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Hopf Bifurcation and Control of a Fractional-Order Delay Stage Structure Prey-Predator Model with Two Fear Effects and Prey Refuge

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Abstract: A generalized delay stage structure prey-predator model with fear effect and prey refuge is considered in this paper via introducing fractional-order and fear effect induced by immature predators. Hopf bifurcation and control of this system are investigated though regarding the delay as the parameter. Firstly, by using the method of linearization and Laplace transform, the roots of the characteristic equation of the linearized system of the original system are discussed, and the sufficient conditions for the system exhibits an unstable state of symmetrical periodic oscillation (Hopf bifurcation) are explored. Secondly, a linear delay feedback controller is added to the system to increase the stability domain successfully. Thirdly, numerical simulations are performed to validate the theoretical analysis, and the various impacts on the dynamical behavior of the system occurring by fear effects, prey refuge, and each fractional-order are illustrated, respectively. Furthermore, the influence of feedback gain on the bifurcation critical point is analyzed. Finally, an analysis based on the results and in-depth research about this system under the biological background is stated in the conclusion.

Keywords: stage structure; fear effect; fractional-order delay system; Hopf bifurcation; feedback control

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1. Introduction

The series of prey-predator models are paid widespread attention because they reflect the ecological phenomenon existing in the real world generally. The dynamic behavior of those models are investigated in-depth and a large number of valuable results have been obtained in the past few decades [1–4]. To keep the ecological balance, it is necessary to increase the survival rate of the prey in some ecosystems. The prey refuge is a suitable method to protect the prey population [5]. Prey-predator models with prey refuge are brought into focus and many worthy results are obtained [6–8].

Generally, the growth of many species are divided into two stages, immaturity and maturity, and the characteristics and behaviors of the different stage are quite distinguishing. For example, the predatory ability of mature predators is stronger than immature predators. In the past few years, researchers found that the prey-predator model with multi-stage structure is more reasonable than the one-stage model for describing the relationship between species [9–11].

Fear effect is another factor that impacts the dynamic behaviors of prey-predator models besides stage structure. For instance, the fear to predators could affect the birth rate of the prey, thereby affecting the population density of the prey [12]. Investigations have shown that the fear induced by the predator has an even greater effect on the prey than the direct killing [13–15]. Biologists discovered that many prey species have the inborn ability to identify predators in addition to acquired learning [16]. This means the fear effect may come from not only mature predators but also immature.

In many fields such as physics, mechanics, biology, chemistry, communication engineering, and control engineering, the evolution of a system not only depends on the current state but also be influenced by the past state. Therefore, it is of great significance to consider the delay in the system [17,18]. Because the population density of predator depends on the consumption of prey in the past partly, it is widely recognized for considering the delay in the prey-predator model. The effect of delay on the system has two sides: on the one hand, the enormous delay of the predator may lead to the extinction of the predator and the prey [19], on the other hand, the proper time delay can also increase the stability of the dynamical system [20,21].

Fractional calculus theory considers the mathematical properties and applications of differential and integral of arbitrary order. Fractional calculus operators have non-locality and are suitable for describing real-life materials and processes with memory and genetic properties [22–25], which are often ignored in classical integer-order models. Fractional calculus has a long history [26,27]. However, in the past three centuries, due to the computational complexity and the lack of practical background, the development of fractional calculus was very slow. Until the past few years, fractional calculus received extensive attention with the rapid development of computer technology and became an active research field. In biological models, the densities of species are in flux. It is related to both the current moment and some past state of the species. Those properties of the biological model coincide with the "memory" characteristic of fractional differentiation. Since each species has different degree of dependence on the past, incommensurate fractional models are more realistic.

Stability and bifurcation are important issues in the research of fractional-order differential equation models. Hopf bifurcation is widely focused on because it reflects the properties of periodic solutions near the steady-state of nonlinear systems [28–31]. However, the appearance of Hopf bifurcation is also a sign that the system appears periodic oscillation and enters an unstable state from a stable state. The Hopf bifurcation control of fractional systems has received more and more attention [32–34]. In Ding et al. [33], the dynamics of a fractional-order memristor-based chaotic system with delay were investigated. The authors confirmed that the delay feedback controller was valid in controlling chaos and Hopf bifurcation in the controlled system. Zheng et al. [34] proposed a linear delay feedback controller to put off the onset of Hopf bifurcation for a fractional-order paddy ecosystem. They observed that the delay could affect the dynamics of the system heavily, and the feedback gain and the fractional-order had significant impacts on the control effect.

For some integer-order delay prey-predator models, the Hopf bifurcation and control of their corresponding fractional-order models have never been studied in-depth, for example, the Crowley–Martin prey-predator model with fear effect and prey refuge [35]. Otherwise, many research considered that only mature predators could give rise to fear effect [35,36]. In fact, the fear effect may come from both mature and immature predators [16]. Inspired by these ideas, the incommensurate fractional-order and the fear factor induced by the immature predators are introduced to the Crowley–Martin prey-predator model [35] in this paper, and Hopf bifurcation and control of the generalized model are investigated by theoretical and numerical method.

The main contributions of this paper include: (1) By adding the fear factor induced by immature predators and introducing the fractional orders, an integer-order delay stage structure prey-predator model with fear effect and prey refuge is generalized. The existence conditions of the coexistence equilibrium point of the proposed system are deduced. (2) The conditions of emergence of Hopf bifurcation for the generalized system are determined. In other words, the critical value of delay that the system switches from asymptotical stability to symmetric periodic oscillation is deduced. (3) A linear delay feedback controller is added to put off the emergence of the Hopf bifurcation for the proposed system, and the stability domain of the system has increased. (4) From an ecological point of view, the effects of two fear factors, prey refuge, three fractional-orders, and the feedback gain to the bifurcation critical value of delay are analyzed in virtue of numeric simulations, respectively.

The organization of this paper is as follows. In Section 2, some definitions of fractional calculus and some basic knowledge are presented. In Section 3, the mathematical model is generalized, and the existence of the coexistence equilibrium point of the model is analyzed. In Section 4, Hopf bifurcation of the generalized system and the control of bifurcation of the controlled system are explored, respectively. In Section 5, numerical simulations are performed to further illustrate our theoretical results, and the influences of two fear effects, prey refuge, and fractional-order to the bifurcation of the system are given. Finally, a necessary conclusion explains the results and in-depth research about this system under the biological background.

2. Preliminary Knowledge

In this section, some basic definitions about fractional-order calculus and Hopf bifurcation used in the following sections are given.

Definition 1 (Riemann–Liouville Fractional Integral [37,38])**.** *Fractional integral of order α for the function $f(t):[a,\infty)\to R$ can be expressed as follows:*

$$ {}_aI_t^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau,\quad t>a, \tag{1}$$

where $\alpha,a\in R,\alpha>0$, $\Gamma(\cdot)$ is Eulers Gamma function.

Definition 2 (Caputo Fractional Derivative [37,38])**.** *The Caputo fractional-order derivative is defined by*

$$ {}_a^CD_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\quad n-1<\alpha\leq n, \tag{2}$$

where $f(t)\in C^n([a,\infty),R)$. In particularly, if $0<\alpha\leq 1$, $a=0$, Equation (2) can be written as:

$$ {}_0^CD_t^{\alpha}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{f^{'}(\tau)}{(t-\tau)^{\alpha}}d\tau,\quad 0<\alpha\leq 1,t>0. \tag{3}$$

Definition 3 (Laplace Transform of Fractional Derivation [39])**.** *The Laplace transform of Caputo fractional derivation of order α $(n-1<\alpha\leq n)$ for the function $f(t)\in C^n([a,\infty),R)$ is*

$$ L\Big\{{}_a^C\mathcal{D}_t^{\alpha}f(t);s\Big\}=s^{\alpha}F(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(a), \tag{4}$$

where $F(s)$ is the Laplace transform of $f(t)$, and $f^k(a)$ $(k=0,1,...,n-1)$ are the initial conditions. Obviously, if $f^k(a)=0$ for $k=0,1,...,n-1$, Equation (4) can be written as

$$ L\Big\{{}_a^C\mathcal{D}_t^{\alpha}f(t);s\Big\}=s^{\alpha}F(s). \tag{5}$$

Definition 4 ([40])**.** *Consider the following n-dimensional fractional-order system with delay*

$$ {}_a^CD_t^{\alpha}u_i(t)=f_i(u_1(t),\cdots,u_n(t);\tau),i=1,2,\cdots,n, \tag{6}$$

where $0<\alpha\leq 1$ and the delay $\tau\geq 0$. System (6) undergos Hopf bifurcation at the equilibrium $u^*=(u_1^*,u_2^*,...,u_n^*)$ when $\tau=\tau_0$ if the following three conditions are satisfied:

C1: All the eigenvalues λ_j $(j=1,2,...,n)$ of the coefficient matrix J of the linearized system of Equation (6) with $\tau=0$ satisfy $|arg(\lambda_j)|>\frac{\alpha\pi}{2}$.

C2: The characteristic equation of the linearized system of Equation (6) has a pair of purely imaginary roots $s=\pm i\omega_0$ when $\tau=\tau_0$.

C3: $Re\Big[\frac{ds(\tau)}{d\tau}\Big]|_{(\tau=\tau_0,\omega=\omega_0)}>0$, where $Re[\cdot]$ denotes the real part of the complex number.

Remark 1. (C3) *in Definition 4 is the so-called transversality condition.*

3. Model Description

Xiao et al. [41] studied the following Beddington–DeAngelis prey-predator model with stage structure and prey refuge

$$\begin{cases} \dot{x}(t) = x(t)(r - cx(t)) - \frac{\alpha(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+by_2(t)}, \\ \dot{y_1}(t) = \frac{\alpha\beta(1-m)x(t-\tau)y_2(t-\tau)}{1+a(1-m)x(t-\tau)+by_2(t-\tau)} - ny_1(t) - d_1y_1(t), \\ \dot{y_2}(t) = ny_1(t) - d_2y_2(t), \end{cases} \tag{7}$$

where $x(t)$, $y_1(t)$, and $y_2(t)$ represent the population densities of prey, immature predator, and mature predator at time t, respectively. r is the birth rate of prey. d_1 and d_2 represent the natural mortality of immature predators and mature predators, respectively. c is the intraspecific competition rate of the prey. $m \in [0,1)$ is the prey refuge rate, and n represents the proportion of immature predators that grow into mature predators. a and b refer to the processing time of the mature predator and the strength of the interaction. α and β refer to the capture rate of the prey and the conversion rate of nutrients into the production of predator species, respectively. τ represents the delay due to the gestation of the mature predator. The authors investigated the local stability of the equilibrium point of the system and the influence of prey refuge on the densities of predator species and prey species.

