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# On the Two Categories of Modules

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**Abstract:** A new category equivalence is proved in this paper, involving the two distinct categories of modules, the covariant and the contravariant, respectively, released by Higgins and Mackenzie. The equivalence of the two categories is given when restricting to almost finitely generated projective modules and their allowed morphisms, defined in the paper. The equivalence is expressed by using generators. In particular, we obtain the well-known equivalence of the two categories of projective finitely generated modules; thus, our main result extends this classical one.

**Keywords:** category; module; covariant; contravariant; equivalence of categories; projective module; almost finitely generated

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## 1. Introduction

The category of modules is of special interest, in the general form (see, for example, [1,2]), or in special forms (see, for example, [3–10]).

A category of modules has the doubles $(A, M)$ as objects, where $A$ is a ring and $M$ is an $A$-module. The usual category of modules has morphisms of $(A, M)$ and $(A', M')$ as doubles $(\varphi, \psi)$, where $\varphi : A \to A'$ is a ring morphism and $\psi : M \to M'$ is an $A$-morphism. One can say it is a *covariant* category of modules $\overrightarrow{Mod}$, since there is another category of modules $\overleftarrow{Mod}$, called here a *contravariant* one, having the same objects, but the morphisms of $(A, M)$ and $(A', M')$ are the set of doubles $(\varphi, \psi)$, where $\varphi : A' \to A$ is a ring morphism and $\psi : M \to A \otimes_{A'} M'$ is an $A$-module morphism. It comes from the categories of vector bundles, where the morphisms and co-morphisms of vector bundles give different categories via the morphisms of modules (see [5]). More specifically, if one considers that, for a smooth vector bundle, $\pi : E \to B$ the ring of real functions $\mathcal{F}(B)$ and the module of sections $\Gamma(E)$, the usual morphism of vector bundles gives rise to a contravariant morphism for modules of sections, while a co-morphism of vector bundles gives rise to a covariant morphism for modules of sections. This fact applies, in general, to module morphisms. These categories of modules are emphasized in [6] (see also [11]).

There is a natural functor $F_{Cnv}$ from $\overleftarrow{Mod}$ to $\overrightarrow{Mod}$ (Proposition 2), while a converse functor can be given only when restricted to some subcategories. We relate the existence of such a functor to the existence of a module morphism $\Psi$, as in Proposition 3. For finitely generated projective modules, $\Psi$ is an isomorphism for every such module and an inverse functor can be considered.

We consider in the paper a more general case, of almost finitely generated projective modules that extend finitely generated projective modules.

Considering also allowed morphisms, we obtain two subcategories: $\overleftarrow{Mod}_{FP}$ and $\overrightarrow{Mod}_{FP}$, of almost finitely generated projective modules, when the morphism $\Psi$ is an isomorphism.

By using generators from condition (Proj), we explicit the morphisms related to two functors $F_{FP} : \overleftarrow{Mod}_{FP} \to \overrightarrow{Mod}_{FP}$ (given from the restriction) and $F_{PF} : \overrightarrow{Mod}_{FP} \to \overleftarrow{Mod}_{FP}$ (constructed using $\Psi$) by using several propositions in Section 3. An essential difference of these two functors, versus the (finitely generated projective) modules, is that the $A$-dual $M^{*_A}$ of a module $M$ is replaced by a restricted dual $M^{+_A}$ (called here a + dual), defined in the paper. In the case of a finitely generated projective module $(A, M)$, we have $M^{+_A} = M^{*_A}$ and the functor $F_{FP}$ is just the restriction of $F_{Cnv}$.

The equivalence of the functors $F_{FP}$ and $F_{PF}$ is known only in the case of finitely generated modules, stated, for example, in [12], without an effective proof.

We summarize the general result explicitly in Theorem 1, where we prove that the functors $F_{FP}$ and $F_{PF}$ are inverse each to other, giving an equivalence of the two categories of modules.

The aim of our paper involves two directions. Besides the equivalence stated above, we claim that the new category of almost finite modules can open many interesting problems involving the properties of finite generated modules and the way they can be extended in this new category. We give now some examples in this line.

A classification of almost finitely generated projective modules can be investigated as in [9].

An infinite sequence of the special kind modules are introduced in [13] for Characteristic Lie rings for Toda type 2 + 1 dimensional lattices, and the author proves that, for known integrable lattices, these modules are finitely generated. We launch the problem that almost finitely generated projective modules can be an alternative for the general case.

The importance of finitely generated projective modules in differential geometry is emphasized in [5,14]. The almost finitely generated projective module case can arise for a numerable direct sum of such objects.

How to involve almost finitely generated projective modules in Lie–Rinehart algebras and modules can be studied as in [4,15].

The authors in [16] show that every finitely generated studied module is associated by a sequence of invariant modules. A similar problem can be investigated in the case of almost finitely generated projective modules.

A relation with the finitly generated projective module can be studied using [17].

## 2. Basic Facts

Let $A$ be a ring (with a unit) and $M$ be an $A$-module; we say also $(A, M)$ is a *module*. We are concerned about the ring of a module, since it is an essential fact in what follows.

