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Some Characteristics of Matrix Operators on Generalized Fibonacci Weighted Difference Sequence Space

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Abstract: The forthcoming property of this manuscript is its calculating of the goal of norms and lower bounds of matrix operators taken from the weighted sequence space $\ell_p(w)$ onto a novel one defined in the present article as the generalized Fibonacci weighted difference sequence space. In this process, first of all the Fibonacci difference matrix $\tilde{F}(r,s)$ and the space composed of sequences of which $\tilde{F}(r,s)$-transforms lie in $\ell_p(\tilde{w})$, where $r, s \in \mathbb{R}$ are defined. Additionaly, since the seminormed space $\ell_p(\tilde{w}, \tilde{F}(r,s))$ has the absolute homogeneous property, the topological characteristics on it are distributed symmetrically everywhere in the space.

Keywords: Fibonacci numbers; sequence spaces; matrix operators; quasi-summable matrices

MSC: J11B39; 46A45; 26D15; 40G05; 47B37

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1. Introduction

Fibonacci numbers, postulated in 1202 by the Italian mathematician, provides the foundation for his book titled Liber Abaci. Fibonacci numbers have a wide range of applications, from the growth of plants and the crystallographic structures of certain solids to the development of computer algorithms written for searching databases. Much has been written and drawn about these numbers so far that these numbers have applications in mathematics, computer science, physics and biology. The simplest method of generating a numeric sequence with a well-defined algorithm is to use one or two kernel values and appropriate recurrence relation. The most well-known sequences, which can be given as an example of such sequences is the Fibonacci sequence. The sequence $f_n = f_{n-1} + f_{n-2}$ (for $n > 2$) is obtained by a relation defined recursively. The sequence (f_n) stars with the kernel values of $f_1 = 1$ and $f_2 = 1$, and each term after the 2nd term is equal to the sum of the two consecutive terms immediately preceding it. The Fibonacci numbers and the Golden ratio are closely related, and after a certain point, the ratio of the larger one to the smaller one of two consecutive Fibonacci numbers gives the Golden ratio. Before diving into the details, we should clarify an issue that will be used frequently in this study about Fibonacci sequences; $\frac{1}{2} \leq \frac{f_n}{f_{n+1}} \leq 1$ for the sequence (f_n) for all $n \in \mathbb{N} := \{1, 2, \ldots\}$. For different approximations using Fibonacci numbers, references [1,2] can be viewed. The collection of all real or complex number sequences forms a vector space which we denote by ω, under the operations of coordinate-wise addition and well-known scalar multiplication. The subspaces of ω are important in such applications because each of them is called a sequence space.

Let us remember the definition of another concept we are going to need in the manuscript. When an infinite matrix $A = (a_{nk})$ is given having complex numbers a_{nk} as entries in which $n, k \in \mathbb{N}$, for a sequence x, it can be written as:

$$(Ax)_n := \sum a_{nk} x_k; \quad (n \in \mathbb{N}, x \in D_{00}(A)),$$

in which $D_{00}(A)$ describes the defined subspace of ω composed of $x \in \omega$ for which the summation exists as a finite sum. For a simple notation, from now on, the summation having no limits ranges from 0 to ∞.

The X_A is known as the matrix domain of an infinite matrix A for any subspace X of the all real-valued sequence space w is described as:

$$X_A := \{x = (x_k) \in \omega : Ax \in X\}$$

which is a sequence space. There are varied techniques for producing new sequence spaces out of old ones such as X. One of them is using any matrix domain produced by an infinite matrix A such as X_A. For a brief explanation about the topic, the sequence spaces, namely X and X_A, may overlap but either of them may contain the other one. For detailed information, the reader can refer to the book "Summability Theory and Its Applications" by Başar [3] and therein.

In recent years, we have seen a dramatic increase in constructing new sequence spaces using matrix domain in summability areas such as sequence spaces.

Many papers [4–8] we have examined so far have something in common, that is, they involve the matrix domain.

In [9], Candan defined the generalized Fibonacci difference matrix $\hat{F}(r,s) = (\hat{f}_{nk}(r,s))$ by:

$$\hat{f}_{nk}(r,s) = \begin{cases} s\frac{f_{n+1}}{f_n}, & k = n-1 \\ r\frac{f_n}{f_{n+1}}, & k = n \\ 0, & 0 \le k < n-1 \text{ or } k > n \end{cases}$$

and built some new difference sequence spaces by using this matrix. Candan revealed many features of the spaces he constructed by taking advantage of the matrix from different angles in his different articles. Fibonacci numbers investigated by different authors have many applications. Where some of these can be found in the references [10–18].

In recent years, via utilizing the matrix described by Kara and Başarır in [19], Talebi and Dehghan [20] proposed a space derived from the Fibonacci weighted sequence space denoted by $F_{w,p}$, which is composed of all the sequences of which F-transforms lie in $\ell_p(w) = \{x = (x_k) \in \omega : \sum_{k=1}^{\infty} w_k |x_k|^p < \infty\}$, for which $1 \le p < \infty$ and also at the same time $w = (w_k)$ is defined as a decreasing non-negative sequence of real numbers.

The finding of the best upper bound for some known matrix operators denoted by T from $\ell_p(w)$ onto $F_{w,p}$ has been tried. In connection with this statement, it should be noted that an upper bound found out for a matrix operator denoted by T defined from a sequence space X into another one denoted by Y can be given by the following value of U:

$$\|Tx\|_Y \le U\|x\|_X,$$

in which $\|.\|_X$ and $\|.\|_Y$ denote the widely known norms prescribed on the spaces X and Y, respectively. Here, U is not dependent on x. Among those, the best value of U can be characterized as the operator norm for T.

Furthermore, several scholars have tried to find out the lower bounds for those matrix operators. This concept was firstly put forward in Ref. [21] about the Cesàro matrix. After that, the other ones such as in Refs. [22–25] have investigated the lower bounds for some matrix operators defined on the sequence space denoted by ℓ_p and at the same time on the weighted sequence space denoted by $\ell_p(w)$ having the Lorentz sequence space. In a similar way, a lower bound of a matrix operator defined as $T : X \to Y$ is defined as the value of L, which satisfies the following inequality:

$$\|Tx\|_Y \ge L\|x\|_X.$$

This inequality can also be utilized for some functional analysis applications. To give an example, finding the necessary and sufficient conditions for which an operator has got its inverse and at the same time finding the operator kernel, including only the zero vector for this case. Because of these reasons, the knowledge of the lower bound for an operator is important. In recent years, Dehghan and Talebi [26] have paid attention to the largest possible lower bound value about some of the matrices on the Fibonacci sequence spaces. Futhermore, Foroutannia and Roopaei [27] take into consideration the problem of calculating both the norm and lower-upper bounds for some operators defined on weighted difference sequence spaces. One can refer to those papers [28–34] and those therein for related problems about some classical sequence spaces.

