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A Novel Approach to the Fuzzification of Fields

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Abstract: There are many symmetries in L-fuzzy algebras. In this paper, a novel approach to the fuzzification of a field is introduced. We define a mapping $\mathcal{F} : L^X \to L$ from the family of all the L-fuzzy sets on a field X to L such that each L-fuzzy set is an L-fuzzy subfield to some extent. Some equivalent characterizations $\mathcal{F}(\mu)$ are given by means of cut sets. It is proved that $\mathcal{F}$ is L-fuzzy convex structure on X, hence $(X, \mathcal{F})$ forms an L-fuzzy convexity space. A homomorphism between fields is exactly an L-fuzzy convexity preserving mapping and an L-fuzzy convex-to-convex mapping. Finally, we discuss some operations of L-fuzzy subsets.

Keywords: L-fuzzy subfield measure; L-fuzzy field homomorphism; L-fuzzy convexity; L-fuzzy convex preserving mapping; L-fuzzy convex-to-convex mapping

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1. Introduction

Symmetry occurs not only in geometry, but also in other branches of mathematics. In particular, there are many symmetries in groups, rings, fields, lattices and L-fuzzy algebras. Given a structured object X of any sort, a symmetry is a mapping of the object onto itself that preserves the structure. In algebra, the notion of field is one of the most basic and important concepts. The fuzzification of fields was first introduced in [1]. Later it was redefined in [2–4]. However, given a fuzzy set in a field, we know that it is either a fuzzy subfield or not. Only one of both cases is true. It does not have any fuzziness. In this paper, our aim is to present a novel approach to the fuzzification of fields. We do this work for two reasons. One is to extend the fuzzy field theory to a more general framework, and the other is to provide new examples for expanding the practical scope of fuzzy convexity theory.

Convexity theory has been accepted to be of increasing importance in recent years in the study of extremum problems in many areas of applied mathematics. As the axiomatization of the properties that usual convex sets fulfill, an abstract convexity on a set was proposed. In fact, convexity exists in so many mathematical areas, such as lattices, groups, rings, fields, metric spaces, graphs, matroids, functional analysis and so on. For example, all subalgebras of a universal algebra form exactly a convex structure [5]. Some more applications can be found in [6].

In 1994, the notions of fuzzy convex spaces and convex closure operators were first proposed by Rosa [7]. Later, Maruyama [8] generalized Rosa's Definition to completely distributive lattice-valued setting, and investigated some combinatorial properties of lattice-valued fuzzy convex sets in Euclidean spaces. In a completely different direction, Shi and Xiu [9] proposed the concept of (L, M)-fuzzy convex structures, which contains L-convex structures and M-fuzzifying convex structures as special cases.

As mentioned above, fuzzy convexity exists in so many algebraic areas, such as hazy lattices, hazy groups, fuzzy rings, fuzzy universal algebras, and so on. For example, Li J. and Shi F.-G. first discovered the close relationship between fuzzy sublattice and fuzzy convex structures [10]. Afterwards, Liu and Shi applied (fuzzifying) convexities into M-hazy lattices [11], and M-fuzzifying groups [12], Mehmood and Shi applied (fuzzifying) convexities into M-hazzy rings [13,14] and An and Shi applied L-fuzzy convexities into L-fuzzy rings [15]. However, the research of the relationship between L-fuzzy convexity and L-fuzzy subfields are hardly available.

In this paper, we define a mapping $\mathcal{F} : L^X \to L$ from the family of all the L-fuzzy sets on a field X to L such that each L-fuzzy set is an L-fuzzy subfield to some extent. It is proved that $\mathcal{F}$ is L-fuzzy convex structure on X, hence $(X, \mathcal{F})$ forms a convexity space. A homomorphism between fields is exactly an L-fuzzy convexity preserving mapping and an L-fuzzy convex-to-convex mapping. Our work shows that L-fuzzy subfield theory can be regarded as a new example for expanding the practical scope of fuzzy convexity theory.

2. Preliminaries

Throughout this paper, L is a completely distributive lattice [16,17], the smallest element and the largest element in L are denoted by $\top$ and $\perp$, respectively.

