}\beta\partial_x^3 u_\varepsilon\right| dx\\
&\leq \frac{\kappa^2 A}{4\beta^2 D_2}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + AD_2\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
|q|\int_{\mathbb{R}} u_\varepsilon^2|\partial_x^2 u_\varepsilon| dx &= 2\int_{\mathbb{R}}\left|\frac{qu_\varepsilon^2}{2\beta\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_x^2 u_\varepsilon\right| dx\\
&\leq \frac{q^2}{4\beta^2 D_1}\int_{\mathbb{R}} u_\varepsilon^4 dx + D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{q^2}{4\beta^2 D_1}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + D_1\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

where D_1, D_2 are two positive constants, which will be specified later. It follows from (31) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\qquad + 2\beta^2(1-3D_1)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + A\beta^2(2-3D_2)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq 2A|\nu|\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{A^2C_0}{D_1\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\nu^2}{\beta^2 D_1}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \frac{A^2C_0}{D_1\beta^2}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \left(\frac{\alpha^2}{AD_2\beta^2} + \frac{q^2}{4D_1\beta^2}\right)\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \frac{\alpha^2 A}{D_1\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\kappa^2 A}{4\beta^2 D_2}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\qquad + \frac{A(AC_0+\kappa^2)}{A\beta^2 D_1}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Choosing $D_1 = \frac{1}{6}$ and $D_2 = \frac{1}{3}$, we have that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\qquad + \beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + A\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq 2A|\nu|\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{6A^2C_0}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{6\nu^2}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \frac{6A^2C_0}{\beta^2}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&+\frac{6\alpha^2+3Aq^2}{2A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&+\frac{6\alpha^2A}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{3\kappa^2A}{4\beta^2}\|\partial_xu_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&+\frac{6A(AC_0+\kappa^2)}{A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Following Lemma 2.3 in [108], we obtain

$$
\|\partial_xu_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\le 9\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \tag{33}
$$

Consequently, by (32),

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\beta^2\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+A\beta^2\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+2\varepsilon\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\varepsilon\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le 2A|\nu|\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{6A^2C_0}{\beta^2}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{6\nu^2}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{6A^2C_0}{\beta^2}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{6\alpha^2+3Aq^2}{2A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{A(24\alpha^2+27\kappa^2)}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{6A(AC_0+\kappa^2)}{A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{34}
$$

Observe that

$$
\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^2u_\varepsilon dx=-\int_{\mathbb{R}}\partial_xu_\varepsilon\partial_x^3u_\varepsilon dx.
$$

Therefore, by the Hölder inequality,

$$
\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\int_{\mathbb{R}}|\partial_xu_\varepsilon||\partial_x^3u_\varepsilon|dx\le\|\partial_xu_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.
$$

Hence, by (34), we obtain that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\beta^2\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+A\beta^2\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+2\varepsilon\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\varepsilon\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le 2A|\nu|\|\partial_xu_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}+\frac{6A^2C_0}{\beta^2}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{6\nu^2}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{6A^2C_0}{\beta^2}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&+\frac{6\alpha^2+3Aq^2}{2A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&+\frac{A(24\alpha^2+27\kappa^2)}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&+\frac{6A(AC_0+\kappa^2)}{A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Due to the Young inequality,

$$
\begin{aligned}
&2A|\nu|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}=A\frac{2|\nu|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}{|\beta|}|\beta|\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad\le\frac{2A\nu^2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})},\\
&\frac{A(24\alpha^2+27\kappa^2)}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad=2A\frac{\sqrt{3}(24\alpha^2+27\kappa^2)\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}{2|\beta|^3}\frac{|\beta|\|\partial_x^3 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}{\sqrt{3}}\\
&\quad\le\frac{3A(24\alpha^2+27\kappa^2)^2}{4\beta^6}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{A\beta^2}{3}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (35) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\qquad+\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{A\beta^2}{6}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2A\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{A(2AC_0+6\nu^2)}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{6\nu^2}{\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad+\frac{A(24A\beta^4C_0+3(24\alpha^2+27\kappa^2)^2)}{4\beta^6}\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad+\frac{6\alpha^2+3Aq^2}{2A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad+\frac{6A(AC_0+\kappa^2)}{A\beta^2}\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{36}
$$

We define

$$
\begin{aligned}
&X_\varepsilon(t):=\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\quad \ell_1:=\frac{2AC_0+12\nu^2+1}{\beta^2},\\
&\ell_2:=\frac{3A(C_0+q^2)+12\kappa^2+6\alpha^2}{A\beta^2},\quad \ell_3:=\frac{24A\beta^4C_0+3(24\alpha^2+27\kappa^2)^2+1}{4\beta^6}.
\end{aligned}
\tag{37}
$$

It follows from (36) that

$$
\frac{dX_\varepsilon(t)}{dt}\le\ell_1X_\varepsilon(t)+\ell_2\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}X_\varepsilon(t)+\ell_3\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})}X_\varepsilon(t).
\tag{38}
$$