Modules over a given ring $A$ is a category $Mod_A$ where objects are $A$-modules and morphisms are the usual morphisms of $A$-modules (i.e., $A$-linear maps). When we are concerned with different rings, then we have to consider some larger categories. Notice that the ring morphisms are very restrictive; for example, if a ring morphism $\varphi : A' \to A$ exists, then the characteristic of $A$ divides the characteristic of $A'$.

First, it is a category $\overrightarrow{Mod}$ that the objects are $(A, M)$ as above and morphisms $\overrightarrow{Hom}((A, M), (A', M'))$ are the set of doubles $(\varphi, \psi)$, where $\varphi : A \to A'$ is a ring morphism and $\psi : M \to M'$ is an $A$-module morphism via $\varphi$. We say that $(\varphi, \psi)$ is a *Cov-morphism*.

Then, it is category $\overleftarrow{Mod}$ that the objects are $(A, M)$, as above, and morphisms $\overleftarrow{Hom}((A, M), (A', M'))$ are the set of doubles $(\varphi, \psi)$, where $\varphi : A' \to A$ is a ring morphism and $\psi : M \to A \otimes_{A'} M'$ is an $A$-module morphism. We say that $(\varphi, \psi)$ is a *Con-morphism*.

There is a natural functor $F_{Cnv} : \overleftarrow{Mod} \to \overrightarrow{Mod}$, defined as follows. We have $F_{Cnv}(A, M) = (A, M^{*_A})$, where $M^{*_A}$ is the $A$-module of $A$-linear maps $\omega : M \to A$; we use (also in the following) a single pair of parenthesis: $F_{Cnv}(A, M)$ instead of $F_{Cnv}((A, M))$. We say that $M^{*_A}$ is the $A$-dual of $M$. Let $(\varphi, \psi)$ be a Con-morphism.

In order to relate chain Cov-morphisms, we have the following true statement.

**Proposition 1.** *Let $(A'', M'')$ be a module, $\varphi : A' \to A$ and $\varphi' : A'' \to A'$ be some ring morphisms. Then, there is a canonical A-module isomorphism $A \otimes_{A''} M'' \xrightarrow{F} A \otimes_{A'} (A' \otimes_{A''} M'')$.*

**Proof.** The morphism $F$ is defined by $F(a \otimes_{A''} m'') = a \otimes_{A'} (1_{A'} \otimes_{A''} m'')$. It has as inverse $A \otimes_{A'} (A' \otimes_{A''} M'') \xrightarrow{F^{-1}} A \otimes_{A''} M''$, given by

$$F^{-1}\big(a \otimes_{A'} \big(a' \otimes_{A''} m''\big)\big) = a\varphi\big(a'\big) \otimes_{A''} m''.$$

□

There is an isomorphism, as given below; when $\varphi$ is a surjection, it is the identity.

**Corollary 1.** *Let $(A', M')$ be a module, $\varphi : A' \to A$ be a ring morphism and denote by $i : A_0 \to A$ the inclusion of the subring $A_0 = \varphi(A') \subset A$. Then, there is a canonical A-module isomorphism $A \otimes_{A'} M' \xrightarrow{F} A \otimes_{A_0} (A_0 \otimes_{A'} M')$.*

**Lemma 1.** *Let $(A', M')$ be a module and $\varphi : A' \to A$ be a ring morphism. Then, there are*

*An $A'$-module morphism $\Phi : (M')^{*_{A'}} \to (A \otimes_{A'} M')^{*_A}$;*
*An A-module morphism $\Psi : A \otimes_{A'} (M')^{*_{A'}} \to (A \otimes_{A'} M')^{*_A}$.*

**Proof.** We define

$$\Phi(\omega')\left(\sum_\alpha b_\alpha \otimes m'_\alpha\right) = \sum_\alpha b_\alpha \varphi(\omega'(m'_\alpha)) \tag{1}$$

and

$$\Psi\left(\sum_i a_i \otimes \omega'_i\right)\left(\sum_\alpha b_\alpha \otimes m'_\alpha\right) = \sum_{i,\alpha} a_i b_\alpha \varphi(\omega'_i(m'_\alpha)).$$

□

Notice that if $A = A'$ and $\varphi = 1_A$, then $\Phi$ and $\Psi$ are identity maps.
Corresponding to the following maps:

$\Phi$ given by Lemma 1;
$\psi^* : (A \otimes_{A'} M')^{*_A} \to M^{*_A}$, the $A$-dual of $\psi : M \to A \otimes_{A'} M'$,

we can consider

$$F_{Cnv}(\varphi, \psi) = (\varphi, \psi_1 = \psi^* \circ \Phi).$$

In addition, $F_{Cnv}(A, M) = (A, M^{*_A})$.
More specifically, if

$$\psi(m) = \sum_\alpha a_\alpha \otimes m'_\alpha,$$

then $\psi_1 : (M')^{*_{A'}} \to M^{*_A}$ is given by

$$\psi_1(\omega')(m) = \sum_\alpha a_\alpha \varphi(\omega'(m'_\alpha)).$$

The validity of the following statement follows by a direct verification.

**Proposition 2.** *There is a natural contravariant functor $F_{Cnv} : \overleftarrow{Mod} \to \overrightarrow{Mod}$.*

Related to application $\Psi$, we have the following statement.

