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Bi-Univalent Problems Involving Certain New Subclasses of Generalized Multiplier Transform on Analytic Functions Associated with Modified Sigmoid Function

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Abstract: The object of the present work is to investigate certain new classes of bi-univalent functions introduced in this paper using the concept of subordination. The research involves a generalized multiplier transform defined in this paper which is a generalization of known operators and the modified sigmoid function. The results contained in the proved theorems refer to coefficient estimates for the functions in the newly introduced classes.

Keywords: analytic; univalent; bi-univalent; multiplier transform; modified sigmoid function

MSC: 30C45

check for updates

Citation: Hamzat, J.O.; Oladipo, A.T.; Oros, G.I. Bi-Univalent Problems Involving Certain New Subclasses of Generalized Multiplier Transform on Analytic Functions Associated with Modified Sigmoid Function. *Symmetry* **2022**, *14*, 1479. https://doi.org/10.3390/sym14071479

Academic Editor: Mihai Postolache

Received: 24 June 2022
Accepted: 18 July 2022
Published: 19 July 2022

1. Introduction

In the recent years, the study of bi-univalent functions has developed nicely with many researchers obtaining new results which proved to have applications in different areas of science. A line of research regarding bi-univalent functions deals with introducing and studying new subclasses containing bi-univalent functions with remarkable properties. Newly defined operators are used for introducing such new subclasses as it is the case for the results presented in this paper.

The present investigation becomes more unique with the introduction of a special function, namely the sigmoid function which is known to be an activation function. Sigmoid function, as an activation function, is inspired by the way the biological nervous system, such as the brain, processes information. The brain is composed of a large number of highly interconnected processing elements, the neurons, working as a unit to solve or process a specific task. The sigmoid function also has a gradient descendant learning algorithm; hence, its evaluation can be done in several ways, including truncated series expansion. The results obtained in the present work have applications in computer science and also in software development for the purpose of information processing and information retrieval due to the property of the sigmoid function of being differentiable, which is important for learning algorithms.

The sigmoid function is referred to as special logistic function and is defined by:

$$g(z) = \frac{1}{1 + e^{-z}}. \tag{1}$$

A sigmoid function is a bounded differentiable real function that is defined for all real input values and has a positive derivative at each point. It is useful in geometric function theory because of the following properties:

It outputs real numbers between 0 and 1.
It maps a large domain to a small range.
It is a one-to-one function; hence, the information is well-preserved.
It increases monotonically.

Just recently, precisely in 2013, Fadipe-Joseph et al. [1] defined the modified Sigmoid function $\phi(z)$ as

$$\phi(z) = 2g(z). \tag{2}$$

They showed among other properties that $\phi(z)$ is a function with positive real part and that $\phi(z) \in P$ (class of Caratheodory functions).

Interestingly, $\phi(z)$ has the following series expansion

$$\phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \dots, \tag{3}$$

(see Hamzat and Makinde [2], Murugusundaramoorthy and Janani [3], Oladipo and Gbolagade [4]).

Let A denote the class of all analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{4}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. In addition, let S denote the subclass of A, consisting of functions which are univalent in U.

It is well known that every function $f \in S$ has an inverse $f^{-1}(z)$ defined as

$$f^{-1}(f(z)) = z, \quad z \in U \tag{5}$$

and

$$f(f^{-1}(\omega)) = \omega, [|\omega| < r_0(f) : r_0(f) \geq \frac{1}{4}]. \tag{6}$$

In addition, one can say that:

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \dots = \omega + \sum_{k=2}^{\infty} b_k\omega^k, \tag{7}$$

where

$$b_2 = -a_2, \;\; b_3 = 2a_2^2 - a_3, \;\; \dots \;. \tag{8}$$

A function $f \in A$ is said to be bi-univalent in U if both f and its inverse, f^{-1}, are univalent.

Recently, the pioneering work of Srivastava et al. [5] has truly resuscitated the study of analytic bi-univalent functions and a rather vast flood of follow-up to this work have resulted in the literature on the study of various subclasses of analytic univalent functions.

