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φ-ψ-Contractions under W-Distances Employing Symmetric Locally T-Transitive Binary Relation

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Abstract: The intent of this paper is to prove the relation-theoretic fixed point results under φ-ψ-contractions involving W-distance on a metric space and equipped with a symmetric locally T-transitive binary relation (not necessarily transitive relation). Our results enrich and improve several fixed point results of the existing literature.

Keywords: $\mathcal{R}$-complete metric spaces; locally T-transitive binary relations; φ-ψ-contractions; $\mathcal{R}$-directed sets

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1. Introduction

The Banach contraction principle (abbreviated as BCP in the sequel) is a simple but very natural and foundational result of metric fixed point theory, which asserts that every contraction self-mapping defined on a complete metric space admits a unique fixed point. A very early and noted generalization of BCP is essentially due to Browder [1], which utilizes a function $\psi : [0, +\infty) \to [0, +\infty)$ satisfying $\psi(t) < t$ for each $t > 0$, wherein ψ is often referred as a control function intended to generalize the term $\alpha\rho(p, q)$, (where $\alpha \in [0, 1)$ and ρ is a metric). While doing so, Browder [1] called a self map T defined on a metric space $(\mathcal{M}, \rho)$ to be a nonlinear contraction if $\rho(Tp, Tq) \leq \psi(\rho(p, q))$ for all $p, q \in \mathcal{M}$, where ψ is hypothesized to be increasing and right continuous. Thereafter, many authors generalized the Browder fixed point theorem by slightly altering the properties of underlying control functions ψ (e.g., Boyd–Wong [2] and Matkowski [3]). Recall that the class of control functions of Boyd and Wong [2] is described as "$\Omega = \Big\{\psi : [0, +\infty) \to [0, +\infty) : \psi(t) < t$ for each $t > 0$ and $\limsup\limits_{r \to t^+} \psi(r) < t$ for each $t > 0\Big\}$". Analogously, Matkowski [3] called a function $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ to be a comparison function if ψ is increasing and $\lim\limits_{n \to +\infty} \psi^n(t) = 0$ for all $t > 0$. Additionally, Matkowski [4] further observed that every comparison function remains a control function. These two classes of nonlinear contractions have been studied extensively in recent years, and by now, there exists considerable literature on such classes of contractions. For more details on metric fixed point theory, one is referred to [5–7].

In the last two decades, the most significant generalizations/extensions of BCP (cf. [8]) have been established by numerous researchers, namely, Ran and Reurings [9], and Nieto and Rodríguez-Lopez [10], to ordered metric spaces. Later, Agarwal et al. [11] extended the results of Ran and Reurings [9] and Nieto and Rodríguez-Lopez via nonlinear contractions, which was later refined by O'Regan and Petruşel [12]. Thereafter, Alam and Imdad [13] derived an analogue of BCP employing an amorphous binary relation, which was further enriched by Alam et al. [14] and Arif et al. [15].

In 1996, Kada et al. [16] discovered the new idea of W-distance on a metric space and utilized the same to prove some fixed point results. Thereafter, many authors improved/generalized the classical BCP using W-distances; see the references [17–19]. In 2009, Razani et al. [20] proved a variant of the classical result under φ-ψ-contractions via W-distance, which deduces several results of Branciari [21] and Banach [8], etc., under suitable considerations on φ and ψ. Most recently, Senapati and Dey [22] obtained a relation-theoretic version of the classical result using an amorphous binary relation involving W-distance.

The intent of this article is to introduce relatively a weaker contractive condition and utilize the same to prove relation-theoretic fixed point theorems for a self-mapping on a metric space equipped with a W-distance and a symmetric locally T-transitive binary relation. Thereafter, we furnish an example which illustrates our results. Additionally, some known related results are noted as consequences of our newly furnished results. Finally, as an application of one of our furnished results, an existence theorem for the nonlinear integral type contractive condition is discussed.

2. Preliminaries

For a subset $\mathcal{R}$ of $\mathcal{M}^2$ (where $\mathcal{M}$ is a nonempty set) is called a binary relation on $\mathcal{M}$. In fact, we often write $(p,q) \in \mathcal{R}$ in place of $p\mathcal{R}q$. Additionally, the term $\mathcal{R}|_E$ refers to the restriction of $\mathcal{R}$ to E and $\mathcal{R}|_E$ defined as $\mathcal{R} \cap E^2$, where $E \subseteq \mathcal{M}$.

To have a precise and self-contained presentation, we borrow the following notions and terms utilized by various mathematicians in their respective investigations.

Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ be a binary relation defined on it. Then $\mathcal{R}$ is called

- "Amorphous";
- "Universal" if $\mathcal{R} = \mathcal{M}^2$;
- "Empty" if $\mathcal{R} = \emptyset$;
- "Reflexive" if $(p,p) \in \mathcal{R}$ for all $p \in \mathcal{M}$;
- "Symmetric" if $(p,q) \in \mathcal{R}$ implies $(q,p) \in \mathcal{R}$;
- "Antisymmetric" if $(p,q) \in \mathcal{R}$ and $(q,p) \in \mathcal{R}$ imply $p = q$;
- "Transitive" if $(p,q) \in \mathcal{R}$ and $(q,z) \in \mathcal{R}$ imply $(p,z) \in \mathcal{R}$;
- "Complete" if $(p,q) \in \mathcal{R}$ *or* $(q,p) \in \mathcal{R}$ for all $p,q \in \mathcal{M}$;
- "Partial order" if $\mathcal{R}$ is "reflexive", "antisymmetric" and "transitive".