Due to (37) and the Hölder inequality,

$$
u_\varepsilon^2(t,x)=2\int_{-\infty}^x u_\varepsilon\partial_x u_\varepsilon dy\le 2\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon|\,dx
$$

$$\begin{aligned}&\le 2\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\&=\frac{2}{\sqrt{A}}\sqrt{\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\sqrt{A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\\&\le\frac{2}{\sqrt{A}}X_\varepsilon(t).\end{aligned}$$

Hence,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\le\frac{2}{\sqrt{A}}X_\varepsilon(t),\quad \text{for every } 0\le t\le T. \tag{39}$$

It follows from (38) and (39) that

$$\frac{dX_\varepsilon(t)}{dt}\le \ell_1 X_\varepsilon(t)+\frac{2\ell_2}{\sqrt{A}}X_\varepsilon^2(t)+\frac{4\ell_3}{A}X_\varepsilon^3(t). \tag{40}$$

Thanks to the Young inequality,

$$\frac{2}{\sqrt{A}}X_\varepsilon^2(t)=\frac{2}{\sqrt{A}}X_\varepsilon^{\frac{1}{2}}(t)X_\varepsilon^{\frac{3}{2}}(t)\le X_\varepsilon+\frac{1}{A}X_\varepsilon^3(t).$$

Consequently, by (40),

$$\frac{dX_\varepsilon(t)}{dt}\le(\ell_1+\ell_2)X_\varepsilon(t)+\frac{4\ell_3+\ell_2}{A}X_\varepsilon^3(t). \tag{41}$$

Define

$$\ell_4:=\ell_1+\ell_2,\quad \ell_5:=4\ell_3+\ell_2. \tag{42}$$

It follows from (41) that

$$\frac{1}{X_\varepsilon^3(t)}\frac{dX_\varepsilon(t)}{dt}\le\frac{\ell_4}{X_\varepsilon^2(t)}+\frac{\ell_5}{A}. \tag{43}$$

Since

$$\frac{d}{dt}\left(\frac{1}{X_\varepsilon^2(t)}\right)=-\frac{2}{X_\varepsilon^3(t)}\frac{dX_\varepsilon(t)}{dt},$$

by (43), we have that

$$\frac{d}{dt}\left(\frac{1}{X_\varepsilon^2(t)}\right)\ge-\frac{2\ell_4}{X_\varepsilon^2(t)}-\frac{2\ell_5}{A},$$

which gives

$$\frac{d}{dt}\left(\frac{1}{X_\varepsilon^2(t)}\right)+\frac{2\ell_4}{X_\varepsilon^2(t)}\ge-\frac{2\ell_5}{A}. \tag{44}$$

Multiplying (44) by $e^{2\ell_4 t}$, we obtain

$$e^{2\ell_4 t}\frac{d}{dt}\left(\frac{1}{X_\varepsilon^2(t)}\right)+\frac{2\ell_4 e^{2\ell_4 t}}{X_\varepsilon^2(t)}\ge-\frac{2\ell_5 e^{2\ell_4 t}}{A}.$$

Therefore,

$$\frac{d}{dt}\left(\frac{e^{2\ell_4 t}}{X_\varepsilon^2(t)}\right)\ge-\frac{2\ell_5 e^{2\ell_4 t}}{A}.$$

Integrating on $(0,t)$, we have that

$$\frac{e^{2\ell_4 t}}{X_\varepsilon^2(t)}-\frac{1}{X_\varepsilon^2(0)}\ge-\frac{\ell_5}{A\ell_4}\left(e^{2\ell_4 t}-1\right),$$

that is,

$$\frac{e^{2\ell_4 t}}{X_\varepsilon^2(t)} + \frac{\ell_5}{A\ell_4}\left(e^{2\ell_4 t} - 1\right) \geq \frac{1}{X_\varepsilon^2(0)}. \tag{45}$$

Using (37) and (42) in (45), thanks to (27), we have (26).

We demonstrate (20). To this end, we begin by observing that, by (19),

$$\|u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} \leq Y_0 + AY_1, \tag{46}$$

where

$$Y_0 := \|u_0\|^2_{L^2(\mathbb{R})}, \quad Y_1 := \|\partial_x u_0\|^2_{L^2(\mathbb{R})}. \tag{47}$$

Consequently, we have that

$$\left(\|u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})}\right)^2 \leq (Y_0 + AY_1)^2,$$

which gives,

$$\frac{1}{\left(\|u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_{\varepsilon,0}\|^2_{L^2(\mathbb{R})}\right)^2} \geq \frac{1}{(Y_0 + AY_1)^2}. \tag{48}$$

Moreover,

$$e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}} \geq e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)t}{A\beta^2}}. \tag{49}$$

It follows from (26), (48), and (49) that

$$\begin{aligned}&\frac{e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}}}{\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)^2} \\ &\qquad + \frac{24C_0A^2\beta^4 + a_2^2A\beta^4 + a_3^2\beta^2 + a_4^2A}{A\beta^4(2C_0A^2 + a_5^2A + a_3^2)}\left(e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}} - 1\right) \\ &\quad \geq \frac{1}{(Y_0 + AY_1)^2},\end{aligned}$$

that is

$$\begin{aligned}&\frac{e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}}}{\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)^2} \\ &\quad \geq -\frac{24C_0A^2\beta^4 + a_2^2A\beta^4 + a_3^2\beta^2 + a_4^2A}{A\beta^4(2C_0A^2 + a_5^2A + a_3^2)}\left(e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}} - 1\right) \\ &\qquad + \frac{1}{(Y_0 + AY_1)^2}.\end{aligned} \tag{50}$$