Let Σ denote the class of all analytic bi-univalent functions in U. The class Σ of bi-univalent functions has been studied intensely in the recent period. In particular, new classes of bi-univalent functions were introduced and investigated regarding coefficient properties as it can be seen, for example, in [6–9]. Coefficient estimates for general subclasses of m-fold symmetric analytic bi-univalent functions were also obtained recently using different types of operators [10], applying subordination properties [11], involving generalization techniques linking the results to previously obtained ones [12], or using the inverse of the square-root transform of the Koebe function [13]. Subclasses of m-fold symmetric bi-bazilevic functions associated with modified Sigmoid functions were considered in [14] and associated with conic domains in [15]. Quantum calculus aspects were also considered in the investigation of subclasses of m-fold symmetric analytic bi-univalent functions in [16]. Extensions, generalizations, and improvements of starlikeness criteria for certain subclasses of analytic and bi-univalent functions were obtained considering coefficient estimates [17] and certain coefficient estimates were also given for particular families of bi-Bazilevic functions of the Ma–Minda type involving the Hohlov operator [18]. Quasi-Subordination was used for investigating new subclasses for bi-univalent functions in [19] and an integral operator based upon Lucas polynomial was also involved in the study of subclasses of bi-univalent functions [20]. New subclasses were introduced and studied considering aspects regarding argument and real part of certain linear combinations involving m-fold symmetric bi-univalent functions in [21].

However, their results seem to lack full flavor addressing the coefficient problems for functions in Σ associated with the Sigmoid function. Consequently, the present work aims at investigating the bi-univalent problems concerning certain classes of analytic function $f(z) \in \Sigma$ as related to the modified Sigmoid function in the open unit disk.

To achieve this, we consider a linear combination

$$\lambda p(z) \;+\; (1-\lambda)\phi(z), \quad (0 \le \lambda \le 1),$$

where $p(z) = 1 \;+\; p_1 z \;+\; p_2 z^2 \;+\; \ldots$ and $\phi(z)$ is as defined in (3).

Let h be univalent in U and f analytic in U, then f is said to be subordinate to h, written as $f \prec h$, if there exists a Schwartz function u which is analytic in U, with $u(0) = 0$ and $|u(z)| < 1$ for all $z \in U$ such that $f(z) = h(u(z))$. In addition, let h be univalent in U, then the following equivalent holds true

$$f \prec h \Leftrightarrow f(0) = h(0) \text{ and } f(U) \subset h(U)$$

(see Miller and Mocanu [22–24]).

2. Preliminary Definitions and Lemmas

For a function $f(z)$ of the form (4), Swamy [25] introduced and studied a multiplier differential operator $I^n_{\alpha,\beta} f(z)$ given by

$$I^n_{\beta,\gamma} f(z) = z \;+\; \sum_{k=2}^{\infty} \left(\frac{\beta \;+\; k\gamma}{\beta \;+\; \gamma}\right)^n a_k z^k. \tag{9}$$

In the next definition, the general multiplier transform used for defining the new classes of bi-univalent functions is introduced.

Definition 1. *Let the function $f(z)$ be of the form (4), for $\beta, \gamma, \sigma \ge 0$, $\beta \;+\; \gamma \;+\; \sigma > 0$ and real β, the following multiplier linear differential operator $L^n_{\beta,\gamma,\sigma} f(z)$ is defined as:*

$$L_{\beta,\gamma,\sigma} f(z) = \frac{\beta f(z) \;+\; \gamma z f'(z) \;+\; \sigma z \big(z f'(z)\big)'}{\beta \;+\; \gamma \;+\; \sigma}$$

$$L^2_{\beta,\gamma,\sigma} f(z) = L_{\beta,\gamma,\sigma} f(z)\Big(L_{\beta,\gamma,\sigma} f(z)\Big)$$

$$L^3_{\beta,\gamma,\sigma} f(z) = L_{\beta,\gamma,\sigma} f(z)\Big(L^2_{\beta,\gamma,\sigma} f(z)\Big)$$

$$\vdots$$

$$L^n_{\beta,\gamma,\sigma} f(z) = L_{\beta,\gamma,\sigma} f(z)\Big(L^{n-1}_{\beta,\gamma,\sigma} f(z)\Big) = z \;+\; \sum_{k=2}^{\infty} \left(\frac{\beta \;+\; \gamma k \;+\; \sigma k^2}{\beta \;+\; \gamma \;+\; \sigma}\right)^n a_k z^k, \tag{10}$$

Remark 1. *Suppose that the function $f(z)$ has the form (4), it is easily verified from (10) that*

$$L^0_{\beta,0,0} f(z) = f(z) \in A,$$

see also [26,27], among others. It is obvious that the operator $L^n_{\beta,\gamma,\sigma} f(z)$ generalizes many existing operators of this kind which were introduced and studied by different authors. For instance:

(i) $L^n_{\beta,\gamma,0} f(z) = I^n_{\beta,\gamma} f(z)$ *studied by Swamy [25].*
(ii) $L^n_{\beta,1,0} f(z) = I^n_{\beta} f(z), \quad \beta > -1$ *studied by Cho and Srivastava [28] and Cho and Kim [29].*
(iii) $L^n_{1,\gamma,0} f(z) = N^n_{\gamma} f(z)$ *studied by Swamy [25].*

Definition 2. *Let $\Omega : U \to \mathbb{C}$ be a convex univalent function in U satisfying the following conditions:*

$$\Omega(0) = 1 \quad \text{and} \quad \Re\{\Omega(z)\} > 0 \quad (z \in U).$$

Further, let $r(s)$ be defined such that

$$r(z) = 1 + \sum_{\kappa=1}^{\infty} B_k z^k. \tag{11}$$

The next definitions give the classes of bi-univalent functions which are further investigated in the section "Main Results".

Definition 3. *Let $\mu(z)$ be an analytic function with positive real part on U which satisfies the conditions $\mu(0) = 1$ and $\mu'(0) > 0$. The function $f(z)$ of the form (4) is said to belong to the class of multiplier starlike function, $\sum_b^*(\beta, \gamma, \sigma; \mu)$, of complex order, with the provision that*

$$1 + \frac{1}{b}\left\{\frac{z\Big[L^n(\beta,\gamma,\sigma)f(z)\Big]'}{\Big[L^n(\beta,\gamma,\sigma)f(z)\Big]} - 1\right\} \prec \mu(z) \tag{12}$$

and

$$1 + \frac{1}{b}\left\{\frac{\omega\Big[L^n(\beta,\gamma,\sigma)g(\omega)\Big]'}{\Big[L^n(\beta,\gamma,\sigma)g(\omega)\Big]} - 1\right\} \prec \mu(\omega), \tag{13}$$

where b is any non-zero complex number, $\prec$ denotes the subordination sign, $n \in \mathbb{N}_0$, $\beta, \gamma, \sigma \geq 0$, $\beta + \gamma + \sigma > 0$ and $z, \omega \in U$. From the above definition, it follows that there exists a unit bound function $u(z)$ satisfying the conditions $u(0) = 0$ and $|u(z)| < 1$. Then, we can say that

$$1 + \frac{1}{b}\left\{\frac{z\Big[L^n(\beta,\gamma,\sigma)f(z)\Big]'}{\Big[L^n(\beta,\gamma,\sigma)f(z)\Big]} - 1\right\} = \mu(u(z)) = \lambda p(z) + (1-\lambda)\phi(z), \quad (0 \leq \lambda \leq 1)$$

and

$$1 + \frac{1}{b}\left\{\frac{\omega\Big[L^n(\beta,\gamma,\sigma)g(\omega)\Big]'}{\Big[L^n(\beta,\gamma,\sigma)g(\omega)\Big]} - 1\right\} = \mu(u(\omega)) = \lambda q(\omega) + (1-\lambda)\phi(\omega), \quad (0 \leq \lambda \leq 1).$$

Definition 4. *Let $\mu(z)$ be an analytic function with positive real part on U which satisfies the conditions $\mu(0) = 1$ and $\mu'(0) > 0$. The function $f(z)$ of the form (4) is said to belong to the class of multiplier convex functions, $\sum_b(\beta, \gamma, \sigma; \mu)$, of complex order, provided*

$$1 + \frac{1}{b}\left\{\frac{z\Big[L^n(\beta,\gamma,\sigma)f(z)\Big]''}{\Big[L^n(\beta,\gamma,\sigma)f(z)\Big]'}\right\} \prec \mu(z) \tag{14}$$

and

$$1 + \frac{1}{b}\left\{\frac{\omega\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]''}{\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]'}\right\} \prec \mu(\omega), \tag{15}$$

where b is any non-zero complex number, $\prec$ denotes the subordination sign, $n \in N_0$, $\beta,\gamma,\sigma \geq 0$, $\beta + \gamma + \sigma > 0$ and $z,\omega \in U$. In addition, from definition (4), it follows that there exists a unit bound function $u(z)$ satisfying the conditions $u(0) = 0$ and $|u(z)| < 1$. Then, we can say that

$$1 + \frac{1}{b}\left\{\frac{z\left[L^n(\beta,\gamma,\sigma)f(z)\right]''}{\left[L^n(\beta,\gamma,\sigma)f(z)\right]'}\right\} = \mu(u(z)) = \lambda p(z) + (1-\lambda)\phi(z), \quad (0 \leq \lambda \leq 1)$$

and

$$1 + \frac{1}{b}\left\{\frac{\omega\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]''}{\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]'}\right\} = \mu(u(\omega)) = \lambda q(\omega) + (1-\lambda)\phi(\omega), \quad (0 \leq \lambda \leq 1).$$