In this article, it is assumed that $w = (w_n)$ and also $\tilde{w} = (\tilde{w}_n)$ are sequences consisting of positive real terms. In the present article, a new space called as the generalized Fibonacci weighted difference sequence space is introduced via the generalized Fibonacci difference matrix. Moreover, some characteristics of this sequence space are investigated. Among others, it has been observed that although this space is a semi-normed one it is not necessarily a normed one. Let us remember that a semi-normed satisfies every axiom of a norm but the semi-norm of a vector must be zero without including the zero vector. Again, it is also a semi-inner product space for the value of $p = 2$. Furthermore, an isomorphism is obtained by utilizing this space. Next, the norm for some matrix operators is defined on the generalized Fibonacci weighted difference sequence space. In the next step, the lower bound problem for the operators described from $\ell_p(w)$ into the generalized Fibonacci weighted difference sequence space. Due to this fact, the seminorm or norm obtained in this study satisfies the absolute homogeneity condition, it shows that the topological characteristics on the seminormed space or normed space are distributed symmetrically all over the space.

2. Fibonacci Weighted Difference Sequence Spaces Generalized with Real Numbers *r* and *s*

We have seen in the previous chapter that many issues lead to constructing new sequence space. Furthermore, the concepts we offered were inherently large. Let us start by defining the following matrix $\hat{F} = (\hat{f}_{nk}(r,s))$, which is similar but different to the matrix presented by Candan [9] earlier:

$$\tilde{f}_{nk}(r,s) = \begin{cases} s\frac{f_n}{f_{n+1}}, & k = n+1 \\ r\frac{f_{n+1}}{f_n}, & k = n \\ 0, & 0 \le k < n \text{ or } k > n+1 \end{cases}$$

where $r, s \in \mathbb{R}$. We will see later that this matrix enables us to construct an efficient structure for solving algebraic and topological properties. By applying the matrix domain definition to this matrix, we define the new sequence space whose result is in the $\ell_p(\tilde{w})$ space, as follows:

$$\ell_p(\tilde{w}, \tilde{F}(r,s)) = \left\{ x = (x_n) \in \omega : \sum_{n=1}^{\infty} \tilde{w}_n \left| r\frac{f_{n+1}}{f_n} x_n + s\frac{f_n}{f_{n+1}} x_{n+1} \right|^p < \infty \right\},$$

in which $1 \le p < \infty$. We note here that the space is a semi-normed space with the semi-norm defined by:

$$\|x\|_{p,\tilde{w},\tilde{F}} = \left(\sum_{n=1}^{\infty} \tilde{w}_n \left| r\frac{f_{n+1}}{f_n} x_n + s\frac{f_n}{f_{n+1}} x_{n+1} \right|^p \right)^{1/p}.$$

To calculate the veracity of this claim, we now give an example. Considering the sequence $x_n = \frac{1}{r}(\frac{-s}{r})^{1-n} f_n^2$, because of $r\frac{f_{n+1}}{f_n} x_n + s\frac{f_n}{f_{n+1}} x_{n+1} = 0$ we get $\|x\|_{p,\tilde{w},\tilde{F}} = 0$, after that, from the definition of the norm, it is seen that $\|.\|_{p,\tilde{w},\tilde{F}}$ defined on $\ell_p(\tilde{w}, \tilde{F}(r,s))$ is not a norm.

Before beginning the general theory, at first, we will state the following fundamental theorem, showing the set just described has an important role in its algebraic structure.

Theorem 1. *The set $\ell_p(\tilde{w}, \tilde{F}(r,s))$ is linear space, namely, sequence space.*

Proof. We omit the proof which can be found in standard process. $\square$

Let us continue with the following theorem regarding an algebraic property of this newly defined sequence space.

Theorem 2. *It is true that the inclusion relation $\ell_p(\tilde{w}) \subset \ell_p(\tilde{w}, \tilde{F}(r,s))$ is strictly valid.*

Proof. If we take an arbitrary $x \in \ell_p(\tilde{w})$, the following calculation demonstrates that inclusion is valid:

$$\begin{aligned}\tilde{w}_n\left|r\frac{f_{n+1}}{f_n}x_n + s\frac{f_n}{f_{n+1}}x_{n+1}\right|^p &\leq \tilde{w}_n 2^{2p-1}(|rx_n|^p + |sx_{n+1}|^p) \\ &\leq 2^{2p-1}max(|r|^p, |s|^p)\tilde{w}_n(|x_n|^p + |x_{n+1}|^p)\end{aligned}$$

by summing n from 1 to ∞, in which $1 \leq p < \infty$.

To illustrate that the inclusion relation is strictly valid, when the sequence $\tilde{w}$ is taken $(1, 1, 1, \dots)$, let us consider again the sequence $(x_n) = \left(\frac{1}{r}(\frac{-s}{r})^{1-n}f_n^2\right) \in \ell_p(\tilde{w}, \tilde{F}(r,s))$. It is easy to deduce from that $(x_n) \notin \ell_p(\tilde{w})$. $\square$

Theorem 3. *When $H = \{x = (x_n) \in \ell_p(\tilde{w}, \tilde{F}(r,s)) : r\frac{f_{n+1}}{f_n}x_n + s\frac{f_n}{f_{n+1}}x_{n+1} = 0$ for all $n \in \mathbb{N}\}$, the quotient space $\ell_p(\tilde{w}, \tilde{F}(r,s))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$.*

Proof. The basic approach to the proof of this theorem is to define a new T transformation from the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$ to $\ell_p(\tilde{w})$ that utilizes the definition of the fundamental matrix transformation, for all $x \in \ell_p(\tilde{w}, \tilde{F}(r,s))$ clearly $Tx = \left(r\frac{f_{n+1}}{f_n}x_n + s\frac{f_n}{f_{n+1}}x_{n+1}\right)$. Since it is quite obvious to show that T is linear, we are first concerned here with showing that T is surjective. One of the ways of doing so for any $y = (y_k) \in \ell_p(\tilde{w})$ is to write $x_n = \frac{1}{r}\sum_{k=n}^{\infty}\left(\frac{-s}{r}\right)^{k-n}\frac{f_n^2}{f_k f_{k+1}}y_k$ for all $n \in \mathbb{N}$ in the norm of $\ell_p(\tilde{w}, \tilde{F}(r,s))$. In this case, we obtain the following equations by simple calculations:

$$\begin{aligned}\|x\|_{p,\tilde{w},\tilde{F}}^p &= \sum_{n=1}^{\infty}\tilde{w}_n\left|r\frac{f_{n+1}}{f_n}\frac{1}{r}\sum_{k=n}^{\infty}\left(\frac{-s}{r}\right)^{k-n}\frac{f_n^2}{f_k f_{k+1}}y_k + s\frac{f_n}{f_{n+1}}\frac{1}{r}\sum_{k=n+1}^{\infty}\left(\frac{-s}{r}\right)^{k-n-1}\frac{f_{n+1}^2}{f_k f_{k+1}}y_k\right|^p \\ &= \sum_{n=1}^{\infty}\tilde{w}_n\left|y_n + f_n f_{n+1}\left[\sum_{k=n+1}^{\infty}\left(\frac{-s}{r}\right)^{k-n}\frac{1}{f_k f_{k+1}}y_k - \sum_{k=n+1}^{\infty}\left(\frac{-s}{r}\right)^{k-n}\frac{1}{f_k f_{k+1}}y_k\right]\right|^p \\ &= \sum_{n=1}^{\infty}\tilde{w}_n|y_n|^p \\ &= \|y\|_{p,\tilde{w}}^p \\ &< \infty\end{aligned}$$

which implies that $x = (x_n) \in \ell_p(\tilde{w}, \tilde{F}(r,s))$. Going back to the T transform described above, it is very straightforward to say that $Tx = y$. Because of the fact that the image of the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$ under the transformation T is $\ell_p(\tilde{w})$ and also $\ker T = H$, we have that $\ell_p(\tilde{w}, \tilde{F}(r,s))/H$ is linearly isomorphic to the space $\ell_p(\tilde{w})$ when considering the first isomorphism theorem. $\square$

Let us give an example to show that the transformation T defined above is not injective. Indeed, for $(x_n) = (\frac{1}{r}(\frac{-s}{r})^{1-n}f_n^2)$ we obtain $Tx = 0$; in other words, $\ker T \neq \{0\}$.

Theorem 4. *Under the conditions that p is different from two and also the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$ is not described as semi-inner product space, $\ell_2(\tilde{w}, \tilde{F}(r,s))$ is a semi-inner product space.*

Proof. First of all, we will answer the question that the semi-norm $\|.\|_{2,\tilde{w},\tilde{F}}$ using a semi-inner product can be induced. It is convenient to introduce at this stage the notation $z_k = \tilde{w}_k^{1/2}\left(r\frac{f_{k+1}}{f_k}x_k + s\frac{f_k}{f_{k+1}}x_{k+1}\right)$ for all $k \in \mathbb{N}$ and $\langle z,z\rangle_2 = \sum_{k=1}^{\infty}|z_k|^2$. In fact taken arbitrary, $x \in \ell_2(\tilde{w}, \tilde{F}(r,s))$, we have:

$$\|x\|_{2,\tilde{w},\tilde{F}} = \sqrt{\langle z,z\rangle_2}.$$

Furthermore, it is easy to check from the following equations that the semi-norm $\|.\|_{p,\tilde{w},\tilde{F}}$ cannot be derived considering a semi-inner product just described:

$$\|x+y\|^2_{p,\tilde{w},\tilde{F}} + \|x-y\|^2_{p,\tilde{w},\tilde{F}} = 4(\tilde{w}_1^{2/p} + \tilde{w}_2^{2/p}) \neq 4(\tilde{w}_1 + \tilde{w}_2)^{2/p} = 2(\|x\|^2_{p,\tilde{w},\tilde{F}} + \|y\|^2_{p,\tilde{w},\tilde{F}}),$$

in which $x = (\frac{2r+s}{2r^2}, -\frac{1}{2r}, 0, 0, \ldots)$, $y = (\frac{2r-s}{2r^2}, \frac{1}{2r}, 0, 0, \ldots)$ and $p \neq 2$. □

3. The Norm of Matrix Operators from $\ell_1(w)$ to $\ell_1(\tilde{w}, \tilde{F}(r,s))$

After defining a function from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, \tilde{F}(r,s))$, in this chapter we will calculate that it is a norm. Before proceeding to develop general theory, let us start up with a very simple definition.

The matrix $A = (a_{nk})$ is known as quasi-summable when A is the upper triangular matrix, namely, $a_{nk} = 0$ for $n > k$. As it can be clearly seen the matrix satisfies $\sum_{n=1}^{k} a_{nk} = 1$ for all $k \in \mathbb{N}$.

Theorem 5. *The matrix $T = (t_{nk})$ is a bounded matrix operator from the space $\ell_1(w)$ to the space $\ell_1(\tilde{w}, \tilde{F}(r,s))$ if $M = \sup_{k\in\mathbb{N}} \frac{s_k}{w_k} < \infty$, in which $\lambda_k = \sum_{n=1}^{\infty} \tilde{w}_n\left|r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right|$. In that case, the norm of operator is obtained as $\|T\|_{1,w,\tilde{w},\tilde{F}} = M$.*

For all $n \in \mathbb{N}$, taking both $w_n = 1$ and $\tilde{w}_n = 1$ specially, the transformation T is a bounded operator from the space ℓ_1 to the space $\ell_1(\tilde{F}(r,s))$ and also $\|T\|_{1,\tilde{F}} = \sup_{k\in\mathbb{N}} s_k$.

Proof. We consider a sequence $x = (x_n)$ in $\ell_1(w)$, therefore:

$$\begin{aligned}
\|Tx\|_{1,\tilde{w},\tilde{F}} &= \sum_{n=1}^{\infty} \tilde{w}_n\left|\sum_{k=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right)x_k\right| \\
&\leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{w}_n\left|r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right||x_k| \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \tilde{w}_n\left|r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right||x_k| \\
&= \sum_{k=1}^{\infty} \lambda_k|x_k| \\
&\leq M\sum_{k=1}^{\infty} w_k|x_k| \\
&= M\|x\|_{1,w}.
\end{aligned}$$

What is seen from these equations is the fact that $\|T\|_{1,w,\tilde{w},\tilde{F}} \leq M$ since $\frac{\|Tx\|_{1,\tilde{w},\tilde{F}}}{\|x\|_{1,w}} \leq M$. We proceed by introducing the sequence $e^i = (0, 0, \ldots, 0, \overset{i.}{1}, 0, \ldots)$ for each $i \in \mathbb{N}$ for computing the converse inequality, and then obtain $\|e^i\|_{1,w} = w_i$ and also $\|Te^i\|_{1,\tilde{w},\tilde{F}} = \lambda_i$. Because of these, it is easy to see that $\|T\|_{1,w,\tilde{w},\tilde{F}} \geq M$ first, and then $\|T\|_{1,w,\tilde{w},\tilde{F}} = M$. □

