

Article

A Variation on Inequality for Quaternion Fourier Transform, Modified Convolution and Correlation Theorems for General Quaternion Linear Canonical Transform

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Abstract: The quaternion linear canonical transform is an important tool in applied mathematics and it is closely related to the quaternion Fourier transform. In this work, using a symmetric form of the two-sided quaternion Fourier transform (QFT), we first derive a variation on the Heisenberg-type uncertainty principle related to this transformation. We then consider the general two-sided quaternion linear canonical transform. It may be considered as an extension of the two-sided quaternion linear canonical transform. Based on an orthogonal plane split, we develop the convolution theorem that associated with the general two-sided quaternion linear canonical transform and then derive its correlation theorem. We finally discuss how to apply general two-sided quaternion linear canonical transform to study the generalized swept-frequency filters.

Keywords: uncertainty principle; general quaternion linear canonical transform; convolution; correlation; generalized swept-frequency filters; Fourier transform



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1. Introduction

The general two-sided quaternion Fourier transform was firstly proposed by Hitzer in [1,2]. As we all know, it is a generalized form of the two-sided quaternion Fourier transform. Hitzer further has defined the two-sided quaternion Fourier transform based on the orthogonal planes split and demonstrated some fundamental properties of this transform. In [3], Hitzer also extended general two-sided quaternion Fourier transform within the context of Clifford algebra, named general two-sided Clifford Fourier transform, and derived its convolution and correlation theorems.

In recent years, many researchers have shown interest in a kind of new signal representation tool, known as the quaternion linear canonical transformations (QLCT). It is well known that the QLCT is a nontrivial generalization of the classical linear canonical transform [4,5]. It also can be regarded as the generalization of the quaternion Fourier transform (QFT) because the QFT is a special case of the QLCT. Several useful properties of the QLCT have been extensively studied (see, e.g., [6–11] and references therein). As in the quaternion Fourier transform case [12–18], there are three different kinds of two-dimensional quaternion linear canonical transforms (QLCTs). They are called left-sided QLCT, right-sided QLCT, and two-sided QLCT, respectively.

In this paper, based on the symmetric decomposition of the two-sided quaternion Fourier transform (QFT), we first provide the derivation of a variation on the Heisenberg-type uncertainty principle related to this transformation. The uncertainty principle describes the interaction between a quaternion function and its QFT. To achieve this, we recall the component-wise uncertainty principle for the QFT and show that it is a special case of the proposed uncertainty principle. We then propose general two-sided quaternion linear canonical transform. We further provide a definition of its convolution operator.

This definition is constructed by combining convolution definitions for the classical linear canonical transform and the quaternion Fourier transform. We establish its convolution theorem, which give a significant result of general two-sided quaternion linear canonical transform. This theorem is inspired by the work of Hitzer [19] who established the convolution theorem associated with general two-sided quaternion Fourier transform. Paper [20,21] proposed similar work for convolution theorem, but different type of the quaternion linear canonical transform which established the convolution theorem for the two-sided quaternion linear canonical transform. We finally derive correlation theorem of continuous quaternion signals associated with the general two-sided quaternion linear canonical transform.

The main content of this article is as follows. In Section 2, we recall the basic knowledge of quaternion algebra and orthogonal planes split that will be needed during the paper. Section 3 is devoted to the derivation of a variation on the Heisenberg-type uncertainty principle related to the two-sided quaternion Fourier transform. In Section 4 we recall the general two-sided quaternion Fourier transform. In Section 5 we discuss convolution definition for general two-sided quaternion linear canonical transform and obtain its convolution theorem. In Section 6, a correlation theorem related to general two-sided quaternion linear canonical transform is presented. In Section 7 an application of general two-sided quaternion linear canonical transform is studied. Lastly, the summary of this article is included in Section 8.

2. Quaternions

For the basic notations and definitions on quaternion algebra, see [22–24]. Quaternions are hypercomplex numbers, which requires an associative noncommutative four-dimensional algebra. They can be expressed as

$$\mathbb{H} = \{h = h_0 + \mathbf{i} h_a + \mathbf{j} h_b + \mathbf{k} h_c ; h_0, h_a, h_b, h_c \in \mathbb{R}\},$$

where the elements $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ have properties:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1. \quad (1)$$

For every quaternion $h \in \mathbb{H}$, the scalar and vector parts of h are denoted by $Sc(h) = h_0$ and $vec(h) = \mathbf{h} = \mathbf{i} h_a + \mathbf{j} h_b + \mathbf{k} h_c$, respectively.

The conjugate element $\bar{h}$ is given by

$$\bar{h} = h_0 - \mathbf{i} h_a - \mathbf{j} h_b - \mathbf{k} h_c, \quad (2)$$

which fulfills

$$\overline{hp} = \bar{p}\bar{h}, \quad \forall h, p \in \mathbb{H}.$$

From (2) we obtain the norm of a quaternion h in the form

$$|h| = \sqrt{h\bar{h}} = \sqrt{h_0^2 + h_a^2 + h_b^2 + h_c^2}. \quad (3)$$

The modulus of a product of two quaternion obeys the property

$$|hp| = |h||p|.$$

It is not difficult to see that

$$Sc(h) \leq |h| \quad \text{and} \quad |\mathbf{h}| \leq |h|. \quad (4)$$

Applying the conjugate (2) and the norm of h gives the inverse of nonzero quaternion h as

$$h^{-1} = \frac{\bar{h}}{|h|^2}.$$

The inner product for two quaternion-valued functions $h, g : \mathbb{R}^2 \rightarrow \mathbb{H}$ is defined as

$$(h, g) = \int_{\mathbb{R}^2} h(z) \overline{g(z)} dz, \quad dz = dz_1 dz_2, \quad (5)$$

for all $z = (z_1, z_2) \in \mathbb{R}^2$. In particular, for $h = g$, we obtain

$$\|h\|_{L^2(\mathbb{R}^2; \mathbb{H})} = \sqrt{(h, h)} = \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^{1/2}. \quad (6)$$

For $1 \leq q < \infty$, (6) becomes

$$\|h\|_q = \left(\int_{\mathbb{R}^2} |h(z)|^q dz \right)^{1/q}. \quad (7)$$

Following Hitzer's work [1], we introduce the definition of orthogonal 2D planes split (OPS) with respect to any two pure unit quaternions I_1, I_2 as follows

