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Order-Theoretic Common Fixed Point Results in R_{mb}-Metric Spaces

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Abstract: In this paper, we prove common fixed point theorems for a pair of self-mappings in the framework of R_{mb}-metric spaces. We also deduce some corollaries of our main result. In order to support our main result, we also set an example.

Keywords: fixed point; common fixed point results; ordered R_{mb}-metric spaces

MSC: 7H10; 54H25

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Citation: Asim, M.; Imdad, M.; Sessa, S. Order-Theoretic Common Fixed Point Results in R_{mb}-Metric Spaces. *Symmetry* **2022**, *14*, 1376. https://doi.org/10.3390/sym14071376

Academic Editors: Alexander Zaslavski and Sergei D. Odintsov

Received: 24 May 2022
Accepted: 1 July 2022
Published: 4 July 2022

1. Introduction

In recent decades, fixed point theory has been extended to numerous abstract spaces, and it has been successfully applied to the study of a wide range of scientific problems, bridging the gap between theoretical and practical techniques and even covering extremely complex computing problems. The study and calculation of solutions to differential equations, integral equations, dynamical systems, models in economics and related subjects, game theory, physics, engineering, computer science, and neural networks are all examples of fixed point theory applications. They are also important tools for researching nonlinear systems because they give a framework for elevating some basic aspects of linear model solutions in order to deduce the response of nonlinear systems whose solutions are located as the fixed points of a specific operator. The most famous and celebrated fixed point theorem, known as the Banach contraction principle [1], was proved in 1922 by Polish mathematician Banach. The fundamental Banach contraction principle, which has been modified and improved in numerous directions, is the core conclusion of metric fixed point theory (see [2–9]).

On the other hand, Matthews [10] introduced the notion of partial metric space in 1994, which is based on the observation that in a metric type definition, the distance of a point from itself may not be essentially zero. In doing so, he was essentially motivated by the study of denotational semantics of data-flow networks. He also proved a Banach-type fixed point theorem on a complete partial metric space, which is an extended version of the Banach contraction principle. In the continuation of this generalization, Asadi et al. [11] gave an idea of M-metric space as a modified version of partial metric space and proved a fixed point theorem for the same in 2014. Later, Mlaiki et al. [12] generalized M-metric space as well as b-metric space by introducing M_b-metric space. Özgür et al. [13] generalized the notion of M-metric space by introducing the notion of M_b-metric space in 2018. In the recent past (in 2019), Asim et al. [2] extended the class of M_b-metric space by introducing the class of rectangular M_b-metric space (denoted by R_{mb}-metric space) and utilized the same to prove a fixed point theorem.

Turinici [14] first proposed the theory of order-theoretic fixed point outcomes in 1986. Ran and Reurings [15] produced a new, more natural formulation of the Banach contraction principle shortly after and used his result to explain the existence and uniqueness of a system of linear equation solutions. Nieto and Rodriguez-Lopez [16,17] closely followed this paper. On such topics, there is currently a great deal of research effort underway, and one might look to these works (see [18–33]) and the papers cited therein.

Inspired by the foregoing observations, we prove some existence and uniqueness common fixed point results in R_{mb}-metric space endowed with an ordered relation. We improve upon the relatively weaker notions of completeness and continuity. The completeness of X is merely required on any subspace of X containing $f(X)$.

2. Preliminaries

The following concepts, definitions, and auxiliary results are required in our upcoming discussions before we can present our results.

In 2012, Asadi et al. [11] introduced the following definition of M-metric space.

Definition 1 ([11])**.** *Let X be a non-empty set. A mapping $m : X \times X \to \mathbb{R}_+$ is called M-metric if m satisfies the following (for all $a, b, c \in X$):*

(1) $m(a,a) = m(a,b) = m(b,b)$ *if and only if* $a = b$,
(2) $m_{a,b} \leq m(a,b)$,
(3) $m(a,b) = m(b,a)$,
(4) $(m(a,b) - m_{a,b}) \leq (m(a,c) - m_{a,c}) + (m(c,b) - m_{c,b})$.

Then, the pair (X, m) is said to be an M-metric space, where $m_{a,b} = \min\{m(a,a), m(b,b)\}$.

After two years, Mlaiki et al. [12] introduced the following definition of M_b-metric space.

Definition 2 ([12])**.** *Let X be a non-empty set. A mapping $m_b : X \times X \to \mathbb{R}_+$ is called M_b-metric with coefficient $s \geq 1$, if m_b satisfies the following (for all $a, b, c \in X$):*

(1) $m_b(a,a) = m_b(a,b) = m_b(b,b)$ *if and only if* $a = b$,
(2) $m_{b_{a,b}} \leq m_b(a,b)$,
(3) $m_b(a,b) = m_b(b,a)$,
(4) $(m_b(a,b) - m_{b_{a,b}}) \leq s[(m_b(a,c) - m_{b_{a,c}}) + (m_b(c,b) - m_{b_{c,b}})] - m_b(c,c)$.

Then, the pair (X, m_b) is said to be an M_b-metric space, where $m_{b_{a,b}} = \min\{m_b(a,a), m_b(b,b)\}$.

Özgür et al. [13] introduced the following definition of rectangular M-metric space.