**Proposition 3.** *An A-module map $\bar{\Psi} : (A \otimes_{A'} M')^{*_A} \to A \otimes_{A'} (M')^{*_{A'}}$ and an $A'$-module map $\psi : M' \to M$ give rise to an A-module map $\bar{\psi} : M^{*_A} \to A \otimes_{A'} (M')^{*_{A'}}$. In particular, if $\Psi$ is an isomorphism, then we can consider $\bar{\Psi} = \Psi^{-1}$.*

**Proof.** One can consider $\tilde{\psi} : A \otimes_{A'} M' \to M$, $\tilde{\psi}(a \otimes_{A'} m') = a\psi(m')$ and its dual $\tilde{\psi}^* : M^{*_A} \to (A \otimes_{A'} M')^{*_A}$. Then, we define $\bar{\psi} = \bar{\Psi} \circ \tilde{\psi}^*$. $\square$

However, there is not a natural functor $\overrightarrow{Mod} \to \overleftarrow{Mod}$; in order to obtain one, we have to consider the restrictions of these categories, imposing supplementary conditions, merely based on projective modules. For projective modules that are finitely generated, $\Psi$ is an isomorphism and one can consider functors between restricted categories giving an equivalence.

If $\mathcal{R} \subset \mathcal{R}ing$ is a subcategory of rings, then we can consider some full subcategories $\overleftarrow{Mod}_{\mathcal{R}}$ and $\overrightarrow{Mod}_{\mathcal{R}}$ of $\overleftarrow{Mod}$ and $\overrightarrow{Mod}$, respectively, considering the rings, morphisms and modules over rings according to $\mathcal{R}$; we call them *restricted categories to* $\mathcal{R}$. If we also restrict the modules, the subcategories can be not full subcategories.

In order to be more clear, we exemplify the above constructions for two cases: the category of scalars and the category of vector bundles.

Let us consider first the full subcategory (of rings) $\mathcal{S}cal$ (of scalars) that has as objects two rings (in fact fields), $\mathcal{O}b(\mathcal{S}cal) = \{\mathbb{R}, \mathbb{C}\}$, and morphisms $\mathcal{H}om_{\mathbb{R},\mathbb{R}} = \{1_{\mathbb{R}}\}$ , $\mathcal{H}om_{\mathbb{C},\mathbb{C}} = \{1_{\mathbb{C}}\}$, $\mathcal{H}om_{\mathbb{R},\mathbb{C}} = \{I\}$ and $\mathcal{H}om_{\mathbb{C},\mathbb{R}} = \oslash$, where $I$ is the natural inclusion of $\mathbb{R}$ in $\mathbb{C}$. The corresponding categories $\overleftarrow{Mod}_{\mathcal{S}cal}\mathcal{M}$ and $\overrightarrow{Mod}_{\mathcal{S}cal}\mathcal{M}$ have the same objects, the real and the complex vector spaces. The morphisms of real vector spaces are real linear ones, morphisms of complex vector spaces are also complex linear ones; there are not covariant morphisms from complex to reals, nor contravariant morphisms from reals to complex, the covariant morphisms from reals to complex are real linear maps, while contravariant morphisms from complex to reals are given by complex linear maps from complex to complexificated real linear spaces.

The map $\Phi$ and $\Psi$ looks as follows in this case:

- For $A = A'$ they are $1_A$;
- For $I : \mathbb{R} \to \mathbb{C}$, $\Phi : (M')^{*\mathbb{R}} \to (\mathbb{C} \otimes M')^{*\mathbb{C}}$, where $M'$ is a real vector space, associates to an $(M')^{*\mathbb{R}} \ni \omega' \to \Phi(\omega') \in (\mathbb{C} \otimes M')^{*\mathbb{C}}$, $\Phi(\omega')\big(m'_1 + im'_2\big) = \omega'(m'_1) + i\omega'(m'_2)$ (i.e., $\Phi(\omega')$ is the natural extension of $\omega'$ to the complex dual). The map $\Psi : \mathbb{C} \otimes (M')^{*\mathbb{R}} \to (\mathbb{C} \otimes M')^{*\mathbb{C}}$ is a natural $\mathbb{C}$-isomorphism that associates to $\omega'_1 + i\omega'_2$ the $\Psi\big(\omega'_1 + i\omega'_2\big)\big(m'_1 + im'_2\big) = \omega'_1(m'_1) - \omega'_2(m'_2) + i\big(\omega'_1(m'_2) + \omega'_2(m'_1)\big)$, which is a complex isomorphism.

More generally, if $K'$ is a field and $f_0 \in K'[X]$ is an irreducible polynomial, then $K = K'[X]/(f_0)$ is an extension of $K'$ and the only (ring) morphism $i' : K' \to K$ is the inclusion. There are not morphisms $\varphi : K \to K'$, because $\widehat{X}$ is a root of $f_0$ in $K$; thus, $\varphi\Big(\widehat{X}\Big)$ would be a root of $f_0$ in $K$, contradicting the irreducibility of $f_0$.