Definition 1. Let $I_1, I_2 \in \mathbb{H}$ be an arbitrary pair of pure quaternions $I_1, I_2, I_1^2 = I_2^2 = -1$, including the cases $I_1 = \pm I_2$. For any $h \in \mathbb{H}$ we introduce the $h_{\pm}$ OPS split parts with respect to the two pure unit quaternion I_1, I_2 as

$$h_{\pm} = \frac{1}{2}(h \pm I_1 h I_2). \quad (8)$$

Moreover one has for $\alpha, \beta \in \mathbb{R}$

$$e^{\alpha I_1} h_{\pm} e^{\beta I_2} = e^{(\alpha \mp \beta) I_1} h_{\pm} = h_{\pm} e^{(\beta \mp \alpha) I_2}. \quad (9)$$

and

$$e^{\beta I_2} \overline{h_{\pm}} e^{\alpha I_1} = e^{(\beta \mp \alpha) I_2} \overline{h_{\pm}} = \overline{h_{\pm}} e^{(\alpha \mp \beta) I_1}. \quad (10)$$

Observe that for $\alpha = \pi/2, \beta = 0$ and $\alpha = 0, \beta = \pi/2$ Equation (9) will lead to

$$I_1 h_{\pm} = \mp h_{\pm} I_2, \quad h_{\pm} I_2 = \mp I_1 h_{\pm}. \quad (11)$$

3. A Variation on Heisenberg's Inequality for Quaternion Fourier Transform

We start this part by introducing the definition of the two-sided quaternion Fourier transform (QFT) and present its useful properties. We use these results to obtain one of the main results in this study.

Definition 2. Given h in $L^1(\mathbb{R}^2; \mathbb{H})$. We define

$$\mathcal{F}_H\{h\}(u) = \int_{\mathbb{R}^2} e^{-iu_1 z_1} h(z) e^{-ju_2 z_2} dz. \quad (12)$$

We call $\mathcal{F}_H\{h\}$ the two-sided quaternion Fourier transform of h .

Definition 3. For any $h \in L^1(\mathbb{R}^2; \mathbb{H})$ for which $\mathcal{F}_H\{h\} \in L^1(\mathbb{R}^2; \mathbb{H})$, its inverse is defined by

$$\mathcal{F}_H^{-1}\{\mathcal{F}_H\{h\}\}(z) = h(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{iu_1 z_1} \mathcal{F}_H\{h\}(u) e^{ju_2 z_2} du. \quad (13)$$

Now we put $u_1 = 2\pi\omega_1$ and $u_2 = 2\pi\omega_2$ on the right-hand sides of (12) and (13), one can easily obtain

$$\mathcal{F}_H\{h\}(\omega) = \int_{\mathbb{R}^2} e^{-i2\pi\omega_1 z_1} h(z) e^{-j2\pi\omega_2 z_2} dz, \quad (14)$$

and

$$h(z) = \int_{\mathbb{R}^2} e^{i2\pi\omega_1 z_1} \mathcal{F}_H\{h\}(\omega) e^{j2\pi\omega_2 z_2} d\omega. \quad (15)$$

By (14) the decomposition of the quaternion function f will lead to

$$\mathcal{F}_H\{h\}(\omega) = \int_{\mathbb{R}^2} e^{i2\pi\omega_1 z_1} (h_o(z) + ih_a(z) + jh_b(z) + kh_c(z)) e^{j2\pi\omega_2 z_2} dz. \quad (16)$$

In the symmetric form, the above identity may be rewritten as

$$\begin{aligned} \mathcal{F}_H\{h\}(\omega) &= \int_{\mathbb{R}^2} e^{i2\pi\omega_1 z_1} (h_o(z) + ih_a(z) + jh_b(z) + ih_c(z)j) e^{j2\pi\omega_2 z_2} dz \\ &= \mathcal{F}_H\{h_o\}(\omega) + i\mathcal{F}_H\{h_a\}(\omega) + \mathcal{F}_H\{h_b\}(\omega)j + i\mathcal{F}_H\{h_c\}(\omega)j. \end{aligned} \quad (17)$$

Now we define the module of $\mathcal{F}_H\{f\}(\omega)$ as

$$|\mathcal{F}_H\{f\}(\omega)|_H = \left(|\mathcal{F}_H\{h_o\}(\omega)|^2 + |\mathcal{F}_H\{h_a\}(\omega)|^2 + |\mathcal{F}_H\{h_b\}(\omega)|^2 + |\mathcal{F}_H\{h_c\}(\omega)|^2 \right)^{1/2}. \quad (18)$$

Furthermore, we obtain the $L^p(\mathbb{R}^2; \mathbb{H})$ -norm

$$\|\mathcal{F}_H\{h\}\|_{H,p} = \left(\int_{\mathbb{R}^2} |\mathcal{F}_H\{h\}(\omega)|_H^p d\omega \right)^{1/p}. \quad (19)$$

Lemma 1. Suppose that $h \in L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H})$. If $\frac{\partial}{\partial \omega_k} h$ with $k = 1, 2$ exists and belongs to $L^2(\mathbb{R}^2; \mathbb{H})$, then for every $n \in \mathbb{N}$ we have

$$\mathcal{F}_H^{-1} \left\{ \frac{\partial^n}{\partial \omega_1^n} h \right\}(z) = (-i2\pi z_1)^n \mathcal{F}_H\{h\}(z), \quad (20)$$

and

$$\mathcal{F}_H^{-1} \left\{ \frac{\partial^n}{\partial \omega_2^n} h \right\}(z) = \mathcal{F}_H\{h\}(z) (-j2\pi z_2)^n. \quad (21)$$

Proof. For $n = 1$ applying (15) results in

$$\begin{aligned} \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_1} h \right\}(z) &= \int_{\mathbb{R}^2} e^{i2\pi\omega_1 z_1} \frac{\partial}{\partial \omega_1} h(\omega) e^{j2\pi\omega_2 z_2} d\omega \\ &= \int_{\mathbb{R}^2} \frac{\partial}{\partial \omega_1} e^{i2\pi\omega_1 z_1} h(\omega) e^{j2\pi\omega_2 z_2} d\omega \\ &= \int_{\mathbb{R}^2} i2\pi z_1 e^{i2\pi\omega_1 z_1} h(\omega) e^{j2\pi\omega_2 z_2} d\omega \\ &= i2\pi z_1 \mathcal{F}_H\{h\}(-z_1, -z_2) \\ &= (-i2\pi(-z_1)) \mathcal{F}_H\{h\}(-z_1, -z_2) \\ &= (-i2\pi y_1) \mathcal{F}_H\{h\}(y), \quad y_i = -z_i, i = 1, 2, \end{aligned} \quad (22)$$

and

$$\begin{aligned}
 \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_2} h \right\} (z) &= \int_{\mathbb{R}^2} e^{j2\pi\omega_1 z_1} h(\omega) \frac{\partial}{\partial \omega_2} e^{j2\pi\omega_2 z_2} d\omega \\
 &= \int_{\mathbb{R}^2} e^{j2\pi\omega_1 z_1} h(\omega) e^{j2\pi\omega_2 z_2} j2\pi z_2 d\omega \\
 &= \mathcal{F}_H \{h\}(-z_1, -z_1)(-j2\pi(-z_2)) \\
 &= \mathcal{F}_H \{h\}(\mathbf{y})(-j2\pi y_2), \quad y_i = -z_i, i = 1, 2.
 \end{aligned} \quad (23)$$