Definition 3 ([13])**.** *Let X be a non-empty set. A mapping $m_r : X \times X \to \mathbb{R}_+$ is said to be a rectangular M-metric if m_r satisfies the following (for all $a, b \in X$ and all distinct $u, v \in X \setminus \{a, b\}$,):*

(1) $m_r(a,a) = m_r(a,b) = m_r(b,b)$ *if and only if* $a = b$,
(2) $m_{r_{a,b}} \leq m_r(a,b)$,
(3) $m_r(a,b) = m_r(b,a)$,
(4) $(m_r(a,b) - m_{r_{a,b}}) \leq (m_r(a,u) - m_{r_{a,u}}) + (m_r(u,v) - m_{r_{u,v}}) + (m_r(v,b) - m_{r_{v,b}})$.

Then, the pair (X, m_r) is said to be a rectangular M-metric space, where $m_{r_{a,b}} = \min\{m_r(a,a), m_r(b,b)\}$.

In 2019, Asim et al. [2] proposed the concept of R_{mb}-metric space as a generalization of both rectangular M-metric space as well as M_b-metric space.

Definition 4 ([2])**.** *Let X be a non-empty set. A mapping $r_{mb} : X \times X \to \mathbb{R}_+$ is said to be M_b-metric with coefficient $s \geq 1$ if r_{mb} satisfies the following (for all $a, b, c \in X$ and all distinct $u, v \in X \setminus \{a, b\}$):*

$(1r_{mb})$ $r_{mb}(a,a) = m_r(a,b) = r_{mb}(b,b)$ *if and only if* $a = b$,
$(2r_{mb})$ $m_{rb_{a,b}} \leq r_{mb}(a,b)$,
$(3r_{mb})$ $r_{mb}(a,b) = r_{mb}(b,a)$,

$(4r_{mb})$ $(r_{mb}(a,b) - r_{mb_{a,b}}) \leq s[(r_{mb}(a,u) - r_{mb_{a,u}}) + (r_{mb}(u,v) - r_{mb_{u,v}}) + (r_{mb}(v,b) - r_{mb_{v,b}})] - r_{mb}(u,u) - r_{mb}(v,v)$.

Then, the pair (X, r_{mb}) is said to be an R_{mb}-metric space, where $r_{mb_{a,b}} = \min\{r_{mb}(a,a), r_{mb}(b,b)\}$.

Remark 1. *If $\xi(a,b) = s \geq 1$, then (X, r_ξ) is said to be a rectangular b-metric space.*

Now, we adopt an example of an R_{mb}-metric space.

Example 1. *Define $r_{mb} : X \times X \to \mathbb{R}_+$ on $X = [0, \infty)$ with any positive integer $p > 1$, by (for all $a, b \in X$):*

$$r_{mb}(a,b) = \max\{a,b\}^p + |a-b|^p.$$

Then, (X, r_{mb}) is an R_{mb}-metric space with coefficient $s = 3^{p-1}$. By routine calculation, one can easily check that (X, r_{mb}) is not rectangular M-metric space.

Definition 5. *Let (X, r_{mb}) be an R_{mb}-metric space. A sequence $\{a_n\} \subseteq X$ is considered to be convergent to $a \in X$ if and only if*

$$\lim_{n\to\infty}(r_{mb}(a_n,a) - m_{rb_{a_n,a}}) = 0.$$

Definition 6. *Let (X, r_{mb}) be an R_{mb}-metric space. A sequence $\{a_n\}$ in (X, r_{mb}) is considered to be Cauchy if and only if*

$$\lim_{n,m\to\infty}(r_{mb}(a_n,a_m) - m_{rb_{a_n,a_m}}) \text{ and } \lim_{n,m\to\infty}(M_{rb_{a_n,a_m}} - m_{rb_{a_n,a_m}})$$

exist and are finite.

Definition 7. *An R_{mb}-metric space (X, r_{mb}) is considered to be complete if and only if every Cauchy in (X, r_{mb}) is convergent to a point in (X, r_{mb}).*

Let (f, g) denote a pair of self-mappings defined on an $X(\neq \emptyset)$ such that $fa = ga = a^*$ for $a, a^* \in X$. Then, the point a is called the coincidence point, a^* is called the point of coincidence, and if $a = a^*$, then a is said to be a common fixed point of (f, g). A binary operation '$\preceq$' on X is said to be partial ordered if it is reflexive, antisymmetric, and transitive. We say 'a' is related to 'b' if $a \preceq b$ (or $b \succeq a$). An ordered set is defined as $X(\neq \emptyset)$ plus '$\preceq$' and is typically expressed by $(X, \preceq)$. The triplet $(X, d, \preceq)$ is said to be a partial ordered metric space or ordered metric space if $(X, \preceq)$ is an ordered set and (X, d) is a metric space. Throughout the paper, the symbols $\uparrow$, $\downarrow$, and $\uparrow\downarrow$ represent increasing, decreasing, and monotonic sequences, respectively.

The following definition is a generalized form of the definition defined in [27].