Throughout this manuscript, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{N}_0$ for the set of whole numbers (i.e., $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) and $\mathbb{R}$ for the set of real numbers. Additionally, we write $\mathcal{R}$ for a binary relation in place of nonempty binary relation.

We adopt the related notions and results, which are needed in our present context. Inspired by partial order relation $(\preceq)$ found in Turinici [23,24], Alam Imdad [13] introduced the following relatively weaker notions.

Definition 1 ([13]). *Let $(\mathcal{M},\rho)$ be a metric space and $\mathcal{R}$ be a binary relation defined on it, then*

(i) *Any p and q in $\mathcal{M}$ are said to be $\mathcal{R}$-comparative if either $(p,q) \in \mathcal{R}$ or $(q,p) \in \mathcal{R}$. We denote it by $[p,q] \in \mathcal{R}$.*
(ii) *A sequence $\{p_n\} \subset \mathcal{M}$ is called $\mathcal{R}$-preserving if $(p_n, p_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$.*
(iii) *$\mathcal{R}$ is called T-closed if for any $p,q \in \mathcal{M}$, $(p,q) \in \mathcal{R} \Rightarrow (Tp,Tq) \in \mathcal{R}$.*
(iv) *$\mathcal{R}$ is called ρ-self-closed, if for any $\mathcal{R}$-preserving sequence $\{p_n\}$ such that $p_n \xrightarrow{\rho} p$, there exists a subsequence $\{p_{n_k}\}$* of $\{p_n\}$ with $[p_{n_k}, p] \in \mathcal{R}$ *for all $k \in \mathbb{N}_0$.*

Definition 2 ([25]). *Let T, $\mathcal{R}$ be a self-mapping and binary relation respectively defined on a nonempty set $\mathcal{M}$. Then*

(i) *$\mathcal{R}$ is called T-transitive if for any $p,q,z \in \mathcal{M}$, $(Tp,Tq),(Tq,Tz) \in \mathcal{R} \Rightarrow (Tp,Tz) \in \mathcal{R}$.*
(ii) *$\mathcal{R}$ is called locally transitive if for each $\mathcal{R}$-preserving sequence $\{p_n\} \subset \mathcal{M}$ (with range $E = \{p_n : n \in \mathbb{N}\}$), $\mathcal{R}|_E$ is transitive.*
(iii) *$\mathcal{R}$ is called locally T-transitive if for each $\mathcal{R}$-preserving sequence $\{p_n\} \subset T(\mathcal{M})$ (with range $E = \{p_n : n \in \mathbb{N}\}$), $\mathcal{R}|_E$ is transitive.*

The following result shows the idea of a class of locally T-transitivity binary relations being relatively larger than other variants of transitivity:

Proposition 1 ([25]). *Let T, $\mathcal{R}$ be a self-mapping and binary relation respectively defined on a metric space $(\mathcal{M}, \rho)$. Then*

(i) $\mathcal{R}$ *is T-transitive* $\Leftrightarrow \mathcal{R}|_{T(\mathcal{M})}$ *is transitive,*
(ii) $\mathcal{R}$ *is locally T-transitive* $\Leftrightarrow \mathcal{R}|_{T(\mathcal{M})}$ *is locally transitive,*
(iii) $\mathcal{R}$ *is transitive* $\Rightarrow \mathcal{R}$ *is locally transitive* $\Rightarrow \mathcal{R}$ *is locally T-transitive,*
(iv) $\mathcal{R}$ *is transitive* $\Rightarrow \mathcal{R}$ *is T-transitive* $\Rightarrow \mathcal{R}$ *is locally T-transitive.*

Definition 3 ([26]). *Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ be a binary relation defined on it, then the dual relation or transpose or inverse of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$, is defined by $\mathcal{R}^{-1} = \{(p,q) \in \mathcal{M}^2 : (q,p) \in \mathcal{R}\}$, whereas the symmetric closure of $\mathcal{R}$ (denoted by $\mathcal{R}^s$) is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$ (i.e., $\mathcal{R}^s := \mathcal{R} \cup \mathcal{R}^{-1}$).*

Proposition 2 ([13]). *Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ be a binary relation defined on it,*

$$(p,q) \in \mathcal{R}^s \iff [p,q] \in \mathcal{R}.$$

Proposition 3 ([25]). *Let T, $\mathcal{R}$ be a self-mapping and binary relation, respectively defined on a nonempty set $\mathcal{M}$. If $\mathcal{R}$ is T-closed, then for all $n \in \mathbb{N}_0$, $\mathcal{R}$ is also T^n-closed, where T^n denotes n^{th}-iterate of T.*

Definition 4 ([27]). *Let $\mathcal{R}$ be a binary relation defined on a nonempty set $\mathcal{M}$. We say that $(\mathcal{M}, \rho)$ is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{M}$ converges.*

