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Geometry and Spectral Theory Applied to Credit Bubbles in Arbitrage Markets: The Geometric Arbitrage Approach to Credit Risk

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Abstract: We apply Geometric Arbitrage Theory (GAT) to obtain results in mathematical finance for credit markets, which do not need stochastic differential geometry in their formulation. The remarkable aspect of the GAT is the gauge symmetry, which can be translated to the financial context, by packaging all of the asset model information into a (stochastic) principal fiber bundle. We obtain closed-form equations involving default intensities and loss-given defaults characterizing the no-free-lunch-with-vanishing-risk condition for government and corporate bond markets while relying on the spread-term structure with default intensity and loss-given default. Moreover, we provide a sufficient condition equivalent to the Novikov condition implying the absence of arbitrage. Furthermore, the generic dynamics for an isolated credit market allowing for arbitrage possibilities (and minimizing the total quantity of potential arbitrage) are derived, and arbitrage credit bubbles for both base credit assets and credit derivatives are explicitly computed. The existence of an approximated risk-neutral measure allowing the definition of fundamental values for the assets is inferred through spectral theory. We show that instantaneous bond returns are serially uncorrelated and centered, that the expected value of credit bubbles remains constant for future times where no coupons are paid, and that the variance of the market portfolio nominals is concurrent with that of the corresponding bond deflators.

Keywords: geometric arbitrage and spectral theory; gauge symmetry and invariance; credit risk; asset bubbles; deflators; interest rate term structures; market portfolio dynamics

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1. Introduction

Following [1–3], this paper utilizes a conceptual structure called Geometric Arbitrage Theory to model arbitrage in financial markets in the presence of credit risks, which are some of the fundamental factors of financial risks. Geometric Arbitrage Theory (GAT) embeds traditional stochastic finance into a modern stochastic differential geometric framework to characterize arbitrage. The main contribution of this approach consists of modeling markets made of basic financial instruments together with their term structures as principal fiber bundles. The financial features of this market—such as the lack of arbitrage and equilibrium—are then characterized in terms of standard differential geometric constructions—such as curvature—associated with a natural connection in this fiber bundle. Principal fiber bundle theory has been heavily exploited in theoretical physics as the language in which the laws of nature can be best formulated by providing an invariant framework to describe physical systems and their dynamics. These ideas can be carried over to mathematical finance and economics. A market is a financial–economic system that can be described by an appropriate principal fiber bundle. A principle such as the invariance of market laws under the change in numéraire can then be seen as gauge invariance.

The fact that gauge theories are the natural language to describe economics was first proposed by Malaney and Weinstein in the context of the economic index problem [4,5]. Ilinski (see [6–8]) proposed to view arbitrage as the curvature of a gauge connection, in analogy to some physical theories. Independently, the authors of [9] further developed the seminal work of Flesaker and Hughston [10] and utilized techniques from differential geometry (indirectly mentioned by allusive wording) to reduce the complexity of asset models before stochastic modeling.

Perhaps due to its borderline nature lying at the intersection between stochastic finance and differential geometry, there was almost no further mathematical research, and the subject, unfairly considered as an exotic topic, remained confined to econophysics (see [11–13]). In more recent years, financial economics and mathematical finance have addressed the challenge of modeling markets and allowing for arbitrage opportunities, such as in [14,15], without utilizing GAT. In [1,2], Geometric Arbitrage Theory was given a rigorous mathematical foundation by utilizing the formal background of stochastic differential geometry, as in [16–21]. GAT already proved itself as a practical portfolio management tool allowing one to construct arbitrage portfolios in a medium-frequency trading set-up, as shown in [22]. Moreover, GAT can bring new insights into mathematical finance by looking at the same concepts from a different perspective, so that the new results can be understood without a stochastic differential geometric background, thus extending the (credit) asset bubble framework for no-arbitrage markets, as described in [23–26]. This is the case for the main contributions of this paper, a characterization of the no-free-lunch-with-vanishing-risk condition for credit markets.

More precisely, we assume that there is a market in one currency for both government and corporate bonds for different maturities, and we choose the government bond as a numéraire. With the formal notation introduced in Section 4.2, we will prove following results.

Theorem 1 (**No Arbitrage Credit Market**). *Let $\lambda = \lambda_t$ and $\mathrm{LGD} = \mathrm{LGD}_t$ be the default intensity and the loss-given default, respectively, of the corporate bond. Let $P^{Corp,\,Gov}$ and $r^{Corp,\,Gov}$ be the term structures and short rate for corporate and government bonds. The following assertions are equivalent:*

(i) *The credit market model satisfies the no-free-lunch-with-vanishing-risk condition.*

(ii) *There exists a positive local martingale $\beta = (\beta_t)_{t\geq 0}$ such that deflators and short rates satisfy, for all times, the condition*
$$r_t^{Corp} - r_t^{Gov} = \beta_t \mathrm{LGD}_t \lambda_t.$$

(iii) *There exists a positive local martingale $\beta = (\beta_t)_{t\geq 0}$ such that deflators and term structures satisfy, for all times, the condition*
$$\frac{P_{t,s}^{Corp}}{P_{t,s}^{Gov}} = \mathbb{E}_t\left[\exp\left(-\int_t^s du\, \beta_u \mathrm{LGD}_u \lambda_u\right)\right].$$

This characterization of no-arbitrage credit markets has been known for a long time (see, for instance, [27], p. 39) and can now be easily inferred as a consequence of Geometric Arbitrage Theory.

Moreover, we obtain what, to our knowledge, is a new result for credit markets.

Theorem 2 (**Novikov's Condition**). *Let the credit market fulfill*
$$r_t^{Corp} - r_t^{Gov} = \beta_t \mathrm{LGD}_t \lambda_t,$$
for a positive semi-martingale $(\beta_t)_t$ and
$$\mathbb{E}_0\left[\exp\left(\int_0^T dt\, \frac{1}{2}\left(\frac{\lambda_t \mathrm{LGD}_t}{1-\mathrm{LGD}_t X_t} - (r_t^{Corp} - r_t^{Gov})\right)^2 \frac{t}{Q_t(K)}\right)\right] < +\infty,$$

where

$$Q_t(K) := \frac{W_t^\dagger W_t}{t} \sim \chi^2(K).$$

Then, the credit market satisfies the no-free-lunch-with-vanishing-risk condition.

Furthermore, applying recently discovered results about the extension of asset bubbles for markets allowing for arbitrage, in Theorem 12, we can explicitly compute the arbitrage bubble for a credit market composed of base assets and credit derivatives, as well as its market dynamics. These are, again, new results.

This paper is structured as follows. Section 2 summarizes traditional stochastic finance and the results of the GAT by reviewing our previous paper [1–3]. A guiding example is provided for a market whose asset prices are Itô processes. Proofs are omitted and can be found in [1–3,28]. Section 3 provides the mathematical background in order to define generalized derivatives of stochastic processes, which are needed in the following, since the typical processes associated with credit risk have jumps and, in particular, do not allow for Nelson's derivatives in the strong sense. Section 4 reviews the fundamentals of credit risk and introduces the two basic model types: the structural and the reduced-form (intensity-based) types. In Section 5, the Geometric Arbitrage Theory toolbox introduced in Section 2 is then utilized to prove results for credit markets allowing for arbitrage. Section 6 concludes.

2. Background of Geometric Arbitrage Theory: Geometrical–Topological and Spectral Theory Reformulation of Mathematical Finance

In this section, we describe the main concepts of GAT introduced in [1,2] and further extended in [28], to which we refer for proofs and examples. This can be considered as the GAT reformulation of market risk.

2.1. The Traditional Market Model

In this subsection, we will summarize the traditional set-up, which will be rephrased in Section 2.2 in differential geometric terminology. The reader is referred to [29] and the ultimate reference [30]. We basically assume continuous-time trading and that the set of trading dates is finite. This assumption is general enough to embed the cases of finite and infinite discrete times, as well as the one with a finite horizon in continuous time. Note that, while it is true that, in the real world, trading occurs at discrete times only, these are not known a priori and can be virtually any points in the time continuum. This motivates the technical effort of continuous-time stochastic finance.

The uncertainty is modeled by a filtered probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathbb{P}$ is the statistical (physical) probability measure, $\mathcal{A} = \{\mathcal{A}_t\}_{t\in[0,+\infty[}$ is an increasing family of sub-σ-algebras of $\mathcal{A}_\infty$ representing the increasing information about the market, and $(\Omega, \mathcal{A}_\infty, \mathbb{P})$ is a probability space. The filtration $\mathcal{A}$ is assumed to satisfy the usual conditions, that is,

- right continuity: $\mathcal{A}_t = \bigcap_{s>t} \mathcal{A}_s$ for all $t \in [0, +\infty[$.
- $\mathcal{A}_0$ contains all null sets of $\mathcal{A}_\infty$.

The market consists of finitely many assets indexed by $j = 1, \ldots, N$, whose nominal prices are given by the vector-valued sem-imartingale $S : [0, +\infty[\times\Omega \to \mathbf{R}^N$ denoted by $(S_t)_{t\in[0,+\infty[}$, which is adapted to the filtration $\mathcal{A}$. The stochastic process $(S_t^j)_{t\in[0,+\infty[}$ describes the price at time t of the j-th asset in terms of the unit of cash *at time* $t = 0$. More precisely, we assume the existence of a 0-th asset, the cash, a strictly positive semi-martingale, which evolves according to $S_t^0 = \exp(\int_0^t du\, r_u^0)$, where the predictable semi-martingale $(r_t^0)_{t\in[0,+\infty[}$ represents the continuous interest rate provided by the cash account: One always knows in advance what the interest rate on one's own bank account is, but this can change from time to time. The cash account is, therefore, considered the locally risk-less asset in contrast to the other assets, the risky ones. In the following, we will mainly utilize discounted prices, defined as $\hat{S}_t^j := S_t^j / S_t^0$, representing the asset prices in terms of the current unit of cash.

We emphasize that there is no need to assume that asset prices are positive. However, there must be at least one strictly positive asset—in our case, the cash. If we want to renormalize the prices by choosing another asset instead of the cash as a reference, i.e., by making it into our numéraire, then this asset must have a strictly positive price process. More precisely, a generic numéraire is an asset whose nominal price is represented by a strictly positive stochastic process $(B_t)_{t\in[0,+\infty[}$ and that is a portfolio of the original assets $j = 0, 1, 2, \ldots, N$. The discounted prices of the original assets are then represented in terms of the numéraire by the semi-martingales $\hat{S}^j_t := S^j_t / B_t$. It is worth emphasizing that the construction depends on the gauge (i.e., the numéraire); however, our new results, such as Theorems 1 and 2, are independent of the choice of the numéraire. In this sense, we can say that the symmetry plays a central role in Geometric Arbitrage Theory.

We assume that there are no transaction costs and that short sales are allowed. We remark that the absence of transaction costs can be a serious limitation for a realistic model. The filtration $\mathcal{A}$ is not necessarily generated by the price process $(S_t)_{t\in[0,+\infty[}$; other sources of information than prices are allowed. All agents have access to the same information structure, that is, to the filtration $\mathcal{A}$.