By repeating this process $n - 1$ additional times, we obtain (20). Using a similar argument as Equation (20), we can obtain the proof for Equation (21). $\square$

By Riesz's interpolation theorem, we obtain the Hausdorff–Young inequality (see [25]), that is

$$\|\mathcal{F}_H \{h\}\|_{H,p'} \leq \|h\|_p, \quad (24)$$

where $1 \leq p \leq 2$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Taking the inversion formula of the QFT on both sides of (24), one has

$$\|h\|_{p'} \leq \|\mathcal{F}_H^{-1} \{h\}\|_{H,p}. \quad (25)$$

Below we state and prove the uncertainty principle associated with the two-sided quaternion Fourier transform. Firstly, we recall the component-wise uncertainty principle for the QFT that will be established in next theorem.

Theorem 1. Suppose that $h \in L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H})$ and that $\frac{\partial}{\partial z_k} h$ exists. Then

$$\int_{\mathbb{R}^2} z_k^2 |h(z)|^2 dz \int_{\mathbb{R}^2} \omega_k^2 |\mathcal{F}_H \{h\}(\omega)|^2 d\omega \geq \frac{1}{16\pi^2} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^2, \quad k = 1, 2. \quad (26)$$

Remark 1. It should be noticed that Theorem 1 is valid for all types of the QFT.

It can be observed that for $1 \leq p \leq 2$ we may change L^2 - norm to L^p - norm on left-hand side of (26) and obtain the next result.

Theorem 2. Under the conditions as above, we have

$$\left(\int_{\mathbb{R}^2} z_k^p |h(z)|^p dz \right)^{1/p} \left(\int_{\mathbb{R}^2} \omega_k^p |\mathcal{F}_H \{h\}(\omega)|_H^p d\omega \right)^{1/p} \geq \frac{1}{4\pi} \int_{\mathbb{R}^2} |h(z)|^2 dz, \quad k = 1, 2. \quad (27)$$

Proof. Due to (4) and the Holder's inequality, we have

$$\begin{aligned}
 \int_{\mathbb{R}^2} |h(z)|^2 dz &= -2Sc \left(\int_{\mathbb{R}} z_k h(z) \overline{\frac{\partial}{\partial z_k} h(z)} dx \right) \\
 &\leq 2 \left| \int_{\mathbb{R}} z_k h(z) \overline{\frac{\partial}{\partial z_k} h(z)} dx \right| \\
 &\leq 2 \left(\int_{\mathbb{R}} |z_k h(z)|^p dz \right)^{1/p} \left(\int_{\mathbb{R}} \left| \frac{\partial}{\partial z_k} h(z) \right|^{p'} dz \right)^{1/p'} \\
 &= 2 \|z_k h\|_p \left\| \frac{\partial}{\partial z_k} h \right\|_{p'}.
 \end{aligned}$$

By virtue of the Hausdorff–Young inequality (25) we obtain

$$\begin{aligned} & \left\| \frac{\partial}{\partial z_k} h \right\|_{p'} \\ & \leq \left\| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_k} h \right\} \right\|_{H,p} \\ & = \left(\int_{\mathbb{R}^2} \left(\left| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_k} h_o \right\} \right|^p + \left| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_k} h_a \right\} \right|^p + \left| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_k} h_b \right\} \right|^p + \left| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_k} h_c \right\} \right|^p \right) dz \right)^{1/p}. \end{aligned}$$

For $k = 1$, we see from (20) that

$$\begin{aligned} & \left\| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_1} h \right\} \right\|_{H,p} \\ & = \left(\int_{\mathbb{R}^2} \left(|(-i2\pi z_1) \mathcal{F}_H \{h_o\}|^p + |(-i2\pi z_1) \mathcal{F}_H \{h_a\}|^p \right. \right. \\ & \quad \left. \left. + |(-i2\pi z_1) \mathcal{F}_H \{h_b\}|^p + |(-i2\pi z_1) \mathcal{F}_H \{h_c\}|^p \right) dz \right)^{1/p} \\ & = 2\pi \left(\int_{\mathbb{R}^2} (z_1^p |\mathcal{F}_H \{h_o\}|^p + z_1^p |\mathcal{F}_H \{h_a\}|^p + z_1^p |\mathcal{F}_H \{h_b\}|^p + z_1^p |\mathcal{F}_H \{h_c\}|^p) dz \right)^{1/p} \\ & = 2\pi \left(\int_{\mathbb{R}^2} z_1^p |\mathcal{F}_H \{f\}|_H^p dz \right)^{1/p}. \end{aligned}$$

For $k = 2$, we can apply similar arguments as above using (21), and obtain

$$\left\| \mathcal{F}_H^{-1} \left\{ \frac{\partial}{\partial \omega_2} h \right\} \right\|_{H,p} = 2\pi \left(\int_{\mathbb{R}^2} z_2^p |\mathcal{F}_H \{h\}|_H^p dz \right)^{1/p}.$$

Hence,

$$\int_{\mathbb{R}^2} |h(z)|^2 dz \leq 4\pi \|z_k h\|_p \left(\int_{\mathbb{R}^2} \omega_k^p |\mathcal{F}_H \{h\}|_H^p d\omega \right)^{1/p}.$$

This ends the proof of the theorem. $\square$

Remark 2. The non-commutativity of the QFT kernel implies that Theorem 2 is slightly different to the right-sided QFT (compare to Theorem 12 of [26]). Below, Theorem 3 is not valid for the right-sided QFT and left-sided QFT.

Theorem 3. If $h \in L^1(\mathbb{R}^2; \mathbb{H}) \cap L^2(\mathbb{R}^2; \mathbb{H})$ and $\mathcal{F}_H \{h\}$ exists and is also in $L^2(\mathbb{R}^2; \mathbb{H})$, then

$$\int_{\mathbb{R}^2} (z_1^p + z_2^p) |h(z)|^p dz \int_{\mathbb{R}^2} (\omega_1^p + \omega_2^p) |\mathcal{F}_H \{h\}(\omega)|^p d\omega \geq \frac{4}{(4\pi)^p} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^p, \quad (28)$$

for $1 \leq p \leq 2$.