Another example is when the category $\mathcal{R}$ is indexed by smooth connected manifolds. In fact, this is the image of the contravariant functor from the category of connected manifolds to the category of rings, which associates to a smooth connected manifold $B$ the ring (in fact a real algebra) $\mathcal{F}(B)$ of smooth real functions on $B$. Instead of all modules, one can consider the projective and also finitely generated ones. According to the Serre–Swan theorem (see [18]), such a module over $\mathcal{F}(B)$ is isomorphic to the $\mathcal{F}(B)$-module of the sections $\Gamma(V)$ of a vector bundle $V$ over the base $B$. A covariant morphism $(f_0^*, f)$ of modules $(\mathcal{F}(B'), \Gamma(V'))$ and $(\mathcal{F}(B), \Gamma(V))$ corresponds to a co-morphism of vector bundles $(f_0, f)$, where $f_0 : B \to B'$ and $f : f_0^* V' \to V$ is an $f_0$ co-morphism of vector bundles, or $f : \Gamma(V') \to \Gamma(V)$ at the section form. A contravariant morphism $(f_0^*, f)$ of modules $(\mathcal{F}(B), \Gamma(V))$ and $(\mathcal{F}(B'), \Gamma(V'))$ corresponds to a morphism of vector bundles $(f_0, f)$, where $f_0 : B \to B'$ and $f : V \to V'$ is an $f_0$ morphism of vector bundles that can be regarded, as well, as the vector bundle morphism $f : V \to f^* V'$, or $f : \Gamma(V) \to \mathcal{F}(B) \otimes_{\mathcal{F}(B')} \Gamma(M')$ at the section form (see [5] for more details).

## 3. The Case of Projective Modules

We are concerned in this section with projective modules, mainly almost finitely generated, defined here. The result concerning the equivalences of categories of finitely generated projective modules is stated, for example in [12], without an effective proof. We prove an analogous result in a more general setting, of almost finitely generated projective modules.

According to the Dual basis lemma (see [2] (Proposition 4.7.5)), the projectivity of a module $(A, M)$ is equivalent to the condition:

(Proj) There exist a set of generators $\{m_i\}_{i\in\mathcal{I}} \subset M$ and a set $\{f_i\}_{i\in\mathcal{I}} \subset M^{*_A}$ such that $(\forall)m \in M$, the set $\{f_i(m)\}_{i\in\mathcal{I}}$ is finite and we have $m = \sum\limits_{i\in\mathcal{I}} f_i(m)m_i$.

If $\mathcal{I}$ is finite, one says the module $M$ is *finitely generated*.

Let $(A, M)$ be projective and condition (Proj) holds, where set $\mathcal{I}$ is not necessarily finite. Let us denote by $M^{+_A} \subset M^{*_A}$ the submodule of $\omega \in M^{*_A}$ such that the set $\{\omega(m_i)\}_{i\in\mathcal{I}}$ is finite. We say that the projective module $(A, M)$ is *almost finitely generated* if there are some $\{f_i\}_{i\in\mathcal{I}} \subset M^{+_A}$ in its definition. These elements of $M^{+_A}$ are not necessarily unique, but their existence will be considered implicitly in what follows.

Notice that a finitely generated module is almost finitely generated, but the converse fact is not true.

Indeed, consider the ring $A = \mathbb{R}[X]$ and the $\mathbb{R}[X]$-module $M = \mathbb{R}[X][Y] = \mathbb{R}[X, Y]$, seen as the module of sequences from $\mathbb{R}[X]$ having a finite support. Then, $M^{*_A}$ is the $A$-module of sequences from $\mathbb{R}[X]$, while $M^{+_A}$ is the module of sequences from $\mathbb{R}[X]$ with a finite support. Notice that this example can be extended to $\mathbb{R}[X_1, \ldots, X_n]$. We have also that $(\mathbb{R}[X], \mathbb{R}[X, Y])$ is an example of an almost finite projective module which is not finitely generated. Indeed, it is a free module, thus, it is projective; then, considering dual bases $m_n = X^n$ and $f_k(X^n) = \delta_k^n$, $n, k \in \mathbb{N}$, the assertion follows easily.

Another example of an almost finite projective module is the direct sum $M = \bigoplus\limits_{i\in I} M_i$ of finitely generated projective modules. When $I$ is not finite, we have again an example of an almost finitely generated projective module which it is not finitely generated.

**Proposition 4.** *If $M$ is projective and almost finitely generated, then $M^{+_A}$ is also projective and almost finitely generated, and the natural morphism $i^{++} : M \to M^{+_A+_A}$, $i^{++}(m)(\omega) = \omega(m) \overset{not.}{=} \tilde{m}(\omega)$, is an isomorphism.*