Theorem 6. *Let us assume that $S = (s_{nk})$ is the upper triangular matrix having the non-negative entries and also assume that (w_n) is an increasing given sequence. When the inequality $s_{nk} \geq s_{n+1,k}$ is valid for each values of $n \in \mathbb{N}$, constant $k \in \mathbb{N}$, at the same time $s = -r < 0$ and $M' = \sup_{k\in\mathbb{N}} \sum_{n=1}^{k} s_{nk} < \infty$, then S is defined as a bounded operator described from $\ell_1(w)$ to $\ell_1(w, \tilde{F}(r,s))$. Furthermore, the norm of this given operator satisfies the inequality given in the form $\|S\|_{1,w,\tilde{F}} \leq rM'$. When the specific condition of S is being a quasi-summable matrix is taken into consideration, the condition $\|S\|_{1,w,\tilde{F}} = r$ is obtained.*

Proof. Due to the hypothesis, we need to say that the matrix $S = (s_{nk})$ that satisfies $s_{nk} \geq s_{n+1,k}$ (for all $n, k = 1, 2, \ldots$) condition is upper triangular and also the sequence (w_n) is increasing. With simple calculations, and taking into consideration $s = -r < 0$, the following is derived:

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} w_n \left| r\frac{f_{n+1}}{f_n} s_{nk} + s\frac{f_n}{f_{n+1}} s_{n+1,k} \right| \\
&= \sum_{n=1}^{k-1} w_n \left(r\frac{f_{n+1}}{f_n} s_{nk} - r\frac{f_n}{f_{n+1}} s_{n+1,k} \right) + w_k r \frac{f_{k+1}}{f_k} s_{kk} \\
&\leq w_k \left[\sum_{n=1}^{k-1} \left(r\frac{f_{n+1}}{f_n} s_{nk} - r\frac{f_n}{f_{n+1}} s_{n+1,k} \right) + r\frac{f_{k+1}}{f_k} s_{kk} \right] \\
&= w_k \left[\left(r\frac{f_2}{f_1} s_{1k} - r\frac{f_1}{f_2} s_{2k} \right) + \cdots + \left(r\frac{f_k}{f_{k-1}} s_{k-1,k} - r\frac{f_{k-1}}{f_k} s_{kk} \right) + r\frac{f_{k+1}}{f_k} s_{kk} \right] \\
&= w_k \left[r\frac{f_2}{f_1} s_{1k} + \left(r\frac{f_3}{f_2} - r\frac{f_1}{f_2} \right) s_{2k} + \cdots + \left(r\frac{f_{k+1}}{f_k} - r\frac{f_{k-1}}{f_k} \right) s_{kk} \right] \\
&= r w_k \sum_{n=1}^{k} s_{nk}.
\end{aligned}$$

Clearly, $\|S\|_{1,w,\tilde{F}} = r \sup_{k\in\mathbb{N}} \frac{\lambda_k}{w_k} \leq r \sup_{k\in\mathbb{N}} \sum_{n=1}^{k} s_{nk} = rM'$ from Theorem 5.

Let us assume that S is a quasi-summable matrix, therefore $M' = 1$ and hence $\|S\|_{1,w,\tilde{F}} \leq r$. To get the inverse inequality, let us take into account the sequence $e^1 = (1, 0, 0, \ldots)$. From this it follows that $\|e^1\|_{1,w} = w_1$ and $\|Se^1\|_{1,w,\tilde{F}} = rw_1$, namely $\|S\|_{1,w,\hat{F}} \geq r$. As a result, $\|S\|_{1,w,\tilde{F}} = r$ is obtained. □

Let us state that, In Theorem 6, if there is no constriction on r and s, then since:

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} w_n \left| r\frac{f_{n+1}}{f_n} s_{nk} + s\frac{f_n}{f_{n+1}} s_{n+1,k} \right| \\
&\leq (|r| + |s|) w_k \sum_{n=1}^{k} \frac{f_{n-1} + f_{n+1}}{f_n} s_{nk} \\
&\leq (|r| + |s|) w_k \sum_{n=1}^{k} \left(\sup_{n\in\mathbb{N}} \frac{f_{n-1} + f_{n+1}}{f_n} \right) s_{nk}
\end{aligned}$$

the inequality is valid, the following is obtained $\|S\|_{1,w,\tilde{F}} \leq 3(|r| + |s|)M'$.

In the light of the above mentioned theorems, we are here concerned with calculating the norm of some specific quasi-summable matrices. Initially, we consider the transpose of the well-known Riesz matrix $\tilde{R} = (\tilde{r}_{nk})$ described as follows:

$$\tilde{r}_{nk} = \begin{cases} \frac{q_n}{Q_k}, & n \leq k \\ 0, & n > k, \end{cases} \tag{1}$$

in which (q_n) is a non-negative sequence with $q_1 > 0$ and $Q_k = q_1 + \cdots + q_k$ for all $k \in \mathbb{N}$. If we take $q_n = 1$ for all $n \in \mathbb{N}$, we derive the transpose of the Cesàro matrix of order one, which is also known as a Copson matrix (see [25]). We indicate this specific matrix by $\tilde{C} = (\tilde{c}_{nk})$, in which:

$$\tilde{c}_{nk} = \begin{cases} \frac{1}{k}, & n \le k \\ 0, & n > k. \end{cases}$$

Corollary 1. *When (q_n) is a decreasing sequence and (w_n) is an increasing sequence, in that case, $\tilde{R}$ is a bounded operator from the space $\ell_1(w)$ into the space $\ell_1(w, \tilde{F}(r,s))$ and, also $\|\tilde{R}\|_{1,w,\tilde{F}} = r$ for $s = -r < 0$.*

Proof. First of all, since (q_n) is a decreasing sequence from the hypothesis, the following inequality $\tilde{r}_{nk} = \frac{q_n}{Q_k} \ge \frac{q_{n+1}}{Q_k} = \tilde{r}_{n+1,k}$ holds for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$. For $\tilde{R}$, is a non-negative upper triangular matrix and (w_n) is an increasing sequence. It follows from Theorem 6 that $\tilde{R}$ is a bounded operator from $\ell_1(w)$ into $\ell_1(w, \tilde{F}(r,s))$. Moreover, due to the fact that $\sum_{n=1}^{k} \tilde{r}_{nk} = 1$ for every $k \in \mathbb{N}$, $\tilde{R}$ is a quasi-summable matrix. If $s = -r < 0$, then it is clear that $\|\tilde{R}\|_{1,w,\tilde{F}} = r$ from Theorem 6. □

Corollary 2. *If $\sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k} < \infty$, then the matrix $\tilde{C}$ defined above is a bounded operator from the space $\ell_1(w)$ into $\ell_1(\tilde{w}, \tilde{F}(r,s))$ and $\|\tilde{C}\|_{1,w,\tilde{w},\tilde{F}} \le (2|r| + |s|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$.*