An element a in L is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ [16]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in L is defined as follows: for $a, b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$ [17]. $\{a \in L \mid a \prec b\}$ is called the greatest minimal family of b, denoted by $\beta(b)$. Moreover, define a binary relation $\lhd$ as follows: for all $a, b \in L$, $b \lhd a$ if and only if for every subset $D \subseteq L$, $\bigwedge D \leqslant b$ implies $d \leqslant a$ for some $d \in D$. The set $\{a \in M \mid b \lhd a\}$, denoted by $\alpha(b)$, is called the greatest maximal family of a in the sense of [17]. In a completely distributive lattice L, α is a $\bigwedge$-$\bigcup$ map, β is a union-preserving map [17].

Theorem 1 ([18])**.** *Let $\mu \in L^X$ and $a \in L$. Define*

$$\begin{aligned} \mu^{[a]} &= \{x \in X \mid a \notin \alpha(\mu(x))\}; & \mu^{(a)} &= \{x \in X \mid \mu(x) \nleqslant a\}; \\ \mu_{[a]} &= \{x \in X \mid a \leqslant \mu(x)\}; & \mu_{(a)} &= \{x \in X \mid a \in \beta(\mu(x))\}. \end{aligned}$$

In a complete Heyting algebra L, there exists an implication operator $\mapsto : L \times L \longrightarrow L$ as the right adjoint for the meet operation $\wedge$ by

$$a \mapsto b = \bigvee \{c \in L \mid a \wedge c \leqslant b\}.$$

We list some properties of the implication operation in the following lemma.

Lemma 1 ([19])**.** *Let L be a complete Heyting algebra and let $\mapsto$ be the implication operator corresponding to $\wedge$. Then for all $a, b, c \in L$, $\{a_i\}_{i \in I} \subseteq L$, the following statements hold:*

(1) $\top \mapsto a = a$;
(2) $c \leqslant a \mapsto b \Leftrightarrow a \wedge c \leqslant b$;
(3) $a \mapsto b = \top \Leftrightarrow a \leqslant b$;
(4) $a \mapsto \left(\bigwedge\limits_{i \in I} a_i \right) = \bigwedge\limits_{i \in I} (a \mapsto a_i)$, *hence* $a \mapsto b \leqslant a \mapsto c$ *whenever* $b \leqslant c$;
(5) $\left(\bigvee\limits_{i \in I} a_i \right) \mapsto b = \bigwedge\limits_{i \in I} (a_i \mapsto b)$, *hence* $b \mapsto c \leqslant a \mapsto c$ *whenever* $a \leqslant b$;
(6) $(a \mapsto c) \wedge (c \mapsto b) \leqslant a \mapsto b$;
(7) $(a \mapsto b) \wedge (c \mapsto d) \leqslant a \wedge c \mapsto b \wedge d$.

Definition 1 ([15])**.** *Let μ be an L-fuzzy subset in a ring R. Then the L-fuzzy subring measure $\mathcal{R}(\mu)$ of μ is defined as*

$$\mathcal{R}(\mu) = \bigwedge_{x,y \in R} [(\mu(x) \wedge \mu(y)) \mapsto (\mu(xy) \wedge \mu(x - y))].$$

Lemma 2 ([19]). *Let $f : S \mapsto S'$ be a set mapping andet $\{\mu_i\}$ and $\{\eta_i\}$ be the families of L-fuzzy subsets in S and S' respectively. Then we have*

(1) $f_L^{\rightarrow}(f_L^{\leftarrow}(\eta_i)) = \eta_i$ *if f is surjective;*
(2) $f_L^{\leftarrow}(f_L^{\rightarrow}(\mu_i)) = \mu_i$ *if f is injective,*

where $f_L^{\rightarrow} : L^S \rightarrow L^{S'}$ *and* $f_L^{\leftarrow} : L^{S'} \rightarrow L^S$ *are defined by*

$$f_L^{\rightarrow}(\mu)(y) = \bigvee\{\mu(x) \mid f(x) = y\}, \;\; f_L^{\leftarrow}(\eta) = \eta \circ f.$$

Definition 2 ([9]). *A mapping $\mathcal{C} : L^X \longrightarrow M$ is called an (L, M)-fuzzy convexity on X if it satisfies the following conditions:*
(LMC1) $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$;
(LMC2) if $\{\mu_i \mid i \in \Omega\} \subseteq L^X$ *is nonempty, then* $\bigwedge\limits_{i\in\Omega} \mathcal{C}(\mu_i) \leqslant \mathcal{C}\left(\bigwedge\limits_{i\in\Omega} \mu_i\right)$;
(LMC3) if $\{\mu_i \mid i \in \Omega\} \subseteq L^X$ *is nonempty and totally ordered, then* $\bigwedge\limits_{i\in\Omega} \mathcal{C}(\mu_i) \leqslant \mathcal{C}\left(\bigvee\limits_{i\in\Omega} \mu_i\right)$.