**Proof.** Using condition (Proj), we have $m = \sum\limits_{i\in\mathcal{I}} f_i(m)m_i$; thus, $\omega \in M^{+_A}$ has the form

$$\omega = \sum_{i\in\mathcal{I}} \omega(m_i) f_i, \tag{2}$$

where one can see $\omega(m_i) = \tilde{m}_i(\omega)$. Let us notice that the generators of $M^{+_A}$ are $\{f_i\}_{i\in\mathcal{I}}$ and the duals are $\{\tilde{m}_i\}_{i\in\mathcal{I}}$. Thus, condition (Proj) holds and the conclusion of the first assertion follows by the Dual basis lemma. For the second assertion, we can see that if $\eta \in M^{+_A+_A}$; then, we have

$$\eta = \sum_{i\in\mathcal{I}} \eta(f_i)\tilde{m}_i, \tag{3}$$

where $\tilde{m}_i = i^{++}(m_i)$. The correspondence $m \longleftrightarrow \eta$ of the isomorphism $i^{++}$ is given by $\eta(f_i) = f_i(m) (= \tilde{m}(f_i))$. If $\eta$ is given by (3), consider $m = \sum\limits_{i\in\mathcal{I}} \eta(f_i)m_i$; then, $\tilde{m} = i^{++}(m) = \sum\limits_{i\in\mathcal{I}} \eta(f_i)\tilde{m}_i = \eta$; thus, $i^{++}$ is a surjection. If $\tilde{m} = 0$, then it follows that

$$m = \sum_{i\in\mathcal{I}} f_i(m)m_i = \sum_{i\in\mathcal{I}} \tilde{m}(f_i)m_i = 0,$$

thus, $i^{++}$ is an injection. □

Since, for a finitely generated module $M$, one has $M^{+_A} = M^{*_A}$, we have the following corollary.

**Corollary 2.** *If $M$ is projective and finitely generated, then $M^{*_A}$ is also projective and finitely generated. In addition, the natural morphism $i^{**} : M \to M^{*_A *_A}$, $i^{**}(m)(\omega) = \omega(m) \overset{not.}{=} \tilde{m}(\omega)$, is an isomorphism.*

**Proposition 5.** *If $M'$ is projective and $\varphi : A' \to A$ is a ring morphism, then the $A$-module $A \otimes_{A'} M'$ is projective. If $M'$ is also almost finitely generated, then $A \otimes_{A'} M'$ and $(A \otimes_{A'} M')^{+_A}$ are projective and almost finitely generated as well.*

**Proof.** Consider $\{m'_i\}_{i \in \mathcal{I}} \subset M'$ and a set $\{f'_i\}_{i \in \mathcal{I}} \subset (M')^{+_{A'}}$ as in condition (Proj). Let $\{1_A \otimes_{A'} m'_i\}_{i \in \mathcal{I}} \subset A \otimes_{A'} M'$ and $\{\tilde{f}_i = \Psi(1_A \otimes_{A'} f'_i)\}_{i \in \mathcal{I}} \subset (A \otimes_{A'} M')^{+_A}$. Then, $a \otimes_{A'} m' = a \otimes_{A'} \sum\limits_{i \in \mathcal{I}} f'_i(m')m'_i = \sum\limits_{i \in \mathcal{I}} (a\varphi(f'_i(m')) \otimes_{A'} m'_i) = \sum\limits_{i \in \mathcal{I}} (\tilde{f}_i(a \otimes_{A'} m') \otimes_{A'} m'_i) = \sum\limits_{i \in \mathcal{I}} \tilde{f}_i(a \otimes_{A'} m') \cdot (1_A \otimes_{A'} m'_i)$.

Since the last sum has a finite number of terms, it follows that $A \otimes_{A'} M'$ is projective.

If $M'$ is almost finitely generated, then $A \otimes_{A'} M'$ is also almost finitely generated, since for every $i \in \mathcal{I}$, the set

$$\left\{\tilde{f}_i\left(1_A \otimes_{A'} m'_j\right)\right\}_{j \in \mathcal{I}} = \left\{1_A \otimes_{A'} f_i\left(m'_j\right)\right\}_{j \in \mathcal{I}}$$

is finite. Using Proposition 4, the final conclusion follows. □

Lemma 1 has the following form for the + duals.

**Lemma 2.** *Let $(A, M)$ and $(A', M')$ be two modules that are projective and almost finitely generated, and $\varphi : A' \to A$ be a morphism of rings. Then, there are*

*An $A'$-module morphism $\Phi : (M')^{+_{A'}} \to (A \otimes_{A'} M')^{+_A}$;*

*An $A$-module morphism $\Psi : A \otimes_{A'} (M')^{+_{A'}} \to (A \otimes_{A'} M')^{+_A}$.*

**Proposition 6.** *If $M'$ and $M$ are projective and almost finitely generated, then the $A$-morphism $\Psi$ given by Lemma 2 is an isomorphism.*

**Proof.** From Propositions 4 and 5, it follows that $\{\tilde{f}_i = \Psi(1_A \otimes_{A'} f'_i)\}_{i \in \mathcal{I}} \subset (A \otimes_{A'} M')^{+_A}$ is a set of generators. The linear extension of the correspondence of generators $\tilde{f}_i \to 1_A \otimes_{A'} f'_i$, $i \in \mathcal{I}$ gives a well-defined inverse $\Psi^{-1}$ of $\Psi$. □

Let us notice that:

- All the sums below (for $i \in \mathcal{I}$ or $i' \in \mathcal{I}'$) are finite ones but, for the sake of simplicity, we do not specify it distinctly;
- The product of two matrices that all lines and all columns have finite support has the same property.

We explicit now the Cov-morphisms and Con-morphisms for almost finitely generated projective modules, using generators and corresponding matrices. In this way, we define the allowed morphisms that give the morphisms of these two categories.