Proof. Simple computation shows that

$$\begin{aligned} & \int_{\mathbb{R}^2} (z_1^p + z_2^p) |h(z)|^p dz \int_{\mathbb{R}^2} (\omega_1^p + \omega_2^p) |\mathcal{F}_H \{h\}(\omega)|^p d\omega \\ & = \int_{\mathbb{R}^2} z_1^p |h(z)|^p dz \int_{\mathbb{R}^2} \omega_1^p |\mathcal{F}_H \{h\}(\omega)|^p d\omega + \int_{\mathbb{R}^2} z_1^p |h(z)|^p dz \int_{\mathbb{R}^2} \omega_2^p |\mathcal{F}_H \{h\}(\omega)|^p d\omega \\ & \quad + \int_{\mathbb{R}^2} z_2^p |h(z)|^p dz \int_{\mathbb{R}^2} \omega_1^p |\mathcal{F}_H \{h\}(\omega)|^p d\omega + \int_{\mathbb{R}^2} z_2^p |h(z)|^p dz \int_{\mathbb{R}^2} \omega_2^p |\mathcal{F}_H \{h\}(\omega)|^p d\omega. \end{aligned}$$

By (26) we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} (z_1^p + z_2^p) |h(z)|^p dz \int_{\mathbb{R}^2} (\omega_1^p + \omega_2^p) |\mathcal{F}_H\{h\}(\omega)|^p d\omega \\ \geq \frac{1}{(4\pi)^p} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^p + \frac{1}{(4\pi)^p} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^p \\ + \frac{1}{(4\pi)^p} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^p + \frac{1}{(4\pi)^p} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^p \\ = \frac{4}{(4\pi)^p} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^p. \end{aligned}$$

The proof is complete. $\square$

It can be observed that for $p = 2$ Theorem 3 changes to

$$\int_{\mathbb{R}^2} |z|^2 |h(z)|^2 dz \int_{\mathbb{R}^2} |\omega|^2 |\mathcal{F}_H\{h\}(\omega)|^2 d\omega \geq \frac{1}{4\pi^2} \left(\int_{\mathbb{R}^2} |h(z)|^2 dz \right)^2, \quad (29)$$

which is directional uncertainty principle for the two-sided quaternion Fourier transform and right-sided quaternion Fourier transform [26].

4. General Two-Sided Quaternion Fourier Transform

In the following we introduce general two-sided quaternion Fourier transform, which is taken from some papers by Hitzer [2,3].

Definition 4. For any $h \in L^1(\mathbb{R}^2; \mathbb{H})$, the general two-sided quaternion Fourier transform of a quaternion function h with respect to two pure quaternions I_1, I_2 such that $I_1^2 = I_2^2 = -1$ is defined as

$$\mathcal{F}^{I_1, I_2}\{h\}(u) = \int_{\mathbb{R}^2} e^{-I_1 z_1 u_1} h(z) e^{-I_2 z_2 u_2} dz, \quad (30)$$

provided that the integral exists.

Especially, when $I_1 = i$ and $I_2 = j$ then (30) becomes

$$\mathcal{F}_H\{h\}(u) = \int_{\mathbb{R}^2} e^{-iz_1 u_1} h(z) e^{-jz_2 u_2} dz, \quad (31)$$

which is two-sided quaternion Fourier transform defined in (12).

For any $\mathcal{F}^{I_1, I_2}\{h\} \in L^1(\mathbb{R}^2; \mathbb{H})$ the general quaternion Fourier transform mentioned above can be inverted by

$$h(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{I_1 z_1 u_1} \mathcal{F}^{I_1, I_2}\{h\}(u) e^{I_2 z_2 u_2} du. \quad (32)$$

From linearity of the integral (30) we obtain the general two-sided quaternion Fourier transform for the OPS split $h = h_- + h_+$ as

$$\begin{aligned} \mathcal{F}^{I_1, I_2}\{h\}(u) &= \mathcal{F}^{I_1, I_2}\{h_- + h_+\}(u) \\ &= \mathcal{F}^{I_1, I_2}\{h_-\}(u) + \mathcal{F}^{I_1, I_2}\{h_+\}(u). \end{aligned} \quad (33)$$

Definition 5. Suppose that $h \in L^2(\mathbb{R}^2; \mathbb{H})$ and $h_{\pm} = \frac{1}{2}(h \pm I_1 h I_2)$. The general quaternion Fourier transform for the $h_{\pm}$ with respect to two linearly independent unit quaternions I_1 and I_2 is defined by

$$\mathcal{F}_{\pm}^{I_1, I_2} \{h\} = \mathcal{F}^{I_1, I_2} \{h_{\pm}\} = \int_{\mathbb{R}^2} e^{-I_1 z_1 u_1} h_{\pm} e^{-I_2 z_2 u_2} dz. \quad (34)$$

By using relation (9), the above identity can be rewritten in the form

$$\mathcal{F}^{I_1, I_2} \{h_{\pm}\} = \int_{\mathbb{R}^2} h_{\pm} e^{-I_2(z_2 u_2 \mp z_1 u_1)} dz \quad (35)$$

$$= \int_{\mathbb{R}^2} e^{-I_1(z_1 u_1 \mp z_2 u_2)} h_{\pm} dz. \quad (36)$$

5. General Two-Sided Quaternion Linear Canonical Transform

Because general two-sided QLCT is a generalization of two-sided QLCT, many useful properties of two-sided QLCT can be extended for general two-sided QLCT such as linearity, time shift, frequency shift, energy conservation, and uncertainty principles with some modifications. Another very important property of general two-sided QLCT is the convolution theorem. We first provide a definition of general two-sided QLCT and its relation to two-sided QLCT. We also present a theorem which describes the relationship between the orthogonal 2D plane split and general two-sided QLCT.

5.1. Definition of General Two-Sided QLCT

Definition 6 (General Two-sided QLCT). Suppose that $B_1 = (a_1, b_1, c_1, d_1) = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in SL(2, \mathbb{R})$ and $B_2 = (a_2, b_2, c_2, d_2) = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in SL(2, \mathbb{R})$. For any $h \in L^1(\mathbb{R}^2; \mathbb{H})$, the general two-sided quaternion linear canonical transform of a quaternion function h with respect to two pure quaternions I_1, I_2 such that $I_1^2 = I_2^2 = -1$ is defined as

$$\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h\}(u) = \begin{cases} \int_{\mathbb{R}^2} K_{B_1}(z_1, u_1) h(z) K_{B_2}(z_2, u_2) dz, & \text{for } b_1 b_2 \neq 0 \\ \sqrt{d_1} e^{I_1 \left(\frac{c_1 d_1}{2}\right) u_1^2} h(d_1 u_1, d_2 u_2) \sqrt{d_2} e^{I_2 \left(\frac{c_2 d_2}{2}\right) u_2^2}, & \text{for } b_1 b_2 = 0, \end{cases} \quad (37)$$

where the kernel functions of the general two-sided QLCT above are given by

$$K_{B_1}(z_1, u_1) = \frac{1}{\sqrt{2\pi b_1}} e^{\frac{I_1}{2} \left(\frac{a_1}{b_1} z_1^2 - \frac{2}{b_1} z_1 u_1 + \frac{d_1}{b_1} u_1^2 - \frac{\pi}{2} \right)}, \quad (38)$$

and

$$K_{B_2}(z_2, u_2) = \frac{1}{\sqrt{2\pi b_2}} e^{\frac{I_2}{2} \left(\frac{a_2}{b_2} z_2^2 - \frac{2}{b_2} z_2 u_2 + \frac{d_2}{b_2} u_2^2 - \frac{\pi}{2} \right)}. \quad (39)$$