Definition 5. *Let $\mu(z)$ be an analytic function with positive real part on U which satisfies the conditions $\mu(0) = 1$ and $\mu'(0) > 0$. The function $f(z)$ of the form (4) is said to belong to the class of multiplier-bounded turning functions, $\sum_b^R(\beta,\gamma,\sigma;\mu)$, of complex order, provided that*

$$1 + \frac{1}{b}\left\{\left[L^n(\beta,\gamma,\sigma)f(z)\right]' - 1\right\} \prec \mu(z) \tag{16}$$

and

$$1 + \frac{1}{b}\left\{\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]' - 1\right\} \prec \mu(\omega), \tag{17}$$

where all parameters involved are as earlier defined. Similarly, from definition (5), there exists a unit bound function $u(z)$ satisfying the conditions $u(0) = 0$ and $|u(z)| < 1$ such that

$$1 + \frac{1}{b}\left\{\left[L^n(\beta,\gamma,\sigma)f(z)\right]' - 1\right\} = \mu(u(z)) = \lambda p(z) + (1-\lambda)\phi(z), \quad (0 \leq \lambda \leq 1)$$

and

$$1 + \frac{1}{b}\left\{\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]' - 1\right\} = \mu(u(\omega)) = \lambda q(\omega) + (1-\lambda)\phi(\omega), \quad (0 \leq \lambda \leq 1).$$

Example 1. *Let*

$$\mu(z) = \frac{1 + Az}{1 + Bz} = \mu_{A,B}, \quad (-1 \leq B < A \leq 1),$$

then

(a)

$$\sum\nolimits_b^*(\beta,\gamma,\sigma;\mu) = \sum\nolimits_b^*\left(\beta,\gamma,\sigma;\frac{1 + Az}{1 + Bz}\right) = \sum\nolimits_b^*(\beta,\gamma,\sigma;\mu_{A,B})$$

(b)

$$\sum\nolimits_b(\beta,\gamma,\sigma;\mu) = \sum\nolimits_b\left(\beta,\gamma,\sigma;\frac{1 + Az}{1 + Bz}\right) = \sum\nolimits_b(\beta,\gamma,\sigma;\mu_{A,B})$$

(c)

$$\sum\nolimits_b^R(\beta,\gamma,\sigma;\mu) = \sum\nolimits_b^R\left(\beta,\gamma,\sigma;\frac{1 + Az}{1 + Bz}\right) = \sum\nolimits_b^R(\beta,\gamma,\sigma;\mu_{A,B})$$

Example 2. *Let*

$$\mu(z) = \frac{1 + Az}{1 + Bz},$$

such that $A = \left(\frac{1-\alpha}{\alpha-1}\right)^2$, $\alpha \leq 0$ *and* $B = -1$. *That is, if*

$$\mu(z) = \frac{1 + \left(\frac{1-\alpha}{\alpha-1}\right)^2 z}{1-z} = \mu_\alpha \quad (-1 \leq B < A \leq 1),$$

then

(a)

$$\sum\nolimits_b^*(\beta,\gamma,\sigma;\mu) = \sum\nolimits_b^*\left(\beta,\gamma,\sigma;\frac{1 + \left(\frac{1-\alpha}{\alpha-1}\right)^2 z}{1-z}\right) = \sum\nolimits_b^*(\beta,\gamma,\sigma;\mu_\alpha)$$

(b)

$$\sum\nolimits_b(\beta,\gamma,\sigma;\mu) = \sum\nolimits_b\left(\beta,\gamma,\sigma;\frac{1 + \left(\frac{1-\alpha}{\alpha-1}\right)^2 z}{1-z}\right) = \sum\nolimits_b(\beta,\gamma,\sigma;\mu_\alpha)$$

(c)

$$\sum\nolimits_b^R(\beta,\gamma,\sigma;\mu) = \sum\nolimits_b^R\left(\beta,\gamma,\sigma;\frac{1 + \left(\frac{1-\alpha}{\alpha-1}\right)^2 z}{1-z}\right) = \sum\nolimits_b^R(\beta,\gamma,\sigma;\mu_\alpha)$$

Before proceeding to the main results, the following Lemmas shall be necessary.