Definition 8. *If (X, r_{mb}) is an R_{mb}-metric space and $(X, \preceq)$ is an ordered set, the triple $(X, r_{mb}, \preceq)$ is termed an ordered R_{mb}- metric space. Moreover, if either $a \preceq b$ or $b \succeq a$, two elements $a, b \in X$ are said to be comparable. We abbreviate this as $a \prec\succ b$. for clarity.*

Definition 9 ([22]). *Let (f, g) be self-mappings on an ordered set $(X, \preceq)$.*

(1) f considered to be g-increasing if $ga \preceq gb \Rightarrow fa \preceq fb$, for all $a, b \in X$,
(2) f considered to be g-decreasing if $ga \preceq gb \Rightarrow fa \succeq fb$, for all $a, b \in X$,
(3) If f is either g-increasing or g-decreasing, f considered to be g-monotone.

The following definitions (Definition 10 and Definition 11) improve the definitions presented in [21].

Definition 10 ([21]). *Let (f, g) be self-mappings on $(X, r_{mb}, \preceq)$ and $a \in X$. Then, f is called $(g,\overline{\text{O}})$- continuous (or $(g,\underline{\text{O}})$-continuous or (g,O)-continuous) at $a \in X$ if $fa_n \xrightarrow{r_{mb}} fa$, for every* *sequence $\{a_n\} \subset X$ with $ga_n \uparrow ga$ (or $ga_n \downarrow ga$ or $ga_n \uparrow\downarrow ga$). Moreover, f is called (g,O)-continuous (or (g,$\overline{\text{O}}$)-continuous or (g,$\underline{\text{O}}$)-continuous) if $fa_n \xrightarrow{r_{mb}} fa$, for every sequence $\{a_n\} \subset X$ with $ga_n \uparrow ga$ (or $ga_n \downarrow ga$ or $ga_n \uparrow\downarrow ga$) at every point of X.*

Remark 2. *In an ordered R_{mb}-metric space, continuity $\Rightarrow$ (g,O)-continuity $\Rightarrow$ (g,$\overline{\text{O}}$)-continuity as well as (g,$\underline{\text{O}}$)-continuity.*

Observe that on setting $g = I_X$, the Definition 10 reduces to the O-continuous (resp. $\overline{\text{O}}$-continuous, $\underline{\text{O}}$-continuous).

Definition 11. *The ordered R_{mb}-metric space $(X, r_{mb}, \preceq)$ stands for $\overline{\text{O}}$-complete (or $\underline{\text{O}}$-complete or O-complete) if every increasing (or decreasing or monotone) Cauchy sequence in X converges to a point of X.*

Remark 3. *From the above definition, it is clear that completeness implies O-completeness, which implies $\overline{\text{O}}$-completeness (together with $\underline{\text{O}}$-completeness).*

The following definitions are a modified version of [21,34,35], respectively.

Definition 12. *Let (f, g) be self-mappings on $(X, r_{mb}, \preceq)$.*

(i) The pair (f, g) is said to be compatible if for a sequence $\{a_n\} \subseteq X$ with $\lim_{n\to\infty} ga_n = \lim_{n\to\infty} fa_n$ implies $\lim_{n\to\infty} r_{mb}(g(fa_n), f(ga_n)) = 0$.

(ii) The pair (f, g) is said to be $\overline{\text{O}}$-compatible (resp. $\underline{\text{O}}$-compatible, O-compatible) if for a sequence $\{a_n\} \subseteq X$ with $\{ga_n\}$ and $\{fa_n\}$ are increasing (resp. decreasing, monotone) sequences such that $\lim_{n\to\infty} ga_n = \lim_{n\to\infty} fa_n$ implies $\lim_{n\to\infty} r_{mb}(g(fa_n), f(ga_n)) = 0$.

(iii) The pair (f, g) is said to be weakly compatible if $fa = ga$ for a in X.

Remark 4. *In $(X, r_{mb}, \preceq)$, compatibility implies O-compatibility, which implies $\overline{\text{O}}$-compatibility (together with $\underline{\text{O}}$-compatibility), which also implies weak compatibility.*

Now, we define the generalized definition due to [20].

Definition 13. *The ordered R_{mb}-metric space $(X, r_{mb}, \preceq)$ is said to have the g-ICU property if the g-image of every increasing convergent sequence $\{a_n\} \subseteq X$ is bounded above by the g-image of its limit (as an upper bound); that is,*

$$a_n \uparrow a \;\Rightarrow\; g(a_n) \preceq g(a) \;\forall\, n \in \mathbb{N}_0.$$

Observe that on setting $g = I_X$, Definition 13 is reduced to the ICU property and still remains a sharpened version of [21].

Definition 14. *Let (f, g) be self-mappings $(X, r_{mb}, \preceq)$.*

(i) $(X, dr_{mb}, \preceq)$ is considered to have the g-ICC property if every g-increasing convergent sequence $\{a_n\} \subseteq X$ has a subsequence $\{a_{n_k}\}$ such that the g-image of every element of $\{a_{n_k}\}$ is comparable with the limit of $\{a_n\}$.

$$a_n \uparrow a \;\Rightarrow\; \exists\, \{a_{n_k}\} \text{ of } \{a_n\} \text{ with } ga_{n_k} \prec\succ ga \;\forall\, k \in \mathbb{N}_0,$$

(ii) $(X, r_{mb}, \preceq)$ is considered to have the g-DCC property if every g-decreasing convergent sequence $\{a_n\} \subseteq X$ has a subsequence $\{a_{n_k}\}$ such that the g-image of every element of $\{a_{n_k}\}$ is comparable with the limit of $\{a_n\}$; that is,

$$a_n \downarrow a \;\Rightarrow\; \exists\, \{a_{n_k}\} \text{ of } \{a_n\} \text{ with } ga_{n_k} \prec\succ ga \;\forall\, k \in \mathbb{N}_0,$$