For almost finitely generated projective modules $(A', M')$ and $(A, M)$, where $\varphi : A' \to A$ and $\psi : M' \to M$, a Cov-morphism $(\varphi, \psi)$ reads as follows. Consider, by condition (Proj) for $M'$ and $M$: $\{m'_{i'}\}_{i'=\overline{1,k'}} \subset M'$, $\{f'_{i'}\}_{i'=\overline{1,k'}} \subset (M')^{+_{A'}}$ and $\{m_i\}_{i'=\overline{1,k}} \subset M$, $\{f_i\}_{i'=\overline{1,k}} \subset M^{+_A}$. Denote $\psi_{i'i} = f_i(\psi(m_{i'}))$; thus,

$$\psi(m_{i'}) = \sum_{i \in \mathcal{I}} \psi_{i'i} m_i. \tag{4}$$

We have

$$\psi(m') = \psi\left(\sum_{i'\in\mathcal{I}'} f'_{i'}(m')m'_{i'}\right) = \sum_{i'\in\mathcal{I}'}\sum_{i\in\mathcal{I}} \varphi(f'_{i'}(m'))\psi_{i'i}m_i. \tag{5}$$

We say that $(\varphi, \psi)$ is *allowed* if all the lines and columns of the matrix $\{\psi_{i'i}\}_{i\in\mathcal{I},i'\in\mathcal{I}'}$ have finite support.

Using the above data for two almost finitely generated projective modules as above, a Con-morphism $(\varphi, \psi)$, reads as follows.

We have that $\varphi : A' \to A$ and $\psi : M \to A \otimes_{A'} M'$; then, $\psi(m_i) = \sum\limits_{\alpha=1}^{n} a_{i\alpha} \otimes_{A'} m''_\alpha = \sum\limits_{\alpha=1}^{n}\sum\limits_{i'\in\mathcal{I}'} a_{i\alpha} \otimes_{A'} (a''_{i'\alpha} m_{i'}) = \sum\limits_{i'\in\mathcal{I}'}\left(\sum\limits_{\alpha=1}^{n} a_{i\alpha}\varphi(a''_{i'\alpha})\right) \otimes_{A'} m_{i'} = \sum\limits_{i'\in\mathcal{I}'} \psi_{i'i} \otimes_{A'} m_{i'}$; thus

$$\psi(m_i) = \sum_{i'\in\mathcal{I}'} \psi_{i'i} \otimes_{A'} m_{i'}. \tag{6}$$

We have

$$\psi(m) = \psi\left(\sum_{i\in\mathcal{I}} f_i(m)m_i\right) = \sum_{i'\in\mathcal{I}'}\left(\sum_{i\in\mathcal{I}} f_i(m)\psi_{i'i}\right) \otimes_{A'} m_{i'}. \tag{7}$$

According to Proposition 4, the modules $(A, M^{+_A})$ and $\left(A', M'^{+_{A'}}\right)$ are almost finitely generated projective modules.

Consider now a Cov-morphism $(\varphi, \psi)$ of almost finitely generated projective modules $(A, M)$ and $(A', M')$; thus, $\varphi : A' \to A$ and $\psi : M' \to M$.

We say that $(\varphi, \psi)$ is *allowed* if all the lines and columns of the matrix $\{\psi_{i'i}\}_{i\in\mathcal{I},i'\in\mathcal{I}'}$ have finite support.

Considering only allowed morphisms, we obtain the following result by a straightforward verification.

**Proposition 7.** *The restrictions of the objects to the finitely generated projective modules and of the morphisms to the allowed ones in the categories $\overrightarrow{Mod}$ and $\overleftarrow{Mod}$ are the objects and the morphisms of two new categories, denoted by $\overrightarrow{Mod}_{FP}$ and $\overleftarrow{Mod}_{FP}$, respectively.*

Let us define, in this case, $F_{FP}(A, M) = (A, M^{+_A})$ and

$$F_{FP}(\varphi, \psi) = (\varphi, \bar{\psi}),$$

where $\bar{\psi}$ is given as in Proposition 3. Specifically, for a given $\psi$, consider $\tilde{\psi} : A \otimes_{A'} M' \to M$, $\tilde{\psi}(a \otimes_{A'} m') = a\psi(m')$ and its dual $\tilde{\psi}^+ : M^{+_A} \to (A \otimes_{A'} M')^{+_A}$; then, $\bar{\psi} = \bar{\Psi} \circ \tilde{\psi}^*$.

According to the proof of Proposition 4, an $\omega \in M^{+_A}$ has the form of (2). Consider $\psi$ having the form (5). Using (6), we can consider $\bar{\psi}$ having the form

$$\bar{\psi}(\omega) = \sum_{i'\in\mathcal{I}'}\left(\sum_{i\in\mathcal{I}} \omega(m_i)\psi_{i'i}\right) \otimes_{A'} f_{i'}. \tag{8}$$

**Proposition 8.** *There is a natural contravariant functor $F_{PF} : \overrightarrow{Mod}_{FP} \to \overleftarrow{Mod}_{FP}$.*