In 2021, Wang and Hu [35] improved this model and discussed a Crowley–Martin prey-predator model with fear effect and prey refuge as follows

$$\begin{cases} \dot{x}(t) = \frac{rx(t)}{1+ky_2(t)} - d_0x(t) - cx^2(t) - \frac{\alpha(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+by_2(t)+ab(1-m)x(t)y_2(t)}, \\ \dot{y_1}(t) = \frac{\beta\alpha(1-m)x(t-\tau)y_2(t-\tau)}{1+a(1-m)x(t-\tau)+by_2(t-\tau)+ab(1-m)x(t-\tau)y_2(t-\tau)} - ny_1(t) - d_1y_1(t), \\ \dot{y_2}(t) = ny_1(t) - d_2y_2(t), \end{cases} \tag{8}$$

where k is the prey's fear factor induced by mature predators and d_0 represents the natural mortality of prey. The existence and stability of the equilibrium point of the system Equation (8) have been established in [35].

Considering that the evolution of prey-predators system related to both the current moment and some past state of the species, and each species has different degree of dependence on the past, incommensurate fractional-orders are added to the system (8). Otherwise, inspired by the reference [16], the prey's fear effect is thought about not only mature predator but also immature predator. Thus, the system Equation (8) is generalized as the follows:

$$\begin{cases} D^{q_1}x(t) = \frac{rx(t)}{1+k_1y_1(t)+k_2y_2(t)} - d_0x(t) - cx^2(t) - \frac{\alpha(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+by_2(t)+ab(1-m)x(t)y_2(t)}, \\ D^{q_2}y_1(t) = \frac{\beta\alpha(1-m)x(t-\tau)y_2(t-\tau)}{1+a(1-m)x(t-\tau)+by_2(t-\tau)+ab(1-m)x(t-\tau)y_2(t-\tau)} - ny_1(t) - d_1y_1(t), \\ D^{q_3}y_2(t) = ny_1(t) - d_2y_2(t), \end{cases} \tag{9}$$

where $q_i \in (0,1]$ $(i = 1,2,3)$ is fractional-order. k_1 and k_2 are the prey's fear factors induced by immature predators and mature predators, respectively.

Obviously, the system has a zero equilibrium point $E_0 = (0,0,0)$. When $r > d_0$, the system has a predator-extinction equilibrium point $E_1 = (\frac{r-d_0}{c}, 0, 0)$.In fact, we are interested in the stability and stability switch at the coexistence equilibrium point of the system (9). Thus, it is necessary to find the conditions in that system (9) has a positive value equilibrium point.

Lemma 1. *When the following four conditions are satisfied, the system* (9) *has a unique coexistence equilibrium point* $E^* = (x^*, y_1^*, y_2^*)$ $(x^* > 0, y_1^* > 0, y_2^* > 0)$

(H1) $c \geq a(r - d_0)(1 - m)$;

(H2) $r > d_0$;

(H3) $n\beta\alpha > d_2 a(n+d_1)$;
(H4) $d_2 c(n+d_1) < (r-d_0)(n\beta\alpha - d_2 a(n+d_1))(1-m)$.

Proof. In fact, system (9) exists an coexistence equilibrium point $E^* = (x^*, y_1^*, y_2^*)$ means the following equation set has positive solution

$$\begin{cases} \frac{r}{1+k_1y_1+k_2y_2} - d_0 - cx - \frac{\alpha(1-m)y_2}{1+a(1-m)x+by_2+ab(1-m)xy_2} = 0, \\ \frac{\beta\alpha(1-m)xy_2}{1+a(1-m)x+by_2+ab(1-m)xy_2} - ny_1 - d_1y_1 = 0, \\ ny_1 - d_2y_2 = 0. \end{cases} \tag{10}$$

It is easy to obtain $y_1 = \frac{d_2}{n}y_2$, and substituting y_1 into the first and second equations of Equation (10), one has

$$\begin{cases} \frac{rn}{n+k_1d_2y_2+nk_2y_2} - d_0 - cx - \frac{\alpha(1-m)y_2}{1+a(1-m)x+by_2+ab(1-m)xy_2} = 0, \\ \frac{\beta\alpha(1-m)x}{1+a(1-m)x+by_2+ab(1-m)xy_2} - d_2 - \frac{d_1d_2}{n} = 0. \end{cases} \tag{11}$$

Let

$$\begin{cases} F(x,y_2) := \frac{rn}{n+k_1d_2y_2+nk_2y_2} - d_0 - cx - \frac{\alpha(1-m)y_2}{1+a(1-m)x+by_2+ab(1-m)xy_2}, \\ G(x,y_2) := \frac{\beta\alpha(1-m)x}{1+a(1-m)x+by_2+ab(1-m)xy_2} - d_2 - \frac{d_1d_2}{n}, \end{cases} \tag{12}$$

if curve $F(x,y_2) = 0$ intersects curve $G(x,y_2) = 0$ in the first quadrant, then system (10) has positive solution. According to the first equation of Equation (12), one has

$$\frac{dy_2}{dx} = -\frac{F_x}{F_{y_2}} = \frac{\frac{\alpha(1-m)^2y_2a(1+by_2)}{(1+a(1-m)x+by_2+ab(1-m)xy_2)^2} - c}{\frac{r(k_2+\frac{k_1d_2}{n})}{(1+k_2y_2+\frac{k_1d_2y_2}{n})^2} + \frac{\alpha(1-m)}{(1+x(1-m)a)(by_2+1)^2}}. \tag{13}$$

In the first quadrant, if $\frac{dy_2}{dx} < 0$ then

$$\frac{\alpha(1-m)^2y_2a(1+by_2)}{(1+a(1-m)x+by_2+ab(1-m)xy_2)^2} < c. \tag{14}$$

From $F(x,y_2) = 0$, one can get

$$\frac{\alpha(1-m)^2y_2a(1+by_2)}{(1+a(1-m)x+by_2+ab(1-m)xy_2)^2} = \frac{(\frac{rn}{n+k_1d_2y_2+nk_2y_2} - d_0 - cx)a(1-m)}{1+a(1-m)x}. \tag{15}$$

Substituting Equation (15) into inequation Equation (14), one has

$$\frac{l_1xy_2 + l_2x + l_3y_2 + l_4}{(1+a(1-m)x)(n+nk_2y_2+k_1d_2y_2)} > 0, \tag{16}$$

where

$$l_1 = 2ac(1-m)(k_1d_2+nk_2),\ l_2 = 2acn(1-m),$$
$$l_3 = (k_1d_2+nk_2)(c+ad_0(1-m)),\ l_4 = n(c+a(d_0-r)(1-m)).$$

For all $x > 0$, $y_2 > 0$, if $l_4 = n(c+a(d_0-r)(1-m)) > 0$, that is

$$c > a(r-d_0)(1-m),$$

then the inequation Equation (16) is established, and it means $\frac{dy_2}{dx} < 0$.

If $y_2 = 0$, then $x^{(1)} = \frac{r-d_0}{c}$, and when $x^{(1)} > 0$, there is $r > d_0$.

When $x = 0$, there is

$$Ay_2^2 + By_2 + C = 0, \tag{17}$$

where

$$\begin{aligned} A &= (d_2k_1 + k_2n)((1-m)\alpha + bd_0), \\ B &= ((1-m)\alpha + d_0(b+k_2) - br)n + d_0d_2k_1, \\ C &= n(d_0 - r). \end{aligned}$$

Obviously, if

$$r > d_0,$$

Equation (17) has only one positive root ${y_2}^{(1)}$.

According to second equation of Equation (12), in the first quadrant, one has

$$\frac{dy_2}{dx} = -\frac{G_x}{G_{y_2}} = \frac{\beta\alpha(1-m)(1+by_2)}{\beta\alpha b(1-m)(1+a(1-m)x)x} > 0. \tag{18}$$

If $y_2 = 0$, then $x^{(2)} = \frac{d_2(n+d_1)}{(1-m)(n\beta\alpha - d_2a(n+d_1))}$. When

$$n\beta\alpha > d_2a(n+d_1),$$

$$d_2c(n+d_1) < (r-d_0)(n\beta\alpha - d_2a(n+d_1))(1-m),$$

it is easy to obtain $0 < x^{(2)} < x^{(1)}$.

Therefore, if (***H1***)$-$(***H4***) are satisfied, $F(x, y_2)$ and $G(x, y_2)$ have a unique intersection $(x^*, {y_2}^*)$ in the first quadrant, and then ${y_1}^* > 0$ can be obtained by the third equation of Equation (10). □

4. Hopf Bifurcation Analysis and Control of System (9)

We are interested in the dynamical properties at the coexistence equilibrium point $(x^*, {y_1}^*, {y_2}^*)$ of system (9). In this section, the Hopf bifurcation and control are analyzed in details.

4.1. Hopf Bifurcation Analysis of System (9)

Using the transformation $u(t) = x(t) - x^*, v(t) = y_1(t) - {y_1}^*, w(t) = y_2(t) - {y_2}^*$, and linearizing the converted system, we can have

$$\begin{cases} D^{q_1}u(t) = a_{11}u(t) + a_{12}v(t) + a_{13}w(t), \\ D^{q_2}v(t) = a_{21}u(t-\tau) + a_{22}v(t) + a_{23}w(t-\tau), \\ D^{q_3}w(t) = a_{31}v(t) + a_{32}w(t), \end{cases} \tag{19}$$

where

$$\begin{aligned} &a_{11} = \tfrac{r}{1+k_1{y_1}^*+k_2{y_2}^*} - d_0 - 2cx^* - \tfrac{\alpha(1-m){y_2}^*}{(1+x^*(1-m)a)^2(b{y_2}^*+1)}, \quad a_{12} = -\tfrac{rx^*k_1}{(1+k_1{y_1}^*+k_2{y_2}^*)^2}, \\ &a_{13} = -\tfrac{rx^*k_2}{(1+k_1{y_1}^*+k_2{y_2}^*)^2} - \tfrac{\alpha(1-m)x^*}{(1+x^*(1-m)a)(b{y_2}^*+1)^2}, \quad a_{21} = \tfrac{\beta\alpha(1-m){y_2}^*}{(1+x^*(1-m)a)^2(b{y_2}^*+1)}, \\ &a_{22} = -n - d_1, \quad a_{23} = \tfrac{\beta\alpha(1-m)x^*}{(1+x^*(1-m)a)(b{y_2}^*+1)^2}, \quad a_{31} = n, \quad a_{32} = -d_2. \end{aligned}$$

According to Definition 4, we analyze the conditions of emergence of Hopf bifurcation for system (9) at the coexistence equilibrium point.