Definition 5 ([27]). *Let T, $\mathcal{R}$ be a self-mapping and binary relation respectively defined on a metric space $(\mathcal{M}, \rho)$, $p \in \mathcal{M}$. Then T is called $\mathcal{R}$-continuous at $\mathcal{M}$ if for any $\mathcal{R}$-preserving sequence $\{p_n\}$ such that $p_n \xrightarrow{\rho} p$, we have $T(p_n) \xrightarrow{\rho} T(p)$. Moreover, T is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of the underlying space $\mathcal{M}$.*

Definition 6 ([28]). *Let $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ be a binary relation defined on it. If E is part of $\mathcal{M}$, then E is called $\mathcal{R}$-directed if for each $p, q \in E$, there exists $z \in \mathcal{M}$ such that $(p,z) \in \mathcal{R}$ and $(q,z) \in \mathcal{R}$.*

Given $\mathcal{M}$ be a nonempty set and $\mathcal{R}$ be a binary relation defined on it, we use the following notations:

(i) $F(T)$:=the set of all fixed points of T,
(ii) $\mathcal{M}(T, \mathcal{R}) := \{p \in \mathcal{M} : (p, Tp) \in \mathcal{R}\}$.
(iii) $\mathcal{M}[T, \mathcal{R}] := \{p \in \mathcal{M} : (p, Tp) \text{ and } (Tp, p) \in \mathcal{R}\}$.

A variant of BCP under amorphous binary relation is contained in [13]:

Theorem 1 ([13]). *Let $(\mathcal{M}, \rho)$ be a metric space endowed with a binary relation $\mathcal{R}$. If T is a self-mapping on $\mathcal{M}$ such that the following conditions are satisfied:*

(i) $(\mathcal{M}, \rho)$ is $\mathcal{R}$-complete,
(ii) $\mathcal{R}$ is T-closed,
(iii) $\mathcal{M}(T, \mathcal{R})$ is nonempty,
(iv) Either T is $\mathcal{R}$-continuous or $\mathcal{R}$ is ρ-self-closed,
(v) There exists $\alpha \in [0,1)$ such that
$\rho(Tp, Tq) \leq \alpha\rho(p,q)$ *for all $p, q \in \mathcal{M}$ with $(p,q) \in \mathcal{R}$,*

then $F(T) \neq \emptyset$. Moreover, if $\mathcal{M}$ is $\mathcal{R}^s$-directed, then $F(T)$ is singleton.

Kada et al. [16] introduced the following notion.

Definition 7 ([16]). *We say that a function $\omega : \mathcal{M} \times \mathcal{M} \to [0, +\infty)$ is called W-distance on a metric space $(\mathcal{M}, \rho)$ if the following properties are satisfied:*

(p_1) *: For any $p, q, z \in \mathcal{M}$, $\omega(p, z) \leq \omega(p, q) + \omega(q, z)$*
(p_2) *: If for any $p \in \mathcal{M}$ and $q_n \to q$ in $\mathcal{M}$, then $\omega(p, q) \leq \liminf_n \omega(p, q_n)$ (i.e., ω is lower semi continuous in its second argument),*
(p_3) *: For each $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z, p) \leq \delta$ and $\omega(z, q) \leq \delta$ implies that $\omega(p, q) \leq \epsilon$.*

Remark 1. *Notice that W-distance is not symmetric, which can be described later by the use of an example.*

The following family of mappings is given by Razani et al. [20].

Let ζ be the class of all continuous functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:

(ζ_1) : φ is increasing,
(ζ_2) : $\varphi(t) > 0$ for all $t > 0$.

Lemma 1. *If φ is a member of ζ such that $\lim\limits_{n \to +\infty} \varphi(a_n) = 0$, then $\lim\limits_{n \to +\infty} a_n = 0$.*

Proof. Suppose, on the contrary, that there exist $\epsilon > 0$ and $\{n_k\}$ for all $k \in \mathbb{N}$ such that

$$a_{n_k} \geq \epsilon > 0$$

according to (ζ_1), and making the limit superior as $k \to 0$, on both sides, we have

$$\limsup_{k \to +\infty} \varphi(a_{n_k}) \geq \varphi(\epsilon) > 0,$$

which is a contradiction, thus $\lim\limits_{n \to +\infty} a_n = 0$. □

The following lemmas are required in the proof of the main results.

Lemma 2 ([16]). *Let ω be a W-distance on a metric space $(\mathcal{M}, \rho)$. If $\{p_n\}$ is a sequence in $\mathcal{M}$ such that $\lim_{n \to +\infty} \omega(p_n, p) = \lim_{n \to +\infty} \omega(p_n, q)$, then $p = q$. In particular if $\omega(z, p) = \omega(z, q) = 0$, then $p = q$.*

Remark 2. *As $\omega(u, v) = \omega(v, u) = 0$ and $\omega(u, u) \leq \omega(u, v) + \omega(v, u) = 0$, implies that $\omega(u, u) = 0$ and due to Lemma 2, we have $u = v$.*

Lemma 3 ([16]). *Let ω be a W-distance defined on a metric space $(\mathcal{M}, \rho)$. We say that $\{p_n\}$ is a Cauchy sequence in $\mathcal{M}$, if for each $\epsilon > 0$ there exists N_ϵ in $\mathbb{N}$ such that $m > n > N_\epsilon$ implies that $\omega(p_n, p_m) < \epsilon$ (or $\lim_{m,n \to +\infty} \omega(p_n, p_m) = 0$).*

The following notion was introduced by Senapati and Dey [22].