On the condition that the general two-sided QLCT parameters satisfy $b_1 b_2 = 0$ or $b_1 = b_2 = 0$, the general two-sided QLCT of a signal is essentially a quaternion chirp multiplication and is of no particular interest for our objective interests. Therefore, without loss of generality we focus mainly on the general two-sided QLCT in the case of $b_1 b_2 \neq 0$. For specific parameter matrices $B_1 = B_2 = (a_i, b_i, c_i, d_i) = (0, 1, -1, 0)$ with $i = 1, 2$, the general two-sided QLCT definition (37) is reduced to general two-sided QFT definition, that is,

$$\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h\}(u) = \int_{\mathbb{R}^2} \frac{e^{-I_1 \frac{\pi}{4}}}{\sqrt{2\pi}} e^{-I_1 u_1 z_1} h(z) e^{-I_2 u_2 z_2} \frac{e^{-I_2 \frac{\pi}{4}}}{\sqrt{2\pi}} dz = \frac{e^{-I_1 \frac{\pi}{4}}}{\sqrt{2\pi}} \mathcal{F}^{I_1, I_2} \{h\}(u) \frac{e^{-I_2 \frac{\pi}{4}}}{\sqrt{2\pi}}. \quad (40)$$

The inversion formula of the general two-sided QLCT is given by

$$\begin{aligned}
 \mathcal{L}_{B_1, B_2}^{-1, I_1, I_2} [\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h\}](z) &= h(z) \\
 &= \int_{\mathbb{R}^2} \overline{K_{B_1}(z_1, u_1)} \mathcal{L}_{B_1, B_2} \{h\}(u) \overline{K_{B_2}(z_2, u_2)} du \\
 &= \int_{\mathbb{R}^2} K_{B_1}^{-1}(u_1, z_1) \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h\}(u) K_{B_2}^{-1}(u_2, z_2) du \\
 &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi b_1}} e^{\frac{I_1}{2} (\frac{a_1}{b_1} z_1^2 - \frac{2}{b_1} z_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2})} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h\}(u) \\
 &\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{\frac{I_2}{2} (\frac{a_2}{b_2} z_2^2 - \frac{2}{b_2} z_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2})} du, \quad (41)
 \end{aligned}$$

provided that the integral exists.

Based on Definition 5 we can easily obtain the following important theorem, which will be required to derive the main result in the sequel.

Theorem 4 (compare to [27]). Let $h \in L^2(\mathbb{R}^2; \mathbb{H})$ be a quaternion function such that $h_{\pm} = \frac{1}{2}(h \pm I_1 h I_2)$. The general two-sided quaternion linear canonical transform for the $h_{\pm}$ with respect to two linearly independent pure quaternions I_1 and I_2 can be expressed of the form

$$\begin{aligned}
 &\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h_{\pm}\} \\
 &= \frac{1}{\sqrt{2\pi b_2}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} h_{\pm} e^{\frac{I_2}{2} (\frac{a_2}{b_2} z_2^2 - \frac{2}{b_2} z_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2}) \mp (\frac{a_1}{b_1} z_1^2 - \frac{2}{b_1} z_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2})} dz, \quad (42)
 \end{aligned}$$

and

$$\begin{aligned}
 &\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h_{\pm}\} \\
 &= \frac{1}{\sqrt{2\pi b_2}} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{\frac{I_1}{2} (\frac{a_1}{b_1} z_1^2 - \frac{2}{b_1} z_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2}) \mp (\frac{a_2}{b_2} z_2^2 - \frac{2}{b_2} z_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2})} h_{\pm} dz, \quad (43)
 \end{aligned}$$

respectively.

5.2. Convolution Theorem for General Two-Sided QLCT

In the classical domain [28], a convolution operator is widely known as a signal processing algorithm in the theory of linear time-invariant systems. The output of any continuous time system is obtained by the convolution of the input signal with the system impulse response. The convolution is also important in designing the filters and in the reconstruction of the signals. According to these facts, the usefulness of convolution can be extended in a new domain. Furthermore, it is very interesting to consider the convolution definition and convolution theorem in the general two-sided QLCT domain. It shall be found that the convolution definition for the general two-sided QLCT is constructed by combining the quaternion convolution and the LCT convolution definitions [29,30].

Definition 7. For any two quaternion functions $h, k \in L^1(\mathbb{R}^2; \mathbb{H})$, we define the convolution and correlation operators related to the general two-sided QLCT as

$$(h \star k)(z) = \int_{\mathbb{R}^2} h(t) e^{I_1 \frac{a_1}{b_1} z_1 (t_1 - \frac{z_1}{2})} k(z - t) e^{I_2 \frac{a_2}{b_2} z_2 (t_2 - \frac{z_2}{2})} dt, \quad (44)$$

and

$$(h \circ k)(z) = \int_{\mathbb{R}^2} h(t) e^{I_1 \frac{a_1}{b_1} z_1 (t_1 - \frac{z_1}{2})} k(t - z) e^{I_2 \frac{a_2}{b_2} z_2 (t_2 - \frac{z_2}{2})} dt, \quad (45)$$

respectively.

The above definition enables us to build the following important theorem which describes how the convolution of two quaternion-valued functions interacts with its general two-sided QLCT.