Lemma 2. *When $\tau = 0$, all the eigenvalues λ_j $(j = 1, 2, 3)$ of the coefficient matrix of the linearized system* (19) *have negative real parts, if the following assumptions* (***H5***)$-$(***H7***) *hold,*

(***H5***) $a_{11} + a_{22} + a_{32} < 0$,

(***H6***) $-a_{11}a_{22}a_{32} + a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{12} > 0$,

(***H7***) $(-a_{11} - a_{22} - a_{32})(a_{11}a_{22} + a_{11}a_{32} - a_{12}a_{21} + a_{22}a_{32} - a_{23}a_{31}) > (-a_{11}a_{22}a_{32} + a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{31})$.

Proof. When $\tau = 0$, the coefficient matrix A of the system (19) is

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{31} & a_{32} \end{pmatrix},$$

the corresponding characteristic equation of A is

$$\begin{aligned} &\lambda^3 + (-a_{11} - a_{22} - a_{32})\lambda^2 + (a_{11}a_{22} + a_{11}a_{32} - a_{12}a_{21} \\ &+ a_{22}k_{32} - a_{23}k_{31})\lambda + (-a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{31}) = 0. \end{aligned} \tag{20}$$

If the assumptions (***H5***)$-$(***H7***) are true, according to the Routh-Hurwitz criterion, all the characteristic roots of Equation (20) have negative real parts, that is, the characteristic root $\lambda_j (j = 1, 2, 3)$ satisfies $|arg(\lambda_j)| > \frac{q\pi}{2}$ $(q = max(q_1, q_2, q_3))$. □

Taking Laplace transform [42] to system (19), one has

$$\begin{cases} s^{q_1} F_1(s) = a_{11}F_1(s) + a_{12}F_2(s) + a_{13}F_3(s), \\ s^{q_2} F_2(s) = a_{21}e^{-s\tau}F_1(s) + a_{22}F_2(s) + a_{23}e^{-s\tau}F_3(s), \\ s^{q_3} F_3(s) = a_{31}F_2(s) + a_{32}F_3(s). \end{cases} \tag{21}$$

The characteristic equation of Equation (21) is

$$\begin{vmatrix} s^{q_1} - a_{11} & -a_{12} & -a_{13} \\ -a_{21}e^{-s\tau} & s^{q_2} - a_{22} & -a_{23}e^{-s\tau} \\ 0 & -a_{31} & s^{q_3} - a_{32} \end{vmatrix} = 0. \tag{22}$$

Equation (22) can be written as

$$U_1(s) + U_2(s)e^{-s\tau} = 0, \tag{23}$$

where

$$\begin{aligned} U_1(s) =& s^{q_1+q_2+q_3} - a_{22}s^{q_1+q_3} - a_{32}s^{q_1+q_2} + a_{22}a_{32}s^{q_1} - a_{11}s^{q_2+q_3} \\ &+ a_{11}a_{22}s^{q_3} + a_{11}a_{32}s^{q_2} - a_{11}a_{22}a_{32}, \\ U_2(s) =& - a_{23}a_{31}s^{q_1} - a_{12}a_{21}s^{q_3} + a_{11}a_{23}a_{31} - a_{13}a_{21}a_{31} + a_{12}a_{21}a_{32}. \end{aligned}$$

In order to find the critical value of delay that the stability of system (19) switches, one can assume

(***A1***) $|U_1(0)| < |U_2(0)|$;

Assume that $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$ $(\omega > 0)$ is a root of Equation (23), substituting it into Equation (23) and separating the real and imaginary parts, we have

$$\begin{cases} \alpha_1 \cos\omega\tau + \alpha_2 \sin\omega\tau = -\alpha_3, \\ \alpha_2 \cos\omega\tau - \alpha_1 \sin\omega\tau = -\alpha_4, \end{cases} \tag{24}$$

where

$$\alpha_1 = Re(U_2(i\omega)),\ \alpha_2 = Im(U_2(i\omega)),\ \alpha_3 = Re(U_1(i\omega)),\ \alpha_4 = Im(U_1(i\omega)),$$

the exact expressions of α_i $(i = 1, 2, 3, 4)$ are defined in Appendix A.

Solving Equation (24), it is easy to obtain the following result,

$$\begin{cases} \sin\omega\tau = \frac{\alpha_1\alpha_4 - \alpha_2\alpha_3}{\alpha_1{}^2 + \alpha_2{}^2} = \frac{Im(U_1(i\omega)\cdot\overline{U_2(i\omega)})}{|U_2(i\omega)|^2}, \\ \cos\omega\tau = -\frac{\alpha_3\alpha_1 + \alpha_2\alpha_4}{\alpha_1{}^2 + \alpha_2{}^2} = -\frac{Re(U_1(i\omega)\cdot\overline{U_2(i\omega)})}{|U_2(i\omega)|^2}, \end{cases} \tag{25}$$

By Equation (23), we can get

$$|U_1(i\omega)| = |U_2(i\omega)|.$$

It is easy to know that

$$\begin{aligned}|U_2(i\omega)| - |U_1(i\omega)| &\le |U_2(i\omega)| - \left(|(i\omega)^{q_1+q_2+q_3}| - |U_1(i\omega) - (i\omega)^{q_1+q_2+q_3}|\right) \\ &= -(i\omega)^{q_1+q_2+q_3} + |U_2(i\omega)| + |U_1(i\omega) - (i\omega)^{q_1+q_2+q_3}|.\end{aligned}$$

Therefore

$$\lim_{\omega\to+\infty}(|U_2(i\omega)| - |U_1(i\omega)|) = -\infty.$$

According to assumption (***A1***), the equation $|U_1(i\omega)| = |U_2(i\omega)|$ has at least one positive root.

Combining with the formula $\sin^2\omega\tau + \cos^2\omega\tau = 1$, the value of ω can be solved. Without loss of generality, we assume that all positive roots are ω_k $(k = 1, 2, ..., K)$. By substituting each ω_k into Equation (25) and the corresponding critical value of τ_k can be obtained (for the exact mathematical expressions, please refer to [40]). In relation to the actual meaning of delay, we only pay attention to the value of τ when Hopf bifurcation occurs firstly, so the bifurcation critical value of delay is

$$\tau_0 = min\{\tau_k\}, k = 0, 1, 2, ...K, \tag{26}$$

the critical value of frequency corresponding to τ_0 is denoted as ω_0.

According to Definition 4, we need to verify the transversality condition at the critical point (τ_0, ω_0). Thus, it is necessary to give the following hypothesis

(***H8***) $\frac{A_1B_1+A_2B_2}{B_1^2+B_2^2} > 0$,

the expressions of A_i, B_i $(i = 1, 2)$ is in Appendix B.

Lemma 3. *If the hypothesis (**H8**) holds, let $s(\tau) = \gamma(\tau) + i\omega(\tau)$ be the root of Equation (23) near $\tau = \tau_j$ satisfying $\gamma(\tau_j) = 0, \omega(\tau_j) = \omega_0$, then the following transversality conditions established*

$$Re\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau=\tau_0,\omega=\omega_0)} > 0. \tag{27}$$

Proof. According to the implicit function derivation rule, deriving τ on both sides of Equation (23) respectively, one gets

$$U_1'(s)\frac{ds}{d\tau} + U_2'(s)e^{-s\tau}\frac{ds}{d\tau} + U_2(s)e^{-s\tau}(-\tau\frac{ds}{d\tau} - s) = 0,$$

where $U_i'(s)$ is the derivative of $U_i(s)$ $(i = 1, 2)$. Hence,

$$\frac{ds}{d\tau} = \frac{A(s)}{B(s)}, \tag{28}$$

where

$$\begin{aligned}A(s) &= U_2(s)se^{-s\tau}, \\ B(s) &= U_1'(s) + U_2'(s)e^{-s\tau} - U_2(s)\tau e^{-s\tau}.\end{aligned}$$

It can be deduced from Equation (28) that

$$Re\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau=\tau_0,\omega=\omega_0)} = \frac{A_1B_1 + A_2B_2}{B_1^2 + B_2^2}, \tag{29}$$

where A_1, A_2 are the real and imaginary parts of $A(s)$, B_1, B_2 are the real and imaginary parts of $B(s)$. In terms of (***H8***), one has

$$Re\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau=\tau_0,\omega=\omega_0)} > 0.$$

The proof of Lemma 3 is finished. □

Based on Lemmas 2 and 3, we can get the following theorem:

Theorem 1. *If (**H1**)−(**H8**) and (**A1**) hold, then the coexistence equilibrium point of system (9) is asymptotically stable when $\tau \in [0, \tau_0)$, and system (9) undergoes Hopf bifurcation at the coexistence equilibrium point when $\tau = \tau_0$. τ_0 is the critical value of delay defined by Equation (26).*

4.2. Hopf Bifurcation Control of System (9)

In this section, we focus on the control of Hopf bifurcation of system (9). From an ecological point of view, it is more effective to control the stability of the system by regulating the population density of mature predators than by regulating the population density of immature predators, as the mature predators play a dominant role in the ecosystem. A linear delay feedback controller $L[y_2(t) - y_2(t-\tau)]$ is added to the third equation of system (9) to control the emergence of Hopf bifurcation, i.e., the stability domain is regulated by controlling the population density of mature predators. The controlled system is

$$\begin{cases} D^{q_1}x(t) = \frac{rx(t)}{1+k_1y_1(t)+k_2y_2(t)} - d_0x(t) - cx^2(t) - \frac{\alpha(1-m)x(t)y_2(t)}{1+a(1-m)x(t)+by_2(t)+ab(1-m)x(t)y_2(t)}, \\ D^{q_2}y_1(t) = \frac{\beta\alpha(1-m)x(t-\tau)y_2(t-\tau)}{1+a(1-m)x(t-\tau)+by_2(t-\tau)+ab(1-m)x(t-\tau)y_2(t-\tau)} - ny_1(t) - d_1y_1(t), \\ D^{q_3}y_2(t) = ny_1(t) - d_2y_2(t) + L[y_2(t) - y_2(t-\tau)], \end{cases} \tag{30}$$

where $L \in R$ is the feedback control gain.