Proof. To show its consistency with the previous work, let us first do the proof for any $k \in \mathbb{N}$ and special case $r \ge -s > 0$ of r and s; under this condition, we get:

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} \tilde{w}_n \left| r\frac{f_{n+1}}{f_n}\tilde{c}_{nk} + s\frac{f_n}{f_{n+1}}\tilde{c}_{n+1,k} \right| \\
&= \left(\sum_{n=1}^{k-1} \tilde{w}_n \left| r\frac{f_{n+1}}{f_n}\frac{1}{k} + s\frac{f_n}{f_{n+1}}\frac{1}{k} \right| \right) + \tilde{w}_k r \frac{f_{k+1}}{f_k}\frac{1}{k} \\
&= \frac{1}{k}\left[\sum_{n=1}^{k-1} \tilde{w}_n \left(r\frac{f_{n+1}}{f_n} + s\frac{f_n}{f_{n+1}} \right) + \tilde{w}_k r \frac{f_{k+1}}{f_k} \right] \\
&\le \frac{r}{k}\sum_{n=1}^{k} \tilde{w}_n \frac{f_{n+1}}{f_n} \le \frac{2r}{k}\sum_{n=1}^{k} \tilde{w}_n.
\end{aligned}$$

Thus, considering the Theorem 5, the norm $\|\tilde{C}\|_{1,w,\tilde{w},\tilde{F}} \le 2r \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$ is obtained. If $r = 1$, $s = -1$ are taken here, the result obtained by İlkhan [14] in Corollary 2 is found.

If we give the proof in general:

$$\begin{aligned}
\lambda_k &= \sum_{n=1}^{\infty} \tilde{w}_n \left| r\frac{f_{n+1}}{f_n}\tilde{c}_{nk} + s\frac{f_n}{f_{n+1}}\tilde{c}_{n+1,k} \right| \\
&\le \frac{1}{k}\left[\sum_{n=1}^{k-1} \tilde{w}_n \left(|r|\frac{f_{n+1}}{f_n} + |s|\frac{f_n}{f_{n+1}} \right) + \tilde{w}_k |r| \frac{f_{k+1}}{f_k} \right] \\
&= \frac{|r|}{k}\sum_{n=1}^{k} \tilde{w}_n \frac{f_{n+1}}{f_n} + \frac{|s|}{k}\sum_{n=1}^{k-1} \tilde{w}_n \frac{f_n}{f_{n+1}} \le \frac{2|r| + |s|}{k}\sum_{n=1}^{k} \tilde{w}_n.
\end{aligned}$$

for $r, s \in \mathbb{R}$, then we obtain that $\|\tilde{C}\|_{1,w,\tilde{w},\tilde{F}} \le (2|r| + |s|) \sup_{k\in\mathbb{N}} \frac{\sum_{n=1}^{k} \tilde{w}_n}{k w_k}$ from Theorem 5. □

When the result obtained for the special cases of r and s is compared with the result obtained without any restrictions, since the general result is greater than the particular ones, there is not any contradictory situation.

Theorem 7. *Let us suppose that $T = (t_{nk})$ is a matrix having the non-negative entries and the inequalities $t_{nk} \ge t_{n+1,k}$ hold for all $n \in \mathbb{N}$ and each fixed $k \in \mathbb{N}$ and $r \ge -s > 0$ are valid. If*

$\sum_{n=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}+s\frac{f_{n-1}}{f_n}\right)t_{nk}<\infty$ for each $k\in\mathbb{N}$ and also $M''=\sup_{k\in\mathbb{N}}\sum_{n=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}+s\frac{f_{n-1}}{f_n}\right)t_{nk}<\infty$, then the matrix T is a bounded operator from the space ℓ_1 to $\ell_1(\tilde{F}(r,s))$ and the norm of operator is $\|T\|_{1,\tilde{F}}\leq M''$. When the fact that the specific condition of T is being a quasi-summable matrix is taken into consideration for $s=-r<0$, then the condition $\|T\|_{1,\tilde{F}}=r$ is derived.

Proof. For arbitrary $k\in\mathbb{N}$, we have:

$$\lambda_k=\sum_{n=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}t_{nk}+s\frac{f_n}{f_{n+1}}t_{n+1,k}\right)=\sum_{n=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}+s\frac{f_{n-1}}{f_n}\right)t_{nk}.$$

If Theorem 5 is used here, it is found that norm $\|T\|_{1,\tilde{F}}\leq M''$. The rest of the proof can be done similarly to the proof of Theorem 6. □

The matrix $H=(h_{nk})$ defined as $h_{nk}=\frac{1}{n+k}$ for all $n,k\in\mathbb{N}$ is called the Hilbert matrix operator. Here, we will discover the norm of the operator just mentioned.

Now, let us give the following integral to be used in the proofs:

$$\int_0^{\infty}\frac{1}{t^{\alpha}(t+c)}dt=\frac{\pi}{c^{\alpha}\sin\alpha\pi},$$

in which $0<\alpha<1$.

Theorem 8. *Let $w_n=\frac{1}{n^{\alpha}}$ for all $n\in\mathbb{N}$, in which $0<\alpha<1$. In this case, the Hilbert matrix operator H just described is bounded from the space $\ell_1(w)$ to the space $\ell_1(w,\tilde{F}(r,s))$ and also the norm $\|H\|_{1,w,\tilde{F}}\leq\frac{\pi}{\sin\alpha\pi}(2|r|+|s|)$.*

Proof. For all $n\in\mathbb{N}$, we have:

$$\begin{aligned}\lambda_n&=\sum_{i=1}^{\infty}w_i\left|r\frac{f_{i+1}}{f_i}h_{in}+s\frac{f_i}{f_{i+1}}h_{i+1,n}\right|\\&\leq\sum_{i=1}^{\infty}\frac{1}{i^{\alpha}}\left(|r|\frac{f_{i+1}}{f_i}\frac{1}{i+n}+|s|\frac{f_i}{f_{i+1}}\frac{1}{i+n+1}\right)\\&\leq\int_0^{\infty}\frac{1}{t^{\alpha}}\left(2|r|\frac{1}{t+n}+|s|\frac{1}{t+n+1}\right)dt\\&=\frac{\pi}{\sin\alpha\pi}\left(\frac{2|r|}{n^{\alpha}}+\frac{|s|}{(n+1)^{\alpha}}\right).\end{aligned}$$

It follows that:

$$n^{\alpha}\lambda_n\leq\frac{\pi}{\sin\alpha\pi}\left[2|r|+|s|\left(\frac{n}{n+1}\right)^{\alpha}\right]\leq\frac{\pi}{\sin\alpha\pi}(2|r|+|s|).$$

Considering Theorem 5, this means that $\|H\|_{1,w,\tilde{F}}\leq\frac{\pi}{\sin\alpha\pi}(2|r|+|s|)$. □

Again in proof, if $r\geq -s>0$ is taken to see the consistency with the result obtained by İlkhan [14], it follows that:

$$n^{\alpha}s_n\leq\frac{\pi}{\sin\alpha\pi}\left[2r+\frac{s}{2}\left(\frac{n}{n+1}\right)^{\alpha}\right]\leq\frac{\pi}{\sin\alpha\pi}\left(2r+\frac{s}{2^{\alpha+1}}\right).$$

Considering Theorem 5, this means that $\|H\|_{1,w,\tilde{F}}\leq\frac{\pi}{\sin\alpha\pi}(2r+\frac{s}{2^{\alpha+1}})$.