(iii) $(X, r_{mb}, \preceq)$ *is said to have a g-monotone-convergence-comparable (in short g-MCC) property if every g-monotone convergent sequence* $\{a_n\}$ *in X has a subsequence* $\{a_{n_k}\}$ *such that the g-image of every term of* $\{a_{n_k}\}$ *is comparable with the limit of* $\{a_n\}$*, i.e.,*

$$a_n \uparrow\downarrow a \Rightarrow \exists \{a_{n_k}\} \text{ of } \{a_n\} \text{ with } ga_{n_k} \prec\succ ga \;\forall\, k \in \mathbb{N}_0.$$

On setting $g = I_X$, Definition 14 (i) (or (ii) or (iii)) reduces to the ICC (or DCC or MCC) property. Moreover, ICC (resp. DCC or MCC) is weaker then ICU (resp. DCL or MCB). Further, Definition 14 (i) is relatively weaker than the notion described in Definition 13.

Definition 15 ([25])**.** *Let* $(X, \preceq)$ *be an ordered,* $Y \subseteq X$ *and g a self-mapping on X. Then Y is said to be g-directed if for every pair* $a, b \in Y$*, there exists* $c \in X$ *with* $a \prec\succ gc$ *and* $b \prec\succ gc$*.*

Lemma 1 ([20])**.** *Let* (f, g) *be a pair of weakly compatible self-mappings defined on* $X(\neq \emptyset)$*. Then, every point of coincidence of the pair* (f, g) *remains a coincidence point.*

Lemma 2 ([36])**.** *Suppose a sequence* $\{a_n\}$ *in* $(X, r_{mb}, \preceq)$ *such that* $\lim_{n\to\infty} r_{mb}(a_n, a_{n+1}) = 0$*. If the sequence* $\{a_n\}$ *is not a Cauchy, then* $\exists\, \varepsilon > 0$*, and* $\{a_{n_k}\}$ *and* $\{a_{m_k}\}$ *of* $\{a_n\}$ *such that*

(1) $n_k > m_k > k$,
(2) $r_{mb}(a_{m_k}, a_{n_k}) \geq \varepsilon$,
(3) $r_{mb}(a_{m_k}, a_{n_k-1}) < \varepsilon$,
(4) *The sequences* $r_{mb}(a_{m_k}, a_{n_k}), r_{mb}(a_{m_k+1}, a_{n_k}), r_{mb}(a_{m_k}, a_{n_k+1}), r_{mb}(a_{m_k+1}, a_{n_k+1})$ *tend to* ε *when* $k \to \infty$*.*

The aim of this article is to prove common fixed point results for a pair of self-mappings satisfying ordered–theoretic contraction in the framework of R_{mb}-metric space. In doing so, we improve Theorem 3.2 from Asim el al. [2] in the following four-respects:

(i) The self-mapping is replaced by a pair of self-mappings to prove unique common fixed point results instead of fixed point results,
(ii) The weaker contraction is utilized—that is, ordered-theoretic contraction,
(iii) Relatively weaker notions of completeness and continuity are utilized,
(iv) The completeness of X is merely required on any subspace Y of X containing $f(X)$.

3. Main Results

Now, we state and prove our main results as follows:

Theorem 1. *Let* $(X, r_{mb}, \preceq)$ *be an ordered* R_{mb}*-metric space with* $s \geq 1$ *and Y an* $\overline{\text{O}}$*-complete subspace of X and* $f, g : X \to X$ *such that f is a g-increasing. Suppose the following conditions hold:*

(i) $\exists\, a_0 \in X$ *with* $ga_0 \preceq fa_0$*,*
(ii) *For all* $a, b \in X$ *such that* $ga \preceq gb$*, there exists* $\lambda \in [0, \frac{1}{s})$ *such that*

$$r_{mb}(fa, fb) \leq \lambda r_{mb}(ga, gb) \;\forall\, a, b \in X. \tag{1}$$

(iii) $f(X) \subseteq Y \subseteq g(X)$,
(iv) *Either*

(a) *f is* $(g, \overline{\text{O}})$*-continuous or*
(b) $(Y, r_{mb}, \preceq)$ *enjoys the g-ICC-property.*

In these conditions, the pair (f, g) *has a coincidence point.*

Proof. Choose a point $a_0 \in X$ such that $ga_0 \preceq fa_0$. Since the mapping f is g-increasing and $f(X) \subseteq g(X)$, we can define increasing sequences $\{ga_n\}$ and $\{fa_n\}$ in Y such that for all $n \in \mathbb{N}_0$

$$ga_{n+1} = fa_n. \tag{2}$$

Notice that, the sequences $\{ga_n\}$ and $\{fa_n\}$ are in Y. If $r_{mb}(ga_n, ga_{n+1}) = 0$ for some $n \in \mathbb{N}_0$, then a_n is a coincidence point, which concludes the proof.