**Proof.** Consider a Cov-morphism $(A, M) \underset{(\varphi,\psi)}{\longrightarrow} (A', M')$, then

$$F_{PF}(A, M) = \left(A, M^+\right),\ F_{PF}(A', M') = \left(A', (M')^+\right),\ F_{PF}(\varphi, \psi) = (\varphi, \bar{\psi}),$$

where $\left(A', (M')^+\right) \xrightarrow{(\varphi,\bar{\psi})} (A, M^+)$, and $\bar{\psi} : (M')^+ \to A' \otimes_{A'} M^+$ is given by (8). It is easy to see that if we consider a Cov-morphism $(A', M') \underset{(\varphi',\psi')}{\longrightarrow} (A'', M'')$, then $F_{PF}((\varphi, \psi) \circ (\varphi', \psi')) = (\varphi \circ \varphi', \bar{\psi} \circ \bar{\psi}')$, where $\bar{\psi} \circ \bar{\psi}'$ is constructed using the isomorphism given by Proposition 1. □

In order to prove that the restriction of the functor considered in Proposition 2 and the functor considered in Proposition 8 give the equivalent categories $\overrightarrow{Mod}_{FP}$ and $\overleftarrow{Mod}_{FP}$, respectively, we also use generators.

**Proposition 9.** *The functor $F_{Cnv} : \overleftarrow{Mod} \to \overrightarrow{Mod}$ induces a natural contravariant functor $F_{FP} : \overleftarrow{Mod}_{FP} \to \overrightarrow{Mod}_{FP}$.*

**Proof.** We check that the functor $F_{FP}$ gives a natural correspondence involving the same matrix $(\psi_{i'i})$.

Indeed, given the Con-morphism $(\varphi, \psi)$, where $\varphi : A' \to A$ and $\psi : M \to A \otimes_{A'} M'$, thus $\psi(m)$ is given by (7), where the matrix $(\psi_{i'i})_{i\in\mathcal{I},i'\in\mathcal{I}'}$ is given by (6). Then

$$F_{FP}(A, M) = (A, M^+), F_{FP}(A', M') = \left(A', (M')^+\right), F_{FP}(\varphi, \psi) = (\varphi, \tilde{\psi}),$$

where $\tilde{\psi} : (M')^+ \to M^+$ is given by

$$\tilde{\psi}(\omega')(m) = \sum_{i'\in\mathcal{I}'} \left(\sum_{i\in\mathcal{I}} f_i(m)\psi_{i'i}\right) \varphi(\omega'(m_{i'})). \tag{9}$$

and the matrix $(\psi_{i'i})_{i\in\mathcal{I},i'\in\mathcal{I}'}$ is given as in (8).

If we consider a Con-morphism $(A', M') \underset{(\varphi',\psi')}{\longrightarrow} (A'', M'')$, then

$$F_{FP}((\varphi, \psi) \circ (\varphi', \psi')) = (\varphi \circ \varphi', \tilde{\psi} \circ \tilde{\psi}'),$$

where the composition $\tilde{\psi} \circ \tilde{\psi}' : M^{++} \to (M'')^{++}$ is constructed as follows:

$$(\tilde{\psi} \circ \tilde{\psi}')(\omega'')(m) = \sum_{i'\in\mathcal{I}'} \left(\sum_{i\in\mathcal{I}} f_i(m)\psi_{i''i}\right) \varphi \circ \varphi'(\omega''(m_{i''})),$$

where, using the definition (6) of the matrix $(\psi_{i'i})_{i\in\mathcal{I},i'\in\mathcal{I}'}$, we have

$$\psi(m_i) = \sum_{i'\in\mathcal{I}'} \psi_{i'i} \otimes_{A'} m_{i'}, \psi'(m'_{i'}) = \sum_{i'\in\mathcal{I}'} \psi_{i''i'} \otimes_{A''} m_{i''}.$$

Using the isomorphism given by Proposition 1, it follows that

$$(\psi_{i''i})_{i\in\mathcal{I},i''\in\mathcal{I}''} = (\varphi(\psi_{i''i'}))_{i'\in\mathcal{I}',i''\in\mathcal{I}''} \cdot (\psi_{i'i})_{i\in\mathcal{I},i'\in\mathcal{I}'}.$$

□

Let $\mathcal{C}$ and $\mathcal{C}'$ be two categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is an *equivalence* of the two categories if there is an *inverse* functor $G : \mathcal{C}' \to \mathcal{C}$, i.e., $F \circ G \approxeq 1_{\mathcal{C}'}$ and $G \circ F \approxeq 1_{\mathcal{C}}$, where $\approxeq$ indicates a category equivalence (or isomorphism). If the functor $F$ is contravariant, then $G$ is assumed contravariant as well. This is the case we are dealing with below.

The natural correspondences involving the same matrices also give the following true statement.

**Theorem 1.** *The functor $F_{PF} : \overrightarrow{Mod}_{FP} \to \overleftarrow{Mod}_{FP}$ in Proposition 8 and the functor $F_{FP} : \overleftarrow{Mod}_{FP} \to \overrightarrow{Mod}_{FP}$ in Proposition 9 give an equivalence of the categories $\overleftarrow{Mod}_{FP}$ and $\overrightarrow{Mod}_{FP}$.*

**Proof.** In order to prove that the categories $\overleftarrow{Mod}_{FP}$ and $\overrightarrow{Mod}_{FP}$ are equivalent, we prove that functors $F_{PF}$ and $F_{FP}$ are inverse each to other. Let us denote by $(A,M) \overset{(\varphi,\psi)}{\longrightarrow} (A',M')$ and $(A,M) \underset{(\varphi,\psi)}{\longrightarrow} (A',M')$ a morphism in the categories $\overrightarrow{Mod}_{FP}$ and $\overleftarrow{Mod}_{FP}$, respectively.