The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex space. An (L, L)-fuzzy convex space is called an L-fuzzy convex space for short.

Definition 3 ([9]). *Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be (L, M)-fuzzy convexity spaces. A mapping $f : X \longrightarrow Y$ is called*

(1) *An (L, M)-fuzzy convexity preserving mapping provided $\mathcal{D}(\lambda) \leqslant \mathcal{C}(f_L^{\leftarrow}(\lambda))$ for all $\lambda \in L^Y$.*
(2) *An (L, M)-fuzzy convex-to-convex mapping provided $\mathcal{C}(\lambda) \leqslant \mathcal{D}(f_L^{\rightarrow}(\lambda))$ for all $\lambda \in L^S$.*
(3) *A mapping $f : X \longrightarrow Y$ is called an (L, M)-fuzzy isomorphism provided f is bijective, (L, M)-fuzzy convex preserving and (L, M)-fuzzy convex-to-convex.*

An (L, L)-fuzzy convex preserving mapping is called an L-fuzzy convex preserving mapping, an (L, L)-fuzzy convex-to-convex mapping is called an L-fuzzy convex-to-convex mapping, and an (L, L)-fuzzy isomorphism is called an L-fuzzy isomorphism for short.

Definition 4 ([2–4]). *An L-fuzzy set μ of a field F is said to be an L-fuzzy subfield of F if for any $x, y \in F$,*

$$\mu(x - y) \geqslant \mu(x) \wedge \mu(y), \;\; \mu(xy^{-1}) \geqslant \mu(x) \wedge \mu(y), \; y \neq 0.$$

Definition 5 ([20]). *Let μ and ν be two L-fuzzy subsets of a field F. Define the L-fuzzy subset $\mu \circ \nu$ of F by $\forall x, y \in F$,*

$$(\mu \circ \nu)(z) = \bigvee_{z=xy} (\mu(x) \wedge \nu(y)).$$

Theorem 2 ([18]). *$\forall \mu \in L^X$, the following conditions are true.*

(1) $\mu = \bigvee\limits_{a\in L} (a \wedge \mu_{[a]}) = \bigvee\limits_{a\in M(L)} (a \wedge \mu_{[a]}) = \bigvee\limits_{a\in L} (a \wedge \mu_{(a)}) = \bigvee\limits_{a\in M(L)} (a \wedge \mu_{(a)})$.
(2) $\mu = \bigwedge\limits_{a\in L} (a \vee \mu_{[a]}) = \bigwedge\limits_{a\in P(L)} (a \vee \mu_{[a]}) = \bigwedge\limits_{a\in L} (a \vee \mu_{(a)}) = \bigwedge\limits_{a\in P(L)} (a \vee \mu_{(a)})$.

3. A Novel Definition of *L*-Fuzzy Subfield

According to Definition 4, we know that an L-fuzzy set in a field F is either an L-fuzzy subfield or not. Only one of both cases is true. It does not have any fuzziness. Next we shall present a novel approach to the fuzzification of subfields.

Definition 6. *Let μ be an L-fuzzy subset in a field F. Then L-fuzzy subfield measure $\mathcal{F}(\mu)$ is defined as*

$$\mathcal{F}(\mu) = \bigwedge_{x,y\in F} \left((\mu(x) \wedge \mu(y)) \mapsto \mu(xy^{-1})\right) \wedge \bigwedge_{x,y\in F} ((\mu(x) \wedge \mu(y)) \mapsto \mu(x - y)).$$

It is obvious that μ is an L-fuzzy subfield of F if and only of $\mathcal{F}(\mu) = \top$.

Example 1. *Let $\mathbb{R}$ be the field of real numbers, and $L = 2^{\{a,b\}}$ with $a \neq b$. Define $\mu, \nu, \lambda : \mathbb{R} \longrightarrow L$ by*

$$\mu(x) = \begin{cases} \{a,b\}, \text{ if } x = 1, \\ \emptyset, \text{ if } x \neq 1. \end{cases} \qquad \nu(x) = \begin{cases} \{a,b\}, \text{if } x \text{ is an integer}, \\ \{b\}, \text{if } x \text{ is not an integer}. \end{cases}$$

$$\lambda(x) = \begin{cases} \{a\}, \text{if } x \text{ is a rational number}, \\ \emptyset, \text{if } x \text{ is an irrational number}. \end{cases}$$

Then we have $\mathcal{F}(\mu) = \perp$, $\mathcal{F}(\nu) = \{b\}$ and $\mathcal{F}(\lambda) = \top$.