A strategy is a predictable stochastic process $x : [0, +\infty[\times\Omega \to \mathbf{R}^N$ describing the portfolio holdings. The stochastic process $(x^j_t)_{t\in[0,+\infty[}$ represents the number of pieces of the j-th asset portfolio held by the portfolio as time goes by. We remark that the Itô stochastic integral

$$\int_0^t x \cdot dS = \int_0^t x_u \cdot dS_u,$$

and the Stratonovich stochastic integral

$$\int_0^t x \circ dS := \int_0^t x \cdot dS + \frac{1}{2}\int_0^t d\langle x, S\rangle = \int_0^t x_u \cdot dS_u + \frac{1}{2}\int_0^t d\langle x, S\rangle_u \tag{1}$$

are well defined for this choice of integrator (S) and integrand (x), as long as the strategy is admissible, meaning that it is v-admissible for some $v \geq 0$, that is, $x = (x_t)_{t\in[0,+\infty[}$ is an S-integrable predictable process for which the Itô integral satisfies $\int_0^t x \cdot dS \geq -v$ almost surely for all $t \geq 0$ with $x_0 = 0$. The bracket $\langle\cdot,\cdot\rangle$ in (1) denotes the quadratic covariation of two processes. In a general context, strategies do not need to be semi-martingales, but if we want the quadratic covariation and, hence, Stratonovich's integral to be well defined, we must require this additional assumption. For details about stochastic integration, we refer to Appendix A in [18], which summarizes Chapter VII of the authoritative [31]. The portfolio value is the process $\{V_t\}_{t\in[0,+\infty[}$ defined by

$$V_t := V^x_t := x_t \cdot S_t.$$

An admissible strategy x is said to be self-financing if and only if the portfolio value at time t is given by

$$V_t = V_0 + \int_0^t x_u \cdot dS_u.$$

This means that the portfolio gain is the Itô integral of the strategy with the price process as integrator; the change in portfolio value is purely due to changes in the assets' values. The self-financing condition can be rewritten in differential form as

$$dV_t = x_t \cdot dS_t.$$

As pointed out in [32], if we want to utilize Stratonovich's integral to rephrase the self-financing condition while maintaining its economical interpretation (which is necessary for the subsequent constructions of mathematical finance), we write

$$V_t = V_0 + \int_0^t x_u \circ dS_u - \frac{1}{2}\int_0^t d\langle x, S\rangle_u$$

or, equivalently,

$$dV_t = x_t \circ dS_t - \frac{1}{2}\, d\langle x, S\rangle_t.$$

An arbitrage strategy for the market model is an admissible self-financing strategy x for which one of the following condition holds for some horizon $T > 0$:

- $\mathbb{P}[V_0^x < 0] = 1$ and $\mathbb{P}[V_T^x \geq 0] = 1$,
- $\mathbb{P}[V_0^x \leq 0] = 1$ and $\mathbb{P}[V_T^x \geq 0] = 1$ with $\mathbb{P}[V_T^x > 0] > 0$.

In Chapter 9 of [30], the no-arbitrage condition is given a topological characterization. In view of the fundamental theorem of asset pricing, the no-arbitrage condition is substituted by a stronger condition, the so-called no-free-lunch-with-vanishing-risk condition. Let us denote by $L^0(\Omega, \mathcal{A}_T, \mathbb{P})$ the set of $\mathcal{A}_T$-measurable random variables and by $L^\infty(\Omega, \mathcal{A}_T, \mathbb{P})$ the set of bounded random variables. Furthermore, L^0_+ and L^∞_+ denote the subsets of positive random variables in L^0 and L^∞, respectively.

Definition 1 (Arbitrage). *Let the process $(S_t)_{t\in[0,+\infty[}$ be a semi-martingale and let $(x_t)_{t\in[0,+\infty[}$ be an admissible self-financing strategy. Let us consider trading up to time $T \leq \infty$. The portfolio wealth at time t is given by $V_t(x) := V_0 + \int_0^t x_u \cdot dS_u$, and we denote by K_0 the subset of $L^0(\Omega, \mathcal{A}_T, \mathbb{P})$ containing all such $V_T(x)$, where x is any admissible self-financing strategy. We define*

- $C_0 := K_0 - L^0_+(\Omega, \mathcal{A}_T, \mathbb{P})$.
- $C := C_0 \cap L^\infty(\Omega, \mathcal{A}_T, \mathbb{P})$.
- $\bar{C}$*: the closure of C in L^∞ with respect to the norm topology.*
- $\mathcal{V}^{V_0} := \left\{ (V_t)_{t\in[0,+\infty[} \,\middle|\, V_t = V_t(x), \text{ where } x \text{ is } V_0\text{-admissible} \right\}$.
- $\mathcal{V}_T^{V_0} := \left\{ V_T \,\middle|\, (V_t)_{t\in[0,+\infty[} \in \mathcal{V}^{V_0} \right\}$*: terminal wealth for V_0-admissible self-financing strategies.*

  We say that S satisfies
- ***the no-arbitrage (NA) condition*** *if and only if $C \cap L^\infty_+(\Omega, \mathcal{A}_T, \mathbb{P}) = \{0\}$.*
- ***the no-free-lunch-with-vanishing-risk (NFLVR) condition*** *if and only if $\bar{C} \cap L^\infty_+(\Omega, \mathcal{A}_T, \mathbb{P}) = \{0\}$.*
- ***the no-unbounded-profit-with-bounded-risk (NUPBR) condition*** *if and only if $\mathcal{V}_T^{V_0}$ is bounded in L^0 for some $V_0 > 0$.*

The relationship between these three different types of arbitrage was elucidated in [33] and in [34] with the proof of the following result.

Theorem 3.

$$\text{(NFLVR)} \Leftrightarrow \text{(NA)} + \text{(NUPBR)}.$$

In 1994, Delbaen and Schachermayer proved (see [30], Chapter 9.4, particularly the main Theorem 9.1.1) the following theorem.

Theorem 4 (First Fundamental Theorem of Asset Pricing in Continuous Time). *Let $(S_t)_{t\in[0,+\infty[}$ and $(\hat{S}_t)_{t\in[0,+\infty[}$ be bounded semi-martingales. There is an equivalent martingale measure $\mathbb{P}^*$ for the discounted prices $\hat{S}$ if and only if the market model satisfies the (NFLVR).*

This is a generalization for continuous time of the Dalang–Morton–Willinger theorem proved in 1990 (see [30], Chapter 6) for the discrete-time case, where the (NFLVR) is relaxed to the (NA) condition. The Dalang–Morton–Willinger theorem generalizes the Harrison and Pliska theorem (see [30], Chapter 2) to arbitrary probability spaces, which holds true in discrete time for finite probability spaces.

An equivalent alternative to the martingale measure approach for asset pricing purposes is given by the pricing kernel (state price deflator) method.

Definition 2. *Let $(S_t)_{t\in[0,+\infty[}$ be a semi-martingale describing the price process for the assets of our market model. The positive semi-martingale $(\beta_t)_{t\in[0,+\infty[}$ is called the pricing kernel (or state price deflator) for S if and only if $(\beta_t S_t)_{t\in[0,+\infty[}$ is a $\mathbb{P}$-martingale.*

As shown in [29] (Chapter 7, definitions 7.18 and 7.47 and Theorem 7.48), the existence of a pricing kernel is equivalent to the existence of an equivalent martingale measure for a specific choice of numéraire. If we want the numéraire to be arbitrary, like the one we originally choose for the model, then we have to additionally assume that the pricing kernel β is a local $\mathbb{P}$-martingale.

Theorem 5. *Let $(S_t)_{t\in[0,+\infty[}$ and $(\hat{S}_t)_{t\in[0,+\infty[}$ be bounded semi-martingales. The process $\hat{S}$ admits an equivalent martingale measure $\mathbb{P}^*$ if and only if there is a pricing kernel β for S (or for $\hat{S}$), which is a local martingale.*

2.2. Geometric Reformulation of the Market Model: Primitives

Following [1–3], we are going to introduce a more general representation of the market model introduced in Section 2.1 that is better suited to the arbitrage modeling task. The underlying theory is that of (stochastic) principal fiber bundles of (stochastic) differential geometry. We restate several definitions, propositions, and theorems without proofs from [1–3,28] for the sake of the readers.

Definition 3. *A **gauge** is an ordered pair of two $\mathcal{A}$-adapted real-valued semi-martingales (D, P), where $D = (D_t)_{t\geq 0} : [0, +\infty[\times\Omega \to \mathbf{R}$ is called the deflator and $P = (P_{t,s})_{t,s} : \mathcal{T} \times \Omega \to \mathbf{R}$, which is called the term structure, is considered as a stochastic process with respect to the time t, termed valuation date, and $\mathcal{T} := \{(t,s) \in [0,+\infty[^2 \,|\, s \geq t\}$. The parameter $s \geq t$ is referred to as the maturity date. The following properties must be satisfied almost surely for all t, s such that $s \geq t \geq 0$:*

(i) $P_{t,s} > 0$,
(ii) $P_{t,t} = 1$.

Deflators and term structures can be considered *outside the context of fixed income.* An arbitrary financial instrument is mapped to a gauge (D, P) with the following economic interpretation:

- Deflator: D_t is the value of the financial instrument at time t expressed in terms of some numéraire. If we choose the cash account and the 0-th asset as the numéraire, then we can set $D^j_t := \hat{S}^j_t = \frac{S^j_t}{S^0_t} \quad (j = 1, \ldots N)$.
- Term structure: $P_{t,s}$ is the value at time t (expressed in units of deflator at time t) of a synthetic zero-coupon bond with maturity s delivering one unit of the financial instrument at time s. It represents a term structure of forward prices with respect to the chosen numéraire.

As we mentioned in [1–3], we point out that there is no unique choice for deflators and term structures describing an asset model. For example, if a set of deflators qualifies, then we can multiply every deflator by the same positive semi-martingale to obtain another suitable set of deflators. Of course, term structures have to be modified accordingly. The term "deflator" is clearly inspired by actuarial mathematics. In the present context, it refers to a nominal asset value up for division by a strictly positive semi-martingale (which can be the state price deflator if this exists and it is made into the numéraire). There is no need to assume that a deflator is a positive process. However, if we want to make an asset into our numéraire, then we have to make sure that the corresponding deflator is a strictly positive stochastic process.

Definition 4. *The term structure can be written as a functional of the instantaneous forward rate f defined as*

$$f_{t,s} := -\frac{\partial}{\partial s} \log P_{t,s}, \quad P_{t,s} = \exp\left(- \int_t^s dh f_{t,h} \right).$$

and

$$r_t := \lim_{s \to t^+} f_{t,s}$$

is called the short rate.