Henceforth, we assume that $r_{mb}(ga_n, ga_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. Now, we have to show that $\lim_{n\to\infty} r_{mb}(ga_{n-1}, ga_n) = 0$. By putting $a = a_n$ and $b = a_{n-1}$ in condition (1), we get

$$\begin{aligned} r_{mb}(ga_{n+1}, ga_n)) &= r_{mb}(fa_n, fa_{n-1})) \\ &\leq \lambda r_{mb}(ga_n, ga_{n-1})), \end{aligned} \tag{3}$$

for all $n \in \mathbb{N}_0$. Therefore, $\{r_{mb}(ga_{n-1}, ga_n)\}$ is a decreasing sequence of non-negative real numbers so that

$$\lim_{n\to\infty} r_{mb}(ga_{n-1}, ga_n) = \alpha \geq 0.$$

By taking the superior limit as $n \to \infty$ in inequality (3), we have

$$\lim_{n\to\infty} r_{mb}(ga_n, ga_{n+1}) \leq \lambda \lim_{n\to\infty} r_{mb}(ga_{n-1}, ga_n)$$

which implies that $\alpha \leq \lambda\alpha$, a contraction unless $\alpha = 0$, so that

$$\lim_{n\to\infty} r_{mb}(ga_n, ga_{n+1}) = 0.$$

Similarly, from condition (1), we get

$$r_{mb}(ga_n, ga_n) = r_{mb}(fa_{n-1}, fa_{n-1}) \leq \lambda r_{mb}(ga_{n-1}, ga_{n-1}) \leq \cdots \leq \lambda^{n-1} r_{mb}(ga_0, ga_0).$$

By taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} r_{mb}(ga_n, ga_n) = 0. \tag{4}$$

Firstly, we show that $a_n \neq a_m$ for any $n \neq m$. On the contrary, if $a_n = a_m$ for some $n > m$, then we have $a_{n+1} = fa_n = fa_m = a_{m+1}$. On using (1) with $a = a_m$ and $b = a_{m+1}$, we have

$$r_{mb}(a_m, a_{m+1}) = r_{mb}(a_n, a_{n+1}) < r_{mb}(a_{n-1}, a_n) < \cdots < r_{mb}(a_m, a_{m+1}),$$

which is a contradiction. This in turn shows that $a_n \neq a_m$ for all $n \neq m$.

Now, we assert that $\{ga_n\}$ is Cauchy sequence. In doing so, we distinguish two cases.

Case 1. Firstly, let p be odd, that is, $p = 2m + 1$ for any $m \geq 1$. Now, using $(4r_{mb})$ for any $n \in \mathbb{N}$, we have

$$\begin{aligned}
\left(r_{mb}(ga_n, ga_{n+p}) - r_{mb_{ga_n,ga_{n+p}}}\right) &\leq s\Big[\left(r_{mb}(ga_n, ga_{n+1}) - r_{mb_{ga_n,ga_{n+1}}}\right) \\
&\quad + \left(r_{mb}(ga_{n+1}, ga_{n+2}) - r_{mb_{ga_{n+1},ga_{n+2}}}\right) \\
&\quad + \left(r_{mb}(ga_{n+2}, ga_{n+p}) - r_{mb_{ga_{n+2},ga_{n+p}}}\right)\Big] \\
&\quad - r_{mb}(ga_{n+1}, ga_{n+1}) - r_{mb}(ga_{n+2}, ga_{n+2}) \\
&\leq s[r_{mb}(ga_n, ga_{n+1}) + r_{mb}(ga_{n+1}, ga_{n+2}) \\
&\quad + r_{mb}(ga_{n+2}, ga_{n+p})] - r_{mb}(ga_{n+1}, ga_{n+1}) \\
&\quad - r_{mb}(ga_{n+2}, ga_{n+2}) \\
&\leq s[\lambda^n r_{mb}(ga_0, ga_1) + \lambda^{n+1} r_{mb}(ga_0, ga_1)] \\
&\quad + s r_{mb}(ga_{n+2}, ga_{n+2m+1}) \\
&\quad - \lambda^{n+1} r_{mb}(ga_0, ga_0) - \lambda^{n+2} r_{mb}(ga_0, ga_0) \\
&= s(\lambda^n + \lambda^{n+1}) r_{mb}(ga_0, ga_1) + s r_{mb}(ga_{n+2}, ga_{n+2m+1}) \\
&\quad - (\lambda^{n+1} + \lambda^{n+2}) r_{mb}(ga_0, ga_0) \\
&\leq s(\lambda^n + \lambda^{n+1}) r_{mb}(ga_0, ga_1) + s^2(\lambda^{n+2} + \\
&\quad \lambda^{n+3}) r_{mb}(ga_0, ga_1) + \cdots + s^m(\lambda^{n+2m-2} + \\
&\quad \lambda^{n+2m-1}) r_{mb}(ga_0, ga_1) + s^m \lambda^{n+2m} r_{mb}(ga_0, ga_1) \\
&\quad - (\lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + \cdots) r_{mb}(ga_0, ga_0) \\
&= \big(s\lambda^n(1 + s\lambda^2 + s^2\lambda^4 + \cdots) + s\lambda^{n+1}(1 + s\lambda^2 + \\
&\quad s^2\lambda^4 + \cdots)\big) r_{mb}(ga_0, ga_1) \\
&= \frac{1+\lambda}{1 - s\lambda^2} s\lambda^n r_{mb}(ga_0, ga_1) - \frac{\lambda^{n+1}}{1-\lambda} r_{mb}(ga_0, ga_0),
\end{aligned}$$

yielding thereby

$$r_{mb}(ga_n, ga_{n+2m+1}) - r_{mb_{ga_n,ga_{n+2m+1}}} \leq \frac{1+\lambda}{1 - s\lambda^2} s\lambda^n r_{mb}(ga_0, ga_1) - \frac{\lambda^{n+1}}{1-\lambda} r_{mb}(ga_0, ga_0). \quad (5)$$