Theorem 5. Let $h, k \in L^1(\mathbb{R}^2; \mathbb{H})$ be two quaternion-valued functions. If the decompositions of h and k are defined by

$$h_{\pm} = \frac{1}{2}(h \pm I_2 h I_1), \quad k_{\pm} = \frac{1}{2}(k \pm I_1 k I_2), \quad (46)$$

then convolution of h and k related to the general two-sided QLCT can be expressed as

$$\begin{aligned} (h \star k)(z) &= \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_2, I_1} \{h_{-}\}(u) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{-}\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \\ &\quad + \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_2, -I_1} \{h_{-}\}(u_1, -u_2) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{+}\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \\ &\quad + \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_2, -I_1} \{h_{+}\}(-u_1, u_2) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{-}\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \\ &\quad + \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-I_2, -I_1} \{h_{+}\}(-u) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{+}\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du. \end{aligned} \quad (47)$$

Proof. Using an inverse transform of the general two-sided QLCT (41), we can rewrite the left-hand side of (47) as

$$\begin{aligned} (h \star k)(z) &= \int_{\mathbb{R}^2} h(t) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{1}{2} \left(\frac{a_1}{b_1} (z_1 - t_1)^2 - \frac{2}{b_1} (z_1 - t_1) u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} e^{I_1 \frac{a_1}{b_1} z_1 (t_1 - \frac{z_1}{2})} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k\}(u) \\ &\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{1}{2} \left(\frac{a_2}{b_2} (z_2 - t_2)^2 - \frac{2}{b_2} (z_2 - t_2) u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} e^{I_2 \frac{a_2}{b_2} z_2 (t_2 - \frac{z_2}{2})} dt du \\ &= \int_{\mathbb{R}^2} h(t) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k\}(u) \\ &\quad \times \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{z_2 u_2}{b_2}} dt du. \end{aligned} \quad (48)$$

An application of two different splits to h and k defined by (46) gives us

$$\begin{aligned} (h \star k)(z) &= \int_{\mathbb{R}^2} (h_{-}(t) + h_{+}(t)) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \\ &\quad \times (\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{-}\}(u) + \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{+}\}(u)) \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_{-}(t) \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \\ &\quad \times \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{-}\}(u) \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\ &\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_{+}(t) \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \\ &\quad \times \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_{+}\}(u) \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\ &\quad + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_{+}(t) \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \end{aligned}$$

$$\begin{aligned}
& \times \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\
& + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_+(t) \frac{1}{\sqrt{2\pi b_1}} e^{I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \\
& \times \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{z_2 u_2}{b_2}} dt du,
\end{aligned}$$

and thus

$$\begin{aligned}
& (h \star k)(z) \\
& = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{a_1 t_1^2}{2b_1}} e^{I_2 \frac{t_1 u_1}{b_1}} e^{I_2 \frac{d_1 u_1^2}{2b_1}} e^{-I_2 \frac{\pi}{4}} h_-(t) \\
& \times \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{a_2 t_2^2}{2b_2}} e^{I_1 \frac{t_2 u_2}{b_2}} e^{I_1 \frac{d_2 u_2^2}{2b_2}} e^{-I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\
& + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{a_1 t_1^2}{2b_1}} e^{I_2 \frac{t_1 u_1}{b_1}} e^{I_2 \frac{d_1 u_1^2}{2b_1}} e^{-I_2 \frac{\pi}{4}} h_-(t) \\
& \times \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{a_2 t_2^2}{2b_2}} e^{-I_1 \frac{t_2 u_2}{b_2}} e^{-I_1 \frac{d_2 u_2^2}{2b_2}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\
& + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{a_1 t_1^2}{2b_1}} e^{-I_2 \frac{t_1 u_1}{b_1}} e^{-I_2 \frac{d_1 u_1^2}{2b_1}} e^{I_2 \frac{\pi}{4}} h_+(t) \\
& \times \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{a_2 t_2^2}{2b_2}} e^{I_1 \frac{t_2 u_2}{b_2}} e^{I_1 \frac{d_2 u_2^2}{2b_2}} e^{-I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} dt du \\
& + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{a_1 t_1^2}{2b_1}} e^{-I_2 \frac{t_1 u_1}{b_1}} e^{-I_2 \frac{d_1 u_1^2}{2b_1}} e^{I_2 \frac{\pi}{4}} h_+(t) \\
& \times \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{a_2 t_2^2}{2b_2}} e^{-I_1 \frac{t_2 u_2}{b_2}} e^{-I_1 \frac{d_2 u_2^2}{2b_2}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} dt du.
\end{aligned}$$

It means that we have

$$\begin{aligned}
& (h \star k)(z) \\
& = \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 - \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} h_-(t) \right. \\
& \times \left. \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 - \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} dt e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \right. \\
& + \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 - \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} h_-(t) \right. \\
& \times \left. \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 + \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} dt e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \right. \\
& + \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 + \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} h_+(t) \right. \\
& \times \left. \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 - \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} dt e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \right. \\
& + \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 + \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} h_+(t) \right. \\
& \times \left. \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 + \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} dt e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \right]
\end{aligned}$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_2, I_1} \{h_-\}(u) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{g, -f} \{h_-\}(u_1, -u_2) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-g, f} \{h_+\}(-u_1, u_2) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-I_2, -f} \{h_+\}(-u) e^{-I_1 \frac{z_1 u_1}{b_1}} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 \frac{z_2 u_2}{b_2}} du.
\end{aligned}$$

The proof is complete. $\square$

From Theorem 5 we know that when $B_1 = B_2 = (0, 1, -1, 0)$, the convolution theorem for the general two-sided QLCT above reduces to convolution theorem for general two-sided QFT [19], that is,

$$\begin{aligned}
(h \star k)(z) &= \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_2, I_1} \{h_-\}(u) e^{-I_1 z_1 u_1} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 z_2 u_2} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_2, -I_1} \{h_-\}(u_1, -u_2) e^{-I_1 z_1 u_1} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 z_2 u_2} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-I_2, I_1} \{h_+\}(-u_1, u_2) e^{-I_1 z_1 u_1} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u) e^{-I_2 z_2 u_2} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-I_2, -I_1} \{h_+\}(-u) e^{-I_1 z_1 u_1} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u) e^{-I_2 z_2 u_2} du. \quad (49)
\end{aligned}$$

6. Correlation Theorem for General Two-Sided QLCT

In the following we derive correlation theorem associated with the general two-sided QLCT. In this case we only apply two same splits to two quaternion function h and k as shown in the following theorem.

Theorem 6. Let $h, k \in L^1(\mathbb{R}^2; \mathbb{H})$ be two quaternion-valued functions. If the decompositions of h and k are defined by

$$h_{\pm} = \frac{1}{2}(h \pm I_1 h I_2), \quad k_{\pm} = \frac{1}{2}(k \pm I_1 k I_2), \quad (50)$$

then the correlation of h and k related to general two-sided QLCT can be expressed as

$$\begin{aligned}
(h \star k)(z) &= \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_1, g} \{h_-\}(u) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 (\frac{z_1 u_1}{b_1} + \frac{z_2 u_2}{b_2})} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{f, -g} \{h_-\}(u_1, u_2) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 (\frac{z_1 u_1}{b_1} - \frac{z_2 u_2}{b_2})} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-f, g} \{h_+\}(u_1, u_2) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 (\frac{z_1 u_1}{b_1} + \frac{z_2 u_2}{b_2})} du \\
&+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-I_1, -g} \{h_+\}(u) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 (\frac{z_1 u_1}{b_1} - \frac{z_2 u_2}{b_2})} du. \quad (51)
\end{aligned}$$