Remark 1. *Since $(P_{t,s})_{t,s}$ is a t-stochastic process (semi-martingale) depending on a parameter $s \geq t$, the s-derivative can be defined deterministically, and the expressions above make sense pathwise in both a classical and a generalized sense. In a generalized sense, we will always have a $\mathcal{D}'$ derivative for any $\omega \in \Omega$; this corresponds to a classic s-continuous derivative if $P_{t,s}(\omega)$ is a C^1-function of s for any fixed $t \geq 0$ and $\omega \in \Omega$.*

Remark 2. *The special choice of vanishing interest rate $r \equiv 0$ or flat-term structure $P \equiv 1$ for all assets corresponds to the classical model, where only asset prices and their dynamics are relevant.*

We now want to introduce transforms of deflators and term structures in order to group gauges containing the same (or less) stochastic information. The reader can refer to [1–3] for details. Therefore, we will consider *deterministic* linear combinations of assets modeled by the same gauge (e.g., zero bonds of the same credit quality with different maturities).

Definition 5. *Let $\pi : [0, +\infty[\longrightarrow \mathbb{R}$ be a deterministic cash-flow intensity (possibly generalized) function. We mean by this that π is, in general, a tempered distribution, which, in many cases, can be chosen to be a classical function. It induces a* ***gauge transform*** $(D, P) \mapsto \pi(D, P) := (D, P)^\pi := (D^\pi, P^\pi)$ *with the formulae*

$$D_t^\pi := D_t \int_0^{+\infty} dh\, \pi_h P_{t,t+h}, \qquad P_{t,s}^\pi := \frac{\int_0^{+\infty} dh\, \pi_h P_{t,s+h}}{\int_0^{+\infty} dh\, \pi_h P_{t,t+h}}.$$

Remark 3. *This gauge transform constructs a bond from the original gauge. The cash-flow intensity π specifies the bond's cash-flow structure. The bond's value at time t expressed in terms of the market model numéraire is given by D_t^π. The term structure of forward prices for the bond's future expressed in terms of the bond's current value is given by $P_{t,s}^\pi$.*

Let us consider a market consisting of N assets and a numéraire in a continuous-time setting. A general portfolio at time t is described by the vector of nominals $x \in \mathfrak{X}$ for an open set $\mathfrak{X} \subset \mathbb{R}^N$. Following Definition 3, for $j = 1, \ldots, N$, the asset model induces the gauge

$$(D^j, P^j) = ((D_t^j)_{t \in [0,+\infty[}, (P_{t,s}^j)_{s \geq t}),$$

where D^j denotes the deflator and P^j denotes the term structure. This can be written as

$$P_{t,s}^j = \exp\left(- \int_t^s f_{t,u}^j du \right),$$

where f^j is the instantaneous forward rate process for the j-th asset, and the corresponding short rate is given by $r_t^j := \lim_{u \to t^+} f_{t,u}^j$. For a portfolio with nominals $x \in \mathfrak{X} \subset \mathbb{R}^N$, we define

$$D_t^x := \sum_{j=1}^N x_j D_t^j \quad f_{t,u}^x := \sum_{j=1}^N \frac{x_j D_t^j}{\sum_{j=1}^N x_j D_t^j} f_{t,u}^j \quad P_{t,s}^x := \exp\left(- \int_t^s f_{t,u}^x du \right).$$

The short rate is written as

$$r_t^x := \lim_{u \to t^+} f_{t,u}^x = \sum_{j=1}^N \frac{x_j D_t^j}{\sum_{j=1}^N x_j D_t^j} r_t^j.$$

The image space of all possible strategies reads

$$M := \{(t, x) \in [0, +\infty[\times \mathfrak{X}\}.$$

As shown in [1,2], the set of all deflators $\{(D_t^x, P_{t,s}^x) \mid s \geq t \geq 0\}$ over $\mathfrak{X}$ can be given the structure of a principal fiber bundle, whose curvature characterizes arbitrage as follows.

Theorem 6 (**No Arbitrage**)**.** *The following assertions are equivalent:*

(i) *The market model satisfies the no-free-lunch-with-vanishing-risk condition.*

(ii) *There exists a positive local martingale $\beta = (\beta_t)_{t \geq 0}$ such that deflators and short rates satisfy, for all times and all portfolio nominals $(t, x) \in M$, the condition*

$$r_t^x = -\mathcal{D} \log(\beta_t D_t^x).$$

(iii) *There exists a positive local martingale $\beta = (\beta_t)_{t \geq 0}$ such that deflators and term structures satisfy, for all times and all portfolio nominals $(t, x) \in M$, the condition*

$$P_{t,s}^x = \frac{\mathbb{E}_t[\beta_s D_s^x]}{\beta_t D_t^x}.$$

This motivates the following definition.

Definition 6. *The market model satisfies the **zero-curvature condition (ZC)** if and only if the curvature vanishes almost surely.*

As proved in [35], the two weaker notions of arbitrage, the zero-curvature and the no-unbounded-profit-with-bounded-risk conditions, are equivalent. Together with the well-known results in [33,34], this leads to

Theorem 7.

$$\text{(NFLVR)} \Leftrightarrow \begin{cases} \text{(NUPBR)} \Rightarrow \text{(ZC)} \\ \text{(NA)} \end{cases}$$

As an example (see [1–3] for details) to demonstrate how the most important geometric concepts of Section 2 can be applied, we consider an asset model whose dynamics are given by a multidimensional Itô process. Let us consider a market consisting of $N + 1$ assets labeled by $j = 0, 1, \ldots, N$, where the 0-th asset is the cash account utilized as a numéraire. Therefore, as explained in the introductory Subsection 2.1, it suffices to model the price dynamics of the other assets $j = 1, \ldots, N$ expressed in terms of the 0-th asset. As vector-valued semi-martingales for the discounted price process $\hat{S} : [0, +\infty[\times \Omega \to \mathbf{R}^N$ and the short rate $r : [0, +\infty[\times \Omega \to \mathbf{R}^N$, we chose the multidimensional Itô processes given by

$$\begin{aligned} d\hat{S}_t &= \hat{S}_t(\alpha_t dt + \sigma_t dW_t) \\ dr_t &= a_t dt + b_t dW_t, \end{aligned} \tag{2}$$

where

- $(W_t)_{t \in [0, +\infty[}$ is a standard $\mathbb{P}$-Brownian motion in $\mathbf{R}^K$, for some $K \in \mathbf{N}$, and,
- $(\sigma_t)_{t \in [0, +\infty[}$, $(\alpha_t)_{t \in [0, +\infty[}$ are, respectively, $\mathbf{R}^{N \times K}$- and $\mathbf{R}^N$-valued stochastic processes,
- $(b_t)_{t \in [0, +\infty[}$, $(a_t)_{t \in [0, +\infty[}$ are, respectively, $\mathbf{R}^{N \times L}$- and $\mathbf{R}^N$-valued stochastic processes.

Proposition 1. *Let the dynamics of a market model be specified by following Itô's processes as in (2), where we additionally assume that the coefficients*

- $(\alpha_t)_t$, $(\sigma_t)_t$, *and* $(r_t)_t$ *satisfy*

$$\lim_{s\to t^+}\mathbb{E}_s[\alpha_t]=\alpha_t,\quad \lim_{s\to t^+}\mathbb{E}_s[r_t]=r_t,\quad \lim_{s\to t^+}\mathbb{E}_s[\sigma_t]=\sigma_t,$$

- $(\sigma_t)_t$ *is an Itô process, and*
- $(\sigma_t)_t$ *and* $(W_t)_t$ *are independent processes.*

Then, the market model satisfies the (ZC) condition if and only if

$$\alpha_t+r_t\in\mathrm{Range}(\sigma_t). \tag{3}$$

Remark 4. *In the case of the classical model, where there are no term structures (i.e.,* $r\equiv 0$*), the condition (3) reads as* $\alpha_t\in\mathrm{Range}(\sigma_t)$*.*

The stronger (NFLVR) condition in this guiding example appears in the following result.

Proposition 2. *For the market model whose dynamics are specified by the SDEs*

$$\begin{aligned} d\hat{S}_t &= \hat{S}_t(\alpha_t dt+\sigma_t dW_t)\\ dr_t &= a_t dt+b_t dW_t, \end{aligned}$$

the no-free-lunch-with-vanishing-risk condition (NFLVR) is satisfied if ***Novikov's condition***

$$\mathbb{E}_0\left[\exp\left(\int_0^T\frac{1}{2}\left|\sigma_t{}^\dagger(\sigma_t\sigma_t{}^\dagger)^{-1}(\alpha_t+r_t)\right|^2du\right)\right]<+\infty, \tag{4}$$

is fulfilled.

2.3. Bubbles in Arbitrage Markets

Asset bubbles were first introduced in [23] for complete markets and were recently extended to and computed for arbitrage markets in [3] thanks to spectral theory, whose main findings we briefly summarize here.

Definition 7. *The* ***cash-flow bundle*** *is defined as*

$$\mathcal{V}:=\underbrace{[0,T]\times\mathfrak{X}}_{=M}\times\mathbf{R}^{[0,+\infty[}.$$

The space of the sections of the cash-flow bundle can be made into a scalar product space by introducing, for stochastic sections $f=f(t,x,\omega)=(f_s(t,x,\omega))_{s\in[0,+\infty[}$ *and* $g=g(t,x,\omega)=(g_s(t,x,\omega))_{s\in[0,+\infty[}$*,*

$$\begin{aligned} (f,g) &:= \int_\Omega d\mathbb{P}\int_X d^N x\int_0^T dt\langle f,g\rangle(t,x,\omega)\\ &= \mathbb{E}_0\left[(f,g)_{L^2(M,\mathbb{R}^{[0,+\infty[})}\right]=(f,g)_{L^2(\Omega,\mathcal{V},\mathcal{A}_0,d\mathbb{P})}, \end{aligned}$$

where

$$\langle f,g\rangle(t,x,\omega):=\int_0^{+\infty}ds f_s(t,x,\omega)g_s(t,x,\omega).$$

The Hilbert space of integrable sections reads

$$\begin{aligned}\mathcal{H} &:= L^2(\Omega, \mathcal{V}, \mathcal{A}_0, d\mathbb{P}) \\ &= \left\{ f = f(t,x,\omega) = (f_s(t,x,\omega))_{s\in[0,+\infty[} \middle| (f,f)_{L^2(\Omega,\mathcal{V},\mathcal{A}_0,d\mathbb{P})} < +\infty \right\}.\end{aligned}$$

Let us extend the coordinate vector $x \in \mathbf{R}^N$ *with a* 0*-th component given by the time* t*. Let* $X = \sum_{j=0}^N X_j \frac{\partial}{\partial x_j}$ *be a vector field over* M *and let* $f = (f_s)_s$ *a section of the cash-flow bundle* $\mathcal{V}$*. Then,*

$$\nabla^{\mathcal{V}}_X f_t := \sum_{j=0}^N \left(\frac{\partial f_t}{\partial x_j} + K_j f_t \right) X_j,$$

where

$$K_0(x) = -r_t^x \qquad K_j(x) = \frac{D_t^j}{D_t^x} \quad (1 \le j \le N)$$

defines a covariant derivative on the cash-flow bundle $\mathcal{V}$*.*

Definition 8 (Spectral Lower Bound). *The highest spectral lower bound of the connection Laplacian on the cash-flow bundle* $\mathcal{V}$ *is given by*

$$\lambda_0 := \inf_{\substack{\varphi \in C^\infty(M,\mathcal{V}) \\ \varphi \neq 0 \\ B_N(\varphi)=0}} \frac{(\nabla^{\mathcal{V}}\varphi, \nabla^{\mathcal{V}}\varphi)_{\mathcal{H}}}{(\varphi,\varphi)_{\mathcal{H}}}$$

and it is assumed on the subspace

$$E_{\lambda_0} := \left\{ \varphi \middle| \varphi \in C^\infty(M,\mathcal{V}) \cap \mathcal{H},\, B_N(\varphi) = 0,\, (\nabla^{\mathcal{V}}\varphi,\, \nabla^{\mathcal{V}}\varphi)_{\mathcal{H}} \ge \lambda_0(\varphi,\varphi)_{\mathcal{H}} \right\}.$$

The space

$$\mathcal{K}_{\lambda_0} := \{ \varphi \in E_{\lambda_0} \,|\, \varphi \ge 0,\, \mathbb{E}[\varphi] = 1 \}$$

contains all candidates for the Radon–Nikodym derivative

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \varphi, \tag{5}$$

for a probability measure $\mathbb{P}^*$ *that is absolutely continuous with respect to the statistical measure* $\mathbb{P}$*.*

Theorem 4, which is the first fundamental theorem of asset pricing, can be reformulated as follows.