We consider condition (Proj) for the modules $(A,M)$ and $(A',M')$ with $\{m_i\}_{i\in\mathcal{I}} \subset M$, $\{f_i\}_{i\in\mathcal{I}} \subset M^{+_A}$ and $\{m'_{i'}\}_{i'\in\mathcal{I}'} \subset M$, $\{f'_{i'}\}_{i'\in\mathcal{I}'} \subset M^{+_A}$, respectively. In order to simplify computations, we denote $m'_{i'}$ and $f'_{i'}$ as $m_{i'}$ and $f_{i'}$, respectively; in addition, similar conventions hold for other cases below.

We check now the equivalence using the constructions performed in the proof of Proposition 9. We use also Formulas (8) and (9).

Consider a Cov-morphism $(A,M) \underset{(\varphi,\psi)}{\longrightarrow} (A',M')$, then $F_{PF}(A,M) = (A,M^+)$, $F_{PF}(A',M') = (A',(M')^+)$ and $F_{PF}(\varphi,\psi) = (\varphi,\bar{\psi})$, as above. Then, $F_{FP} \circ F_{PF}(A,M) = (A,M^{++})$, $F_{FP} \circ F_{PF}(A',M') = \left(A',(M')^{++}\right)$ and $F_{FP} \circ \circ F_{PF}(\varphi,\psi) = \left(\varphi,\tilde{\bar{\psi}}\right)$, where $\tilde{\bar{\psi}} : M^{++} \to (M')^{++}$

$$\tilde{\bar{\psi}}(\tilde{m})(\omega') = \sum_{i'\in\mathcal{I}'}\sum_{i\in\mathcal{I}} \tilde{m}_{i'}(\omega')\psi_{i'i}\varphi(\tilde{m}(f_i))$$

has the same action as $\psi$, via the canonical isomorphisms coming from Proposition 4, $i^{++} : M \to M^{++}$ and $(i')^{++} : M' \to (M')^{++}$. Thus, $F_{FP} \circ F_{PF} \approxeq 1_{\overrightarrow{Mod}_{FP}}$.

Consider now a Con-morphism $(A,M) \overset{(\varphi,\psi)}{\longrightarrow} (A',M')$.

Then, $F_{PF} \circ F_{FP}(A,M) = (A,M^{++})$, $F_{PF} \circ F_{FP}(A',M'') = \left(A',(M')^{++}\right)$ and $F_{PF} \circ F_{FP}(\varphi,\psi) = \left(\varphi,\bar{\bar{\psi}}\right)$, where $\bar{\bar{\psi}} : M^{++} \to A \otimes_{A'} (M')^{++}$

$$\bar{\bar{\psi}}(\tilde{m}) = \sum_{i'\in\mathcal{I}'}\sum_{i\in\mathcal{I}} \tilde{m}(f_i)\psi_{i'i} \otimes_{A'} \tilde{m}_{i'}$$

has the same action as $\psi$, via the canonical isomorphism coming from Proposition 4. As in the previous case, $F_{PF} \circ F_{FP} \approxeq 1_{\overleftarrow{Mod}_{FP}}$. □

**Corollary 3.** *The restrictions of categories $\overrightarrow{Mod}_{FP}$ and $\overleftarrow{Mod}_{FP}$ to the finitely generated projective modules give two categories $\overrightarrow{Mod}_{FGP}$ and $\overleftarrow{Mod}_{FGP}$, respectively. The restrictions of functors $F_{PF}$ and $F_{FP}$ in Theorem 1 to $\overrightarrow{Mod}_{FGP}$ and $\overleftarrow{Mod}_{FGP}$ give a equivalence of these categories.*

## 4. Conclusions

The equivalence of the two categories of modules (covariant and contravariat) in the case of finitely generated projective modules $\overrightarrow{Mod}_{FGP}$ and $\overleftarrow{Mod}_{FGP}$ is well-known. In this paper, we extended the above result to the more general case of the categories of almost finitely generated projective modules, $\overrightarrow{Mod}_{FP}$ and $\overleftarrow{Mod}_{FP}$, defined in the paper. In the general case of the module categories, this equivalence does not take place (there are no natural functors to give the equivalence of the two general categories $\overrightarrow{Mod}$ and $\overleftarrow{Mod}$). That is why we considered the restrictions $\overrightarrow{Mod}_{FP}$ and $\overleftarrow{Mod}_{FP}$ of the two general categories so that there are natural functors that ensure their equivalence.

We claim that the study of both categories of modules can give interesting results in the future. The non-equivalence of the two categories of modules, in the general case, can give different contributions. According to the results in the paper, the two categories of modules can have other equivalent subcategories, other than finitely generated projective modules, but using some different functors. A study of the (pre-)shaves of such modules can be performed as, for example, in [12].

We expect that the new category of almost finite modules, defined in the paper, will raise many interesting problems involving the properties of finite generated modules and the way they can be extended in this new category.

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