We search A such that

$$\frac{1}{(Y_0 + AY_1)^2} - \frac{24C_0A^2\beta^4 + a_2^2A\beta^4 + a_3^2\beta^2 + a_4^2A}{A\beta^4(2C_0A^2 + a_5^2A + a_3^2)}\left(e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}} - 1\right) > 0, \tag{51}$$

that is,

$$e^{\frac{2(2A^2C_0 + a_5^2 A + a_3^2)T}{A\beta^2}} < 1 + \frac{A\beta^4(2C_0A^2 + a_5^2A + a_3^2)}{(Y_0 + AY_1)^2(24C_0A^2\beta^4 + a_2^2A\beta^4 + a_3^2\beta^2 + a_4^2A)}$$

which gives

$$e < \left(1 + \frac{A\beta^4(2C_0A^2 + a_5^2A + a_3^2)}{(Y_0 + AY_1)^2(24C_0A^2\beta^4 + a_2^2A\beta^4 + a_3^2\beta^2 + a_4^2A)}\right)^{\frac{A\beta^2}{2(2A^2C_0 + a_5^2A + a_3^2)T}}. \tag{52}$$

Taking β as in

$$|\beta| = A^n, \quad n > 2. \tag{53}$$

Equation (52) reads as follows

$$e < \left(1 + \frac{A^{1+4n}(2C_0A^2 + a_5^2A + a_3^2)}{(Y_0 + AY_1)^2(24C_0A^{2+4n} + a_2^2A^{1+4n} + a_3^2A^{2n} + a_4^2A)}\right)^{\frac{A^{1+2n}}{2(2A^2C_0 + a_5^2A + a_3^2)T}}.$$

Thanks to (53), we have that

$$\lim_{A\to\infty}\left(1 + \frac{A^{1+4n}(2C_0A^2 + a_5^2A + a_3^2)}{(Y_0 + AY_1)^2(24C_0A^{2+4n} + a_2^2A^{1+4n} + a_3^2A^{2n} + a_4^2A)}\right)^{\frac{A^{1+2n}}{2(2A^2C_0 + a_5^2A + a_3^2)T}} = \infty. \tag{54}$$

In fact,

$$\begin{aligned}
&\lim_{A\to\infty}\left(1 + \frac{A^{1+4n}(2C_0A^2 + a_5^2A + a_3^2)}{(Y_0 + AY_1)^2(24C_0A^{2+4n} + a_2^2A^{1+4n} + a_3^2A^{2n} + a_4^2A)}\right)^{\frac{A^{1+2n}}{2(2A^2C_0 + a_5^2A + a_3^2)T}} \\
&\quad = \lim_{A\to\infty}\left[\left(1 + \frac{1}{a_7^2(A)}\right)^{a_7^2(A)}\right]^{a_8^2(A)},
\end{aligned}$$

where

$$\begin{aligned}
a_7^2(A) &:= \frac{(Y_0 + AY_1)^2(24C_0A^{2+4n} + a_2^2A^{1+4n} + a_3^2A^{2n} + a_4^2A)}{A^{1+4n}(2C_0A^2 + a_5^2A + a_3^2)}, \\
a_8^2(A) &:= \frac{A^{6n+2}}{2(24C_0A^{2+4n} + a_2^2A^{1+4n} + a_3^2A^{2n} + a_4^2A)(Y_0 + AY_1)^2T}.
\end{aligned} \tag{55}$$

Observe, thanks to (55),

$$\lim_{A\to\infty}\left(1 + \frac{1}{a_7^2(A)}\right)^{a_7^2(A)} = e, \tag{56}$$

while, by (53) and (55),

$$\lim_{A\to\infty}\frac{A^{6n+2}}{2(24C_0A^{2+4n} + a_2^2A^{1+4n} + a_3^2A^{2n} + a_4^2A)(Y_0 + AY_1)^2T} = \infty. \tag{57}$$

Equations (55)–(57) give (54).

Therefore, thanks to (54), (51), which is equivalent to (52), holds; taking A very large and up to rescaling, we can have $|\beta| = A^n$, with n defined in (53).

Consequently, by (53), (50), (51) or (52), and (54), there exists a constant $C(T) > 0$, independent of ε, such that

$$\frac{e^{\frac{2(2A^2C_0 + a_5^2A + a_3^2)T}{A\beta^2}}}{\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)^2} \geq C(T).$$

Hence,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \sqrt{\frac{e^{\frac{2(2A^2C_0+a_5^2A+a_3^2)T}{A\beta^2}}}{C(T)}},$$

which gives (20).

We prove (21). Thanks to (20) and (37) with $A=1$, and (39) with $A=1$,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \le 2\Big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big) \le C(T),$$

which gives (21).