Proposition 3. *The market model satisfies the (NFLVR) condition if and only if* $\lambda_0 = 0$*. Any probability measure defined by (5) with* $\varphi \in \mathcal{K}_0$ *is a risk-neutral measure, that is,* $(D_t)_{t\in[0,T]}$ *is a vector-valued martingale with respect to* P^**, i.e.,*

$$\mathbb{E}_t^*[D_s] = D_t \qquad \text{for all } s \ge t \text{ in } [0,T].$$

The market is complete if and only if $\lambda_0 = 0$ *and* $\dim E_0 = 1$*.*

For arbitrage markets, we have that $\lambda_0 > 0$, and there exist no risk-neutral probability measures. Nevertheless, it is possible to define a fundamental value, but not in a unique way. The relationships between spectral theory and topology given by the Atiyah–Singer (Gauss–Bonnet–Chern) theorem were explored in [3] to construct topological obstructions to the no-free-lunch-with-vanishing-risk condition given by the non-vanishing of the Euler characteristic of the space of nominals $\mathfrak{X}$ and the non-vanishing of the homology group of the cash-flow bundle $\mathcal{V}$ over M.

Definition 9 (Fundamental Prices and Bubbles of Basic Assets' Arbitrage). *Let $(C_t)_{t\in[0,T]}$ be the $\mathbf{R}^N$ cash-flow stream stochastic process associated with the N assets of the market model with the given spectral lower bound λ_0 and Radon–Nikodym candidates' subspace $\mathcal{K}_{\lambda_0}$. For a given choice of $\varphi \in \mathcal{K}_{\lambda_0}$, the approximated fundamental value of the assets with the stochastic $\mathbf{R}^N$-valued price process $(S_t)_{t\in[0,T]}$ is defined as*

$$S_t^{*,\varphi} := \mathbb{E}_t\left[\varphi\left(\int_t^\tau dC_u\, \exp\left(-\int_t^u r_s^0 ds\right) + S_\tau \exp\left(-\int_t^\tau r_s^0 ds\right) 1_{\{\tau<+\infty\}}\right)\right] 1_{\{t<\tau\}},$$

where τ denotes the maturity time of all risky assets in the market model, and the approximated bubble is defined as

$$B_t^\varphi := S_t - S_t^{*,\varphi}.$$

The fundamental price vector for the assets and their asset bubble prices are defined as

$$\begin{aligned} S_t^* &:= S_t^{*,\varphi_0} \\ B_t &:= B_t^{\varphi_0} \\ \varphi_0 &:= \arg\min_{\varphi\in\mathcal{K}_{\lambda_0}} \mathbb{E}_0\left[\int_0^T ds\, |B_s^\varphi|^2\right]. \end{aligned} \tag{6}$$

The probability measure $\mathbb{P}^*$ with the Radon–Nikodym derivative

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \varphi_0$$

is called the ***minimal arbitrage measure****.*

Proposition 4. *The assets' fundamental values can be expressed as conditional expectations with respect to the minimal arbitrage measure as*

$$S_t^* := \mathbb{E}_t^*\left[\int_t^\tau dC_u\, \exp\left(-\int_t^u r_s^0 ds\right) + S_\tau \exp\left(-\int_t^\tau r_s^0 ds\right) 1_{\{\tau<+\infty\}}\right] 1_{\{t<\tau\}}. \tag{7}$$

Formula (7) can be reformulated in terms of the curvature, by means of which we can extend Jarrow, Protter, and Shimbo's results in [23] to the following bubble decomposition and classifications theorems proved in [3].

Theorem 8 (Bubble decomposition and types). *Let $T = +\infty$ and τ denote the maturity time of all risky assets in the market model. S_t admits a unique (up to a $\mathbb{P}$-evanescent set) decomposition*

$$S_t = \tilde{S}_t + B_t,$$

where $B = (B_t)_{t\in[0,T]}$ is a càdlàg process satisfying, for all $j = 1, \ldots, N$,

$$B_t^j = S_t^j - \mathbb{E}_t^*\left[\int_t^\tau dC_u^j\, \exp\left(-\int_t^u ds\, r_s^0\right) + \exp\left(\int_t^\tau ds\, r_s^0\right) S_\tau^j 1_{\{\tau<+\infty\}}\right] 1_{\{t<\tau\}}, \tag{8}$$

into a sum of fundamental and bubble values.

If there exists a non-trivial bubble B_t^j in an asset's price for $j = 1, \ldots, N$, then there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$, for which we have three and only three possibilities:

Type1: *B_t^j is a local super- or sub-martingale with respect to both $\mathbb{P}$ and $\mathbb{P}^*$ if $\mathbb{P}[\tau = +\infty] > 0$.*

Type2: *B_t^j is a local super- or sub-martingale with respect to both $\mathbb{P}$ and $\mathbb{P}^*$, but is not a uniformly integrable super- or sub-martingale if B_t^j is unbounded, but with $\mathbb{P}[\tau < +\infty] = 1$.*

Type3: *B_t^j is a strict local super- or sub- $\mathbb{P}$- and $\mathbb{P}^*$-martingale if τ is a bounded stopping time.*

Next, we analyze the situation for derivatives.

Definition 10 (Fundamental Prices and Bubbles of a Contingent Claim's Arbitrage). *Let us consider, in the context of Definition (9), a European-style derivative (e.g., put, call) given by the contingent claim with a unique payoff $G(S_T)$ at time T for an appropriate real-valued function G of N real variables. The contingent claim's fundamental price and its corresponding arbitrage bubble are defined in the case of base assets paying no dividends as*

$$\begin{aligned} V_t^*(G) &:= \mathbb{E}_t\left[\varphi_0 \exp\left(-\int_t^T r_s^0 ds\right) G(S_T)\, 1_{\{T<+\infty\}}\right] 1_{\{t<T\}} \\ &= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s^0 ds\right) G(S_T)\, 1_{\{T<+\infty\}}\right] 1_{\{t<T\}} \\ B_t(G) &:= V_t(G) - V_t^*(G), \end{aligned} \tag{9}$$

where φ_0 is the minimizer for the basic assets' bubbles defined in (6), $\mathbb{P}^*$ is the minimal arbitrage measure, and $(V_t(G))_{t\in[0,T]}$ is the price process of the European option.

In the case of base assets paying dividends, the definition becomes

$$\begin{aligned} V_t^*(G) &:= \mathbb{E}_t\left[\varphi_0 \exp\left(-\int_t^T r_s^0 ds\right) G\left(S_T \exp\left(\frac{C_T}{S_T}(T-t)\right)\right) 1_{\{T<+\infty\}}\right] 1_{\{t<T\}} \\ &= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s^0 ds\right) G\left(S_T \exp\left(\frac{C_T}{S_T}(T-t)\right)\right) 1_{\{T<+\infty\}}\right] 1_{\{t<T\}} \\ B_t(H) &:= V_t(G) - V_t^*(G), \end{aligned} \tag{10}$$

where $\frac{C_t^j}{S_t^j}$ is the instantaneous dividend rate for the j-th asset.

Remark 5. *If the market is complete, then $\lambda_0 = 0$ and $\mathcal{K}_{\lambda_0} = \{\varphi_0\}$, where φ_0 is the Radon–Nikodym derivative of the unique risk-neutral probability measure with respect to the statistical probability measure. The definitions in (9) and in (8) coincide for the complete market with the definitions of the fundamental value and asset bubble price for both the base asset and contingent claim introduced by Jarrow, Protter, and Shimbo in [23], proving that they are a natural extension to markets allowing for arbitrage opportunities.*

Corollary 1. *The bubble discounted values for the base assets in Definition 9 and for the contingent claim on the base assets paying dividends in Definition 8*

$$\widehat{B}_t := \exp\left(-\int_0^t r_s^0 ds\right) B_t \qquad \widehat{B}(G)_t := \exp\left(-\int_0^t r_s^0 ds\right) B(G)_t$$

satisfy the equalities

$$\begin{aligned} \widehat{B}_t^j &= D_t^j - \left(\mathbb{E}_t^*\left[D_\tau^j 1_{\{\tau<+\infty\}}\right] + \mathbb{E}_t^*\left[\widehat{C}_\tau^j 1_{\{\tau<+\infty\}}\right] - \widehat{C}_t^j\right) 1_{\{t<\tau\}} \\ \widehat{B}_t(G) &= \widehat{V}_t(G) - \mathbb{E}_t^*\left[\widehat{G}\left(S_T \exp\left(\frac{C_T}{S_T}(T-t)\right)\right) 1_{\{T<+\infty\}}\right] 1_{\{t<T\}}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \widehat{C}_t^j &:= \exp\left(-\int_0^t r_s^0 ds\right) C_t^j, \qquad \widehat{G} := \exp\left(-\int_0^T r_s^0 ds\right) G \\ \widehat{V}_t(G) &:= \exp\left(-\int_0^t r_s^0 ds\right) V_t(G) \end{aligned}$$

are the discounted cash flow for the j-th asset, the discounted contingent claim payoff, and the discounted value of the derivative.

As shown in [28], we have the following theorem.