Making a transformation $u(t) = x(t) - x^*$, $v(t) = y_1(t) - y_1^*$, $w(t) = y_2(t) - y_2^*$, and doing the linearization at zero equilibrium point to Equation (30), the linearization system of controlled system (30) can be achieved

$$\begin{cases} D^{q_1}u(t) = a_{11}u(t) + a_{12}v(t) + a_{13}w(t), \\ D^{q_2}v(t) = a_{21}u(t-\tau) + a_{22}v(t) + a_{23}w(t-\tau), \\ D^{q_3}w(t) = a_{31}v(t) + a_{32}w(t) + L[w(t) - w(t-\tau)], \end{cases} \tag{31}$$

where a_{ij} $(i,j = 1,2,3)$ is same as Equation (19).

Taking Laplace transform to system (31), one can get the characteristic equation as following

$$\begin{vmatrix} s^{q_1} - a_{11} & -a_{12} & -a_{13} \\ -a_{21}e^{-s\tau} & s^{q_2} - a_{22} & -a_{23}e^{-s\tau} \\ 0 & -a_{31} & s^{q_3} - a_{32} - L + Le^{-s\tau} \end{vmatrix} = 0. \tag{32}$$

Obviously, Equation (32) is equivalent to

$$V_1(s) + V_2(s)e^{-s\tau} + V_3e^{-2s\tau} = 0, \tag{33}$$

where

$$\begin{aligned} V_1(s) =& s^{q_1+q_2+q_3} - a_{22}s^{q_1+q_3} - a_{32}s^{q_1+q_2} - a_{11}s^{q_2+q_3} - Ls^{q_1+q_2} + La_{22}s^{q_1} \\ &+ La_{11}s^{q_2} + a_{22}a_{32}s^{q_1} + a_{11}a_{32}s^{q_2} + a_{11}a_{22}s^{q_3} - a_{11}a_{22}a_{32} - La_{11}a_{22}, \\ V_2(s) =& Ls^{q_1+q_2} - La_{22}s^{q_1} - La_{11}s^{q_2} - a_{23}a_{31}s^{q_1} - a_{12}a_{21}s^{q_3} + La_{11}a_{22} \\ &+ La_{12}a_{21} + a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{31}, \\ V_3 =& -La_{12}a_{21}. \end{aligned}$$

Multiplying $e^{s\tau}$ on both sides of Equation (33), one gets

$$V_1(s)e^{s\tau} + V_2(s) + V_3e^{-s\tau} = 0. \tag{34}$$

In order to find the critical value of delay that the stability of system (30) switches, one can assume

(***A2***) $|V_1(0)| - |V_2(0) + V_3| < 0$;

Let $s = i\omega = \omega(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})(\omega > 0)$ as a root of Equation (34), substituting it into Equation (34) and separating the real and imaginary parts, one has:

$$\begin{cases} (\beta_1 + \beta_3)\cos\omega\tau - \beta_2\sin\omega\tau = -\gamma_1, \\ \beta_2\cos\omega\tau + (\beta_1 - \beta_3)\sin\omega\tau = -\gamma_2, \end{cases} \tag{35}$$

where

$$\beta_1 = Re(V_1(i\omega)),\ \beta_2 = Im(V_1(i\omega)),\ \beta_3 = V_3, \gamma_1 = Re(V_2(i\omega)), \gamma_2 = Im(V_2(i\omega)),$$

the exact expressions of β_i $(i = 1, 2, 3)$ and γ_i $(i = 1, 2)$ are given in Appendix C.

By Equation (34), it can get

$$|V_1(i\omega)| = |V_2(i\omega) + V_3e^{-i\omega\tau}|.$$

Set $G(\omega) = |V_1(i\omega)| - |V_2(i\omega) + V_3e^{-i\omega\tau}|$, then

$$\begin{aligned} G(\omega) &= |V_1(i\omega) - (i\omega)^{q_1+q_2+q_3} + (i\omega)^{q_1+q_2+q_3}| - |V_2(i\omega) + V_3e^{-i\omega\tau}| \\ &\geq |(i\omega)^{q_1+q_2+q_3}| - |V_1(i\omega) - (i\omega)^{q_1+q_2+q_3}| - |V_2(i\omega) + V_3e^{-i\omega\tau}| \\ &= \omega^{q_1+q_2+q_3} - |V_1(i\omega) - (i\omega)^{q_1+q_2+q_3}| - |V_2(i\omega) + V_3e^{-i\omega\tau}|. \end{aligned}$$

Therefore,

$$\lim_{\omega\to+\infty} G(\omega) = \infty.$$

According to assumption (***A2***), the equation $G(\omega) = 0$ has at least one positive root.

Same as Section 4.1, we can obtain the minimum bifurcation critical point (τ_0^*, ω_0^*) of the controlled system (30).

It is necessary to get the transversality condition, thus we make the following assumption

(***H9***) $\frac{C_1D_1+C_2D_2}{D_1^2+D_2^2} > 0$,

the expressions of C_i, D_i $(i = 1, 2)$ are in Appendix D.

Lemma 4. *If the hypothesis (**H9**) holds, let $s(\tau) = \delta(\tau) + i\omega(\tau)$ be the root of Equation (33) near $\tau = \tau_j$ satisfying $\delta(\tau_j) = 0, \omega(\tau_j) = \omega_0^*$, then the following transversality condition satisfied*

$$Re\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau=\tau_0^*, \omega=\omega_0^*)} > 0.$$

Proof. Deriving on both sides of Equation (33) for the variable τ, one gets

$$V_1'(s)\frac{ds}{d\tau} + V_2'(s)e^{-s\tau}\frac{ds}{d\tau} + V_2(s)e^{-s\tau}(-\tau\frac{ds}{d\tau} - s) + V_3e^{-2s\tau}(-2\tau\frac{ds}{d\tau} - 2s) = 0,$$

where $V_i'(s)$ is the derivative of $V_i(s)$ $(i = 1, 2)$. Hence,

$$\frac{ds}{d\tau} = \frac{C(s)}{D(s)}, \tag{36}$$

where

$$C(s) = s[V_2(s)e^{-s\tau} + 2V_3e^{-2s\tau}],$$
$$D(s) = V_1'(s) + [V_2'(s) - V_2(s)\tau]e^{-s\tau} - 2V_3\tau e^{-2s\tau}.$$

It can be deduced from Equation (36) that

$$Re\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau=\tau_0^*,\omega=\omega_0^*)} = \frac{C_1D_1 + C_2D_2}{D_1^2 + D_2^2}, \tag{37}$$

where C_1, C_2 are the real and imaginary parts of $C(s)$, D_1, D_2 are the real and imaginary parts of $D(s)$.

Obviously, if hypothesis (***H9***) is true, then the transversality condition is true. The proof of Lemma 4 is finished. □

Based on Lemmas 2 and 4, we can get the following theorem:

Theorem 2. *If (**H1**)−(**H7**), (**H9**) and (**A2**) hold, the coexistence equilibrium point of controlled system (30) is asymptotically stable when $\tau \in [0, \tau_0^*)$, controlled system (30) undergoes Hopf bifurcation at the coexistence equilibrium point when $\tau = \tau_0^*$.*

Remark 2. *The influence of the linear delay feedback controller on the stability domain of system (9) is direct because the formula for calculating the delay critical value includes L (see Appendixes C and D).*

5. Numerical Simulations

In this section, we use the Adama-Bashforth-Moulton predictive correction method [43] to validate the feasibility of theoretical analysis.

5.1. Example 1

For better comparison, the parameters of system (9) refer to the literature [35]: $r = 0.34$, $a = 1$, $b = 1$, $c = 0.3$, $\alpha = 1$, $\beta = 0.8$, $m = 0.05$, $n = 0.8$, $d_0 = 0.1$, $d_1 = 0.1$, $d_2 = 0.1$, $k_1 = 0.05$, $k_2 = 0.1$, and fractional-orders are chosen as $q_1 = 0.98$, $q_2 = 0.92$, $q_3 = 0.95$, then system (9) is

$$\begin{cases} D^{0.98}x(t) = \frac{0.3x(t)}{1+0.05y_1(t)+0.1y_2(t)} - 0.1x(t) - 0.3x^2(t) - \frac{0.95x(t)y_2(t)}{1+0.95x(t)+y_2(t)+0.95x(t)y_2(t)}, \\ D^{0.92}y_1(t) = \frac{0.76x(t-\tau)y_2(t-\tau)}{1+0.95x(t-\tau)+y_2(t-\tau)+0.95x(t-\tau)y_2(t-\tau)} - 0.9y_1(t), \\ D^{0.95}y_2(t) = 0.8y_1(t) - 0.1y_2(t). \end{cases} \tag{38}$$

It can be verified that (***H1***)−(***H8***) and (***A1***) hold. It is easy to obtain that the coexistence equilibrium point is $(x^*, y_1^*, y_2^*) = (0.2130, 0.0246, 0.1968)$, the bifurcation critical point is $(\tau_0 = 35.533, \omega_0 = 0.038858)$, and transversality condition $Re\left[\frac{ds(\tau)}{d\tau}\right]|_{(\tau_0=35.533,\omega_0=0.038858)} = 0.0000899288 > 0$. By means of Theorem 1, the coexistence equilibrium point (x^*, y_1^*, y_2^*) is asymptotically stable when $\tau \in [0, \tau_0)$, and Hopf bifurcation occurs when $\tau \geq \tau_0$. These results are illustrated in Figures 1 and 2 by choosing $\tau = 34.99$ and $\tau = 36.01$, respectively. Moreover, we can see from Figure 2 that the system is in an unstable state of symmetrical periodic oscillation.