Letting $n \to \infty$ in (5), we conclude that

$$\lim_{n,m \to \infty} r_{mb}(ga_n, ga_{n+2m+1}) - r_{mb_{ga_n,ga_{n+2m+1}}} = 0.$$

Case 2. Secondly, assume that p is even, that is, $p = 2m$ for any $m \geq 1$. Then,

$$\begin{aligned}
\left(r_{mb}(ga_n, ga_{n+p}) - r_{mb_{ga_n,ga_{n+p}}}\right) &\leq s\Big[\Big(r_{mb}(ga_n, ga_{n+1}) - r_{mb_{ga_n,ga_{n+1}}}\Big) \\
&\quad + \Big(r_{mb}(ga_{n+1}, ga_{n+2}) - r_{mb_{ga_{n+1},ga_{n+2}}}\Big) \\
&\quad + \Big(r_{mb}(ga_{n+2}, ga_{n+p}) - r_{mbga_{n+2},ga_{n+p}}\Big)\Big] \\
&\quad - r_{mb}(ga_{n+1}, ga_{n+1}) - r_{mb}(ga_{n+2}, ga_{n+2}) \\
&\leq s[r_{mb}(ga_n, ga_{n+1}) + r_{mb}(ga_{n+1}, ga_{n+2}) \\
&\quad + r_{mb}(ga_{n+2}, ga_{n+p})] - r_{mb}(ga_{n+1}, ga_{n+1}) \\
&\quad - r_{mb}(ga_{n+2}, ga_{n+2}) \\
&\leq s[\lambda^n r_{mb}(ga_0, ga_1) + \lambda^{n+1} r_{mb}(ga_0, ga_1)] \\
&\quad + s r_{mb}(ga_{n+2}, ga_{n+2m}) - \lambda^{n+1} r_{mb}(ga_0, ga_0) \\
&\quad - \lambda^{n+2} r_{mb}(ga_0, ga_0) \\
&= s(\lambda^n + \lambda^{n+1}) r_{mb}(ga_0, ga_1) + s r_{mb}(ga_{n+2}, ga_{n+2m}) \\
&\quad - (\lambda^{n+1} + \lambda^{n+2}) r_{mb}(ga_0, ga_0) \\
&\leq s(\lambda^n + \lambda^{n+1}) r_{mb}(ga_0, ga_1) + s^2(\lambda^{n+2} + \\
&\quad \lambda^{n+3}) r_{mb}(ga_0, ga_1) + \cdots + s^{m-1}(\lambda^{n+2m-4} + \\
&\quad \lambda^{n+2m-3}) r_{mb}(ga_0, ga_1) + s^{m-1}\lambda^{n+2m-2} r_{mb}(ga_0, ga_2) \\
&\quad + s^{m-1}\lambda^{n+2m-2} r_{mb}(ga_0, ga_2) \\
&\quad - (\lambda^{n+1} + \lambda^{n+2} + \lambda^{n+3} + \cdots) r_{mb}(ga_0, ga_0) \\
&= \big(s\lambda^n(1 + s\lambda^2 + s^2\lambda^4 + \cdots) \\
&\quad + s\lambda^{n+1}(1 + s\lambda^2 + s^2\lambda^4 + \cdots)\big) r_{mb}(ga_0, ga_1) \\
&= \frac{1+\lambda}{1 - s\lambda^2} s\lambda^n r_\xi(ga_0, ga_1) + s^{m-1}\lambda^{n+2m-2} r_{mb}(ga_0, ga_2) \\
&\quad - \frac{\lambda^{n+1}}{1-\lambda} r_{mb}(ga_0, ga_0),
\end{aligned}$$

so that

$$\begin{aligned}
r_{mb}(ga_n, ga_{n+2m}) - r_{mb_{ga_n,ga_{n+2m}}} &\leq \frac{1+\lambda}{1 - s\lambda^2} s\lambda^n r_{mb}(ga_0, ga_1) + s^{m-1}\lambda^{n+2m-2} r_{mb}(ga_0, ga_2) \\
&\quad - \frac{\lambda^{n+1}}{1-\lambda} r_{mb}(ga_0, ga_0).
\end{aligned} \tag{6}$$

Taking the limit as $n \to \infty$ in the inequality (6), we conclude that

$$\lim_{n,m \to \infty} r_{mb}(ga_n, ga_{n+2m}) - r_{mb_{ga_n,ga_{n+2m}}} = 0.$$

Therefore, in both the cases, we have

$$\lim_{n,m \to \infty} (r_{mb}(ga_n, ga_m) - m_{rb_{ga_n,ga_m}}) = 0.$$

On the other hand, without loss of generality, we may assume that

$$M_{rb_{ga_n,ga_m}} = r_{mb}(ga_n, ga_n).$$

Hence, we obtain

$$\begin{aligned} M_{rb_{ga_n,ga_m}} - m_{rb_{ga_n,ga_m}} &\leq M_{rb_{ga_n,ga_m}} \\ &= r_{mb}(ga_n, ga_n) \\ &\leq \lambda^n r_{mb}(ga_0, ga_0). \end{aligned}$$