Theorem 9. *The following statements hold true for any market model with $T \leq +\infty$ allowing for arbitrage:*

(*a*) *The market portfolio, asset values, and term structures solving the minimal arbitrage problem are identically distributed over time, and their returns are centered and serially uncorrelated:*

$([x_t; D_t; r_t]))_{t\in[0,T]}$ *is an i.d. process with respect to the statistical probability measure $\mathbb{P}$, and* $([\mathcal{D}x_t; \mathcal{D}D_t; \mathcal{D}r_t])_{t\in[0,T]}$ *is centered and has a vanishing autocovariance function.*

In particular, conditional and total expectations of asset values, nominals, and term structures are constant over time:

$$\begin{array}{lll} \mathbb{E}_0[x_t] \equiv const & \mathbb{E}_0[D_t] \equiv const & \mathbb{E}_0[r_t] \equiv const \\ \mathbb{E}_0[\mathcal{D}x_t] \equiv 0 & \mathbb{E}_0[\mathcal{D}D_t] \equiv 0 & \mathbb{E}_0[\mathcal{D}r_t] \equiv 0. \end{array}$$

The variances of portfolio nominals are concurrent with those of the deflators:

$$\mathrm{Var}_0\left(D_t^j\right)\mathrm{Var}_0\left(\frac{x_t^j}{x_t \cdot D_t}\right) \geq \frac{1}{4}, \tag{12}$$

for all indices $j = 1, \ldots, N$.

(*b*) *The expectation and variance of the bubble discounted value for the j-th asset read*

$$\begin{aligned} \mathbb{E}_0[\widehat{B}_t^j] &= \mathbb{E}_0\left[D_t^j - \widehat{C}_t^j\right] - \mathbb{E}_0^*\left[D_T^j - \widehat{C}_T^j\right] \\ \mathrm{Var}_0(\widehat{B}_t^j) &= \mathrm{Var}_0\left(D_t^j - \widehat{C}_t^j\right) + \mathrm{Var}_0^*\left(D_T^j - \widehat{C}_T^j\right) - 2\mathrm{Cov}_0^*\left(D_t^j - \widehat{C}_t^j, D_T^j - \widehat{C}_T^j\right). \end{aligned}$$

(*c*) *The expectation and variance of the bubble discounted value for the contingent claim $G(S_T)$ on the base assets read*

$$\begin{aligned} \mathbb{E}_0[\widehat{B}_t(G)] &= \mathbb{E}_0[\widehat{V}_t(G)] - \mathbb{E}_0^*\left[\widehat{G}\left(S_T \exp\left(\frac{C_T}{S_T}(T-t)\right)\right)\right] \\ \mathrm{Var}_0(\widehat{B}_t(G)) &= \mathrm{Var}_0(\widehat{V}_t(G)) + \mathrm{Var}_0^*\left(\widehat{G}\left(S_T \exp\left(\frac{C_T}{S_T}(T-t)\right)\right)\right). \end{aligned}$$

3. Generalized Derivatives of Stochastic Processes

In stochastic differential geometry, one would like to lift the constructions of stochastic analysis from open subsets of $\mathbf{R}^N$ to N-dimensional differentiable manifolds. To that aim, chart-invariant definitions are needed, and hence, a stochastic calculus satisfying the usual chain rule and not Itô's Lemma is required (cf. [19], Chapter 7, and the remark in Chapter 4 at the beginning of page 200). That is why papers about Geometric Arbitrage Theory are mainly concerned with stochastic integrals and derivatives meant in *Stratonovich*'s sense and not in *Itô*'s. Of course, at the end of the computation, Stratonovich integrals can be transformed into Itô's. Note that, in a fundamental portfolio equation, the self-financing condition cannot be directly formally expressed with Stratonovich integrals, but first with Itô's and then transformed into Stratonovich's because it is a non-anticipative condition. We refer to [1–3] and restate the definitions for the readers' convenience.

Definition 11. *Let I be a real interval and let $Q = (Q_t)_{t\in I}$ be an $\mathbf{R}^N$-valued stochastic process on the probability space $(\Omega, \mathcal{A}, P)$. The process Q determines three families of σ-subalgebras of the σ-algebra $\mathcal{A}$:*

(i) *"Past" $\mathcal{P}_t$, generated by the pre-images of Borel sets in $\mathbf{R}^N$ by all mappings $Q_s : \Omega \to \mathbf{R}^N$ for $0 < s < t$.*

(ii) *"Future" $\mathcal{F}_t$, generated by the pre-images of Borel sets in $\mathbf{R}^N$ by all mappings $Q_s : \Omega \to \mathbf{R}^N$ for $0 < t < s$.*

(iii) *"Present" $\mathcal{N}_t$, generated by the pre-images of Borel sets in $\mathbf{R}^N$ by the mapping $Q_s : \Omega \to \mathbf{R}^N$.*

Let $Q = (Q_t)_{t\in I}$ be continuous. Assuming that the following limits exist, ***Nelson's stochastic derivatives*** *are defined as*

$$\begin{aligned} DQ_t &:= \lim_{h\to 0^+} \mathbb{E}\Big[\frac{Q_{t+h}-Q_t}{h}\Big|\mathcal{P}_t\Big]\text{: forward derivative,} \\ D_*Q_t &:= \lim_{h\to 0^+} \mathbb{E}\Big[\frac{Q_t-Q_{t-h}}{h}\Big|\mathcal{F}_t\Big]\text{: backward derivative,} \\ \mathcal{D}Q_t &:= \frac{DQ_t+D_*Q_t}{2}\text{: mean derivative.} \end{aligned}$$

Let $\mathcal{S}^1(I)$ be the set of all processes Q such that $t\mapsto Q_t$, $t\mapsto DQ_t$, and $t\mapsto D_*Q_t$ are continuous mappings from I to $L^2(\Omega,\mathcal{A})$. Let $\mathcal{C}^1(I)$ be the completion of $\mathcal{S}^1(I)$ with respect to the norm

$$\|Q\| := \sup_{t\in I}\big(\|Q_t\|_{L^2(\Omega,\mathcal{A})} + \|DQ_t\|_{L^2(\Omega,\mathcal{A})} + \|D_*Q_t\|_{L^2(\Omega,\mathcal{A})}\big).$$

Remark 6. *The stochastic derivatives D, D_*, and $\mathcal{D}$ correspond to Itô's, the anticipative, and, Stratonovich's integral, respectively (cf. [36]). The process space $\mathcal{C}^1(I)$ contains all Itô processes. If Q is a Markov process, then the sigma algebras $\mathcal{P}_t$ ("past") and $\mathcal{F}_t$ ("future") in the definitions of forward and backward derivatives can be substituted by the sigma algebra $\mathcal{N}_t$ ("present"); see Chapters 6.1 and 8.1 in [36].*

Stochastic derivatives can be defined pointwise in $\omega\in\Omega$ outside the class $\mathcal{C}^1$ in terms of generalized functions.

Definition 12. *Let $Q : I\times\Omega \to \mathbf{R}^N$ be a continuous linear functional in the test processes $\varphi : I\times\Omega\to\mathbf{R}^N$ for $\varphi(\cdot,\omega)\in C_c^\infty(I,\mathbf{R}^N)$. We mean by this that, for a fixed $\omega\in\Omega$, the functional $Q(\cdot,\omega)\in\mathcal{D}(I,\mathbf{R}^N)$ is the topological vector space of continuous distributions. We can then define* ***Nelson's generalized stochastic derivatives:***

$$\begin{aligned} &DQ(\varphi_t) := -Q(D\varphi_t)\text{: forward generalized derivative,} \\ &D_*Q(\varphi_t) := -Q(D_*\varphi_t)\text{: backward generalized derivative,} \\ &\mathcal{D}(\varphi_t) := -Q(\mathcal{D}\varphi_t)\text{: mean generalized derivative.} \end{aligned}$$

If the generalized derivative is regular, then the process has a derivative in the classic sense. This construction is nothing else than a straightforward pathwise lift of the theory of generalized functions to a wider class of stochastic processes that do not allow a priori for Nelson's derivatives in the strong sense. We will utilize this feature in the treatment of credit risk, where many processes with jumps occur.

4. Credit Risk

After having introduced the machinery of Geometric Arbitrage Theory, we can tackle the modeling of assets' defaults and their recoveries.

4.1. Classical Credit Risk Models

Here, we summarize the standard ways to model credit risk. There are basically two possibilities for modeling defaults: structural model types, on one hand, and reduced-form (intensity-based) model types on the other. We refer to [37,38] to understand the differences in concepts of these models as follows: The difference between them can be characterized in terms of the information assumed to be known by the observer. Structural models assume that the observer has the same information set as the firm's manager, i.e., the complete knowledge of all firms' assets and liabilities. In most situations, this knowledge leads to a predictable default time. In contrast, reduced-form models assume that the observer has the same information set as the market, i.e., an incomplete knowledge of the firm's condition. In most cases, this imperfect knowledge leads to an inaccessible default time.

As highlighted in [37], these models are not disconnected and disjoint model types, as was commonly supposed, but rather, they are really the same model containing different informational assumptions. Furthermore, the insightful idea of [37] is that the key distinction between structural and reduced-form models is not in the characteristic of the default time (predictable vs. inaccessible), but in the information set available to the observer. Indeed, structural models can be transformed into reduced-form models as the information set changes and becomes less refined, from that observable by the firm's manager to that which is observed by the market.

Rather than comparing model types on the basis of their forecasting performance, the choice of model type should be based on the information set available to the observer. For general risk management purposes, the relevant set is the information available in the market; hence, a structural model is to be preferred. By contrast, if one is interested in a firm's risky debt or related credit derivatives, then reduced-form models are the better approach.

Let us introduce the standard setup by utilizing the market model introduced in Section 2 to account for defaults and different information sets. Credit risk management investigates an entity (corporation, bank, individual) that borrows funds, promises to return these funds under a prespecified contractual agreement, and may default before the funds (in their entirety) are repaid. Therefore, we introduce a market allowing for two kinds of assets (besides the cash account): non-defaultable (e.g., government bonds) and defaultable assets (e.g., corporate bonds).