Lemma 1 ([30]). *Let a function* $p \in P$ *be given by*

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \quad z \in U.$$

Then,

$$|p_k| \leq 2, \quad k \in \mathbb{N},$$

where p *is the family of functions analytic in* U *for which*

$$p(0) = 1, Re\{p(z)\} > 0, \quad z \in U.$$

Lemma 2 ([31]). *Let the function* $r(z)$ *given by*

$$r(z) = 1 + \sum_{k=1}^{\infty} C_k z^k, \quad z \in U$$

be convex in U. *In addition, let the function* $h(z)$ *given by*

$$l(z) = 1 + \sum_{k=1}^{\infty} L_k z^k,$$

be holomorphic in U. *If*

$$l(z) \prec r(z), \quad z \in U,$$

then

$$|L_k| \leq |C_1|, \quad k \in \mathbb{N},$$

(see also [27,32], among others).

3. Main Results

In the theorem that follows, we give the coefficient estimates for the functions in the class $\sum_b^* (\beta, \gamma, \sigma; \mu)$.

Theorem 1. *Let $f(z) \in \sum_b^* (\beta, \gamma, \sigma; \mu)$. Then for $\beta, \gamma, \sigma \geq 0$, $\beta + \gamma + \sigma > 0$ and $0 \leq \lambda \leq 1$*

$$|a_2| \leq \sqrt{\frac{\lambda |b| |B_1|}{2\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n - \left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}}} \tag{18}$$

and

$$|a_3| \leq \frac{\lambda |b| |B_1|}{2\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n} + \frac{|b|^2 \Big(1 + \lambda\Big(2|B_1| - 1\Big)\Big)^2}{4\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}}. \tag{19}$$

Proof. Suppose that $f(z) \in \sum_b^* (\beta, \gamma, \sigma; \mu)$, then from the Definition 3, it follows that

$$1 + \frac{1}{b}\left\{\frac{z\Big[L^n(\beta, \gamma, \sigma) f(z)\Big]'}{\Big[L^n(\beta, \gamma, \sigma) f(z)\Big]} - 1\right\} = \mu(u(z)) = \lambda p(z) + (1 - \lambda)\phi(z)$$

and

$$1 + \frac{1}{b}\left\{\frac{\omega\Big[L^n(\beta, \gamma, \sigma) g(\omega)\Big]'}{\Big[L^n(\beta, \gamma, \sigma) g(\omega)\Big]} - 1\right\} = \mu(u(\omega)) = \lambda q(\omega) + (1 - \lambda)\phi(\omega).$$

It implies that

$$\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{\beta + \gamma k + \sigma k^2}{\beta + \gamma + \sigma}\right)^n a_k z^{k-1}}{1 + \sum_{k=2}^{\infty}\left(\frac{\beta + \gamma k + \sigma k^2}{\beta + \gamma + \sigma}\right)^n a_k z^{k-1}} = b\Big[\lambda p_1 + \frac{1-\lambda}{2}\Big] z + b\lambda p_2 z^2 + b\Big[\lambda p_3 - \frac{1-\lambda}{2}\Big] z^3 + \ldots \tag{20}$$

and

$$\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{\beta + \gamma k + \sigma k^2}{\beta + \gamma + \sigma}\right)^n b_k \omega^{k-1}}{1 + \sum_{k=2}^{\infty}\left(\frac{\beta + \gamma k + \sigma k^2}{\beta + \gamma + \sigma}\right)^n b_k \omega^{k-1}} = b\Big[\lambda q_1 + \frac{1-\lambda}{2}\Big]\omega + b\lambda q_2 \omega^2 + b\Big[\lambda q_3 - \frac{1-\lambda}{2}\Big]\omega^3 + \ldots, \tag{21}$$

where $p, q, \phi \in P$ (class of Caratheodory functions),

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \quad (z \in U), \tag{22}$$

$$q(\omega) = 1 + \sum_{k=1}^{\infty} q_k \omega^k, \quad (\omega \in U) \tag{23}$$

and ϕ is as earlier defined in (3).