Definition 8. *Let $\mathcal{R}$ be a binary relation defined on a metric space $(\mathcal{M}, \rho)$. We say that a mapping $f : \mathcal{M} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is $\mathcal{R}$-lower semi continuous (or, in short, $\mathcal{R}$-LSC) at p in $\mathcal{M}$ if for every $\mathcal{R}$-preserving sequence $\{p_n\}$ converging to $\mathcal{M}$, we have $\liminf_{n \to +\infty} f(p_n) \geq f(p)$.*

A variant of BCP under a W-distance and amorphous binary relation is contained in [22]:

Theorem 2 ([22]). *Let T be a self-mapping defined on a metric space $(\mathcal{M}, \rho)$, wherein $(\mathcal{M}, \rho)$ equipped with a binary relation $\mathcal{R}$ and a W-distance ω. Let ω be $\mathcal{R}$-lower semi-continuous in its second argument and Z an $\mathcal{R}$-complete subspace of $\mathcal{M}$ with $T(\mathcal{M}) \subset Z$. Assume that the conditions (ii), (iii) and (iv) of Theorem 1 along with the following condition holds:*

(v) *There exists $\alpha \in [0,1)$ such that*

$$\omega(Tp, Tq) \leq \alpha\omega(p,q) \text{ for all } p, q \in \mathcal{M} \text{ with } (p,q) \in \mathcal{R}.$$

Then $F(T) \neq \emptyset$. Moreover, if $T(\mathcal{M})$ is $\mathcal{R}^s$-directed, then $F(T)$ is singleton.

The attempted improvements in our results are based on the following motivations, which are as follows:

- To extend linear contractions due to Senapati and Dey ([22], Theorems 2.1 and 2.2) to nonlinear contractions on a metric space endowed via W-distances and symmetric binary relations.
- To give an example which demonstrate the utility of our presented results herein.
- To discuss several sharpened versions of our main results by considering suitable assumptions.
- To utilize our results and obtain a result for the integral type contractive condition in relational metric space involving W-distance.

3. Main Results

Before presenting our main theorems, firstly, we refine the class of control functions, which is indicated in Razani et al. [20]. Let ϑ be the collection of all mappings $\psi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

(ϑ_1) : ψ is increasing,
(ϑ_2) : $\limsup_{r\to t^+} \psi(r) < t$,
(ϑ_3) : $\psi(t) < t$ for all $t > 0$.

Now, we propose two suitable properties of a member ψ lies in ϑ.

Lemma 4. *Let ψ be in ϑ, then $\lim\limits_{n\to+\infty} \psi^n(t) = 0$ for each $t > 0$.*

Proof. For each $0 < t$, in view of (ϑ_1), we have that $\{\psi^n(t)\}$ is a decreasing sequence of non-negative numbers, and thus there exists $\epsilon \geq 0$ such that

$$\lim_{n\to+\infty} \psi^n(t) = \epsilon^+. \tag{1}$$

Let, if possible, $\epsilon > 0$, and we set $\gamma := \limsup\limits_{n\to+\infty} \psi^n(t)$. Clearly, $\gamma \geq \epsilon$ In view of (1), $\epsilon = \lim\limits_{n\to+\infty} \psi^n(t) = \limsup\limits_{n\to+\infty} \psi^n(t) = \limsup\limits_{\gamma\to\epsilon^+} \psi(\epsilon) < \epsilon$, which is impossible, hence $\lim\limits_{n\to+\infty} \psi^n(t) = 0$ for each $t > 0$. □

Proposition 4. *If ψ is in ϑ, then $\psi(0) = 0$.*

Proof. Suppose on contrary that $\psi(0) = t$ for some $t > 0$. As $0 < t$ and according to (ϑ_1), we have $\psi(0) \leq \psi(t)$ and also, utilizing (ϑ_3), it allows that $t = \psi(0) \leq \psi(t) < t$, which is impossible, hence $\psi(0) = 0$. □

Now, we are equipped to prove an existence result under φ-ψ-contractions employing binary relation via the W-distance.

Theorem 3. *Let T be a self-mapping defined on a metric space $(\mathcal{M}, \rho)$, wherein $(\mathcal{M}, \rho)$ equipped with a symmetric locally T-transitive binary relation $\mathcal{R}$ and a W-distance ω. Let ω be $\mathcal{R}$-lower semi-continuous in its second argument. Assume that the conditions (i), (ii), (iii) and (iv) of Theorem 1 along with the following condition holds:*

(v) *There exist $\varphi \in \zeta$ and $\psi \in \vartheta$ such that*
$\varphi(\omega(Tp, Tq)) \leq \psi(\varphi(\omega(p,q)))$ for all $p, q \in \mathcal{M}$ with $(p,q) \in \mathcal{R}$.

Then $F(T) \neq \emptyset$.