4. The Norm of Matrix Operators from $\ell_p(w)$ to $\ell_p(w,\tilde{F}(r,s))$

In this section, we discuss calculating the norm of some matrix operators from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w},\tilde{F}(r,s))$. Now, we are going to present an essential lemma, which is obtained by Jameson and Lashkaripour, since this important result is used in the proofs.

Lemma 1. *[25] Let us suppose that $S = (s_{nk})$ is a matrix operator having the nonnegative entries $s_{nk} \geq 0$, also suppose that (u_n) and (t_k) are positive sequences given such that:*

$$u_n^{1/p} \sum_{k=1}^{\infty} \frac{s_{nk}}{t_k^{1/p}} \leq A \qquad (\text{for } n \in \mathbb{N},\ A \in \mathbb{R})$$

and,

$$\frac{1}{t_k^{(1-p)/p}} \sum_{n=1}^{\infty} u_n^{(1-p)/p} s_{nk} \leq B \qquad (\text{for } k \in \mathbb{N},\ B \in \mathbb{R})$$

in that case, that inequality $\|S\|_p \leq \frac{B^{1/p}}{A^{(1-p)/p}}$ is valid, in which $p > 1$.

Now, let us state and prove another necessary lemma.

Lemma 2. *Let us assume that the equality $s_{nk} = \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p} \left(r \frac{f_{n+1}}{f_n} t_{nk} + s \frac{f_n}{f_{n+1}} t_{n+1,k}\right)$ is valid for the matrix operators $T = (t_{nk})$ and $S = (s_{nk})$. At the same time, we have $\|T\|_{p,w,\tilde{w},\tilde{F}} = \|S\|_p$, for $p \geq 1$. Under the conditions of this hypothesis, T is bounded operator from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$ iff S is bounded operator onto the space ℓ_p.*

Proof. If a sequence $x = (x_k)$ lying in the space $\ell_p(w)$ is taken as arbitrary, and a sequence $y = (y_k)$ is defined as $y_k = w_k^{1/p} x_k$ for all $k \in \mathbb{N}$ by making use of it, then we derive the equality $\|x\|_{p,w} = \|y\|_p$. Therefore, the proof should be clear with the following rudimentary calculations:

$$\begin{aligned}
\|T\|_{p,w,\tilde{w},\tilde{F}}^p &= \sup_{x \in \ell_p(w), x \neq 0} \frac{\|Tx\|_{p,\tilde{w},\tilde{F}}^p}{\|x\|_{p,w}^p} \\
&= \sup_{x \in \ell_p(w), x \neq 0} \frac{\sum_{n=1}^{\infty} \tilde{w}_n \left| \sum_{k=1}^{\infty} \left(r \frac{f_{n+1}}{f_n} t_{nk} + s \frac{f_n}{f_{n+1}} t_{n+1,k}\right) x_k \right|^p}{\sum_{k=1}^{\infty} w_k |x_k|^p} \\
&= \sup_{y \in \ell_p} \frac{\sum_{n=1}^{\infty} \left| \sum_{k=1}^{\infty} \left(\frac{\tilde{w}_n}{w_k}\right)^{1/p} \left(r \frac{f_{n+1}}{f_n} t_{nk} + s \frac{f_n}{f_{n+1}} t_{n+1,k}\right) y_k \right|^p}{\sum_{k=1}^{\infty} |y_k|^p} \\
&= \sup_{y \in \ell_p} \frac{\sum_{n=1}^{\infty} |\sum_{k=1}^{\infty} s_{nk} y_k|^p}{\sum_{k=1}^{\infty} |y_k|^p} = \sup_{y \in \ell_p} \frac{\|Sy\|_p^p}{\|y\|_p^p} = \|S\|_p^p.
\end{aligned}$$

□

Theorem 9. *Let us assume that the matrix operator $\tilde{R}$ is as defined in (1), and also assume that (q_n) is a decreasing sequence having $q_1 = q_2 = 2$ and $\lim_{n\to\infty} Q_n = \infty$. For all $n \in \mathbb{N}$, if the sequence (w_n) is taken as $\left(\frac{2Q_{n-1}}{q_n}\right)^p$ with $Q_0 = 1$, in that case, $\tilde{R}$ is bounded operator from the space $\ell_p(w)$ to the space $\ell_p(\tilde{F}(r,s))$ and $\|\tilde{R}\|_{p,w,\tilde{F}} = r$ for $p > 1$ and $s = -r < 0$.*

Proof. In Lemma 2, we are going to utilize the matrix $\tilde{R}$ in place of T. So, the matrix $S = (s_{nk})$ is described by:

$$s_{nk} = \begin{cases} \frac{rq_k}{2Q_{k-1}Q_k}\left(\frac{f_{n+1}}{f_n}q_n - \frac{f_n}{f_{n+1}}q_{n+1}\right), & n < k \\ \frac{1}{2}r\frac{f_{k+1}}{f_k}\frac{q_k^2}{Q_{k-1}Q_k}, & n = k \\ 0, & n > k \end{cases}$$

and besides that, $\|\tilde{R}\|_{p,w,\tilde{F}} = \|S\|_p$ is obtained.