Definition 13 (**Information Structures**)**.** *To model uncertainty, there are two filtrations for* $(\Omega, \mathcal{A}, \mathbb{P})$*:*

- ***Market Filtration:*** *This is the* $\mathcal{A} = \{\mathcal{A}_t\}_{t\in[0,+\infty[}$ *used so far for market risk, representing the information available to all market participants.*
- ***Global Filtration:*** *This is the* $\mathcal{G} = \{\mathcal{G}_t\}_{t\in[0,+\infty[}$ *representing the information available to the management of the bond issuer's company.*

The global filtration is postulated to contain the market filtration, i.e., $\mathcal{A}_t \subset \mathcal{G}_t$ *for all* $t \geq 0$*. Unless otherwise specified, conditional probabilities and expectations refer to the market filtration, i.e.,* $\mathbb{P}_t[\cdot] = \mathbb{P}[\cdot|\mathcal{A}_t]$ *and* $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{A}_t]$*.*

Definition 14 (**Default and Recovery Models**)**.** *Let* D_t^{Corp} *be the market value of a defaultable asset.*

- ***Default indicator:***

$$X_t := \begin{cases} 1, & \textit{corporate bond in default state at time } t \\ 0, & \textit{corporate bond in non-default state at time } t. \end{cases}$$

- ***Time to default:***

$$\tau := \inf\{t \geq 0 \,|\, X_t = 1\}.$$

- ***Conditional default probability:***

$$p_{t,s}^{\mathcal{A}} := \mathbb{P}_t[\tau \leq s \,|\, \tau > t].$$

- ***Structural model:*** *Let* $(E_t)_{t\geq 0}$ *be the corporate equity process with default threshold* $E_{\min}$*. The structural model for default is the following specification for the default indicator:*

$$X_t := 1_{\{E_t \leq E_{\min}\}}.$$

The corporate equity dynamics are observable in the market, i.e., $\mathcal{A}_t \supset \sigma(\{E_s \,|\, s \leq t\})$*, and they are typically given by an Itô diffusion with respect to the market filtration:*

$$dE_t = E_t(\alpha_t^E(E_t) + \sigma_t^E(E_t))dW_t.$$

- ***Intensity model:*** *The global filtration $\mathcal{G}$ contains the filtration $\sigma\big(\{\tau, Y_s \mid s \leq t\}\big)$ generated by the time to default and by a vector of state variables Y_t, which follows an Itô diffusion. The default indicator is a Cox process induced by τ with a positive intensity process $(\lambda_t)_{t\geq 0}$, which corresponds to the following specification:*

$$X_t := 1_{\{\Lambda^{-1}(E)\leq t\}}, \tag{13}$$

where $\Lambda_t := \int_0^t dh\lambda_h$ and $E \sim Exp(1)$ is an exponentially distributed random variable.

- ***Loss-given default:*** *If there is default at time t, then the recovered value at time t^+ is given by $(1-\mathrm{LGD}_t)D^{Corp}_{t^-}$. The stochastic process $(\mathrm{LGD}_t)_{t\geq 0}$ is observable in the market filtration.*

Proposition 5. *The default probabilities in the two models read:*

- *Structural model:*

$$p^{\mathcal{A}}_{t,s} = \mathbb{P}_t[E_s \leq E_{\min} \,|\, E_t \geq E_{\min}].$$

- *Intensity model:*

$$p^{\mathcal{G}}_{t,s} = 1 - \mathbb{E}\left[\exp\left(-\int_t^s dh\lambda_h\right)\middle|\mathcal{G}_t\right].$$

A known fact about structural credit risk models is summarized by the following proposition.

Proposition 6. *In the structural models, the time to default is a predictable stopping time and corresponds to the first hitting time of the barrier*

$$\tau = \inf\{t \geq 0 \,|\, E_t \leq E_{\min}\}.$$

Remark 7. *A stopping time τ is a non-negative random variable such that the event $\{\tau \leq t\} \in \mathcal{A}_t$ for every $t \geq 0$. A stopping time is predictable if there exists a sequence of stopping times $(\tau_n)_{n\geq 0}$ such that τ_n is increasing with n, $\tau_n < \tau$ for all $n \geq 0$ and $\lim_{n\to+\infty}\tau_n = \tau$ almost surely. Intuitively, as mentioned in [37], an event described by a predictable stopping time is "known" to occur "just before" it happens, since it is announced by an increasing sequence of stopping times. This is certainly the situation for structural models with respect to the market filtration. In essence, although default is an uncertain event and, thus, technically a surprise, it is not a "true surprise" to the global observer because it can be anticipated with almost certainty by watching the path of the company's equity value. The key characteristic of a structural model is the observability of the market information set $\mathcal{A}_t \supset \sigma(\{E_s \,|\, s \leq t\})$ and not the fact that default is predictable.*

Another known fact about reduced-form credit risk models (cf. [37]) is the following.

Proposition 7. *In reduced-form models, the time to default is a totally inaccessible stopping time, i.e., for every predictable stopping time S, the event $\{\omega \in \Omega \,|\, \tau(\omega) = S(\omega) < +\infty\}$ vanishes almost surely.*

Now, what are the relationships between structural and reduced-form models? The reason for the transformation of the default time τ from a predictable stopping time in Proposition 6 into an inaccessible stopping time in Proposition 7 is that, between the times of observations of the company equity value, we do not know how the equity value has evolved. Consequently, prior to our next observation, a default could occur unexpectedly (as a complete surprise). If one changes the information set held by the observer from more to less information from $\mathcal{G}$ to $\mathcal{A}$, then a structural model with default being a predictable stopping time can be transformed into a hazard rate model with default being an inaccessible stopping time:

$$\mathbb{E}\Big[p^{\mathcal{G}}_{t,s}\Big|\mathcal{A}_t\Big] = 1 - \mathbb{E}_t\Big[\exp\Big(-\int_t^s du\,\lambda_u\Big)\Big] = 1 - \exp\Big(-\int_t^s du\,h_u\Big),$$

where h denotes the deterministic hazard function. Thus, the overall relevant structure is that of the two filtrations and how stopping times behave in them. The structural models play a role in the determination of the structure generating the default time. However, as soon as the information available to the observer is reduced or obscured, one needs to project onto a smaller filtration; then, the default time becomes totally inaccessible, and the compensator Λ of the one-jump point process $1 - X_t$ becomes the object of interest. If the compensator can be written in the form $\Lambda_t = \int_0^t dh\lambda_h$, then the process $(\lambda_t)_{t\geq 0}$ can be interpreted as the instantaneous rate of default given the observer's information set. In that case, from Proposition 5, we derive the following.

Proposition 8. *Structural and intensity models are related by the following relationship:*

$$\lambda_t = \lim_{s\to t^+} \frac{\partial}{\partial s}\mathbb{P}_t[E_s \leq E_{\min} \,|\, E_t > E_{\min}].$$

Proposition 9. *For both structural and reduced-form credit model, if the market model satisfies the no-arbitrage-with-vanishing-risk condition, the risk-free discounted value of the corporate bond reads, for any $s \geq t$,*

$$D_t^{Corp} = \mathbb{E}_t^*\Big[\Big((1-\mathrm{LGD}_\tau)1_{\{\tau\leq s\}} + 1_{\{\tau>s\}}\Big)D_s^{Corp}\Big].$$

Proof. Let S_t denote the value of the corporate bond and c_t its cash-flow intensity. Then,

$$S_t = \mathbb{E}_t^*\Big[\int_t^{+\infty} dh\,c_h\exp\Big(-\int_t^h du\,r_u^0\Big)\Big].$$

Therefore,

$$\begin{aligned}
D_t^{\mathrm{Corp}} &= \mathbb{E}_t^*\Big[\int_t^{+\infty} dh\,c_h\exp\Big(-\int_t^h du\,r_u^0\Big)\Big] \\
&= \mathbb{E}_t^*\Big[\int_t^{+\infty} dh\,\delta(h-\tau)\Big((1-\mathrm{LGD}_h)1_{\{h\leq s\}} + 1_{\{h>s\}}\Big)D_s^{\mathrm{Corp}}\Big] \\
&= \mathbb{E}_t^*\Big[\Big((1-\mathrm{LGD}_\tau)1_{\{\tau\leq s\}} + 1_{\{\tau>s\}}\Big)D_s^{\mathrm{Corp}}\Big]].
\end{aligned}$$

□

Is it possible to characterize the model type on the basis of Nelson's differentiation property of the default indicator?

Proposition 10. *In the structural model, the generalized Nelson forward derivative of the default indicator reads*

$$D^{\mathcal{A}}X_t = \lim_{s\to t^+} \frac{\partial}{\partial s}\mathbb{P}_t[E_s \leq E_{\min} \,|\, E_t > E_{\min}].$$

Proof. The default probability can be developed as

$$\begin{aligned}
\mathbb{P}_t[E_s \leq E_{\min} \,|\, E_t > E_{\min}] &= \frac{\mathbb{E}_t\Big[1_{\{E_s\leq E_{\min}\}}1_{\{E_t>E_{\min}\}}\Big]}{\mathbb{E}_t\Big[1_{\{E_t>E_{\min}\}}\Big]} \\
&= \mathbb{E}_t\Big[1_{\{E_s\leq E_{\min}\}}\Big].
\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
&\lim_{s\to t^+}\frac{\partial}{\partial s}\mathbb{P}_t[E_s\le E_{\min}\,|\,E_t>E_{\min}]\\
&=\lim_{s\to t^+}\lim_{h\to 0^+}\frac{\mathbb{P}_t[E_{s+h}\le E_{\min}\,|\,E_t>E_{\min}]-\mathbb{P}_t[E_s\le E_{\min}\,|\,E_t>E_{\min}]}{h}\\
&=\lim_{s\to t^+}\lim_{h\to 0^+}\mathbb{E}_t\left[\frac{1_{\{E_{s+h}\le E_{\min}\,|\,E_t>E_{\min}\}}-1_{\{E_s\le E_{\min}\,|\,E_t>E_{\min}\}}}{h}\right]\\
&=\lim_{s\to t^+}D^{\mathcal{A}}1_{\{E_s\le E_{\min}\,|\,E_t>E_{\min}\}}=D^{\mathcal{A}}X_t,
\end{aligned}
$$

where Nelson's derivative D has to be understood in the generalized sense. □

Proposition 11. *For the intensity model, with respect to the global filtration, the Nelson forward generalized derivative of the default indicator reads*

$$D^{\mathcal{G}}X_t=\lambda_t.$$

Proof. We can reformulate (13) as a structural model as

$$X_t=1_{\{\Lambda^{-1}(E)\le t\}}=1_{\{-\Lambda_t+(E)\le 0\}},$$

and therefore,

$$p^{\mathcal{G}}_{t,s}=\mathbb{P}[-\Lambda_s+E\le 0\,|\,\{-\Lambda_t+E>0\}\cap\mathcal{G}_t].$$

By mimicking the proof of Proposition 10 with these adaptations, we have

$$
\begin{aligned}
D^{\mathcal{G}}X_t&=\lim_{s\to t^+}\frac{\partial}{\partial s}\mathbb{P}[-\Lambda_s+E\le 0\,|\,\{-\Lambda_t+E>0\}\cap\mathcal{G}_t]\\
&=\lim_{s\to t^+}\frac{\partial}{\partial s}\left(1-\mathbb{E}\left[\exp\left(-\int_t^s dh\lambda_h\right)\middle|\mathcal{G}_t\right]\right)\\
&=\lim_{s\to t^+}\mathbb{E}\left[\exp\left(-\int_t^s dh\lambda_h\right)\lambda_s\middle|\mathcal{G}_t\right]=\mathbb{E}[\lambda_t|\mathcal{G}_t]=\lambda_t.
\end{aligned}
\tag{14}
$$

□

Therefore, by comparing Propositions 8, 10, and 11, we can conclude with the following theorem.

Theorem 10. *Structural models admit an intensity formulation where the intensity is given by the Nelson forward derivative with respect to the market filtration.*

4.2. Geometric Arbitrage Theory Credit Risk Model

Now, we can carry out the analysis of credit markets described in Section 4.1 by utilizing the tools of Geometric Arbitrage Theory introduced in Section 2 and, in particular, Proposition 2.