Proof. Taking into account inverse transform of the general two-sided QLCT (41), we have

$$\begin{aligned}
&(h \star k)(z) \\
&= \int_{\mathbb{R}^2} h(t) \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{1}{2} \left(\frac{a_2}{b_2} (t_2 - z_2)^2 - \frac{2}{b_2} (t_2 - z_2) u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} e^{-I_2 \frac{a_2}{b_2} z_2 (t_2 - \frac{z_2}{2})} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k\}(u)} \\
&\times \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{1}{2} \left(\frac{a_1}{b_1} (t_1 - z_1)^2 - \frac{2}{b_1} (t_1 - z_1) u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} e^{-I_1 \frac{a_1}{b_1} z_1 (t_1 - \frac{z_1}{2})} dt du \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h(t) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{-I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} e^{-I_2 \frac{z_2 u_2}{b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k\}(u)}
\end{aligned}$$

$$\times \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{-I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{a_1 z_1^2}{2b_1}} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du. \quad (52)$$

Subsequent calculation yields

$$\begin{aligned} & (h \star k)(z) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (h_-(t) + h_+(t)) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{-I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} e^{-I_2 \frac{z_2 u_2}{b_2}} \\ & \times (\overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} + \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)}) \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{-I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{a_1 z_1^2}{2b_1}} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_-(t) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{-I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} e^{-I_2 \frac{z_2 u_2}{b_2}} \\ & \times \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{-I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{a_1 z_1^2}{2b_1}} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ & + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_-(t) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{-I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} e^{-I_2 \frac{z_2 u_2}{b_2}} \\ & \times \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{-I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{a_1 z_1^2}{2b_1}} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ & + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_+(t) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{-I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} e^{-I_2 \frac{z_2 u_2}{b_2}} \\ & \times \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{-I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{a_1 z_1^2}{2b_1}} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ & + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} h_+(t) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{a_2 t_2^2}{2b_2}} e^{I_2 \frac{t_2 u_2}{b_2}} e^{-I_2 \frac{d_2 u_2^2}{2b_2}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} e^{-I_2 \frac{z_2 u_2}{b_2}} \\ & \times \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{a_1 t_1^2}{2b_1}} e^{I_1 \frac{t_1 u_1}{b_1}} e^{-I_1 \frac{d_1 u_1^2}{2b_1}} e^{I_1 \frac{\pi}{4}} e^{-I_1 \frac{a_1 z_1^2}{2b_1}} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du. \end{aligned}$$

This leads to

$$\begin{aligned} & (h \star k)(z) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{a_2 t_2^2}{2b_2}} e^{I_1 \frac{t_2 u_2}{b_2}} e^{-I_1 \frac{d_2 u_2^2}{2b_2}} e^{I_1 \frac{\pi}{4}} h_-(t) \\ & \times \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{a_1 t_1^2}{2b_1}} e^{I_2 \frac{t_1 u_1}{b_1}} e^{-I_2 \frac{d_1 u_1^2}{2b_1}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ & + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{a_2 t_2^2}{2b_2}} e^{I_1 \frac{t_2 u_2}{b_2}} e^{-I_1 \frac{d_2 u_2^2}{2b_2}} e^{I_1 \frac{\pi}{4}} h_-(t) \\ & \times \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{a_1 t_1^2}{2b_1}} e^{-I_2 \frac{t_1 u_1}{b_1}} e^{I_2 \frac{d_1 u_1^2}{2b_1}} e^{-I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ & + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{a_2 t_2^2}{2b_2}} e^{-I_1 \frac{t_2 u_2}{b_2}} e^{I_1 \frac{d_2 u_2^2}{2b_2}} e^{-I_1 \frac{\pi}{4}} h_+(t) \\ & \times \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{a_1 t_1^2}{2b_1}} e^{I_2 \frac{t_1 u_1}{b_1}} e^{-I_2 \frac{d_1 u_1^2}{2b_1}} e^{I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du \\ & + \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{a_2 t_2^2}{2b_2}} e^{-I_1 \frac{t_2 u_2}{b_2}} e^{I_1 \frac{d_2 u_2^2}{2b_2}} e^{-I_1 \frac{\pi}{4}} h_+(t) \\ & \times \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{a_1 t_1^2}{2b_1}} e^{-I_2 \frac{t_1 u_1}{b_1}} e^{I_2 \frac{d_1 u_1^2}{2b_1}} e^{-I_2 \frac{\pi}{4}} e^{-I_2 \frac{a_2 z_2^2}{2b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} dt du. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
 & (h \star k)(z) \\
 &= \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{-I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 - \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} h_-(t) \right. \\
 &\quad \times \left. \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 - \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} \right] dt e^{-I_2 \frac{z_2 u_2}{b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} du \\
 &+ \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_1}} e^{-I_1 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 - \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} h_-(t) \right. \\
 &\quad \times \left. \frac{1}{\sqrt{2\pi b_2}} e^{I_2 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 - \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} \right] dt e^{-I_2 \frac{z_2 u_2}{b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} du \\
 &+ \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 - \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} h_+(t) \right. \\
 &\quad \times \left. \frac{1}{\sqrt{2\pi b_1}} e^{-I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 - \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} \right] dt e^{-I_2 \frac{z_2 u_2}{b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} du \\
 &+ \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{1}{\sqrt{2\pi b_2}} e^{I_1 \frac{1}{2} \left(\frac{a_2}{b_2} t_2^2 - \frac{2}{b_2} t_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} h_+(t) \right. \\
 &\quad \times \left. \frac{1}{\sqrt{2\pi b_1}} e^{I_2 \frac{1}{2} \left(\frac{a_1}{b_1} t_1^2 - \frac{2}{b_1} t_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} \right] dt e^{-I_2 \frac{z_2 u_2}{b_2}} \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 \frac{z_1 u_1}{b_1}} du \\
 &= \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{I_1, g} \{h_-\}(u) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 \left(\frac{z_1 u_1}{b_1} + \frac{z_2 u_2}{b_2} \right)} du \\
 &+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{f, -g} \{h_-\}(u_1, u_2) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 \left(\frac{z_1 u_1}{b_1} - \frac{z_2 u_2}{b_2} \right)} du \\
 &+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-f, g} \{h_+\}(u_1, u_2) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_-\}(u)} e^{-I_1 \left(\frac{z_1 u_1}{b_1} + \frac{z_2 u_2}{b_2} \right)} du \\
 &+ \int_{\mathbb{R}^2} \mathcal{L}_{B_1, B_2}^{-I_1, -g} \{h_+\}(u) \overline{\mathcal{L}_{B_1, B_2}^{I_1, I_2} \{k_+\}(u)} e^{-I_1 \left(\frac{z_1 u_1}{b_1} - \frac{z_2 u_2}{b_2} \right)} du.
 \end{aligned}$$