Proof. As $\mathcal{M}(T,\mathcal{R}) \neq \emptyset$. Let $p_0 \in \mathcal{M}(T,\mathcal{R})$. Construct a sequence $\{p_n\}$ with an initial point p_0, that is

$$p_n = T^n(p_0) \text{ for all } n \in \mathbb{N}_0. \tag{2}$$

As $(p_0, Tp_0) \in \mathcal{R}$, using T is $\mathcal{R}$-closed and Proposition 3, we get

$$(T^n p_0, T^{n+1} p_0) \in \mathcal{R}$$

so that

$$(p_n, p_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N}_0. \tag{3}$$

Thus, the sequence $\{p_n\}$ is $\mathcal{R}$-preserving. Due to symmetry of $\mathcal{R}$, we have

$$(p_{n+1}, p_n) \in \mathcal{R} \text{ for all } \ n \in \mathbb{N}_0. \tag{4}$$

Applying the contractivity condition (v) to (3) and (ϑ_1), we deduce, for all $n \in \mathbb{N}_0$

$$\begin{aligned} \varphi(\omega(p_n, p_{n+1})) &\leq \psi(\varphi(\omega(p_{n-1}, p_n))) \\ &\leq \psi^2(\varphi(\omega(p_{n-2}, p_{n-1}))) \\ &\leq \cdots \\ &\leq \psi^n(\varphi(\omega(p_0, p_1))). \end{aligned}$$

Making $n \to +\infty$, and employing Lemma 4, we get $\lim\limits_{n\to+\infty} \varphi(\omega(p_n, p_{n+1})) = 0$. Using this fact and in lieu of Lemma 1, we have

$$\lim_{n\to+\infty} \omega(p_n, p_{n+1}) = 0. \tag{5}$$

Now, if $(p_n, p_{n+1}) \in \mathcal{R}$ so $(p_{n+1}, p_n) \in \mathcal{R}$ for all $n \in \mathbb{N}_0$ (due to symmetric property of $\mathcal{R}$). Similarly, we have

$$\lim_{n\to+\infty} \omega(p_{n+1}, p_n) = 0. \tag{6}$$

We claim that $\{p_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{p_n\}$ is not Cauchy. Then there exist two subsequences $\{p_{m_k}\}$, $\{p_{n_k}\}$ of $\{p_n\}$ and $\epsilon > 0$ with $k \leq m_k < n_k$ such that

$$\omega(p_{m_k}, p_{n_k}) \geq \epsilon. \tag{7}$$

Using (7), there exists $k_0 \in \mathbb{N}$ such that $m_k > k_0$, which implies that

$$\omega(p_{m_k}, p_{m_{k+1}}) < \epsilon. \tag{8}$$

If $m_k > k_0$, and in view of (7) and (8), $n_k \neq m_{k+1}$, we can choose n_k as a minimal index such that $\omega(p_{m_k}, p_{n_k}) \geq \epsilon$ but $\omega(p_{m_k}, p_{p_k}) < \epsilon$ for $p_k \in \{m_{k+1}, m_{k+2}, \cdots, n_{k-1}\}$. Now, using (5), we have

$$\begin{aligned} \epsilon &\leq \omega(p_{m_k}, p_{n_k}) \\ &\leq \omega(p_{m_k}, p_{n_{k-1}}) + \omega(p_{n_{k-1}}, p_{n_k}) \\ &< \epsilon + \omega(p_{n_{k-1}}, p_{n_k}) \to \epsilon^+ \ \text{(as } k \to +\infty). \end{aligned}$$

Letting $k \to +\infty$, we have $\lim_{k\to+\infty} \omega(p_{m_k}, p_{n_k}) = \epsilon^+$. Therefore, for any $\{k_r\}_{r=1}^{+\infty}$ such that $\omega(p_{m_{k_r}}, p_{n_{k_r}})$ tends to ϵ^+. Denote $\delta_{k_r} := \varphi(\omega(p_{m_{k_r}}, p_{n_{k_r}}))$ for all $r \in \mathbb{N}$, and utilizing this fact and the continuity of φ, we have

$$\delta_{k_r} \to \varphi(\epsilon)^+ \quad as \quad r \to +\infty. \tag{9}$$

If $\gamma := \limsup_{k\to+\infty} \omega(p_{m_{k+1}}, p_{n_{k+1}}) \geq \epsilon$, then there exists $\{k_r\}_{r=1}^{+\infty}$ such that

$$\omega(p_{m_{k_r+1}}, p_{n_{k_r+1}}) \to \gamma \geq \epsilon \text{ as } r \to +\infty.$$

Employing locally T-transitivity of $\mathcal{R}$ and (3), we have $(p_{m_{k_r+1}}, p_{n_{k_r+1}}) \in \mathcal{R}$ for all $r \in \mathbb{N}$. Now, employing continuity and increasing property of φ, (ϑ_2), contractive condition (v) and (9), we obtain

$$\begin{aligned} \varphi(\epsilon) \leq \varphi(\gamma) &= \lim_{r\to+\infty} \varphi(\omega(p_{m_{k_r+1}}, p_{n_{k_r+1}})) \\ &\leq \lim_{r\to+\infty} \psi\varphi(\omega(p_{m_{k_r}}, p_{n_{k_r}})) \\ &= \limsup_{\delta_{k_r}\to\varphi(\epsilon)^+} \psi(\delta_{k_r}) < \varphi(\epsilon) \quad (as\ \varphi(\epsilon) > 0, \text{for all } \epsilon > 0), \end{aligned}$$

which is a contradiction and hence $\limsup\limits_{k\to+\infty} \omega(p_{m_{k+1}}, p_{n_{k+1}}) < \epsilon$. Now, we have

$$\begin{aligned} \epsilon &\leq \omega(p_{m_k}, p_{n_k}) \\ &\leq \omega(p_{m_k}, p_{m_{k+1}}) + \omega(p_{m_{k+1}}, p_{n_{k+1}}) + \omega(p_{n_{k+1}}, p_{n_k}). \end{aligned}$$