Definition 15 (Credit Market). *A (simple) credit market consists of a government asset $(S^{Gov}_t)_{t\in[0,T]}$ with cash flow $(C^{Gov}_t)_{t\in[0,T]}$ and a corporate asset $(S^{Corp}_t)_{t\in[0,T]}$ with cash flow $(C^{Corp}_t)_{t\in[0,T]}$. The credit asset is defined as a portfolio consisting of a long position in the corporate asset and a short position in the government asset: $S^{Cred}_t:=S^{Corp}_t-S^{Cred}_t$ and $C^{Cred}_t:=C^{Corp}_t-C^{Cred}_t$. Following Definition 3, let (D^{Gov},P^{Gov}) and (D^{Corp},P^{Corp}) be the gauges corresponding to the government and the corporate asset, respectively, with their corresponding term structures. The credit gauge (D^{Cred},P^{Cred}) is defined as*

- ***Deflator:*** $D^{Cred}_t:=D^{Corp}_t-D^{Gov}_t:=\exp\left(\int_0^t ds\,r^0_s\right)(S^{Corp}_t-S^{Gov}_t)$,

- ***Discounted cash flow:*** $\widehat{C}^{Cred}_t := \exp\left(\int_0^t ds\, r^0_s\right)(\widehat{C}^{Corp}_t - \widehat{C}^{Gov}_t)$,
- ***Instantaneous forward rate:*** $f^{Cred}_{t,s} := f^{Corp}_{t,s} - f^{Gov}_{t,s}$,
- ***Short rate:*** $r^{Cred}_t := \lim_{s\to t^+} f^{Cred}_{t,s}$,
- ***Term structure:*** $P^{Cred}_{t,s} := \exp\left(-\int_t^s dh f^{Cred}_{t,h}\right)$.

The credit gauge represents all relevant information necessary to model a credit market for bonds with arbitrary maturities and of a given rating in one currency. Different ratings correspond to different credit gauges. In the vector notation of Definitions 3 and 4, we have, with the choice $x^{Cred} := [-1,+1]^\dagger$,

$$\begin{aligned} D_t &:= [D^{Gov}_t, D^{Corp}_t]^\dagger & r_t &:= [r^{Gov}_t, r^{Corp}_t]^\dagger \\ D^{Cred}_t &= D^{x^{Cred}}_t & r^{Cred}_t &= r^{x^{Cred}}_t. \end{aligned}$$

Proposition 12. *The credit asset gauge satisfies the following properties:*

- *Deflator:*
$$D^{Cred}_t = (1 - \mathrm{LGD}_t X_t) D^{Corp}_0 - D^{Gov}_t.$$
- *Term structure:*
$$P^{Cred}_{t,s} = \frac{P^{Corp}_{t,s}}{P^{Gov}_{t,s}}.$$
- *Short rate:*
$$r^{Cred}_t = r^{Corp}_t - r^{Gov}_t.$$

We can apply Theorem 6 to the credit market to characterize the no-arbitrage condition.

Theorem 11 (**No-Arbitrage Credit Market**)**.** *Let* $\lambda = \lambda_t$ *and* $\mathrm{LGD} = \mathrm{LGD}_t$ *be the default intensity and the loss-given default, respectively, of the corporate bond. The following assertions are equivalent:*

(i) *The credit market model satisfies the no-free-lunch-with-vanishing-risk condition.*

(ii) *There exists a positive local martingale* $\beta = (\beta_t)_{t\geq 0}$ *such that deflators and short rates satisfy, for all times, the condition*
$$r^{Cred}_t = \beta_t \mathrm{LGD}_t \lambda_t.$$

(iii) *There exists a positive local martingale* $\beta = (\beta_t)_{t\geq 0}$ *such that deflators and term structures satisfy, for all times, the condition*
$$P^{Cred}_{t,s} = \mathbb{E}_t\left[\exp\left(-\int_t^s du\, \beta_u \mathrm{LGD}_u \lambda_u\right)\right].$$

Proof. By Theorem 6 (iii) for $N = 2$, with government ($x = [1,0]^\dagger$) and corporate ($x = [0,-1]^\dagger$) values, the (NFLVR) condition reads

$$\begin{cases} P^{\mathrm{Gov}}_{t,s} = \dfrac{\mathbb{E}_t[\beta_s D^{\mathrm{Gov}}_s]}{\beta_t D^{\mathrm{Gov}}_t} \\ P^{\mathrm{Corp}}_{t,s} = \dfrac{\mathbb{E}_t[\beta_s D^{\mathrm{Corp}}_s]}{\beta_t D^{\mathrm{Corp}}_t} \end{cases}$$

By making the government asset into the numéraire (i.e., $D^{\mathrm{Gov}}_t \equiv 1$), we obtain, from the first equation, $P^{\mathrm{Gov}}_{t,s} \equiv 1$ and, hence, $r^{\mathrm{Gov}}_t \equiv 0$. We define an equivalent martingale measure for $(D_t)_t$ by setting the Radon–Nikodym derivative as

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} := \beta_T,$$

and rewrite the second equation utilizing $\beta_t = \mathbb{E}_t\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}\right]$ as

$$P_{t,s}^{\mathrm{Corp}} = \frac{\mathbb{E}_t[\beta_s D_s^{\mathrm{Corp}}]}{\beta_t D_t^{\mathrm{Corp}}} = \mathbb{E}_t^*[1 - \mathrm{LGD}_s X_s].$$

By taking on both sides of the equation $-\frac{\partial}{\partial s}\Big|_{s:=t}$, we obtain

$$r_t^{\mathrm{Corp}} = \mathrm{LGD}_t \lambda_t^*,$$

because, by rewriting the default indicator by means of the Heaviside function and the time to default,

$$\begin{aligned}
&\mathrm{LGD}_t = \mathrm{LGD}_\tau \Theta(t-\tau)\\
&X_t = \Theta(t-\tau)\\
&\mathbb{E}_t^*[D^{\mathcal{A}}(\mathrm{LGD}_t X_t)] = \mathbb{E}_t^*[\underbrace{D^{\mathcal{A}}(\mathrm{LGD}_t)X_t)}_{=\delta(t-\tau)\Theta(t-\tau)=0} + \mathrm{LGD}_t D^{\mathcal{A}}(X_t)] = \mathrm{LGD}_t \lambda_t^*,
\end{aligned}$$

where λ_t^* is the default intensity with respect to $\mathbb{P}^*$. Therefore, with the government asset as a numéraire,

$$r_t^{\mathrm{Cred}} = r_t^{\mathrm{Corp}} = \mathrm{LGD}_t \lambda_t^* = \beta_t \mathrm{LGD}_t \lambda_t,$$

which is (ii), to which (iii) is equivalent. The proof is completed.
□

Theorem 1 follows directly from Theorem 11. We can now apply Proposition 1 and Proposition 2 to the credit market to find the dynamics satisfying the no-free-lunch-with-vanishing-risk condition.

We have a version where the equivalence of (NFLVR) with (ZC) holds true if Novikov's growth condition for the instantaneous Sharpe ratio is satisfied.

Corollary 2. *For the market with the government bond chosen as a numéraire and corporate bond dynamics* $(D_t^{Corp})_t$ *specified by the weak limits* $D_t^{Corp} = \mathcal{D}' - \lim_{\epsilon\to 0} D_t^{Corp,\epsilon}$ *and* $r_t^{Corp} = \mathcal{D}' - \lim_{\epsilon\to 0} r_t^{Corp,\epsilon}$ *of Itô processes* $(D_t^{Corp,\epsilon})_t$ $(r_t^{Corp,\epsilon})_t$ *and satisfying SDE*

$$\begin{aligned}
dD_t^{Corp,\epsilon} &= D_t^{Corp,\epsilon}(\alpha_t^{Corp,\epsilon}dt + \sigma_t^{Corp,\epsilon}dW_t)\\
dr_t^{Corp,\epsilon} &= a_t^{Corp,\epsilon}dt + b_t^{Corp,\epsilon}dW_t
\end{aligned} \tag{15}$$

where

- $(W_t)_{t\in[0,+\infty[}$ *is a standard* $\mathbb{P}$*-Brownian motion in* $\mathbf{R}^K$*, for some* $K \in \mathbf{N}$*,*
- $(\alpha_t^{Corp,\epsilon})_{t\in[0,+\infty[}$*,* $(\sigma_t^{Corp,\epsilon})_{t\in[0,+\infty[}$*, and* $(r_t^{Corp,\epsilon})_{t\in[0,+\infty[}$ *are, respectively,* $\mathbf{R}$*-,* $\mathbf{R}^K$*-, and* $\mathbf{R}^K$*-valued predictable stochastic processes,*
- $(\alpha_t)_t$*,* $(\sigma_t)_t$*, and* $(r_t)_t$ *satisfy*

$$\begin{aligned}
&\lim_{s\to t^+} \mathbb{E}_s[\alpha_t^{Corp,\epsilon}] = \alpha_t^{Corp,\epsilon}, \quad \lim_{s\to t^+} \mathbb{E}_s[r_t^{Corp,\epsilon}] = r_t^{Corp,\epsilon},\\
&\lim_{s\to t^+} \mathbb{E}_s[\sigma_t^{Corp,\epsilon}] = \sigma_t^{Corp,\epsilon},
\end{aligned}$$

- $(\sigma_t^{Corp,\epsilon})_t$ *is an Itô process, and*
- $(\sigma_t^{Corp,\epsilon})_t$ *and* $(W_t)_t$ *are independent processes.*

The no-free-lunch-with-vanishing-risk condition is satisfied if Novikov's condition is satisfied, which is the case if and only if the zero-curvature condition is satisfied and

$$\mathbb{E}_0\left[\exp\left(\int_0^T dt\,\frac{1}{2}\left(\frac{\lambda_t \mathrm{LGD}_t}{1-\mathrm{LGD}_t X_t} - r_t^{Corp}\right)^2 \frac{t}{Q_t(K)}\right)\right] < +\infty, \tag{16}$$

where

$$Q_t(K) := \frac{W_t^\dagger W_t}{t} \sim \chi^2(K)$$

is a chi-squared distributed real random variable.

Theorem 2 follows from Corollary 2 because any $\mathcal{D}'$ process can be regularized by a sequence of Itô processes satisfying the assumptions of the corollary as

$$\lim_{\epsilon\to 0} D_t^{\mathrm{Corp},\epsilon}(\varphi) = D_t^{\mathrm{Corp}}(\varphi) \quad \text{and} \quad \lim_{\epsilon\to 0} r_t^{\mathrm{Corp},\epsilon}(\varphi) = r_t^{\mathrm{Corp}}(\varphi)$$

for all $\varphi(\cdot,\omega) \in C_c^\infty([0,T],\mathbf{R})$.