(1) *In fact, if $x = y = 1$, then $x - y = 0$, in this case,*

$$\mu(x) \wedge \mu(y) \mapsto \mu(0) = \{a,b\} \mapsto \emptyset = \emptyset,$$

therefore we obtain $\mathcal{F}(\mu) = \emptyset (= \perp)$.

(2) *If $x = 2, y = 3$, then xy^{-1} is not an integer, in this case,*

$$\nu(x) \wedge \nu(y) \mapsto \nu(xy^{-1}) = \{a,b\} \mapsto \{b\} = \{b\},$$

therefore we obtain $\mathcal{F}(\nu) = \{b\}$.

(3) *If both x and y are rational numbers, then $x - y, xy^{-1}$ are rational numbers, in this case,*

$$\lambda(x) \wedge \lambda(y) \mapsto \lambda(x-y) \wedge \lambda(xy^{-1}) = \{a,b\} \mapsto \{a,b\} = \{a,b\}(= \top).$$

If one of x and y is irrational number, then both $x - y$ and xy^{-1} are irrational numbers, in this case,

$$\lambda(x) \wedge \lambda(y) \mapsto \lambda(x-y) \wedge \lambda(xy^{-1}) = \emptyset \mapsto \emptyset = \{a,b\}(= \top).$$

However we can obtain $\mathcal{F}(\lambda) = \top$.

Example 2. *Let $\mathbb{R}$ be the field of real numbers, and $L = [0,1]$, define $\mu, \nu, \lambda : \mathbb{R} \longrightarrow L$ by*

$$\mu(x) = \begin{cases} 0, \text{if } x = 0,1, \\ 1, \text{if } x \neq 0,1. \end{cases} \qquad \nu(x) = \begin{cases} 0.5, \text{ if } x \text{ is a rational number}, \\ 1, \text{ if } x \text{ is an irrational number}. \end{cases} \qquad \lambda(x) = 0.5, \forall x \in \mathbb{R}.$$

Then we have $\mathcal{F}(\mu) = 0$, $\mathcal{F}(\nu) = 0.5$ and $\mathcal{F}(\lambda) = 1$.

(1) *In fact, if $x = y \neq 0,1$, then $x - y = 0$ and $xy^{-1} = 1$, in this case,*

$$\mu(x) \wedge \mu(y) \mapsto \mu(0) \wedge \mu(1) = 1 \mapsto 0 = 0.$$

So we can get $\mathcal{F}(\mu) = 0$.

(2) *If $x = y$ are irrational numbers, then $x - y = 0, xy^{-1} = 1$ are rational numbers, in this case,*

$$\nu(x) \wedge \nu(y) \mapsto \nu(0) \wedge \nu(1) = 1 \mapsto 0.5 = 0.5.$$

So $\mathcal{F}(\nu) = 0.5$.

(3) *$\mathcal{F}(\lambda) = 1$ is obvious.*

Theorem 3. *Let μ be an L-fuzzy set in a field F. Then*

$$\mathcal{F}(\mu) = \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy)) \wedge \bigwedge_{y\in F, y\neq 0} \left(\mu(y) \mapsto \mu(y^{-1})\right) \\ \wedge \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x+y)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(-y)).$$

Proof. From Definition 6 we can obtain that $\forall x, y \in F$, ($y \neq 0$ when considering y^{-1})

$$\mathcal{F}(\mu) \wedge \mu(x) \wedge \mu(y) \leqslant \mu(xy^{-1}), \quad \mathcal{F}(\mu) \wedge \mu(y) \leqslant \mu(1), \\ \mathcal{F}(\mu) \wedge \mu(x) \wedge \mu(y) \leqslant \mu(x-y), \quad \mathcal{F}(\mu) \wedge \mu(y) \leqslant \mu(0).$$