Making the limit $k \to +\infty$, in the above in-equation besides using (5) and (6), we obtain

$$\begin{aligned} \epsilon &\leq \lim_{k\to+\infty} \omega(p_{m_k}, p_{n_k}) \\ &\leq \lim_{k\to+\infty} \omega(p_{m_k}, p_{m_{k+1}}) + \limsup_{k\to+\infty} \omega(p_{m_{k+1}}, p_{n_{k+1}}) + \lim_{k\to+\infty} \omega(p_{n_{k+1}}, p_{n_k}) < \epsilon, \end{aligned}$$

which is again a contradiction. Hence, we conclude that

$$\lim_{m,n\to+\infty} \omega(p_m, p_n) = 0. \tag{10}$$

In lieu of Lemma 3, $\{p_n\}$ is a Cauchy sequence in $\mathcal{M}$, which is $\mathcal{R}$-preserving by virtue of $(\mathcal{M}, \rho)$ being $\mathcal{R}$-complete. We now demonstrate that p is fixed on T. To accomplish this, firstly assume that T is $\mathcal{R}$-continuous. Since $\{p_n\}$ is $\mathcal{R}$-preserving with $p_n \xrightarrow{\rho} p$, implies that $p_{n+1} = T(p_n) \xrightarrow{\rho} T(p)$ ($\mathcal{R}$-continuity of T). Due to the limit's uniqueness, we obtain $T(p) = p$, that is $F(T) \neq \emptyset$. Alternately, suppose that $\mathcal{R}$ is ρ-self-closed. Since $\{p_n\}$ is $\mathcal{R}$-preserving such that $p_n \xrightarrow{\rho} p$, then there is a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with $[p_{n_k}, p] \in \mathcal{R}$, for all $k \in \mathbb{N}_0$ (the ρ-self-closedness of $\mathcal{R}$). Utilizing the assumption (v), $[p_{n_k}, p] \in \mathcal{R}$ with $p_{n_k} \xrightarrow{\rho} p$, Proposition 4 and (ϑ_3), (either $\varphi(\omega(p_{n_k}, p))$ is zero or nonzero)), we have

$$\varphi(\omega(p_{n_{k+1}}, Tp)) \quad \leq \quad \psi\varphi(\omega(p_{n_k}, p)) \leq \varphi(\omega(p_{n_k}, p)) \text{ for all } k \in \mathbb{N}. \tag{11}$$

Considering (10), for each $\epsilon > 0$ there exists K_ϵ with $n > K_\epsilon$ such that $\omega(p_{K_\epsilon}, p_n) < \epsilon$. As $p_n \xrightarrow{\rho} p$ and $\omega(p, .)$ is $\mathcal{R}$-lower semi continuous, we have

$$\omega(p_{K_\epsilon}, p) \leq \liminf_{k\to+\infty} \omega(p_{K_\epsilon}, p_n) \leq \epsilon.$$

Thus $\omega(p_{K_\epsilon}, p) \leq \epsilon$. Set $\epsilon = \frac{1}{k}$ and $K_\epsilon = n_k$, we have

$$\lim_{k\to+\infty} \omega(p_{n_k}, p) = 0. \tag{12}$$

Letting $k \to +\infty$ in (11), using the continuity of φ and using (12), we have $\lim_{k\to+\infty} \omega(p_{n_{k+1}}, Tp) = 0$. according to this fact and (4), we have

$$\omega(p_{n_k}, Tp) \leq \omega(p_{n_k}, p_{n_{k+1}}) + \omega(p_{n_{k+1}}, Tp) \to 0 \ (as\ k \to +\infty),$$

so that

$$\lim_{k\to+\infty} \omega(p_{n_k}, Tp) = 0. \tag{13}$$

Making use of (12), (13) and Lemma 2, we have $T(p) = p$. Hence, p is a fixed point of T. □

Combining Proposition 1 and Theorem 3, we deduce the following corollary.

Corollary 1. *Theorem 3 remains valid if locally, the T-transitivity of $\mathcal{R}$ is replaced by any one of the following hypotheses besides assuming the rest of the hypotheses:*

(i) *$\mathcal{R}$ is transitive;*
(ii) *$\mathcal{R}$ is T-transitive;*
(iii) *$\mathcal{R}$ is locally transitive.*

4. Uniqueness Result

Now, in regard of Theorem 3, we state and prove the following uniqueness theorem.