We prove (22)–(24). Thanks to (20), (21), and (36) with $A=1$, we have that

$$\begin{aligned}
&\frac{d}{dt}\Big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\Big)\\
&\qquad + \beta^2\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{6}\Big\|\partial_x^3 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})}\\
&\qquad + 2\varepsilon\Big\|\partial_x^3 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})} + 2\varepsilon\Big\|\partial_x^4 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})}\\
&\le C(T).
\end{aligned}$$

Integrating on $(0,t)$ by (19), we obtain

$$\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad + \beta^2\int_0^t\Big\|\partial_x^2 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds + \frac{\beta^2}{6}\int_0^t\Big\|\partial_x^3 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds\\
&\qquad + 2\varepsilon\int_0^t\Big\|\partial_x^3 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds + 2\varepsilon\int_0^t\Big\|\partial_x^4 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0 + C(T)t \le C(T),
\end{aligned}$$

which gives (22)–(24).

Finally, we prove (25). Observe that, by (21) and (33), we have that

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \le C(T)\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})}.$$

Consequently, by an integration on $\mathbb{R}$ and (22)–(24), we have (25). □

Lemma 2. *The following inequalities hold*

$$\|\partial_x u_\varepsilon\|_{L^\infty(\mathbb{R})} \le C(T), \tag{58}$$

$$\begin{aligned}
&\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t\Big\|\partial_x^4 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds\\
&\qquad + 2\varepsilon\int_0^t\Big\|\partial_x^5 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds \le C(T),
\end{aligned} \tag{59}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (18) by $2\partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}}\partial_x^4 u_\varepsilon \partial_t u_\varepsilon\, dx\\
=& -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\nu\int_{\mathbb{R}}\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\delta\int_{\mathbb{R}}\partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon\, dx
\end{aligned}$$

$$
\begin{aligned}
&-2\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx-2\kappa\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\, dx\\
&-2q\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx-2\gamma\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^5 u_\varepsilon\, dx\\
&\qquad-2\beta^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^6 u_\varepsilon\, dx\\
=&-2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx+2\nu\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&+(\alpha+2\kappa)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx-2q\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon\, dx\\
&-2\beta^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=-2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx+2\nu\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+(\alpha+2\kappa)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx-2q\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\quad\le 2\int_{\mathbb{R}}|f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|dx+2|q|\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx\\
&\quad+\left(2|\nu|+|\alpha+2\kappa|\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{60}
$$

Due to (20), (21), and the Young inequality,

$$
\begin{aligned}
2\int_{\mathbb{R}}|f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|dx&\le 2\|f'\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|\, dx\\
&\le C(T)\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon|dx=\int_{\mathbb{R}}\left|\frac{C(T)\partial_x u_\varepsilon}{\beta}\right|\left|\beta\partial_x^4 u_\varepsilon\right|\, dx\\
&\le C(T)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)+\frac{\beta^2}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|q|\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2|\partial_x^4 u_\varepsilon|dx&=\int_{\mathbb{R}}\left|\frac{2q(\partial_x u_\varepsilon)^2}{\beta}\right|\left|\beta\partial_x^4 u_\varepsilon\right|\, dx\\
&\le\frac{2q^2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (60) that

$$
\begin{aligned}
\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&+\beta^2\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)+\frac{2q^2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\quad+\left(2|\nu|+|\alpha+2\kappa|\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0,t)$, by (19), (22)–(25), we have that

$$
\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+2\varepsilon\int_0^t\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds
$$

$$\begin{aligned}&\leq C_0 + C(T)t + \frac{2q^2}{\beta^2}\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})}ds \\&\quad + \Big(2|\nu| + |\alpha+2\kappa|\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\Big)\int_0^t \Big\|\partial_x^3 u_\varepsilon(s,\cdot)\Big\|^2_{L^2(\mathbb{R})}ds \\&\leq C(T)\Big(1 + \|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\Big).\end{aligned} \tag{61}$$

We prove (58). Thanks to (20), (61), and the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,x))^2 =&2\int_{-\infty}^x \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \leq 2\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|\,dx \\ \leq&2\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|_{L^2(\mathbb{R})} \leq C(T)\sqrt{1+\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}}.\end{aligned}$$

Hence,

$$\|\partial_x u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} - C(T) \leq 0.$$

Arguing as in [103], Lemma 2.3 or [104], Lemma 2.4, we have (58).
Finally, (59) follows from (58) and (61). □

Lemma 3. *The following estimates hold*

$$\int_0^t \Big\|\partial_x^2 u_\varepsilon(s,\cdot)\Big\|^4_{L^4(\mathbb{R})} ds \leq C(T), \tag{62}$$

$$\int_0^t \Big\|\partial_x^2 u_\varepsilon(s,\cdot)\Big\|^6_{L^6(\mathbb{R})} ds \leq C(T), \tag{63}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by proving (62). We observe that

$$\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^4_{L^4(\mathbb{R})} = \int_{\mathbb{R}} \partial_x^2 u_\varepsilon(\partial_x^2 u_\varepsilon)^3 dx = -3\int_{\mathbb{R}} \partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 \partial_x^3 u_\varepsilon dx. \tag{64}$$