In what follows, the influences on bifurcation critical value of delay coursed by the fear factors k_1, k_2, and the prey refuge rate m are discussed through numerical simulations, respectively. Furthermore, numerical simulations show that the fractional-order q_i $(i = 1, 2, 3)$ has different effects on the stability region of system (38).

Case 1. The influences of fear factors on the stability region

In this paper, the fear factor is considered as two cases caused by mature predators and immature predators, respectively, i.e., k_1 and k_2. We are interested in which one makes an important role in the stability of system (38). When all parameters and fractional-orders remain unchanged except k_1, let k_1 increases continuously, we can get different bifurcation critical points (τ_0, ω_0) presented in Table 1:

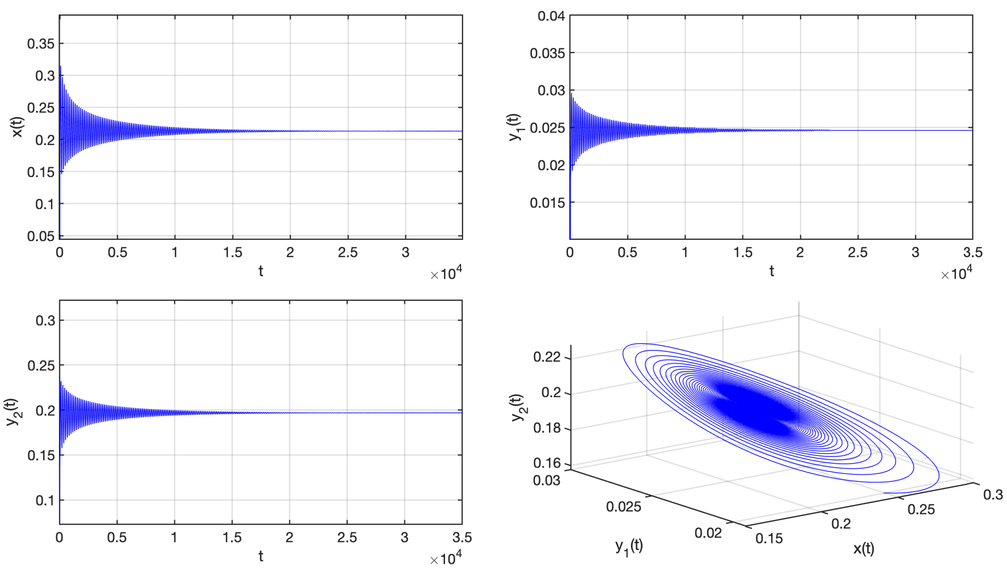

Figure 1. Waveform plots and phase portrait of system (38) with $\tau = 34.99 < \tau_0 = 35.533$.

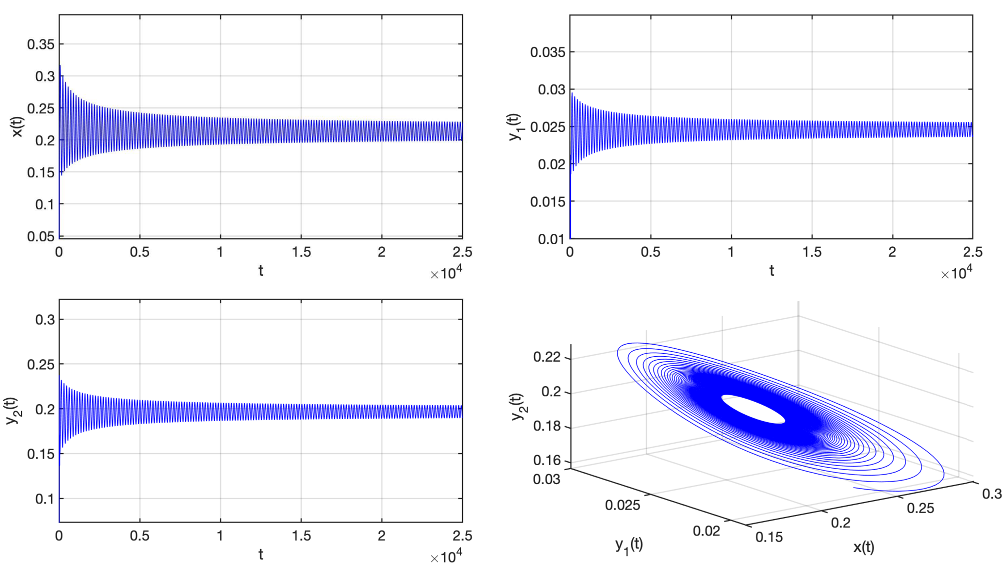

Figure 2. Waveform plots and phase portrait of system (38) with $\tau = 36.01 > \tau_0 = 35.533$.

Next, in a similar way, remain all parameters and fractional-orders unchanged except k_2, and let k_2 increases continuously, we also get different bifurcation critical points (τ_0, ω_0) presented in Table 2:

It can be viewed in Figure 3 that the occurrence of the Hopf bifurcation is put off slightly as k_1 increases. However, the relationship between the critical value of delay τ_0 and fear factor k_2 shows a U-shaped curve. What calls for special attention is when k_2 increases from 0.1 to 0.6, τ_0 descends by 9%. From the perspective of ecology, if the fear of predators is greater, the instability of the system will be more obvious, and the critical value of delay will decrease. In other words, the occurrence of Hopf bifurcation is advanced and the stability state is broken. However, when the intensity of the fear effect reaches a certain level, the limit effect will be produced, and the critical value of delay will decrease more and more weakly. On the other hand, Figure 3 shows that the fear effect on the stability region of the system mainly comes from mature predators.

Table 1. The relationship between k_1 and τ_0.

k_1	ω_0	τ_0	Transversality Condition
0.05	0.038858	35.533	0.0000899288
0.1	0.038850	35.608	0.0000900460
0.2	0.038833	35.757	0.0000902623
0.3	0.038815	35.906	0.0000904556
0.4	0.038796	36.054	0.0000906266
0.5	0.038776	36.202	0.0000907763
0.6	0.038755	36.348	0.0000909056
0.7	0.038732	36.494	0.0000910151
0.8	0.038709	36.640	0.0000911057
0.9	0.038685	36.784	0.0000911781
0.99	0.038663	36.914	0.0000912283

Table 2. The relationship between k_2 and τ_0.

k_2	ω_0	τ_0	Transversality Condition
0.1	0.038858	35.533	0.0000899288
0.2	0.039765	34.178	0.0000966910
0.3	0.040387	33.304	0.0001012262
0.4	0.040800	32.752	0.0001040106
0.5	0.041056	32.429	0.0001054188
0.6	0.041191	32.276	0.0001057446
0.7	0.041229	32.255	0.0001052195
0.8	0.041192	32.338	0.0001040259
0.9	0.041092	32.507	0.0001023092
0.99	0.040958	32.721	0.0001004132

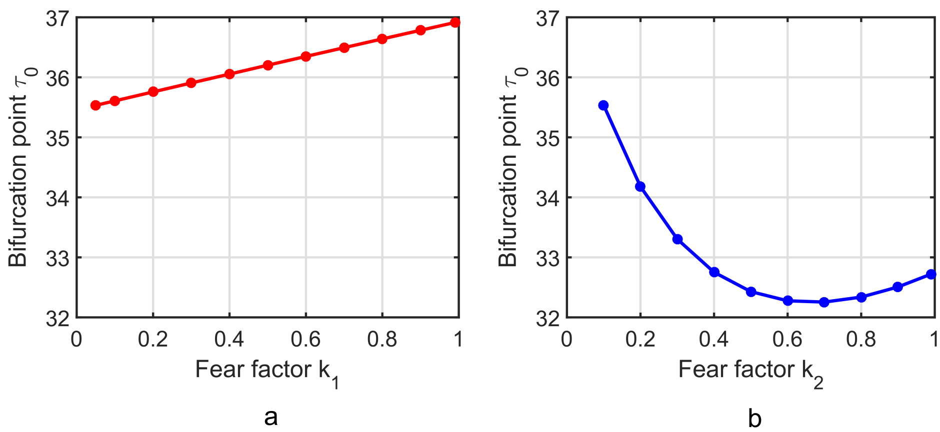

Figure 3. (**a**) $k_2 = 0.1$, τ_0 varies with the increase of k_1. (**b**) $k_1 = 0.05$, τ_0 varies with the increase of k_1.

Case 2. The influence of prey refuge rate on the stability region

Prey refuge is an effective measure of ecosystem regulation. We are interested in how the prey refuge rate m influences the stability of system (38). Same as Case 1, remain all the parameters and fractional-orders unchanged except m, let m increases continuously, we can get different bifurcation critical points (τ_0, ω_0) presented in Table 3.

It can be noticed easily from Table 3 and Figure 4 that when m increases from 0.05 to 0.12, the critical value of delay τ_0 increases from 35.533 to 262.802, i.e., the stability region of the system becomes 7.4 times the original. That is to say, to system (38), m has extremely influence on stability at the coexistence

Table 3. The relationship between m and τ_0.

m	ω_0	τ_0	Transversality Condition
0.05	0.038858	35.533	0.0000899288
0.06	0.036354	40.349	0.0000686956
0.07	0.033570	46.673	0.0000499478
0.08	0.030428	55.381	0.0000338849
0.09	0.026812	68.210	0.0000207164
0.10	0.022526	89.220	0.0000106555
0.11	0.017200	130.850	0.0000038985
0.12	0.009920	262.802	0.0000005387

equilibrium point, and it is a useful method to keep the ecosystem (38) stable development by changing the degree of prey refuge.