Theorem 4. *If in the hypotheses of Theorem 3, the assumption $\mathcal{R}$-directedness of $T(\mathcal{M})$ or $\mathcal{R}$-completeness of $T(\mathcal{M})$ is added, then $F(T)$ is singleton.*

Proof. Firstly assume that the $\mathcal{R}$-directedness of $T(\mathcal{M})$. In lieu of Theorem 3, we have $F(T) \neq \emptyset$. Let $l, m \in F(T)$, we need to show that $l = m$. As $l, m \in F(T) \subseteq T(\mathcal{M})$, there exists $z \in \mathcal{M}$, such that (l, z) and $(m, z) \in \mathcal{R}$. Since $\mathcal{R}$ is T-closed and in view of Proposition 3 and symmetry of $\mathcal{R}$ (for all $n \in \mathbb{N}_0$), we have $(T^n(z), T^n(l)) \in \mathcal{R}$ and $(T^n(z), T^n(m)) \in \mathcal{R}$. Applying the contractive condition (v) to $(T^n(z), T^n(l)) \in \mathcal{R}$, we have

$$\begin{aligned} \varphi(\omega(T^n(z), l)) &\leq \psi(\varphi(\omega(T^{n-1}(z), l))) \\ &\leq \psi^2(\varphi(\omega(T^{n-2}(z), l))) \\ &\leq \cdots \\ &\leq \psi^n(\varphi(\omega(z, l))). \end{aligned}$$

Due to Lemma 4, we obtain $\lim_{n\to+\infty} \varphi(\omega(T^n(z), l)) = 0$ and in lieu of Lemma 1, we have

$$\lim_{n\to+\infty} \omega(T^n(z), l)) = 0. \tag{14}$$

Similarly, we can obtain

$$\lim_{n\to+\infty} \omega(T^n(z), m)) = 0. \tag{15}$$

Hence, due to Lemma 2, (14) and (15), we have

$$l = m.$$

Secondly, assume that $\mathcal{R}|_{T(\mathcal{M})}$ is complete. Then for any $l, m \in T(\mathcal{M})$, $(l, m) \in \mathcal{R}$ or $(m, l) \in \mathcal{R}$; therefore, $(l, m), (m, l) \in \mathcal{R}$ (symmetry of $\mathcal{R}$). Now, using the contractive condition (v) to $(l, m) \in \mathcal{R}$, we have

$$\varphi(\omega(m, l)) \leq \psi\varphi(\omega(m, l)) < \varphi(\omega(m, l)).$$

Thus $\varphi(\omega(m, l)) = 0$, (as $\varphi(t) > 0$, for all $t > 0$) implies that $\omega(m, l) = 0$. Similarly, $\omega(l, m) = 0$. By Remark 2 we have $l = m$. Thus, in both the assumptions of $\mathcal{R}|_{T(\mathcal{M})}$, $F(T)$ is a singleton. □

Remark 3. *Observe that the requirement of the symmetry of a binary relation is not necessary if condition $\mathcal{M}(T, \mathcal{R})$ (utilized in the assumption (iii)) is replaced by $\mathcal{M}[T, \mathcal{R}]$ in Theorem 4.*

5. Illustrative Example

Finally, we furnish an example to validate the utility of Theorems 3 and 4 over corresponding earlier known results.

Example 1. *Let $\mathcal{M} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. For each p, q in $\mathcal{M}$, define a metric ρ by*

$$\rho(p, q) = \begin{cases} p + q, & p \neq q \\ 0, & p = q, \end{cases}$$

and define ω by $\omega(p, q) = q$. It can be easily seen that w is a W-distance on $(\mathcal{M}, \rho)$. On $\mathcal{M}$, define a self-mapping T by $T(p) = \frac{p}{2p+1}$ for all $p \in \mathcal{M}$ and also $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ by $\varphi(t) = t$ and $\psi(t) = \frac{t}{t+1}$ for all $t \in [0, +\infty)$. Clearly, $\varphi \in \zeta$ and $\psi \in \vartheta$. Let $\mathcal{R} := \{(p, q) \in \mathcal{M}^2 : p - q > 0 \text{ or } q - p < 0\}$ be symmetric closure of a binary relation $\mathcal{B} := \{(p, q) \in \mathcal{M}^2 : p - q < 0\}$, which is a locally T-transitive. It is easy to verify that $\mathcal{R}$ is T-closed, $\mathcal{M}(T, \mathcal{R}) \neq \emptyset$ and $(\mathcal{M}, \rho)$ is $\mathcal{R}$-complete. Now, for each $(p, q) \in \mathcal{R}$

$$\varphi(\omega(Tp, Tq)) = Tq = \frac{q}{2q + 1} \leq \frac{q}{q + 1} = \psi(\varphi(\omega(p, q))).$$

Thus remaining assumptions of Theorem 4 can be easily verified. Observe that $p = 0$, is the unique fixed point of T.

Notice that, $\mathcal{M}(T, \mathcal{R}) \neq \emptyset$, T is $\mathcal{R}$-continuous and $\mathcal{R}$ is T-closed. However, sufficiently small positive $\epsilon > 0$, choose $q = \epsilon$ and $p = 0$. Now, $(p, q) = (0, \epsilon) \in \mathcal{R}$,

$$\omega(Tp, Tq) = \omega(T0, T\epsilon) = Tq = \frac{\epsilon}{2\epsilon + 1} \leq \alpha\epsilon = \alpha\omega(p, q)).$$

Therefore, $\alpha \geq \frac{1}{2\epsilon+1}$ as ϵ was very small, ϵ tends to zero, implies that $\alpha \geq 1$, which is a contradiction. Thus contraction condition of Theorem 2 of Senapati and Dey [22] does not work for the present example.

Incidentally, also, for $(p, q) = (0, \epsilon) \in \mathcal{B}$,

$$\rho(Tp, Tq) = \rho(T0, T\epsilon) = T\epsilon = \frac{\epsilon}{2\epsilon + 1} \leq \alpha\epsilon = \alpha\rho(p, q)).$$

Same as above, we obtain a contradiction (that is $\alpha \geq 1$). Thus contractive condition of Theorem 1 of Alam and Imdad [13] does not work for the present example under the metrical consideration.