Due to (58) and the Young inequality,

$$\begin{aligned}3\int_{\mathbb{R}} |\partial_x u_\varepsilon|(\partial_x^2 u_\varepsilon)^2|\partial_x^3 u_\varepsilon| dx \leq&3\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2|\partial_x^3 u_\varepsilon|\,dx \\ \leq&C(T)\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2|\partial_x^3 u_\varepsilon| dx = \int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2|C(T)\partial_x^3 u_\varepsilon|\,dx \\ \leq&\frac{1}{2}\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^4_{L^4(\mathbb{R})} + C(T)\Big\|\partial_x^3 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (64) that

$$\frac{1}{2}\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^4_{L^4(\mathbb{R})} \leq C(T)\Big\|\partial_x^3 u_\varepsilon(t,\cdot)\Big\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, thanks to (22)–(24), we have (62).
Finally, we prove (63). We begin by observing that

$$\Big\|\partial_x^2 u_\varepsilon(t,\cdot)\Big\|^6_{L^6(\mathbb{R})} = \int_{\mathbb{R}} \partial_x^2 u_\varepsilon(\partial_x^2 u_\varepsilon)^5 dx = -5\int_{\mathbb{R}} \partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^4 \partial_x^3 u_\varepsilon dx. \tag{65}$$

Due to (58), (59), and the Young inequality,

$$\begin{aligned}&5\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|^4|\partial_x^3 u_\varepsilon|\,dx \\&\leq 5\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon|^4|\partial_x^3 u_\varepsilon|\,dx\end{aligned}$$

$$\begin{aligned}
&\leq C(T)\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon|^5|\partial_x^3 u_\varepsilon|\,dx\\
&=\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon|^3|C(T)\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon|\,dx\\
&\leq \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}+C(T)\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2(\partial_x^3 u_\varepsilon)^2\,dx\\
&\leq \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}+C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}+C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Therefore, by (65),

$$\frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\leq C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}. \tag{66}$$

Due to the Hölder inequality,

$$\begin{aligned}
(\partial_x^3 u_\varepsilon(t,x))^2 =&2\int_{-\infty}^{x}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dy\leq 2\int_{\mathbb{R}}|\partial_x^3 u_\varepsilon||\partial_x^4 u_\varepsilon|\,dx\\
\leq&2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Hence,

$$\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leq 2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Thanks to the Young inequality,

$$\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\leq \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (66) that

$$\frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\leq C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (22)–(24) and (59), we have (63). □

3. Theorem 1's Proof Based on the Aubin–Lions Lemma

This section is devoted to the proof of Theorem 1.

Our compactness argument is based on the Aubin–Lions Lemma (see [105,106]).

Lemma 4 (Aubin–Lions)**.** *Let X, B, Y be Banach spaces such that*

$$X\hookrightarrow\hookrightarrow B\hookrightarrow Y.$$

If $1\leq p\leq\infty$, F is bounded in $L^p(0,T;X)$, and

$$\|\tau_h f-f\|_{L^p(0,T-h;Y)}\to 0 \quad \text{as } h\to 0 \text{ uniformly for } f\in F,$$

where τ_h is the translation operator. Then, F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p=\infty$).

We begin by proving the following lemma.

Lemma 5. *Fix γ, α, κ, δ, T. Then,*

$$\text{the sequence } \{u_\varepsilon\}_{\varepsilon>0} \text{ is compact in } L^2_{loc}((0,\infty)\times\mathbb{R}). \tag{67}$$

Consequently, there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and* $u \in L^2_{loc}((0,\infty)\times\mathbb{R})$ *such that, for each compact subset* K *of* $(0,\infty)\times\mathbb{R})$,

$$u_{\varepsilon_k} \to u \text{ in } L^2(K) \text{ and a.e.} \tag{68}$$

Moreover, u is a solution of (1), *satisfying* (16).

Proof. We begin by proving (67). To prove (67), we rely on the Aubin–Lions Lemma (see Lemma 4). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second is continuous. Owing to the Aubin–Lions Lemma (see Lemma 4), to prove (67), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{loc}(\mathbb{R})), \tag{69}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{loc}(\mathbb{R})). \tag{70}$$

We prove (69). Thanks to Lemmas 1 and 2,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^2(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^1(\mathbb{R})),$$

which gives (69).

We prove (70). We begin by observing that

$$\partial_x\left(u_\varepsilon\partial_x^2 u_\varepsilon\right) = \frac{1}{2}\partial_x\left((\partial_x u_\varepsilon)^2\right) + u_\varepsilon\partial_x^3 u_\varepsilon. \tag{71}$$

Therefore, by (18) and (71), we have that

$$\partial_t u_\varepsilon = \partial_x(G(u_\varepsilon)) - f'(u_\varepsilon)\partial_x u_\varepsilon - q(\partial_x u_\varepsilon)^2,$$

where

$$G(u_\varepsilon) = \frac{\alpha-\kappa}{2}(\partial_x u_\varepsilon)^2 - \alpha u_\varepsilon\partial_x^2 u_\varepsilon - \nu\partial_x u_\varepsilon - \delta\partial_x^2 u_\varepsilon - \gamma\partial_x^4 u_\varepsilon - \beta^2\partial_x^3 u_\varepsilon + \varepsilon\partial_x^5 u_\varepsilon. \tag{72}$$