We derive:

$$\begin{aligned} \sum_{k=1}^{\infty} s_{nk} &= \frac{1}{2}r\frac{f_{n+1}}{f_n}q_n\frac{q_n}{Q_{n-1}Q_n} + \frac{1}{2}\left(r\frac{f_{n+1}}{f_n}q_n - r\frac{f_n}{f_{n+1}}q_{n+1}\right)\sum_{k=n+1}^{\infty}\frac{q_k}{Q_{k-1}Q_k} \\ &= \frac{1}{2}r\frac{f_{n+1}}{f_n}q_n\left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right) + \frac{1}{2}\left(r\frac{f_{n+1}}{f_n}q_n - r\frac{f_n}{f_{n+1}}q_{n+1}\right)\frac{1}{Q_n} \\ &= \frac{1}{2}r\frac{f_{n+1}}{f_n}\frac{q_n}{Q_{n-1}} - \frac{1}{2}r\frac{f_n}{f_{n+1}}\frac{q_{n+1}}{Q_n} \\ &\leq r \end{aligned}$$

for all $n \in \mathbb{N}$. We also derive:

$$\begin{aligned} \sum_{n=1}^{\infty} s_{nk} &= \frac{1}{2}\frac{q_k}{Q_{k-1}Q_k}\left[\sum_{n=1}^{k-1}\left(r\frac{f_{n+1}}{f_n}q_n - r\frac{f_n}{f_{n+1}}q_{n+1}\right)\right] + \frac{1}{2}r\frac{q_k}{Q_{k-1}Q_k}\frac{f_{k+1}}{f_k}q_k \\ &= \frac{1}{2}\frac{q_k}{Q_{k-1}Q_k}r\sum_{n=1}^{k}q_n \leq r \end{aligned}$$

for all $n \in \mathbb{N}$. Now, In Lemma 1, if we take $u_n = t_n = 1$ for all $n \in \mathbb{N}$, we get $A \leq r$ and $B \leq r$ which requires that $\|\tilde{R}\|_{p,w,\tilde{F}} \leq r$. Now, for $e^1 = (1,\ 0,\ 0,\ \dots)$, we get $\|e^1\|_{p,w} = \frac{2Q_0}{q_1} = 1$ and $\|\tilde{R}e^1\|_{p,\tilde{F}} = \left(\left(r\frac{f_2}{f_1}\frac{q_1}{Q_1}\right)^p\right)^{\frac{1}{p}} = r$ and then $\|\tilde{R}\|_{p,w,\tilde{F}} \geq r$. $\square$

Theorem 10. *Let $w_n = \frac{1}{n^\alpha}$ for all $n \in \mathbb{N}$, in which $1 - p < \alpha < 1$ and $p > 1$. In that case, the Hilbert matrix operator H is a bounded operator from the space $\ell_p(w)$ to the space $\ell_p(w, \tilde{F}(r,s))$ also following inequality:*

$$\|H\|_{p,w,\tilde{F}} \leq \max\left\{\frac{\pi}{\sin\beta\pi}(2|r| + |s|), \frac{\pi}{\sin\gamma\pi}(2|r| + |s|)\right\},$$

is valid, in which $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$.

Proof. Let us define the matrix $S = (s_{nk})$ as follows:

$$s_{nk} = \left(\frac{k}{n}\right)^{\alpha/p}\left(\frac{f_{n+1}}{f_n}\frac{r}{n+k} + \frac{f_n}{f_{n+1}}\frac{s}{n+k+1}\right)$$

for all $n, k \in \mathbb{N}$. In this case, $\|H\|_{p,w,\tilde{F}} = \|S\|_p$ which obtained by using Lemma 2. Specifically, when we choose $u_n = t_n = n$ in Lemma 1 for all $n \in \mathbb{N}$, we find that:

$$
\begin{aligned}
u_n^{\frac{1}{p}}\sum_{k=1}^{\infty}\frac{s_{nk}}{t_k^{\frac{1}{p}}} &= n^{1/p}\sum_{k=1}^{\infty}\frac{1}{k^{1/p}}\left(\frac{k}{n}\right)^{\alpha/p}\left(\frac{f_{n+1}}{f_n}\frac{r}{n+k}+\frac{f_n}{f_{n+1}}\frac{s}{n+k+1}\right)\\
&\leq n^{\beta}\sum_{k=1}^{\infty}\frac{1}{k^{\beta}}\left(\frac{2|r|}{n+k}+\frac{|s|}{n+k+1}\right)\\
&\leq n^{\beta}\int_{t=0}^{\infty}\frac{1}{t^{\beta}}\left(\frac{2|r|}{t+n}+\frac{|s|}{t+(n+1)}\right)dt\\
&= n^{\beta}\left(\frac{2|r|\pi}{n^{\beta}\sin\beta\pi}+\frac{|s|\pi}{(n+1)^{\beta}\sin\beta\pi}\right)\\
&\leq \frac{\pi}{\sin\beta\pi}(2|r|+|s|)
\end{aligned}
$$

for all $n \in \mathbb{N}$ also:

$$
\begin{aligned}
\frac{1}{t_k^{\frac{1-p}{p}}}\sum_{n=1}^{\infty}u_n^{\frac{1-p}{p}}s_{nk} &= \frac{1}{k^{(1-p)/p}}\sum_{n=1}^{\infty}n^{(1-p)/p}\left(\frac{k}{n}\right)^{\alpha/p}\left(\frac{f_{n+1}}{f_n}\frac{r}{n+k}+\frac{f_n}{f_{n+1}}\frac{s}{n+k+1}\right)\\
&\leq k^{\gamma}\sum_{n=1}^{\infty}\frac{1}{n^{\gamma}}\left(\frac{2|r|}{n+k}+\frac{|s|}{n+k+1}\right)\\
&\leq k^{\gamma}\int_{t=0}^{\infty}\frac{1}{t^{\gamma}}\left(\frac{2|r|}{t+k}+\frac{|s|}{t+(k+1)}\right)dt\\
&= k^{\gamma}\left(\frac{2|r|\pi}{k^{\gamma}\sin\gamma\pi}+\frac{|s|\pi}{(k+1)^{\gamma}\sin\gamma\pi}\right)\\
&\leq \frac{\pi}{\sin\gamma\pi}(2|r|+|s|)
\end{aligned}
$$

for all $k \in \mathbb{N}$, where $\beta = \frac{1-\alpha}{p}$ and $\gamma = \frac{p-1+\alpha}{p}$. We therefore obtain that:

$$
\|H\|_{p,w,\tilde{F}} \leq \max\left\{\frac{\pi}{\sin\beta\pi}(2|r|+|s|), \frac{\pi}{\sin\gamma\pi}(2|r|+|s|)\right\}.
$$

from Lemma 1. □

5. Lower Bounds of Matrix Operators from $\ell_p(w)$ to $\ell_p(\tilde{w}, \tilde{F}(r,s))$

An important problem that arises in this work is how to calculate the lower bound of an operator T defined from the space $\ell_p(w)$ to space $\ell_p(\tilde{w}, \tilde{F}(r,s))$. Therefore, here we obtain the lower bound of the operator T for the largest L value that satisfies the following inequality

$$
\|Tx\|_{p,\tilde{w},\tilde{F}} \geq L\|x\|_{p,w}
$$

for every decreasing sequence $x = (x_k)$ with $x_k \geq 0$.

We need the following Lemma to perform the calculations in the proofs in this section.