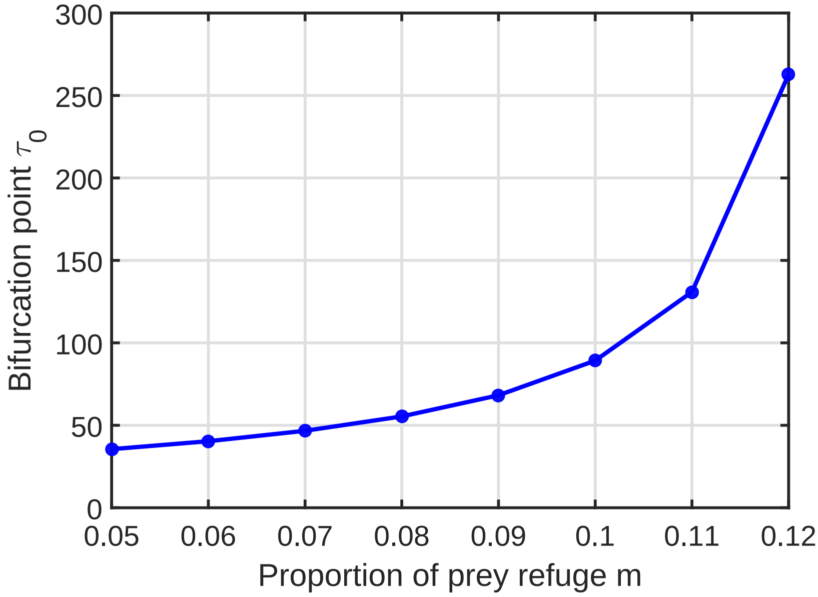

Figure 4. τ_0 varies with the increase of m.

Case 3. The influence of fractional-orders on the stability region

Let $q_1 = 1, q_2 = 1, q_3 = 1$ and $k_1 = 0$, the system (38) becomes an integer-order system corresponding to system (8). The coexistence equilibrium point is $(0.2131, 0.0247, 0.1974)$, and the bifurcation critical point is $(\tau_0 = 22.77, \omega_0 = 0.0501)$, which is consistent with the results in Wang [35]. If $q_1 = 0.98, q_2 = 0.92, q_3 = 0.95$ and $k_1 = 0$, the bifurcation critical point is $(\tau_0 = 35.46, \omega_0 = 0.0389)$. These results validate that when other parameters of the model are consistent, the delay critical value of the emergence of Hopf bifurcation in the fractional-order system is obviously larger than that in the integer-order system, and the stability domain of the system expands from $[0, 22.77)$ to $[0, 35.46)$. Otherwise, Tables 4–6 further illustrate that fractional-order can effectively expand the stability domain of the system.

Next, we want to know which fractional-order has the obvious effect on the stability of the system (38). The main idea is to keep two fractional-orders unchanged and vary the third one. Tables 4–6 show the different bifurcation critical points (τ_0, ω_0) along with q_i $(i = 1, 2, 3)$ varying, respectively. The varying curves are drawn in Figure 5 to compare the distinguishing influences to the stability region.

Table 4. The relationship between q_1 and τ_0 ($q_2 = 0.92, q_3 = 0.95$).

q_1	ω_0	τ_0	**Transversality Condition**
0.63	0.001594	1641.098	0.0000000631
0.67	0.002666	951.636	0.0000001883
0.71	0.004259	573.477	0.0000005115
0.75	0.006541	355.943	0.0000012877
0.79	0.009698	226.110	0.0000030445
0.83	0.013900	146.548	0.0000068265
0.87	0.019235	96.960	0.0000145882
0.91	0.025648	65.706	0.0000296835
0.95	0.032949	45.759	0.0000571911
0.98	0.038858	35.533	0.0000899288
0.99	0.040891	32.769	0.0001037719
1	0.042951	30.268	0.0001192876

Table 5. The relationship between q_2 and τ_0 ($q_1 = 0.98, q_3 = 0.95$).

q_2	ω_0	τ_0	**Transversality Condition**
0.63	0.033009	46.423	0.0000585174
0.67	0.034213	43.729	0.0000646623
0.71	0.035273	41.558	0.0000702717
0.75	0.036199	39.808	0.0000752912
0.79	0.037000	38.400	0.0000796967
0.83	0.037686	37.275	0.0000834898
0.87	0.038266	36.382	0.0000866934
0.91	0.038751	35.681	0.0000893461
0.95	0.039149	35.141	0.0000914972
0.98	0.039398	34.824	0.0000928146
0.99	0.039472	34.733	0.0000932021
1	0.039542	34.649	0.0000935654

Table 6. The relationship between q_3 and τ_0 ($q_1 = 0.98, q_2 = 0.92$).

q_3	ω_0	τ_0	**Transversality Condition**
0.63	0.010074	236.261	0.0000029780
0.67	0.013125	170.912	0.0000057800
0.71	0.016401	128.390	0.0000102249
0.75	0.019836	99.378	0.0000167187
0.79	0.023394	78.732	0.0000256033
0.83	0.027064	63.492	0.0000371544
0.87	0.030851	51.884	0.0000515967
0.91	0.034772	42.804	0.0000691252
0.95	0.038858	35.533	0.0000899288
0.98	0.042054	30.971	0.0001078058
0.99	0.043148	29.589	0.0001142184
1	0.044259	28.269	0.0001208652

It can be seen from Tables 4–6 and Figure 5 that q_1 has an important influence over q_2 and q_3 on the stability of system (38). That is to say, in an ecosystem such as model Equation (38), the prey is the main fact that affects the stability of the ecosystem. It can be expressed that the change of prey affects the population density not only of prey but also of predator, which intensifies the turbulence of the ecosystem. Moreover, immature predators have more influence than mature predators when fractional-order is less than 0.8 in this ecosystem. It can be seen that when the three fractional-orders tend to 1, respectively, the critical value of delay changes gradually converge. This further shows that it is more practical to use fractional-order to explain the evolution of the ecosystem.

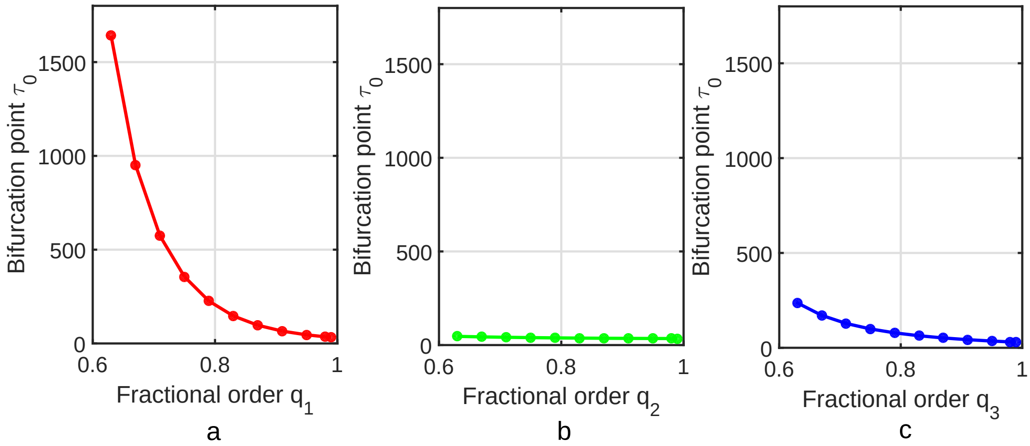

Figure 5. (**a**) $q_2 = 0.92$, $q_3 = 0.95$, τ_0 varies with the increase of q_1. (**b**) $q_1 = 0.98$, $q_3 = 0.95$, τ_0 varies with the increase of q_1. (**c**) $q_1 = 0.98$, $q_2 = 0.92$, τ_0 varies with the increase of q_1.

5.2. Example 2

Now, we add a linear delay feedback controller to the system to control the emergence of Hopf bifurcation. All the parameters of the controller system are the same as system (38), and the feedback gain is selected as $L = -0.01$, then the controlled system can be described as

$$\begin{cases} D^{0.98}x(t) = \frac{0.3x(t)}{1+0.05y_1(t)+0.1y_2(t)} - 0.1x(t) - 0.3x^2(t) - \frac{0.95x(t)y_2(t)}{1+0.95x(t)+y_2(t)+0.95x(t)y_2(t)}, \\ D^{0.92}y_1(t) = \frac{0.76x(t-\tau)y_2(t-\tau)}{1+0.95x(t-\tau)+y_2(t-\tau)+0.95x(t-\tau)y_2(t-\tau)} - 0.9y_1(t), \\ D^{0.95}y_2(t) = 0.8y_1(t) - 0.1y_2(t) + L(y_2(t) - y_2(t-\tau)). \end{cases} \tag{39}$$

The bifurcation critical point of controlled system (39) is $(\omega_0^* = 0.028071, \tau_0^* = 61.544)$. This means the emergence of Hopf bifurcation is put off obviously, and the stability region is enlarged successfully. The influence of feedback gain L on the bifurcation critical point is illustrated by numeric simulations in Table 7 and Figure 6.

Table 7. The relationship between L and τ_0^*.

L	ω_0^*	τ_0^*	Transversality Condition
−0.003	0.036053	40.542	0.0000654495
−0.005	0.034009	44.827	0.0000512848
−0.007	0.031796	50.214	0.0000388544
−0.010	0.028071	61.544	0.0000234772
−0.013	0.023680	80.183	0.0000120318
−0.015	0.020215	101.259	0.0000065762
−0.018	0.013707	172.017	0.0000015764
−0.020	0.007774	344.276	0.0000002167

It can be seen from Figure 6 that as the feedback gain decreases, the system converges to a steady state faster. In other words, the smaller the feedback gain, the better the control effect of the controller to the Hopf bifurcation.