We claim that

$$\int_0^T\int_{\mathbb{R}} u_\varepsilon^2(\partial_x^2 u_\varepsilon)^2 dtdx \le C(T). \tag{73}$$

Thanks to (21)–(24)

$$\kappa^2\int_0^T\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dtdx \le \kappa^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}dt \le C(T).$$

Moreover, since $0<\varepsilon<1$, by (20), (22)–(25), and (59),

$$\begin{aligned}
&\frac{(\alpha-\kappa)^2}{4}\|\partial_x u_\varepsilon\|^4_{L^4((0,T)\times\mathbb{R})},\ \nu^2\|\partial_x u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})},\ \delta^2\left\|\partial_x^2 u_\varepsilon\right\|_{L^2((0,T)\times\mathbb{R})} \le C(T),\\
&\gamma^2\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})},\ \beta^4\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T),\ \varepsilon^2\left\|\partial_x^5 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T).
\end{aligned} \tag{74}$$

Therefore, by (72)–(74), we have that

$$\{\partial_x(G(u_\varepsilon))\}_{\varepsilon>0} \quad \text{is bounded in } H^1((0,T)\times\mathbb{R}). \tag{75}$$

We have that
$$\int_0^T\int_{\mathbb{R}}(f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 dtdx \le C(T). \tag{76}$$
Thanks to (20) and (21),
$$\int_0^T\int_{\mathbb{R}}(f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 dtdx \le \|f'\|^2_{L^\infty(-C(T),C(T))}\int_0^T \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}dt \le C(T).$$
Moreover, thanks to (25),
$$(\delta+\kappa)^2\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4 dtdx \le C(T). \tag{77}$$

Consequently, (70) follows from (75)–(77). Thanks to the Aubin–Lions Lemma, (67) and (68) hold. Therefore, u is solution of (1) and, thanks to Lemmas 1, 2, and 3, (16) holds. □

Now, we prove Theorem 1.

Proof of Theorem 1. We begin by observing that, by (71), (1) reads:
$$\begin{cases} \partial_t u+\partial_x f(u)+\nu\partial_x^2 u+\delta\partial_x^3 u+\alpha\partial_x\left(u\partial_x^2 u\right) & \\ \qquad +\frac{\kappa-\alpha}{2}\partial_x\left((\partial_x u)^2\right)+q(\partial_x u)^2 & \\ \qquad +\gamma\partial_x^5 u+\beta^2\partial_x^4 u=0, & 0<t<T, \quad x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}. \end{cases} \tag{78}$$

Lemma 5 gives the existence of a solution (78) satisfying (16). We prove (17). Let u_1 and u_2 be two solutions of (78) that verify (16), that is
$$\begin{cases} \partial_t u_i+\partial_x f(u_i)+\nu\partial_x^2 u_i+\delta\partial_x^3 u_i+\alpha\partial_x\left(u_i\partial_x^2 u_i\right) & \\ \qquad +\frac{\kappa-\alpha}{2}\partial_x\left((\partial_x u_i)^2\right)+q(\partial_x u_i)^2 & \\ \qquad +\gamma\partial_x^5 u_i+\beta^2\partial_x^4 u_i=0, & 0<t<T, \quad x\in\mathbb{R}, \\ u_i(0,x)=u_{i,0}(x), & x\in\mathbb{R}, \end{cases} \qquad i=1,2.$$

Then, the function
$$\omega(t,x)=u_1(t,x)-u_2(t,x), \tag{79}$$
is the solution of the following Cauchy problem:
$$\begin{cases} \partial_t\omega+\partial_x(f(u_1)-f(u_2))+\nu\partial_x^2\omega+\delta\partial_x^3\omega & \\ \qquad +\alpha\partial_x\left(u_1\partial_x^2 u_1-u_2\partial_x^2 u_2\right) & \\ \qquad +\frac{\kappa-\alpha}{2}\partial_x\left((\partial_x u_1)^2-(\partial_x u_2)^2\right) & \\ \qquad +q\left((\partial_x u_1)^2-(\partial_x u_2)^2\right) & \\ \qquad +\gamma\partial_x^5\omega+\beta^2\partial_x^4\omega=0, & 0<t<T, \quad x\in\mathbb{R}, \\ \omega(0,x)=u_{1,0}(x)-u_{2,0}(x), & x\in\mathbb{R}. \end{cases} \tag{80}$$

Fixe $T>0$. Since $u_1,\, u_2\in H^2(\mathbb{R})$, for every $0\le t\le T$, we have that
$$\begin{aligned} &\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|u_2\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T), \\ &\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T), \\ &\|\partial_x u_2(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C(T). \end{aligned} \tag{81}$$

Since $f \in C^1(\mathbb{R})$, thanks to (79), there exists ξ between u_1 and u_2, such that

$$f(u_1) - f(u_2) = f'(\xi)(u_1 - u_2) = f'(\xi)\omega, \quad u_1 < \xi < u_2, \text{ or, } u_2 < \xi < u_1. \tag{82}$$