The proof is complete. $\square$

7. An Application

In this part, we aim to consider a simple application of the general two-sided QLCT for studying the generalized swept-frequency filters. The output of swept-frequency filters (compare to [31,32]) is then defined as

$$\begin{aligned}
 y(z) &= e^{I_1 \frac{c_1}{2} z_1^2} \left[(e^{-I_1 \frac{c_1}{2} z_1^2} h(z) e^{-I_2 \frac{c_2}{2} z_2^2}) *_q r(z) \right] e^{I_2 \frac{c_2}{2} z_2^2} \\
 &= e^{I_1 \frac{c_1}{2} z_1^2} \int_{\mathbb{R}^2} [(e^{-I_1 \frac{c_1}{2} \tau_1^2} h(\tau) e^{-I_2 \frac{c_2}{2} \tau_2^2}) r(z - \tau)] d\tau e^{I_2 \frac{c_2}{2} z_2^2}, \quad (53)
 \end{aligned}$$

where $*_q$ stands for quaternion convolution operator in the QFT domain. Assume that $r(z)$ is real impulse response of the shift-invariant filter. By taking the general two-sided QLCT on both sides of (53), we see that

$$\begin{aligned}
 & \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{y\}(u) \\
 &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{1}{2} \left(\frac{a_1}{b_1} z_1^2 - \frac{2}{b_1} z_1 u_1 + \frac{d_1}{b_1} u_1^2 + \frac{\pi}{2} \right)} \\
 &\quad \times \left[(e^{I_1 \frac{c_1}{2} (z_1^2 - \tau_1^2)} h(\tau) e^{I_2 \frac{c_2}{2} (z_2^2 - \tau_2^2)}) r(z - \tau) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{1}{2} \left(\frac{a_2}{b_2} z_2^2 - \frac{2}{b_2} z_2 u_2 + \frac{d_2}{b_2} u_2^2 + \frac{\pi}{2} \right)} dz \right] d\tau.
 \end{aligned}$$

If the matrix parameters are chosen as $B_1 = (a_1, b_1, c_1, d_1) = (c_1, 1, -1, 0)$ and $B_2 = (a_2, b_2, c_2, d_2) = (c_2, 1, -1, 0)$, then the above identity can be expressed as

$$\begin{aligned} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{y\}(u) &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} z_1^2} e^{-I_1 z_1 u_1} e^{-I_1 \frac{\pi}{4}} \\ &\times \left[(e^{I_1 \frac{c_1}{2} (z_1^2 - \tau_1^2)} h(\tau) e^{I_2 \frac{c_2}{2} (z_2^2 - \tau_2^2)}) r(z - \tau) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{c_2}{2} z_2^2} e^{-I_2 z_2 u_2} e^{-I_2 \frac{\pi}{4}} dz \right] d\tau \\ &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} \tau_1^2} e^{-I_1 z_1 u_1} e^{-I_1 \frac{\pi}{4}} h(\tau) r(z - \tau) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{c_2}{2} z_2^2} e^{-I_2 z_2 u_2} e^{-I_2 \frac{\pi}{4}} dz d\tau. \end{aligned}$$

By changing the variables with $z - \tau = y$ in the above expression we have

$$\begin{aligned} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{y\}(u) &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} \tau_1^2} e^{-I_1 \tau_1 u_1} e^{-I_1 \frac{\pi}{4}} e^{-I_1 y_1 u_1} h(\tau) r(y) \\ &\times \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \tau_2 u_2} e^{-I_2 y_2 u_2} e^{-I_2 \frac{c_2}{2} \tau_2^2} e^{-I_2 \frac{\pi}{4}} dy d\tau. \end{aligned} \quad (54)$$

Rewriting (54) as

$$\begin{aligned} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{y\}(u) &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} \tau_1^2} e^{-I_1 \tau_1 u_1} e^{-I_1 \frac{\pi}{4}} e^{-I_1 y_1 u_1} h_-(\tau) r(y) \\ &\times \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \tau_2 u_2} e^{-I_2 y_2 u_2} e^{-I_2 \frac{c_2}{2} \tau_2^2} e^{-I_2 \frac{\pi}{4}} dy d\tau \\ &+ \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} \tau_1^2} e^{-I_1 \tau_1 u_1} e^{-I_1 \frac{\pi}{4}} e^{-I_1 y_1 u_1} h_+(\tau) r(y) \\ &\times \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \tau_2 u_2} e^{-I_2 y_2 u_2} e^{-I_2 \frac{c_2}{2} \tau_2^2} e^{-I_2 \frac{\pi}{4}} dy d\tau \\ &= \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} \tau_1^2} e^{-I_1 \tau_1 u_1} e^{-I_1 \frac{\pi}{4}} h_-(\tau) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{c_2}{2} \tau_2^2} e^{-I_2 \tau_2 u_2} e^{-I_2 \frac{\pi}{4}} \\ &\times e^{-I_2 y_1 u_1} r(y) e^{-I_2 y_2 u_2} d\tau dy \\ &+ \frac{1}{\sqrt{2\pi b_1}} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-I_1 \frac{c_1}{2} \tau_1^2} e^{-I_1 \tau_1 u_1} e^{-I_1 \frac{\pi}{4}} h_+(\tau) \frac{1}{\sqrt{2\pi b_2}} e^{-I_2 \frac{c_2}{2} \tau_2^2} e^{-I_2 \tau_2 u_2} e^{-I_2 \frac{\pi}{4}} \\ &\times e^{I_2 y_1 u_1} r(y) e^{-I_2 y_2 u_2} d\tau dy. \end{aligned} \quad (55)$$

Consequently, we finally arrive at

$$\begin{aligned} \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{y\}(u) &= \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h_-\}(u) \int_{\mathbb{R}^2} e^{-I_2 y_1 u_1} r(y) e^{-I_2 y_2 u_2} dy \\ &+ \mathcal{L}_{B_1, B_2}^{I_1, I_2} \{h_+\}(u) \int_{\mathbb{R}^2} e^{I_2 y_1 u_1} r(y) e^{-I_2 y_2 u_2} dy. \end{aligned} \quad (56)$$

8. Conclusions

In this work, based on the symmetric form of the two-sided quaternion Fourier transform, we have proved a variation on the Heisenberg-type uncertainty principle related to the two-sided quaternion Fourier transform. We also have introduced the general two-sided QLCT and provided its convolution definitions. In view of orthogonal plane split, we have derived convolution and correlation theorems related to general two-sided QLCT. We also have discussed its application to study the generalized swept-frequency filters.

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