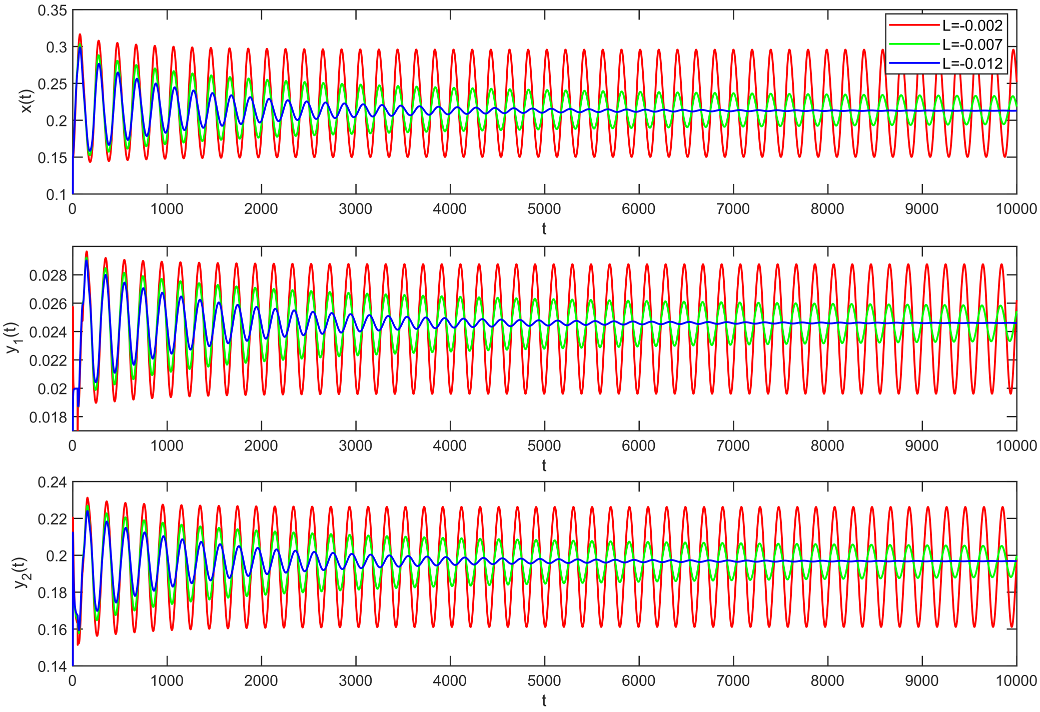

Figure 6. Waveform plots of system (39) with $\tau = 50$, feedback gains are $L = -0.002$, $L = -0.007$ and $L = -0.012$, respectively. The control effect increases as the feedback gain decreases.

6. Conclusions

In this paper, the dynamic behaviors of a fractional-order delay stage structure prey-predator model with two fear effects and prey refuge are explored by the linearized method and Laplace transform of fractional-order delay differential equation. Firstly, the conditions for the existence of the coexistence equilibrium point of the system (9) is deduced through the implicit function derivation rule and the function monotonicity theory. Secondly, the stability of the coexistence equilibrium point of the system (9) is investigated with the delay as parameter, and sufficient conditions for the emergence of Hopf bifurcation of the system (9) are obtained. Thirdly, a linear delay feedback controller is added to the system (9) to control the emergence of Hopf bifurcation, and the result states that the system can be controlled successfully by selecting an appropriate feedback gain. Finally, two examples are introduced to validate the theoretical results with the help of numerical simulation.

Moreover, some numerical simulations are performed to explore the influence facts of stability of system (9). The results show that fear factors k_1, k_2, prey refuge rate m and fractional-orders q_i $(i = 1, 2, 3)$ have distinguish effects to the bifurcation critical value of delay τ, and then affect the stability region of the system. These results have some implications for the regulation and management of ecosystems described in system (9).

However, because of lacking the complete theory of fractional-order differential equation, all the theoretical analyses in this paper are performed on the linearization system of original system (9) and the rationality of theoretical analysis is verified by numerical simulation. We are trying to study the stability of fractional differential equations theoretically in the next work as it is a challenging problem.

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Appendix A

Computation of the expressions $\alpha_1, \alpha_2, \alpha_3$ and α_4 in Equation (24)

$$
\begin{aligned}
\alpha_1 =& -\omega^{q_1}\cos(\frac{q_1\pi}{2})a_{23}a_{31} - \omega^{q_3}\cos(\frac{q_3\pi}{2})a_{12}a_{21} + a_{11}a_{23}a_{31} + a_{12}a_{21}a_{32} - a_{13}a_{21}a_{31},\\
\alpha_2 =& -\omega^{q_1}\sin(\frac{q_1\pi}{2})a_{23}a_{31} - \omega^{q_3}\sin(\frac{q_3\pi}{2})a_{12}a_{21},\\
\alpha_3 =& \omega^{q_1+q_2+q_3}\cos(\frac{\pi(q_1+q_2+q_3)}{2}) - \omega^{q_1+q_2}\cos(\frac{\pi(q_1+q_2)}{2})a_{32} + \omega^{q_1}\cos(\frac{q_1\pi}{2})a_{22}a_{32}\\
& - \omega^{q_1+q_3}\cos(\frac{\pi(q_1+q_3)}{2})a_{22} + a_{11}(\omega^{q_2}\cos(\frac{q_2\pi}{2})a_{32} + a_{22}(\omega^{q_3}\cos(\frac{q_3\pi}{2}) - a_{32}))\\
& - \omega^{q_2+q_3}\cos(\frac{\pi(q_2+q_3)}{2})a_{11},\\
\alpha_4 =& \omega^{q_1}\sin(\frac{q_1\pi}{2})a_{22}a_{32} + \omega^{q_3}\sin(\frac{q_3\pi}{2})a_{11}a_{22} + \omega^{q_2}\sin(\frac{q_2\pi}{2})a_{11}a_{32}\\
& - \omega^{q_1+q_3}\sin(\frac{\pi(q_1+q_3)}{2})a_{22} - \omega^{q_1+q_2}\sin(\frac{\pi(q_1+q_2)}{2})a_{32}\\
& - \omega^{q_2+q_3}\sin(\frac{\pi(q_2+q_3)}{2})a_{11} + \omega^{q_1+q_2+q_3}\sin(\frac{\pi(q_1+q_2+q_3)}{2}).
\end{aligned}
$$

Appendix B

Computation of the expressions A_1, B_1, A_2 and B_2 in Equation (29)

$$
\begin{aligned}
A_1 =& \omega_0[-\omega_0{}^{q_1}a_{23}a_{31}\sin(-\frac{q_1\pi}{2} + \omega_0\tau_0) + ((a_{12}a_{32} - a_{13}a_{31})a_{21} + a_{11}a_{23}a_{31})\sin(\omega_0\tau_0)\\
& - \omega_0{}^{q_3}a_{12}a_{21}\sin(-\frac{q_3\pi}{2} + \omega_0\tau_0)],\\
B_1 =& \tau_0a_{31}a_{23}\omega_0{}^{q_1}\cos(-\frac{q_1\pi}{2} + \omega_0\tau_0) + \tau_0a_{21}a_{12}\omega_0{}^{q_3}\cos(-\frac{q_3\pi}{2} + \omega_0\tau_0)\\
& + q_1a_{31}a_{23}\omega_0{}^{q_1-1}\sin(-\frac{q_1\pi}{2} + \omega_0\tau_0) + q_3a_{21}a_{12}\omega_0{}^{q_3-1}\sin(-\frac{q_3\pi}{2} + \omega_0\tau_0)\\
& + \omega_0{}^{q_1-1+q_2+q_3}(q_1+q_2+q_3)\sin(\frac{\pi(q_1+q_2+q_3)}{2})\\
& - a_{32}\omega_0{}^{q_1-1+q_2}(q_1+q_2)\sin(\frac{\pi(q_1+q_2)}{2}) - \omega_0{}^{q_1-1+q_3}a_{22}(q_1+q_3)\sin(\frac{\pi(q_1+q_3)}{2})\\
& - \omega_0{}^{q_2-1+q_3}a_{11}(q_2+q_3)\sin(\frac{\pi(q_2+q_3)}{2}) + \omega_0{}^{q_1-1}\sin(\frac{q_1\pi}{2})q_1a_{22}a_{32}\\
& + \omega_0{}^{q_2-1}\sin(\frac{q_2\pi}{2})q_2a_{11}a_{32} + \omega_0{}^{q_3-1}\sin(\frac{q_3\pi}{2})q_3a_{11}a_{22}\\
& - \tau_0(a_{11}a_{23}a_{31} - a_{21}(-a_{12}a_{32} + a_{13}a_{31}))\cos(\omega_0\tau_0),\\
A_2 =& \omega_0[-\omega_0{}^{q_1}a_{23}a_{31}\cos(-\frac{q_1\pi}{2} + \omega_0\tau_0) + ((a_{12}a_{32} - a_{13}a_{31})a_{21} + a_{11}a_{23}a_{31})\cos(\omega_0\tau_0)\\
& - \omega_0{}^{q_3}a_{12}a_{21}\cos(-\frac{q_3\pi}{2} + \omega_0\tau_0)],\\
B_2 =& q_1a_{31}a_{23}\omega_0{}^{q_1-1}\cos(-\frac{q_1\pi}{2} + \omega_0\tau_0) + q_3a_{21}a_{12}\omega_0{}^{q_3-1}\cos(-\frac{q_3\pi}{2} + \omega_0\tau_0)\\
& - \tau_0a_{31}a_{23}\omega_0{}^{q_1}\sin(-\frac{q_1\pi}{2} + \omega_0\tau_0) - \tau_0a_{21}a_{12}\omega_0{}^{q_3}\sin(-\frac{q_3\pi}{2} + \omega_0\tau_0)
\end{aligned}
$$

$$
\begin{aligned}
& - {\omega_0}^{q_1-1+q_2+q_3}(q_1+q_2+q_3)\cos(\frac{\pi(q_1+q_2+q_3)}{2}) \\
& + a_{32}{\omega_0}^{q_1-1+q_2}(q_1+q_2)\cos(\frac{\pi(q_1+q_2)}{2}) + {\omega_0}^{q_1-1+q_3}a_{22}(q_1+q_3)\cos(\frac{\pi(q_1+q_3)}{2}) \\
& + {\omega_0}^{q_2-1+q_3}a_{11}(q_2+q_3)\cos(\frac{\pi(q_2+q_3)}{2}) - {\omega_0}^{q_1-1}\cos(\frac{q_1\pi}{2})q_1a_{22}a_{32} \\
& - {\omega_0}^{q_2-1}\cos(\frac{q_2\pi}{2})q_2a_{11}a_{32} - {\omega_0}^{q_3-1}\cos(\frac{q_3\pi}{2})q_3a_{11}a_{22} \\
& + \tau_0(a_{11}a_{23}a_{31} - a_{21}(-a_{12}a_{32} + a_{13}a_{31}))\sin(\omega_0\tau_0).
\end{aligned}
$$