Moreover, by (81), we have that

$$|f'(\xi)| \leq \|f'\|_{L^\infty(-C(T),C(T))} \leq C(T). \tag{83}$$

Observe that, thanks to (79),

$$\begin{aligned}
u_1\partial_x^2 u_1 - u_2\partial_x^2 u_2 =& u_1\partial_x^2 u_1 - u_1\partial_x^2 u_2 + u_1\partial_x^2 u_2 - u_2\partial_x^2 u_2 \\
=& u_1\partial_x^2\omega - \partial_x^2 u_2\omega, \\
(\partial_x u_1)^2 - (\partial_x u_2)^2 =& (\partial_x u_1 + \partial_x u_2)(\partial_x u_1 - \partial_x u_2) = \partial_x u_1\partial_x\omega + \partial_x u_2\partial_x\omega.
\end{aligned} \tag{84}$$

Thanks to (82) and (84), (80) reads

$$\begin{aligned}
\partial_t\omega =& -\partial_x(f'(\xi)\omega) - \nu\gamma\partial_x^2\omega - \delta\partial_x^3\omega - \alpha\partial_x\left(u_1\partial_x^2\omega\right) \\
& - \alpha\partial_x\left(\partial_x^2 u_2\omega\right) + \frac{\alpha-\kappa}{2}\partial_x(\partial_x u_1\partial_x\omega) + \frac{\alpha-\kappa}{2}\partial_x(\partial_x u_2\partial_x\omega) \\
& + q\partial_x u_1\partial_x\omega + q\partial_x u_2\partial_x\omega - \gamma\partial_x^5\omega - \beta^2\partial_x^4\omega.
\end{aligned} \tag{85}$$

Multiplying (85), an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}}\omega\partial_t\omega\, dx \\
=& -2\int_{\mathbb{R}}\omega\partial_x(f'(\xi)\omega) - 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx - 2\delta\int_{\mathbb{R}}\omega\partial_x^3\omega\, dx \\
& - 2\alpha\int_{\mathbb{R}}\omega\partial_x\left(u_1\partial_x^2\omega\right)dx - 2\alpha\int_{\mathbb{R}}\omega\partial_x\left(\partial_x^2 u_2\omega\right)dx \\
& + (\alpha-\kappa)\int_{\mathbb{R}}\omega\partial_x(\partial_x u_1\partial_x\omega)dx + (\alpha-\kappa)\int_{\mathbb{R}}\omega\partial_x(\partial_x u_2\partial_x\omega)dx \\
& + 2q\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx + 2q\int_{\mathbb{R}}\partial_x u_2\omega\partial_x\omega dx - 2\gamma\int_{\mathbb{R}}\omega\partial_x^5\omega\, dx \\
& - 2\beta^2\int_{\mathbb{R}}\omega\partial_x^4\omega\, dx \\
=& 2\int_{\mathbb{R}} f'(\xi)\omega\partial_x\omega dx - 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx + 2\delta\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega\, dx \\
& + 2\alpha\int_{\mathbb{R}} u_1\omega\partial_x^2\omega dx + 2\alpha\int_{\mathbb{R}}\omega\partial_x\omega\partial_x^2 u_2\, dx \\
& + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2 dx + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_2(\partial_x\omega)^2\, dx \\
& + 2q\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx + 2q\int_{\mathbb{R}}\partial_x u_2\omega\partial_x\omega dx + 2\gamma\int_{\mathbb{R}}\partial_x\omega\partial_x^4\omega\, dx \\
& + 2\beta^2\int_{\mathbb{R}}\partial_x\omega\partial_x^3\omega\, dx \\
=& 2\int_{\mathbb{R}} f'(\xi)\omega\partial_x\omega dx - 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx - 2\alpha\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega\, dx \\
& - 2\alpha\int_{\mathbb{R}} u_1(\partial_x\omega)^2 dx + 2\alpha\int_{\mathbb{R}}\omega\partial_x\omega\partial_x^2 u_2\, dx \\
& + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2 dx + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_2(\partial_x\omega)^2\, dx \\
& + 2q\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx + 2q\int_{\mathbb{R}}\partial_x u_2\omega\partial_x\omega dx - 2\gamma\int_{\mathbb{R}}\partial_x^2\omega\partial_x^3\omega\, dx \\
& - 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
\end{aligned}$$