In particular we have that

$$\begin{aligned} &\mathcal{F}(\mu) \wedge \mu(y) \leqslant \mathcal{F}(\mu) \wedge \mu(e) \wedge \mu(y) \leqslant \mu(y^{-1}), \\ &\mathcal{F}(\mu) \wedge \mu(y) \leqslant \mathcal{F}(\mu) \wedge \mu(0) \wedge \mu(y) \leqslant \mu(-y), \\ &\mathcal{F}(\mu) \wedge \mu(y) = \mathcal{F}(\mu) \wedge \mu(y^{-1}), \\ &\mathcal{F}(\mu) \wedge \mu(y) = \mathcal{F}(\mu) \wedge \mu(-y), \\ &\mathcal{F}(\mu) \wedge \mu(x) \wedge \mu(y) = \mathcal{F}(\mu) \wedge \mu(x) \wedge \mu(y^{-1}), \\ &\mathcal{F}(\mu) \wedge \mu(x) \wedge \mu(y) = \mathcal{F}(\mu) \wedge \mu(x) \wedge \mu(-y). \end{aligned}$$

This shows that

$$\begin{aligned} \mathcal{F}(\mu) \leqslant & \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(y^{-1})), \\ \mathcal{F}(\mu) \leqslant & \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x+y)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(-y)). \end{aligned}$$

So

$$\begin{aligned} \mathcal{F}(\mu) \leqslant & \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(y^{-1})) \\ & \wedge \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x+y)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(-y)). \end{aligned}$$

can be proved.

Analogously we can prove

$$\begin{aligned} \mathcal{F}(\mu) \geqslant & \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(xy)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(y^{-1})) \\ & \wedge \bigwedge_{x,y\in F} (\mu(x) \wedge \mu(y) \mapsto \mu(x+y)) \wedge \bigwedge_{y\in F} (\mu(y) \mapsto \mu(-y)). \end{aligned}$$

To sum up, it is available. □

The following lemma is obvious.

Lemma 3. *Let μ be an L-fuzzy set in a field F. Then $\mathcal{F}(\mu) \geqslant a$ if and only if for any $x, y \in F$,*

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), y \neq 0, \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

The next Theorem presents some equivalent descriptions of L-fuzzy subfield measure.

Theorem 4. *Let μ be an L-fuzzy set in a field F. Then*

(1) $\mathcal{F}(\mu) = \bigvee\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\}$.

(2) $\mathcal{F}(\mu) = \bigvee\{a \in L \mid \forall b \leqslant a, \mu_{[b]}$ *is a subfield of* $F\}$.

(3) $\mathcal{F}(\mu) = \bigvee\{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]}$*is a subfield of* $F\}$.

(4) $\mathcal{F}(\mu) = \bigvee\left\{a \in L \mid \forall b \in P(L), b \ngeq a, \mu^{(b)} \text{ is a subfield of } F\right\}$.

(5) $\mathcal{F}(\mu) = \bigvee\{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{ is a subfield of } F\}$ *if* $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ *for any* $a, b \in L$.

Proof. $(1) \Longleftrightarrow (2)$ Suppose that

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F.$$

Then for any $b \leqslant a$ and for any $x, y \in \mu_{[b]}$, we have

$$\mu(xy^{-1}) \geqslant \mu(x) \wedge \mu(y) \wedge a \geqslant \mu(x) \wedge \mu(y) \wedge b \geqslant b,$$
$$\mu(x-y) \geqslant \mu(x) \wedge \mu(y) \wedge a \geqslant \mu(x) \wedge \mu(y) \wedge b \geqslant b,$$

this shows $xy^{-1} \in \mu_{[b]}, x - y \in \mu_{[b]}$. Therefore $\mu_{[b]}$ is a subfield of F. Hence

$$\begin{aligned}\mathcal{F}(\mu) &= \bigvee\left\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\right\}\\ &\leqslant \bigvee\left\{a \in L \mid \forall b \leqslant a, \mu_{[b]} \text{is a subfield of } F\right\}.\end{aligned}$$

Conversely, assume that $a \in L$ and $\forall b \leqslant a$, $\mu_{[b]}$ is a subfield of F.