Appendix C

Computation of the expressions $\beta_1, \beta_2, \beta_3, \gamma_1$ and γ_2 in Equation (35)

$$
\begin{aligned}
\beta_1 =& \omega^{q_1+q_2+q_3}\cos(\frac{\pi(q_1+q_2+q_3)}{2}) - \omega^{q_1+q_2}(L+a_{32})\cos(\frac{\pi(q_1+q_2)}{2}) \\
& - \omega^{q_1+q_3}\cos(\frac{\pi(q_1+q_3)}{2})a_{22} - \omega^{q_2+q_3}\cos(\frac{\pi(q_2+q_3)}{2})a_{11} \\
& + (\omega^{q_2}(L+a_{32})\cos(\frac{q_2\pi}{2}) - a_{22}(-\omega^{q_3}\cos(\frac{q_3\pi}{2}) + L + a_{32}))a_{11} \\
& + \omega^{q_1}a_{22}(L+a_{32})\cos(\frac{q_1\pi}{2}), \\
\beta_2 =& - \omega^{q_1+q_2}(L+a_{32})\sin(\frac{\pi(q_1+q_2)}{2}) - \omega^{q_1+q_3}\sin(\frac{\pi(q_1+q_3)}{2})a_{22} \\
& + \omega^{q_1}a_{22}(L+a_{32})\sin(\frac{q_1\pi}{2}) + \sin(\frac{q_2\pi}{2})(L+a_{32})a_{11}\omega^{q_2} + \sin(\frac{q_3\pi}{2})\omega^{q_3}a_{11}a_{22} \\
& + \omega^{q_1+q_2+q_3}\sin(\frac{\pi(q_1+q_2+q_3)}{2}) - \omega^{q_2+q_3}\sin(\frac{\pi(q_2+q_3)}{2})a_{11}, \\
\beta_3 =& - La_{12}a_{21}, \\
\gamma_1 =& - \omega^{q_1}(La_{22}+a_{23}a_{31})\cos(\frac{q_1\pi}{2}) - L\omega^{q_2}\cos(\frac{q_2\pi}{2})a_{11} - \omega^{q_3}\cos(\frac{q_3\pi}{2})a_{12}a_{21} \\
& + (La_{22}+a_{23}a_{31})a_{11} + a_{21}(L+a_{32})a_{12} + \omega^{q_1+q_2}\cos(\frac{\pi(q_1+q_2)}{2})L - a_{13}a_{21}a_{31}, \\
\gamma_2 =& L\omega^{q_1+q_2}\sin(\frac{\pi(q_1+q_2)}{2}) - \omega^{q_1}(La_{22}+a_{23}a_{31})\sin(\frac{q_1\pi}{2}) - L\omega^{q_2}\sin(\frac{q_2\pi}{2})a_{11} \\
& - \omega^{q_3}\sin(\frac{q_3\pi}{2})a_{12}a_{21}.
\end{aligned}
$$

Appendix D

Computation of the expressions C_1, D_1, C_2 and D_2 in Equation (37)

$$
\begin{aligned}
C_1 =& - \omega_0^*[-L(\omega_0^*)^{q_1+q_2}\sin(\frac{(-q_1-q_2)\pi}{2} + \omega_0^*\tau_0^*) + L(\omega_0^*)^{q_2}a_{11}\sin(-\frac{q_2\pi}{2} + \omega_0^*\tau_0^*) \\
& + (\omega_0^*)^{q_1}(La_{22}+a_{23}a_{31})\sin(-\frac{q_1\pi}{2} + \omega_0^*\tau_0^*) + (\omega_0^*)^{q_3}a_{12}a_{21}\sin(-\frac{q_3\pi}{2} + \omega_0^*\tau_0^*) \\
& + 2\sin(2\omega_0^*\tau_0^*)La_{12}a_{21} - \sin(\omega_0^*\tau_0^*)((a_{11}a_{22}+a_{12}a_{21})L + (a_{12}a_{32} - a_{13}a_{31})a_{21} \\
& + a_{11}a_{23}a_{31})], \\
D_1 =& - L(\omega_0^*)^{q_1-1+q_2}(q_1+q_2)\sin(\frac{(-q_1-q_2)\pi}{2} + \omega_0^*\tau_0^*) + (\omega_0^*)^{q_3-1}\sin(\frac{q_3\pi}{2})q_3a_{11}a_{22} \\
& - L(\omega_0^*)^{q_1+q_2}\cos(\frac{(-q_1-q_2)\pi}{2} + \omega_0^*\tau_0^*)\tau_0^* + a_{12}a_{21}(\omega_0^*)^{q_3-1}q_3\sin(-\frac{q_3\pi}{2} + \omega_0^*\tau_0^*) \\
& + (La_{22}+a_{23}a_{31})\tau_0^*(\omega_0^*)^{q_1}\cos(-\frac{q_1\pi}{2} + \omega_0^*\tau_0^*) + (L+a_{32})(\omega_0^*)^{q_2-1}a_{11}q_2\sin(\frac{q_2\pi}{2}) \\
& + (La_{22}+a_{23}a_{31})(\omega_0^*)^{q_1-1}q_1\sin(-\frac{q_1\pi}{2} + \omega_0^*\tau_0^*) + \tau_0^*(\omega_0^*)^{q_2}La_{11}\cos(-\frac{q_2\pi}{2} + \omega_0^*\tau_0^*) \\
& + a_{12}a_{21}\tau_0^*(\omega_0^*)^{q_3}\cos(-\frac{q_3\pi}{2} + \omega_0^*\tau_0^*) + (\omega_0^*)^{q_2-1}q_2La_{11}\sin(-\frac{q_2\pi}{2} + \omega_0^*\tau_0^*)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\omega_0^*)^{q_1-1+q_2+q_3}(q_1+q_2+q_3)\sin(\frac{\pi(q_1+q_2+q_3)}{2}) \\
&- (\omega_0^*)^{q_1-1+q_2}(q_1+q_2)(L+a_{32})\sin(\frac{\pi(q_1+q_2)}{2}) \\
&- a_{22}(\omega_0^*)^{q_1-1+q_3}(q_1+q_3)\sin(\frac{\pi(q_1+q_3)}{2}) + a_{22}(\omega_0^*)^{q_1-1}(L+a_{32})q_1\sin(\frac{q_1\pi}{2}) \\
&- (\omega_0^*)^{q_2-1+q_3}a_{11}(q_2+q_3)\sin(\frac{\pi(q_2+q_3)}{2}) \\
&- \tau_0^*(\cos(\omega_0^*\tau_0^*)((a_{11}a_{22}+a_{12}a_{21})L + a_{11}a_{23}a_{31} - a_{21}(-a_{12}a_{32}+a_{13}a_{31})) \\
&- 2\cos(2\omega_0^*\tau_0^*)La_{12}a_{21}),
\end{aligned}
$$

$$
\begin{aligned}
C_2 =& -\omega_0^*[-L(\omega_0^*)^{q_1+q_2}\cos(\frac{(-q_1-q_2)\pi}{2} + \omega_0^*\tau_0^*) + 2\cos(2\omega_0^*\tau_0^*)La_{12}a_{21} \\
&+ L(\omega_0^*)^{q_2}a_{11}\cos(-\frac{q_2\pi}{2} + \omega_0^*\tau_0^*) + (\omega_0^*)^{q_3}a_{12}a_{21}\cos(-\frac{q_3\pi}{2} + \omega_0^*\tau_0^*) \\
&- \cos(\omega_0^*\tau_0^*)((a_{11}a_{22}+a_{12}a_{21})L + (a_{12}a_{32}-a_{13}a_{31})a_{21} + a_{11}a_{23}a_{31}) \\
&+ (\omega_0^*)^{q_1}(La_{22}+a_{23}a_{31})\cos(-\frac{q_1\pi}{2} + \omega_0^*\tau_0^*)],
\end{aligned}
$$

$$
\begin{aligned}
D_2 =& -L(\omega_0^*)^{q_1-1+q_2}(q_1+q_2)\cos(\frac{(-q_1-q_2)\pi}{2} + \omega_0^*\tau_0^*) \\
&+ L(\omega_0^*)^{q_1+q_2}\sin(\frac{(-q_1-q_2)\pi}{2} + \omega_0^*\tau_0^*)\tau_0^* - a_{12}a_{21}\tau_0^*(\omega_0^*)^{q_3}\sin(-\frac{q_3\pi}{2} + \omega_0^*\tau_0^*) \\
&+ (La_{22}+a_{23}a_{31})(\omega_0^*)^{q_1-1}q_1\cos(-\frac{q_1\pi}{2} + \omega_0^*\tau_0^*) \\
&- (La_{22}+a_{23}a_{31})\tau_0^*(\omega_0^*)^{q_1}\sin(-\frac{q_1\pi}{2} + \omega_0^*\tau_0^*) + (\omega_0^*)^{q_2-1}q_2La_{11}\cos(-\frac{q_2\pi}{2} + \omega_0^*\tau_0^*) \\
&+ a_{12}a_{21}(\omega_0^*)^{q_3-1}q_3\cos(-\frac{q_3\pi}{2} + \omega_0^*\tau_0^*) - \tau_0^*(\omega_0^*)^{q_2}La_{11}\sin(-\frac{q_2\pi}{2} + \omega_0^*\tau_0^*) \\
&- (\omega_0^*)^{q_1-1+q_2+q_3}(q_1+q_2+q_3)\cos(\frac{\pi(q_1+q_2+q_3)}{2}) \\
&+ (\omega_0^*)^{q_1-1+q_2}(q_1+q_2)(L+a_{32})\cos(\frac{\pi(q_1+q_2)}{2}) - (\omega_0^*)^{q_3-1}\cos(\frac{q_3\pi}{2})q_3a_{11}a_{22} \\
&+ a_{22}(\omega_0^*)^{q_1-1+q_3}(q_1+q_3)\cos(\frac{\pi(q_1+q_3)}{2}) - (L+a_{32})(\omega_0^*)^{q_2-1}a_{11}q_2\cos(\frac{q_2\pi}{2}) \\
&+ (\omega_0^*)^{q_2-1+q_3}a_{11}(q_2+q_3)\cos(\frac{\pi(q_2+q_3)}{2}) - a_{22}(\omega_0^*)^{q_1-1}(L+a_{32})q_1\cos(\frac{q_1\pi}{2}) \\
&+ \tau_0^*(\sin(\omega_0^*\tau_0^*)((a_{11}a_{22}+a_{12}a_{21})L + a_{11}a_{23}a_{31} - a_{21}(-a_{12}a_{32}+a_{13}a_{31})) \\
&- 2\sin(2\omega_0^*\tau_0^*)La_{12}a_{21}).
\end{aligned}
$$

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