$$
\begin{aligned}
=&2\int_{\mathbb{R}} f'(\xi)\omega\partial_x\omega dx - 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx + 2\delta\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega\, dx\\
&+2\alpha\int_{\mathbb{R}} u_1\omega\partial_x^2\omega dx + 2\alpha\int_{\mathbb{R}}\omega\partial_x\omega\partial_x^2 u_2\, dx\\
&+(\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2 dx + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_2(\partial_x\omega)^2\, dx\\
&+2q\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx + 2q\int_{\mathbb{R}}\partial_x u_2\omega\partial_x\omega dx + 2\gamma\int_{\mathbb{R}}\partial_x\omega\partial_x^4\omega\, dx\\
&+2\beta^2\int_{\mathbb{R}}\partial_x\omega\partial_x^3\omega\, dx\\
=&2\int_{\mathbb{R}} f'(\xi)\omega\partial_x\omega dx - 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx - 2\alpha\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega\, dx\\
&-2\alpha\int_{\mathbb{R}} u_1(\partial_x\omega)^2 dx + 2\alpha\int_{\mathbb{R}}\omega\partial_x\omega\partial_x^2 u_2\, dx\\
&+(\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2 dx + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_2(\partial_x\omega)^2\, dx\\
&+2q\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx + 2q\int_{\mathbb{R}}\partial_x u_2\omega\partial_x\omega dx - 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad= 2\int_{\mathbb{R}} f'(\xi)\omega\partial_x\omega dx - 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega dx - 2\alpha\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx\\
&\quad\quad - 2\alpha\int_{\mathbb{R}} u_1(\partial_x\omega)^2 dx + 2\alpha\int_{\mathbb{R}}\omega\partial_x\omega\partial_x^2 u_2 dx + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2 dx\\
&\quad\quad + (\kappa-\alpha)\int_{\mathbb{R}}\partial_x u_2(\partial_x\omega)^2 dx + 2q\int_{\mathbb{R}}\partial_x u_1\omega\partial_x\omega dx + 2q\int_{\mathbb{R}}\partial_x u_2\omega\partial_x\omega dx.
\end{aligned}
\tag{86}
$$

Due to (81), (83), and the Young inequality,

$$
\begin{aligned}
&2\int_{\mathbb{R}}|f'(\xi)||\omega||\partial_x\omega|dx \le 2\|f'\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\omega||\partial_x\omega|\, dx\\
&\quad\le C(T)\int_{\mathbb{R}}|\omega||\partial_x\omega|dx \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&2|\nu|\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx = 2\int_{\mathbb{R}}\left|\frac{\nu\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|\, dx\\
&\quad\le \frac{\nu^2}{\beta^2}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2|\alpha|\int_{\mathbb{R}}|\partial_x u_1|(\partial_x\omega)^2 dx \le 2|\alpha|\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&2|\alpha|\int_{\mathbb{R}}|u_1|(\partial_x\omega)^2 dx \le 2|\alpha|\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&2|\alpha|\int_{\mathbb{R}}|\omega||\partial_x\omega||\partial_x^2 u_2|dx \le \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \alpha^2\int_{\mathbb{R}}\omega^2(\partial_x^2 u_2)^2\, dx\\
&\quad\le \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \alpha^2\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_2(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})},\\
&|\kappa-\alpha|\int_{\mathbb{R}}|\partial_x u_1|(\partial_x\omega)^2 dx \le |\kappa-\alpha|\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&|\kappa-\alpha|\int_{\mathbb{R}}|\partial_x u_2|(\partial_x\omega)^2 dx \le |\kappa-\alpha|\|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|q| \int_{\mathbb{R}} |\partial_x u_1||\omega||\partial_x \omega| dx \leq & 2|q| \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\omega||\partial_x \omega| dx \leq C(T) \int_{\mathbb{R}} |\omega||\partial_x \omega| \, dx \\
\leq & C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
2|q| \int_{\mathbb{R}} |\partial_x u_2||\omega||\partial_x \omega| dx \leq & 2|q| \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\omega||\partial_x \omega| dx \leq C(T) \int_{\mathbb{R}} |\omega||\partial_x \omega| \, dx \\
\leq & C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (86) that

$$
\begin{aligned}
& \frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \qquad \leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{87}
$$

Thanks to the Hölder inequality,

$$
\omega^2(t,x) = 2\int_{-\infty}^{x} \omega \partial_x \omega dy \leq \int_{\mathbb{R}} |\omega||\partial_x \omega| dx \leq 2\|\omega(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x \omega(t,\cdot)\|_{L^2(\mathbb{R})}.
$$

Hence,

$$
\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq 2\|\omega(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x \omega(t,\cdot)\|_{L^2(\mathbb{R})}.
$$

Due to the Young inequality,

$$
\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \leq \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
$$

Therefore, by (87), we have that

$$
\begin{aligned}
& \frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
& \qquad \leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{88}
$$

Observe that

$$
C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} = C(T)\int_{\mathbb{R}} \partial_x \omega \partial_x \omega dx = -C(T)\int_{\mathbb{R}} \omega \partial_x^2 \omega dx.
$$

Hence, by the Young inequality,

$$
\begin{aligned}
C(T)\|\partial_x \omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq & C(T)\int_{\mathbb{R}} |\omega||\partial_x^2 \omega| dx = \int_{\mathbb{R}} \left|\frac{C(T)\omega}{\beta}\right| \left|\beta \partial_x^2 \omega\right| \, dx \\
\leq & C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently, by (88),

$$
\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2 \omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
$$

The Gronwall Lemma and (80) give

$$
\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2 e^{C(T)t}}{2}\int_0^t e^{-C(T)s}\left\|\partial_x^2 \omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq e^{C(T)t}\|\omega_0\|^2_{L^2(\mathbb{R})}. \tag{89}
$$

Equation (17) follows from (79) and (89).
□

4. Discussion

In this paper we studied the Cauchy problem of the fifth order Kudryashov–Sinelshchikov equation. Our argument is based on several a priori estimates on a sixth order approximation of the equation. The compactness of such approximations is obtained using the Aubin–Lions Lemma. Finally, we obtained the uniqueness of solutions proving a stability estimate with respect to the initial conditions.

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