Taking the limit of the above inequality as $n \to \infty$, we deduce that

$$\lim_{n,m\to\infty} (M_{rb_{ga_n,ga_m}} - m_{rb_{ga_n,ga_m}}) = 0.$$

Therefore, the sequence, $\{ga_n\}$ is Cauchy in Y. Since Y is $\overline{\text{O}}$-complete, then there exists some $a \in Y$ such that

$$ga_n \uparrow a. \tag{7}$$

Owing to condition (1), there exists some $z \in X$ such that $a = gc$, meaning that

$$ga_n \uparrow gc. \tag{8}$$

We can now show that z is a coincidence point of the pair (f, g) by using the condition (iv). Consider f to be $(g, \overline{\text{O}})$-continuous. We find this as a result of condition (8), in which we have $fa_n \to fc$, which (as a result of (2)) gives rise to $gc = fc$.

Alternately, assume that $(Y, d, \preceq)$ has the g-ICC-property. Then, there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $ga_{n_k} \preceq gc, \ \forall\, k \in \mathbb{N}$. On setting $a = a_{n_k}$, $b = c$ in (1), we have (for all $k \in \mathbb{N}_0$)

$$r_{mb}(ga_{n_k+1}, fc) \quad = \quad r_{mb}(fa_{n_k}, fc) \leq \lambda r_{mb}(ga_{n_k}, gc)), \tag{9}$$

On using Equations (2) and (8) and taking the superior limit in (9) as $k \to \infty$, we have

$$r_{mb}(gc, fc) \leq \lambda r_{mb}(gc, gc),$$

which is a contradiction unless $gc = fc$. This concludes the proof. □

By setting $Y = g(X)$ in Theorem 1, we deduce a new result for the ordered–theoretic coincidence point.

Corollary 1. *Let $(X, d, \preceq)$ be an ordered R_{mb}-metric space and $f, g : X \to X$ such that f is a g-increasing. Suppose the following conditions hold:*

(i) *There exists an $a_0 \in X$ such that $ga_0 \preceq fa_0$,*
(ii) *For all $a, b \in X$ such that $ga \preceq gb$, there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$r_{mb}(fa, fb) \leq \lambda r_{mb}(ga, gb)\ \forall\, a, b \in X.$$

(iii) *$f(X) \subseteq g(X)$,*
(iv) *Either*
 (a) *f is $(g, \overline{\text{O}})$-continuous or*
 (b) *$g(X)$ is complete and enjoys the ICC-property.*

Then, the pair (f, g) has a coincidence point.

Choosing $g = I_X$ (where I_X is an identity mapping) in the Theorem 1, we deduce the generalized version of the Theorem 3.2 due to Asim el al. [2].

Corollary 2. *Let $(X, r_{mb}, \preceq)$ be an ordered complete R_{mb}-metric space with $s \geq 1$ and $f : X \to X$ such that f is increasing. Suppose the following conditions hold:*

(1) *There exists an $a_0 \in X$ such that $a_0 \preceq fa_0$,*

(2) *For all $a, b \in X$ such that $a \preceq b$, there exists $\lambda \in [0, \frac{1}{s})$ such that*

$$r_{mb}(fa, fb) \leq \lambda r_{mb}(a, b)\ \forall\ a, b \in X.$$

(3) *Either*

(a) *f is $\overline{\mathrm{O}}$-continuous or*

(b) *$(Y, r_{mb}, \preceq)$ enjoys the ICC-property.*

Then, f has a fixed point.

Example 2. *Consider $X = (-1, 0]$. Define $r_{mb} : X \times X \to \mathbb{R}_+$ by (for all $a, b \in X$):*

$$r_{mb}(a, b) = \max\{a, b\}^2 + |a - b|^2.$$

Note that every increasing Cauchy sequence is convergent in X. Therefore, $(X, r_{mb}, \preceq)$ is an $\overline{\mathrm{O}}$-complete R_{mb}-metric space with coefficient $s = 3$.

Now, we define an ordered relation on X:

$$a, b \in X,\ a \preceq b \Leftrightarrow\ a = b \text{ or } \left(a, b \in \{0\} \cup \left\{\frac{-1}{n} : n = 2, 3, \cdots\right\} \text{ and } a \leq b\right),$$

where $\leq$ is the usual order. Define the mappings $f, g : X \to X$ as follows:

$$fa = \begin{cases} 0, & \text{if } a = 0 \\ \frac{-1}{4n}, & \text{if } a = -1/n, n = 2, 3, \cdots \\ -0.5, & \text{otherwise} \end{cases} \quad ga = \begin{cases} 0, & \text{if } a = 0 \\ \frac{-1}{2n}, & \text{if } a = -1/n, n = 2, 3, \cdots. \\ -0.5, & \text{otherwise} \end{cases}$$

Observe that f is g-increasing and X has the g-ICC-property.