Now, we deduce some special cases from our main result (that is, Theorem 4).

(i) On setting $Z = \mathcal{M}$, $\varphi(t) = t$ and $\psi(t) = \alpha t$ ($\alpha \in [0, 1)$) in Theorem 4, we obtain Theorem 2 due to Senapati and Dey [22](without utilizing the symmetric property and locally T-transitivity of $\mathcal{R}$).

(ii) On setting $\omega = \rho$, $\varphi(t) = t$ and $\psi(t) = \alpha t$ $(\alpha \in [0,1))$ in Theorem 4, we obtain Theorem 1 due to Alam and Imdad [13] (without utilizing the symmetry and locally T-transitivity of $\mathcal{R}$).
(iii) On setting $\omega = \rho$, $\varphi(t) = t$ and $\psi \in \vartheta$ (condition (ϑ_2) is replaced by a continuous control function or a upper semi continuous control function) in Theorem 4, we obtain a Corollary 1 contained in [14].
(iv) On setting $\omega = \rho$, $\varphi(t) = t$ and $\psi \in \vartheta$ (without using condition (ϑ_1)) in Theorem 4, we obtain Theorem 4 contained in [15].
(v) Choosing $\mathcal{R} = \mathcal{M}^2$ (the universal relation) and ψ in ϑ in Theorem 4 (wherein (ϑ_2) is replaced by upper semi-continuous from the right), the main theorem of Razani et al. [20] is deduced.
(vi) On taking $\mathcal{R} = \mathcal{M}^2$, (the universal relation), $\omega = \rho$, $\varphi(t) = t$ and $\psi(t) = \alpha t$ $(\alpha \in [0,1))$ in Theorem 4, we deduce the classical Banach fixed point theorem.
(vii) "Choosing $\mathcal{R} = \mathcal{M}^2$, (universal relation), $\varphi(t) = \int_0^{\epsilon} \theta(t)dt$ and $\psi(t) = \alpha t$ $(\alpha \in [0,1))$, where $\theta : [0,+\infty) \to [0,+\infty)$ is Lebesgue-integrable mapping, which is summable and $\int_0^{\epsilon} \theta(t)dt > 0$ for each $\epsilon > 0$. Clearly, $\varphi \in \zeta$ and $\psi \in \vartheta$. Henceforth in view of preceding hypotheses, the main theorem of Branciari [21] can be deduced".

6. An Application

As an application of Theorem 3, we prove a result under φ-ψ type integral contraction in a metric space equipped with binary relation via W-distance. Suppose $\mathcal{C}$ be the set of mappings $\mu : [0,+\infty) \to [0,+\infty)$ satisfying the following:

(I) Consider a mapping μ on each compact subset of $[0,+\infty)$ along with Lebesgue-integrablilty of μ,
(II) $\int_0^{\epsilon} \mu(t)dt > 0$ for all$\epsilon > 0$.

Theorem 5. *Let $(\mathcal{M},\rho)$ be a metric space endowed with a binary relation $\mathcal{R}$, where $\mathcal{R}$ is a symmetric and locally T-transitive. Let ω be a W-distance on $(\mathcal{M},\rho)$, which is $\mathcal{R}$-lower semicontinuous in its second arguments. Let T be a self-mapping on $\mathcal{M}$ such that the following condition is satisfied: For every $p,q \in \mathcal{M}$ with $(p,q) \in \mathcal{R}$ and $\mu \in \mathcal{C}$ such that*

$$\varphi(\omega(Tp,Tq)) \leq \int_0^{\gamma\circ\varphi(\omega(p,q))} \mu(t)dt, \tag{16}$$

where $\varphi \in \zeta$ and $\gamma \in \vartheta$. Further, if the assumptions, (i)-(iv) of Theorem 3 are satisfied, then T has a fixed point.

Proof. Consider a mapping $\Gamma : [0,+\infty) \to [0,+\infty)$ defined by $\Gamma(s) = \int_0^s \mu(t)dt$, then Γ is a continuous increasing function. Therefore, (16) can be written as

$$\varphi(\omega(Tp,Tq)) \leq (\Gamma \circ \gamma) \circ \varphi(\omega(p,q)).$$

Clearly, $\Gamma \circ \gamma$ is increasing due to the composition of the two increasing functions namely: Γ and γ and $(\Gamma \circ \gamma)(t) < t$, for each $t > 0$ (due to increasingness of $\Gamma \circ \gamma$ and using the fact that $\lim_{n\to\infty}(\Gamma \circ \gamma)^n(t) = 0$, for each $t > 0$). Now, we need to show that $\Gamma \circ \gamma$ verify the property of ϑ_2. Let $\{t_n\}$ be a sequence in $[0,\infty)$ such that t_n is decreasing with $t_n \to t^+$ as $n \to \infty$. Then

$$\begin{aligned} \limsup_{t_n \to t^+}(\Gamma \circ \gamma)(t_n) &= \Gamma\big(\limsup_{t_n \to t^+} \varphi(t_n)\big) \\ &\leq (\Gamma \circ \varphi)(t) < t. \end{aligned}$$

Thus, $\Gamma \circ \gamma \in \vartheta$. This ends the proof. □

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