Equating the coefficients of the same powers of z and ω in (20) and (21), respectively, then we obtain

$$\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^n a_2 = \frac{1}{2} b\Big(1 + \lambda(2p_1 - 1)\Big), \tag{24}$$

$$2\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n a_3 - \left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n} a_2^2 = \lambda b p_2, \tag{25}$$

$$\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^n a_2 = -\frac{1}{2} b\Big(1 + \lambda(2q_1 - 1)\Big) \tag{26}$$

and

$$\left[4\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n - \left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}\right] a_2^2 - 2\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n a_3 = \lambda b q_2, \tag{27}$$

From (24) and (26), it is observed that

$$a_2 = \frac{b\Big(1 + \lambda(2p_1 - 1)\Big)}{\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^n} = -\frac{b\Big(1 + \lambda(2q_1 - 1)\Big)}{\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^n}$$

which can easily be verified (with $\lambda = 1$) that

$$p_1 = -q_1.$$

If we square both sides of (24) and (26) and add together, then we can easily obtain

$$a_2^2 = \frac{b^2\left[\Big(1 + \lambda(2p_1 - 1)\Big)^2 + \Big(1 + \lambda(2q_1 - 1)\Big)^2\right]}{8\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}}. \tag{28}$$

Now, the sum of (25) and (27) yields

$$a_2^2 = \frac{b\lambda(p_2 + q_2)}{2\left[2\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n - \left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}\right]}. \tag{29}$$

Using Lemma 2 in (29), having considered (11), (22), and (23), we have

$$|p_k| = \left|\frac{p^k(0)}{k!}\right| \leq |B_1| \quad and \quad |q_k| = \left|\frac{q^k(0)}{k!}\right| \leq |B_1|. \tag{30}$$

Now, the application of (30) in (29) yields

$$|a_2|^2 \leq \frac{|b|\lambda|B_1|}{2\left[2\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n - \left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}\right]}$$

from where the inequality (18) is obtained.

To obtain the bound on $|a_3|$ as contained in (19), subtract (24) from (22) and then use (29) in the result of the difference, then

$$a_3 = \frac{b\lambda(p_2 - q_2)}{4\left(\frac{\beta + 3\gamma + 9\sigma}{\beta + \gamma + \sigma}\right)^n} + \frac{b^2\left[\Big(1 + \lambda(2p_1 - 1)\Big)^2 + \Big(1 + \lambda(2q_1 - 1)\Big)^2\right]}{8\left(\frac{\beta + 2\gamma + 4\sigma}{\beta + \gamma + \sigma}\right)^{2n}}. \tag{31}$$

Using Lemma 1 and (30) in (31), the inequality (19) is obtained and this completes the proof of Theorem 1. □

The next result is the consequence of Theorem 1 corresponding to Example 1a.

Consequence 1. *Let* $f(z) \in \sum_b^*(\beta,\gamma,\sigma;\mu_{A,B})$*. Then, for* $\beta,\gamma,\sigma \geq 0$*,* $\beta + \gamma + \sigma > 0$*,* $-1 \leq B < A \leq 1$ *and* $0 \leq \lambda \leq 1$

$$|a_2| \leq \sqrt{\frac{\lambda|b|(A-B)}{2\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n - \left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}} \tag{32}$$

and

$$|a_3| \leq \frac{\lambda|b|(A-B)}{2\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\Big(1 + \lambda\big(2(A-B)-1\big)\Big)^2}{4\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{33}$$

The consequence of Theorem 1 corresponding to Example 2a.

Consequence 2. *Let* $f(z) \in \sum_b^*(\beta,\gamma,\sigma;\mu_\alpha)$*. Then, for* $\beta,\gamma,\sigma \geq 0$*,* $\beta + \gamma + \sigma > 0$ *and* $0 \leq \lambda \leq 1$

$$|a_2| \leq \sqrt{\frac{2\lambda|b|}{2\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n - \left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}} \tag{34}$$

and

$$|a_3| \leq \frac{\lambda|b|}{\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\big(1 + 3\lambda\big)^2}{4\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{35}$$

Theorem 2. *Let* $f(z) \in \sum_b(\beta,\gamma,\sigma;\mu)$*. Then, for* $\beta,\gamma,\sigma \geq 0$*,* $\beta + \gamma + \sigma > 0$ *and* $0 \leq \lambda \leq 1$*,*

$$|a_2| \leq \sqrt{\frac{\lambda|b||B_1|}{2\left[3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n - 2\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}\right]}} \tag{36}$$

and

$$|a_3| \leq \frac{\lambda|b||B_1|}{6\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\Big(1 + \lambda\big(2|B_1|-1\big)\Big)^2}{16\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{37}$$