Lemma 3 ([25]). *Let us assume that both (q_n) and (x_n) are non-negative sequences, and that (x_n) is also a decreasing sequence satisfying condition $\lim_{n\to\infty} x_n = 0$. For $Q_n = \sum_{i=1}^{n} q_i$ with $Q_0 = 1$ also $R_n = \sum_{i=1}^{n} q_i x_i$, the following statements holds, in which $p \geq 1$ and $n \in \mathbb{N}$.*

1. $R_n^p - R_{n-1}^p \geq (Q_n^p - Q_{n-1}^p)x_n^p$.
2. *When the series $\sum_{i=1}^{\infty} q_i x_i$ converges, the following inequality is satisfied.*

$$
\left(\sum_{i=1}^{\infty} q_i x_i\right)^p \geq \sum_{n=1}^{\infty} Q_n^p (x_n^p - x_{n+1}^p).
$$

Theorem 11. *When $T = (t_{nk})$ is a matrix operator of which entries are non-negative and defined from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$, in which $p \geq 1$, the following inequality $t_{nk} \geq$*

$t_{n+1,k}$ is valid for all $n \in \mathbb{N}$, each fixed $k \in \mathbb{N}$, also the series $\sum_{n=1}^{\infty} w_n$ diverges to infinity, in that case, for every decreasing sequence $x = (x_k)$ having $x_k \geq 0$, we have:

$$\|Tx\|_{p,\tilde{w},\tilde{F}} \geq L\|x\|_{p,w},$$

in which $L^p = \inf_{n\in\mathbb{N}} \frac{S_n}{W_n}$, $W_n = \sum_{k=1}^{n} w_k$ and $S_n = \sum_{i=1}^{\infty} \tilde{w}_i \left(\sum_{k=1}^{n}\left(r\frac{f_{i+1}}{f_i}t_{ik} + s\frac{f_i}{f_{i+1}}t_{i+1,k}\right)\right)^p$ for $r \geqslant -s > 0$.

Proof. Under the conditions of the hypothesis expressed in the theorem, we can make the proof as follows. Since $\sum_{n=1}^{\infty} w_n = \infty$, we get $\lim_{k\to\infty} x_k = 0$, and at the same time, it can be obtained that the series $\sum_{k=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right)x_k$ is convergent for all $n \in \mathbb{N}$. On the other hand, by using Lemma 3 and also using Abel summation, we have:

$$\begin{aligned}
\|Tx\|^p_{p,\tilde{w},\tilde{F}} &= \sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{\infty}\left(r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right)x_k\right)^p \\
&\geq \sum_{n=1}^{\infty} \tilde{w}_n \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i}\left(r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right)\right)^p (x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty}\left[\sum_{n=1}^{\infty} \tilde{w}_n \left(\sum_{k=1}^{i}\left(r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k}\right)\right)^p\right](x_i^p - x_{i+1}^p) \\
&= \sum_{i=1}^{\infty} S_i(x_i^p - x_{i+1}^p) \geq L^p \sum_{i=1}^{\infty} W_i(x_i^p - x_{i+1}^p) = L^p\|x\|^p_{p,w}
\end{aligned}$$

which completes the proof. □

The following Lemma can be verified in a similar technique with the proof of Proposition 1 in [25].

Lemma 4. *Let us assume that $T = (t_{nk})$ be a matrix operator of which entries are non-negative and defined from the space $\ell_p(w)$ to the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$, in which $p \geq 1$. If the following inequality:*

$$r\frac{f_{n+1}}{f_n}t_{nk} + s\frac{f_n}{f_{n+1}}t_{n+1,k} \geq r\frac{f_{n+1}}{f_n}t_{n,k+1} + s\frac{f_n}{f_{n+1}}t_{n+1,k+1}$$

is valid also $t_{nk} \geq t_{n+1,k}$ for all $k \in \mathbb{N}$, each fixed $n \in \mathbb{N}$ and $r \geq -s > 0$, if the series $\sum_{n=1}^{\infty} w_n$ is divergent the infinity, then we have:

$$L^p \geq \inf_{n\in\mathbb{N}} [n^p - (n-1)^p]\frac{t_n}{w_n},$$

in which $t_n = \sum_{i=1}^{\infty} \tilde{w}_i \left(r\frac{f_{i+1}}{f_i}t_{in} + s\frac{f_i}{f_{i+1}}t_{i+1,n}\right)^p$.

Theorem 12. *Let $H = (h_{nk})$ is the Hilbert matrix operator, $w_n = \frac{1}{n^{p+\alpha}}$ and $\tilde{w}_n = \frac{1}{n^\alpha}$ for every $n \in \mathbb{N}$, in which $p \geq 1$, $0 \leq p + \alpha \leq 1$ and $r \geq -s > 0$. For every decreasing sequences $x = (x_k)$ that are not negative terms, we have:*

$$\|Hx\|_{p,\tilde{w},\tilde{F}} \geq L\|x\|_{p,w}$$

in which $L^p \geq \sum_{i=1}^{\infty} \frac{1}{i^\alpha (i+1)^p (i+2)^p}$.

Proof. It is clear that both the Hilbert matrix $H = (h_{nk})$ and the sequence (w_n) fulfill the conditions listed in Lemma 4, therefore, we obtain:

$$
\begin{aligned}
L^p &\geq \inf_{n\in\mathbb{N}} [n^p - (n-1)^p] \frac{t_n}{w_n} \\
&\geq \inf_{n\in\mathbb{N}} n^{p-1} n^{p+\alpha} \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \left(r \frac{f_{i+1}}{f_i} \frac{1}{i+n} + s \frac{f_i}{f_{i+1}} \frac{1}{i+n+1} \right)^p \\
&\geq \inf_{n\in\mathbb{N}} n^{2p+\alpha-1} \sum_{i=1}^{\infty} \frac{1}{i^\alpha} \left(\frac{r}{i+n} + \frac{s}{i+n+1} \right)^p.
\end{aligned}
$$

The rest of the proof can be derived in the same way as in the proof of Theorem 4.3 in [27]. □

Conclusions

In the present article, the norm of matrix operators described among the weighted sequence space $\ell_p(w)$ and the Fibonacci weighted difference sequence space denoted by $\ell_p(\tilde{w}, \tilde{F}(r,s))$ for $1 \leq p < \infty$ has been put forward. During the process for this aim, several special matrices like quasi-summable matrices (the transposes of both Riesz and the transpose of Cesàro matrices having order one) and also Hilbert matrix have been utilized. First of all, the space $\ell_p(\tilde{w}, \tilde{F}(r,s))$ has been introduced and several characteristics of the space have been investigated. Namely, an isomorphism is obtained by utilizing this space. Later, the norm for some matrix operators is defined on the generalized Fibonacci weighted difference sequence space. Finally, the lower bound for the matrix operator defined from $\ell_p(w)$ to $\ell_p(\tilde{w}, \tilde{F}(r,s))$ has been calculated.

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