We distinguish two cases:

Case 1. *Taking $a = 1/n$, (wherein $n = 3, 4, \cdots$) and $b = 0$. Then, from (1), we have*

$$\begin{aligned} r_{mb}(fa, fb) &= \max\left\{\frac{-1}{4n}, 0\right\}^2 + \left|\frac{-1}{4n} - 0\right|^2 \\ &= \frac{1}{4}\left\{\max\left\{\frac{-1}{2n}, 0\right\}^2 + \left|\frac{-1}{2n} - 0\right|^2\right\} \\ &= \frac{1}{4} r_{mb}(ga, gb) \end{aligned}$$

Case 2. *Taking $a = 1/n, b = 1/m\ m > n \geq 3$. Then, we have*

$$\begin{aligned} r_{mb}(fa, fb) &= \max\left\{\frac{-1}{4n}, \frac{-1}{4m}\right\}^2 + \left|\frac{-1}{4n} - \frac{-1}{4m}\right|^2 \\ &= \frac{1}{4}\left\{\max\left\{\frac{-1}{2n}, \frac{-1}{2m}\right\}^2 + \left|\frac{-1}{2n} - \frac{-1}{2m}\right|^2\right\} \\ &= \frac{1}{4} r_{mb}(ga, gb) \end{aligned}$$

If $a = b$, then condition (1) holds trivially. Thus, all the conditions of Theorems 1 are satisfied, and also the pair (f, g) has a unique common fixed point (namely $a = 0$).

Now, one can conclude that the present example is not applicable for the fixed point results of Asim et al. [2], as the space $(X, r_{mb}, \preceq)$ is not complete but $\overline{\mathrm{O}}$-complete R_{mb}-metric space. Moreover, it is easy to check that the contraction condition used in [2] does not hold for any $\lambda \in [0, \frac{1}{s})$.

Now, we prove the result for a unique point of coincidence as follows:

Theorem 2. *In Theorem 1, if we consider that $f(X)$ is g-directed, then (f,g) has a unique point of coincidence.*

Proof. Suppose that the mapping f has two coincidence points, say a and b, i.e., $fa = ga$ and $fb = gb$. We have shown that $ga = gb$. Since $f(X)$ is g-directed, there exists $c \in X$ such that gc is comparable to both fa and fb. Now, we assume that $fa \preceq gc$ and $fb \preceq gc$. Set $c = c_0$. Since $f(X) \subseteq g(X)$, one can define a sequence $c_n \subset X$ such that

$$gc_{n+1} = fc_n \text{ and } ga \preceq gc_n \text{ for all } n \in \mathbb{N}.$$

Using condition (1), we have (for all $n \in \mathbb{N}$)

$$r_{mb}(ga, gc_{n+1}) = r_{mb}(fa, fc_n) \leq \lambda r_{mb}(ga, gc_n) < r_{mb}(ga, gc_n) \tag{10}$$

Now, $\{r_{mb}(ga, gc_n)\}$ is a decreasing sequence of non-negative real numbers. On the contrary, assume that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} r_{mb}(ga, gc_n) = r.$$

Again, by employing the contraction condition (1), we obtain

$$\lim_{n\to\infty} r_{mb}(ga, gc_n) = \lim_{n\to\infty} \lambda^n r_{mb}(ga, gc) = 0. \tag{11}$$

Similarly, it is possible to demonstrate that

$$\lim_{n\to\infty} r_{mb}(gb, gc_n) = 0. \tag{12}$$

On using Equations (11) and (12), we have

$$\begin{aligned} r_{mb}(ga, gb) \quad \leq \quad & s[r_{mb}(ga, gc_n) + r_{mb}(gc_n, gc_{n+1}) + r_{mb}(gc_{n+1}, gb) \\ & - r_{mb}(gc_n, gc_n) - r_{mb}(gc_{n+1}, gc_{n+1}) \end{aligned}$$

Limiting as $n \to \infty$, we get $ga = gb$. Hence, the pair (f,g) has a unique point of coincidence. □

Theorem 3. *In Theorem 2, if we consider that the pair (f,g) is weakly compatible, then (f,g) has a unique common fixed point.*

Proof. Allow $a \in X$ to be an arbitrary coincidence point of the pair (f,g). There is a unique point of coincidence $a^* \in X$, for example, such that $fa = ga = a^*$ according to Theorem 2. As per Lemma 1, a^* is a coincidence point, i.e., $fa^* = ga^*$. Theorem 2 provides $fa^* = ga^* = a^*$, i.e., a^* is a unique common fixed point of f and g. □

Theorem 4. *If we replace the conditions $\overline{\mathrm{O}}$-complete, $(g, \overline{\mathrm{O}})$-continuous, and g-ICC with $\underline{\mathrm{O}}$-complete (or O-complete), $(g, \underline{\mathrm{O}})$-continuous (or (g, O)-continuous), and g-DCC (or g-MCC) and the property $ga_0 \preceq fa_0$ is followed by $ga_0 \succeq fa_0$ (or $ga_0 \prec\succ fa_0$), then the results of Theorems 1–3 remain true.*

4. Conclusions

This paper consists of ordered–theoretic coincidence point results, point of coincidence results, and common fixed point results in the framework of rectangular M_b-metric spaces endowed with an ordered relation. Some corollaries are also deduced from the existing literature. An example is also constructed to demonstrate the utility of one of the main results.

Author Contributions: All the authors contributed equally in the preparation of this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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