Proof. Suppose that $f(z) \in \sum_b(\beta,\gamma,\sigma;\mu)$, then from the Definition 4, it follows that

$$1 + \frac{1}{b}\left\{\frac{z\big[L^n(\beta,\gamma,\sigma)f(z)\big]''}{\big[L^n(\beta,\gamma,\sigma)f(z)\big]'}\right\} = \mu(u(z)) = \lambda p(z) + (1-\lambda)\phi(z), \quad (0 \leq \lambda \leq 1)$$

and

$$1 + \frac{1}{b}\left\{\frac{\omega\big[L^n(\beta,\gamma,\sigma)g(\omega)\big]''}{\big[L^n(\beta,\gamma,\sigma)g(\omega)\big]'}\right\} = \mu(u(\omega)) = \lambda q(\omega) + (1-\lambda)\phi(\omega), \quad (0 \leq \lambda \leq 1).$$

It implies that

$$\frac{\sum_{k=2}^{\infty} k(k-1)\left(\frac{\beta+\gamma k+\sigma k^2}{\beta+\gamma+\sigma}\right)^n a_k z^{k-1}}{1+\sum_{k=2}^{\infty} k\left(\frac{\beta+\gamma k+\sigma k^2}{\beta+\gamma+\sigma}\right)^n a_k z^{k-1}} = b\left[\lambda p_1 + \frac{1-\lambda}{2}\right]z + b\lambda p_2 z^2 + b\left[\lambda p_3 - \frac{1-\lambda}{2}\right]z^3 + \ldots \tag{38}$$

and

$$\frac{\sum_{k=2}^{\infty} k(k-1)\left(\frac{\beta+\gamma k+\sigma k^2}{\beta+\gamma+\sigma}\right)^n b_k \omega^{k-1}}{1+\sum_{k=2}^{\infty} k\left(\frac{\beta+\gamma k+\sigma k^2}{\beta+\gamma+\sigma}\right)^n b_k \omega^{k-1}} = b\left[\lambda q_1 + \frac{1-\lambda}{2}\right]\omega + b\lambda q_2 \omega^2 + b\left[\lambda q_3 - \frac{1-\lambda}{2}\right]\omega^3 + \ldots, \tag{39}$$

where $p, q, \phi \in P$ (class of Caratheodory functions). Following the same process as in Theorem 1, we obtain the required results as contained in (36) and (37). □

Next is the consequence of Theorem 2 corresponding to Example 1b

Consequence 3. *Let $f(z) \in \sum_b(\beta,\gamma,\sigma;\mu_{A,B})$. Then, for $\beta,\gamma,\sigma \geq 0$, $\beta+\gamma+\sigma > 0$, $-1 \leq B < A \leq 1$ and $0 \leq \lambda \leq 1$,*

$$|a_2| \leq \sqrt{\frac{\lambda|b|(A-B)}{2\left[3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n - 2\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}\right]}} \tag{40}$$

and

$$|a_3| \leq \frac{\lambda|b|(A-B)}{6\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\Big(1+\lambda\big(2(A-B)-1\big)\Big)^2}{16\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{41}$$

Next is the consequence of Theorem 2 corresponding to Example 2b.

Consequence 4. *Let $f(z) \in \sum_b(\beta,\gamma,\sigma;\mu_\alpha)$. Then, for $\beta,\gamma,\sigma \geq 0$, $\beta+\gamma+\sigma > 0$ and $0 \leq \lambda \leq 1$,*

$$|a_2| \leq \sqrt{\frac{\lambda|b|}{\left[3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n - 2\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}\right]}} \tag{42}$$

and

$$|a_3| \leq \frac{\lambda|b|}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2(1+3\lambda)^2}{16\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{43}$$

Theorem 3. *Let $f(z) \in \sum_b^R(\beta,\gamma,\sigma;\mu)$. Then, for $\beta,\gamma,\sigma \geq 0$, $\beta+\gamma+\sigma > 0$ and $0 \leq \lambda \leq 1$,*

$$|a_2| \leq \sqrt{\frac{\lambda|b||B_1|}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n}} \tag{44}$$

and

$$\left|a_3\right| \leq \frac{\lambda|b||B_1|}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\left(1+\lambda\left(2|B_1|-1\right)\right)^2}{16\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{45}$$