Proof. The only thing to prove is inequality (16). On one hand,

$$\begin{cases} D_t^{\mathrm{Gov}} \equiv 1 \\ D_t^{\mathrm{Corp}} = (1-\mathrm{LGD}_t X_t) D_0^{\mathrm{Corp}}. \end{cases} \tag{17}$$

On the other, the solution of (15) reads

$$D_t^{\mathrm{Corp},\epsilon} = D_0^{\mathrm{Corp},\epsilon} \exp\left(\int_0^t \left(\alpha_u^{\mathrm{Corp},\epsilon} + \frac{1}{2}\sigma_u^{\mathrm{Corp},\epsilon}\sigma_u^{\mathrm{Corp},\epsilon\dagger}\right)du + \int_0^t \sigma_u^{\mathrm{Corp},\epsilon} dW_u\right),$$

and, hence,

$$\mathcal{D}\log(D_t^{\mathrm{Corp},\epsilon}) = \alpha_t^{\mathrm{Corp},\epsilon} + \frac{1}{2}\sigma_t^{\mathrm{Corp},\epsilon}\sigma_t^{\mathrm{Corp},\epsilon\dagger} + \sigma_t^{\mathrm{Corp},\epsilon}\frac{W_t}{2t}, \tag{18}$$

because

$$\mathcal{D}W_t = \frac{W_t}{2t} \quad \text{and} \quad \left\langle \sigma^{\mathrm{Corp},\epsilon}, W\right\rangle_t \equiv 0. \tag{19}$$

Since the zero-curvature condition is satisfied, we infer that

$$\mathcal{D}\log(D_t^{\mathrm{Corp},\epsilon}) + r_t^{\mathrm{Corp},\epsilon} = \mathcal{D}\log(\underbrace{D_t^{\mathrm{Gov}}}_{\equiv 1}) + \underbrace{r_t^{\mathrm{Gov}}}_{=0} = 0,$$

which, inserted into Equation (19), leads to

$$\alpha_t^{\mathrm{Corp},\epsilon} + r_t^{\mathrm{Corp},\epsilon} + \frac{1}{2}\sigma_t^{\mathrm{Corp},\epsilon}\sigma_t^{\mathrm{Corp},\epsilon\dagger} + \sigma_t^{\mathrm{Corp},\epsilon}\frac{W_t}{2t} = 0.$$

By applying $\lim_{s\to t^+}\mathbb{E}_s[\cdot]$ on both sides, we conclude that

$$\alpha_t^{\mathrm{Corp},\epsilon} + r_t^{\mathrm{Corp},\epsilon} + \frac{1}{2}\sigma_t^{\mathrm{Corp},\epsilon}\sigma_t^{\mathrm{Corp},\epsilon\dagger} = 0. \tag{20}$$

Hence, as the government asset is the numéraire, the integrand in formula (4) reads

$$\left|\sigma_t^{\mathrm{Corp},\epsilon\dagger}\left(\sigma_t^{\mathrm{Corp},\epsilon}\sigma_t^{\mathrm{Corp},\epsilon\dagger}\right)^{-1}\left(\alpha_t^{\mathrm{Corp},\epsilon} + r_t^{\mathrm{Corp},\epsilon}\right)\right|^2 = \frac{1}{4}\sigma_t^{\mathrm{Corp},\epsilon}\sigma_t^{\mathrm{Corp},\epsilon\dagger}.$$

Equation (18) becomes, after having inserted (20),

$$\left(\mathcal{D}\log(D_t^{\mathrm{Corp},\epsilon}) + r_t^{\mathrm{Corp},\epsilon}\right)^2 = \sigma_u^{\mathrm{Corp},\epsilon}\sigma_u^{\mathrm{Corp},\epsilon\dagger}\frac{W_t^\dagger W_t}{4t^2}.$$

On the other hand, the second equation in (17) leads to

$$\mathcal{D}\log(D_t^{\mathrm{Corp}}) = -\frac{\mathrm{LGD}_t\lambda_t}{1-\mathrm{LGD}_t X_t},$$

and, therefore,

$$\lim_{\epsilon\to 0}\left|\sigma_t^{\mathrm{Corp},\epsilon\dagger}\left(\sigma_t^{\mathrm{Corp},\epsilon}\sigma_t^{\mathrm{Corp},\epsilon\dagger}\right)^{-1}\left(\alpha_t^{\mathrm{Corp},\epsilon}+r_t^{\mathrm{Corp},\epsilon}\right)\right|^2 = \left(\frac{\mathrm{LGD}_t\lambda_t}{1-\mathrm{LGD}_t X_t} - r_t^{\mathrm{Corp}}\right)^2 \frac{t}{Q_t(K)},$$

which, inserted into Novikov's condition (4), proves inequality (16). □

5. Credit Arbitrage Dynamics and Bubbles

We now apply the results for arbitrage market bubbles as recalled in Section 2.3 to our simple credit market model consisting of a two-base asset—a government and a corporate bond. We allow arbitrage and just assume that the total quantity of potential arbitrage is minimized by the market forces, as explained in detail in [2,28].

Theorem 12 (Credit Arbitrage Dynamics and Arbitrage Bubbles for Credit Markets). *The following statements hold true for the credit market model of Definition 15:*

$$D_t := [D_t^{Gov}, D_t^{Cred}]^\dagger \qquad r_t := [r_t^{Gov}, r_t^{Cred}]^\dagger \qquad x_t := [x_t^{Gov}, x_t^{Cred}]^\dagger$$

with $T \leq +\infty$ allowing for arbitrage:

(*a*) *The market portfolio, asset values, and term structures solving the minimal arbitrage problem are identically distributed over time, and their returns are centered and serially uncorrelated:*

$$\begin{aligned} &(([x_t; D_t; r_t]))_{t\in[0,T]} \text{ is an i.d. process with respect to } \mathbb{P}, \text{ and} \\ &([\mathcal{D}x_t; \mathcal{D}D_t; \mathcal{D}r_t])_{t\in[0,T]} \text{ is centered and has a vanishing autocovariance function.} \end{aligned} \tag{21}$$

In particular, the conditional and total expectations of asset values, nominals, and term structures are constant over time:

$$\begin{array}{lll} \mathbb{E}_0[x_t] \equiv const & \mathbb{E}_0[D_t] \equiv const & \mathbb{E}_0[r_t] \equiv const \\ \mathbb{E}_0[\mathcal{D}x_t] \equiv 0 & \mathbb{E}_0[\mathcal{D}D_t] \equiv 0 & \mathbb{E}_0[\mathcal{D}r_t] \equiv 0. \end{array}$$

The variances of the portfolio nominals are concurrent with those of the deflators:

$$\mathrm{Var}_0\left(D_t^j\right)\mathrm{Var}_0\left(\frac{x_t^j}{x_t\cdot D_t}\right) \geq \frac{1}{4}, \tag{22}$$

for all indices $j = 1, 2$. For the credit component, we have:

$$\mathbb{E}_0[x_t^{Cred}] \equiv const \qquad \mathbb{E}_0[D_t^{Corp}] \equiv const \qquad \mathbb{E}_0[\lambda_t \mathrm{LGD}_t] \equiv const.$$

The variance of the portfolio credit nominal is concurrent with that of the credit deflator:

$$\mathrm{Var}_0\left(D_t^{Cred}\right)\mathrm{Var}_0\left(\frac{x_t^{Cred}}{x_t^{Gov}D_t^{Gov} + x_t^{Cred}D_t^{Cred}}\right) \geq \frac{1}{4}. \tag{23}$$

(*b*) *The expectation and variance of the discounted value for the credit bubble read*

$$\begin{aligned}\mathbb{E}_0[\widehat{B}_t^{Cred}] &= \mathbb{E}_0\left[D_t^{Cred} - \widehat{C}_t^{Cred}\right] - \mathbb{E}_0^*\left[D_T^{Cred} - \widehat{C}_T^{Cred}\right] \\ \mathrm{Var}_0(\widehat{B}_t^{Cred}) &= \mathrm{Var}_0\left(D_t^{Cred} - \widehat{C}_t^{Cred}\right) + \mathrm{Var}_0^*\left(D_T^{Cred} - \widehat{C}_T^{Cred}\right) \\ &\quad - 2Cov_0^*\left(D_t^{Cred} - \widehat{C}_t^{Cred}, D_T^{Cred} - \widehat{C}_T^{Cred}\right).\end{aligned} \tag{24}$$

(*c*) *The expectation and variance of the discounted value for the credit derivative $G(S_T^{Cred})$ on the credit asset read*

$$\begin{aligned}\mathbb{E}_0[\widehat{B}_t^{Cred}(G)] &= \mathbb{E}_0[\widehat{V}_t^{Cred}(G)] - \mathbb{E}_0^*\left[\widehat{G}\left(S_T^{Cred}\exp\left(\frac{C_T^{Cred}}{S_T^{Cred}}(T-t)\right)\right)\right] \\ \mathrm{Var}_0(\widehat{B}_t^{Cred}(G)) &= \mathrm{Var}_0(\widehat{V}_t^{Cred}(G)) + \mathrm{Var}_0^*\left(\widehat{G}\left(S_T\exp\left(\frac{C_T^{Cred}}{S_T^{Cred}}(T-t)\right)\right)\right).\end{aligned} \tag{25}$$

Theorem 12 can be extended to portfolios of government and corporate coupon bonds with an appropriate cash-flow structure by means of gauge transforms, as in Definition 5 of $(D^{\mathrm{Gov}}, P^{\mathrm{Gov}})$ and $(D^{\mathrm{Corp}}, P^{\mathrm{Corp}})$. Its financial interpretation for the arbitrage-minimizing dynamics of an isolated market comprising government and corporate bonds is the following:

- The instantaneous bond returns have zero expectation and are serially uncorrelated.
- For any corporate bond, the product of default intensity and loss-given default have a time-constant expectation.
- The bigger the variance of the market portfolio nominals, the smaller the variance of the bond values, and vice versa.
- The expected values of credit bubbles during future periods with no coupons are constant.
- The expected values of credit derivative bubbles during future periods can be computed and are typically not constant.

The main limitation of Theorem 12 is given by an isolated market assumption because many bond markets correlate with other markets and macroeconomic factors. In some sense, this justifies the econometric research on general asset dynamics.

6. Conclusions

By introducing an appropriate stochastic differential geometric formalism, the classical theory of stochastic finance can be embedded into a conceptual framework called Geometric Arbitrage Theory, where the market is modeled with a principal fiber bundle, arbitrage corresponds to its curvature, and the existence of an approximated risk-neutral measure allowing the definition of fundamental values for the assets is inferred through spectral theory. The tools developed can be applied to default risk and recovery modeling, leading to arbitrage and no-arbitrage characterizations for credit markets, as well as the explicit computation for arbitrage credit bubbles and credit market dynamics.

In particular, we provide a no-free-lunch-with-vanishing-risk characterization of bond markets by relying on the spread-term structure with default intensity and loss-given default, as well as a sufficient condition equivalent to the Novikov condition implying the absence of arbitrage. Moreover, for isolated government and corporate bond markets, we construct the market portfolio and asset dynamics that minimize the quantity of potential arbitrage, showing that instantaneous bond returns are serially uncorrelated and centered, that the expected value of credit bubbles remains constant for future times where no coupons are paid, and that the variance of the market portfolio nominals is concurrent with that of the corresponding bond deflators.

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