Proof. Suppose that $f(z) \in \sum_b^R(\beta,\gamma,\sigma;\mu)$, then from the Definition 5, it follows that

$$1 + \frac{1}{b}\{z\left[L^n(\beta,\gamma,\sigma)f(z)\right]' - 1\} = \mu(u(z)) = \lambda p(z) + (1-\lambda)\phi(z), \quad (0 \leq \lambda \leq 1)$$

and

$$1 + \frac{1}{b}\{\omega\left[L^n(\beta,\gamma,\sigma)g(\omega)\right]' - 1\} = \mu(u(\omega)) = \lambda q(\omega) + (1-\lambda)\phi(\omega), \quad (0 \leq \lambda \leq 1).$$

It implies that

$$\sum_{k=2}^{\infty} k\left(\frac{\beta+\gamma k+\sigma k^2}{\beta+\gamma+\sigma}\right)^n a_k z^{k-1} = b\left[\lambda p_1 + \frac{1-\lambda}{2}\right]z + b\lambda p_2 z^2 + b\left[\lambda p_3 - \frac{1-\lambda}{2}\right]z^3 + \ldots \tag{46}$$

and

$$\sum_{k=2}^{\infty} k\left(\frac{\beta+\gamma k+\sigma k^2}{\beta+\gamma+\sigma}\right)^n b_k \omega^{k-1} = b\left[\lambda q_1 + \frac{1-\lambda}{2}\right]\omega + b\lambda q_2 \omega^2 + b\left[\lambda q_3 - \frac{1-\lambda}{2}\right]\omega^3 + \ldots, \tag{47}$$

where $p, q, \phi \in P$ (class of Caratheodory functions). Following the same process as in Theorem 1, we obtain the required results as revealed in (44) and (45). □

Further, we present the consequence of Theorem 3 corresponding to Example 1c.

Consequence 5. *Let* $f(z) \in \sum_b^R(\beta,\gamma,\sigma;\mu_{A,B})$*. Then, for* $\beta,\gamma,\sigma \geq 0$, $\beta+\gamma+\sigma > 0$, $-1 \leq B < A \leq 1$ *and* $0 \leq \lambda \leq 1$,

$$\left|a_2\right| \leq \sqrt{\frac{\lambda|b|(A-B)}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n}} \tag{48}$$

and

$$\left|a_3\right| \leq \frac{\lambda|b|(A-B)}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\left(1+\lambda\left(2(A-B)-1\right)\right)^2}{16\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{49}$$

Finally, we present the consequence of Theorem 3 corresponding to Example 2c.

Consequence 6. *Let* $f(z) \in \sum_b^R(\beta,\gamma,\sigma;\mu_\alpha)$*. Then, for* $\beta,\gamma,\sigma \geq 0$, $\beta+\gamma+\sigma > 0$ *and* $0 \leq \lambda \leq 1$,

$$\left|a_2\right| \leq \sqrt{\frac{2\lambda|b|}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n}} \tag{50}$$

and

$$\left|a_3\right| \leq \frac{2\lambda|b|}{3\left(\frac{\beta+3\gamma+9\sigma}{\beta+\gamma+\sigma}\right)^n} + \frac{|b|^2\left(1+3\lambda\right)^2}{16\left(\frac{\beta+2\gamma+4\sigma}{\beta+\gamma+\sigma}\right)^{2n}}. \tag{51}$$

4. Conclusions

The study conducted in this paper involves three classes of bi-univalent functions associated with the modified sigmoid function in the open unit disk. They are introduced in Definitions 3–5 using the general multiplier transform given in Definition 1 and the concept of subordination. Two interesting examples are constructed for particular values of the parameters involved in the definition of the classes. The main results are contained in three theorems in which coefficient estimates are obtained for each of the newly defined classes of bi-univalent functions. Interesting consequences of the theorems follow when applying the proved results considering the two examples earlier mentioned.

The new results presented in this paper are interesting for researchers since the coefficient estimates obtained in this work could be used in the future to investigate the Fekete–Szegö relation as well as the Hankel determinants for the newly introduced classes as seen in the previously cited papers [33,34], among others.

Author Contributions: Conceptualization, J.O.H.; methodology, G.I.O., A.T.O. and J.O.H.; software, G.I.O.; validation, G.I.O., A.T.O. and J.O.H.; formal analysis, G.I.O., A.T.O. and J.O.H.; investigation, G.I.O., A.T.O. and J.O.H.; resources, G.I.O. and J.O.H.; data curation, G.I.O. and J.O.H.; writing—original draft preparation, J.O.H.; writing—review and editing, G.I.O., A.T.O. and J.O.H.; visualization, G.I.O.; supervision, J.O.H.; project administration, G.I.O.; funding acquisition, G.I.O. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the reviewers for their valuable suggestions which have improved the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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