For any $x, y \in F$, let $b = \mu(x) \wedge \mu(y) \wedge a$. Then $b \leqslant a$ and $x, y \in \mu_{[b]}$, thus $xy^{-1}, x - y \in \mu_{[b]}$, i.e.,

$$\mu(xy^{-1}) \geqslant b = \mu(x) \wedge \mu(y) \wedge a, \mu(x-y) \geqslant b = \mu(x) \wedge \mu(y) \wedge a.$$

This means that

$$\begin{aligned}\mathcal{F}(\mu) &= \bigvee\left\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\right\}\\ &\geqslant \bigvee\left\{a \in L \mid \forall b \leqslant a, \mu_{[b]} \text{is a subfield of } F\right\}.\end{aligned}$$

So $(1) \Longleftrightarrow (2)$ is clearly established;

$(1) \Longleftrightarrow (3)$ Suppose that $a \in L$ and for any $x, y \in F$,

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Then for any $b \notin \alpha(a)$and $x, y \in \mu^{[b]}$, we have

$$b \notin \alpha(\mu(x)) \cup \alpha(\mu(y)) \cup \alpha(a) = \alpha(\mu(x) \wedge \mu(y) \wedge \mu(a)).$$

By

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y),$$

we know

$$\alpha(\mu(xy^{-1})) \subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a), \alpha(\mu(x-y)) \subseteq \alpha(\mu(x) \wedge \mu(y) \wedge a).$$

Hence $b \notin \alpha(\mu(xy^{-1})), b \notin \alpha(\mu(x-y))$, i.e. $xy^{-1}, x - y \in \mu^{[b]}$.

This means that $\mu^{[b]}$ is a subfield of F and $a \in \{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]}\text{is a subfield of } F\}$. This shows that

$$\begin{aligned}\mathcal{F}(\mu) &= \bigvee\left\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\right\}\\ &\leqslant \bigvee\left\{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]} \text{ is a subfield of } F\right\}.\end{aligned}$$

Conversely, assume that

$$a \in \{a \in L \mid \forall b \notin \alpha(a), \mu^{[b]}\text{is a subfield of } F\}.$$

Now we prove for any $x, y \in F$,

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Suppose that $b \notin \alpha(\mu(x) \wedge \mu(y) \wedge a)$. By

$$\alpha(\mu(x) \wedge \mu(y) \wedge a) = \alpha(\mu(x)) \cup \alpha(\mu(x)) \cup \alpha(a),$$

we know that $b \notin \alpha(a)$ and $x, y \in \mu^{[b]}$. Since $\mu^{[b]}$ is a subfield of F, it holds that xy^{-1}, $x - y \in \mu^{[b]}$, i.e., $b \notin \alpha(\mu(xy^{-1}))$ and $b \notin \alpha(\mu(x-y))$. This shows that

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Therefore

$$\begin{aligned}\mathcal{F}(\mu) &= \bigvee \Big\{ a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \Big\} \\ &\geqslant \bigvee \Big\{ a \in L \mid b \notin \alpha(a), \mu^{[b]} \text{is a subfield of } F \Big\}.\end{aligned}$$

So (1) $\Longleftrightarrow$ (3) is clearly established;
(1) $\Longleftrightarrow$ (4) Suppose that $a \in L$ and for any $x, y \in F$,

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Let $b \in P(L), b \ngeqslant a$ and $x, y \in \mu^{(b)}$. Now we prove $xy^{-1}, x - y \in \mu^{(b)}$. If xy^{-1}, $x - y \notin \mu^{(b)}$, i.e., $\mu(xy^{-1}) \leqslant b$ and $\mu(x-y) \leqslant b$, then

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}) \leqslant b, \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y) \leqslant b.$$

By $b \in P(L)$ and $x, y \in \mu^{(b)}$, we have $a \leqslant b$, which contradicts $b \ngeqslant a$. Hence $xy^{-1}, x - y \in \mu^{(b)}$. This shows that $\mu^{(b)}$ is a subfield of F. Therefore

$$\begin{aligned}\mathcal{F}(\mu) &= \bigvee \Big\{ a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F \Big\} \\ &\leqslant \bigvee \Big\{ a \in L \mid \forall b \in P(L), b \ngeqslant a, \mu^{(b)} \text{is a subfield of } F \Big\}.\end{aligned}$$

Conversely, assume that

$$a \in \{a \in L \mid \forall b \in P(L), b \ngeqslant a, \mu^{(b)} \text{is a subfield of F}\}.$$

Now we prove that for any $x, y \in F$,

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Let $b \in P(L)$ and $\mu(x) \wedge \mu(y) \wedge a \nleqslant b$. Then $\mu(x) \nleqslant b$, $\mu(y) \nleqslant b$ and $a \nleqslant b$, i.e., $x, y \in \mu^{(b)}$. Since $\mu^{(b)}$is a subfield of F, it holds that $xy^{-1} \in \mu^{(b)}, x - y \in \mu^{(b)}$, i.e. $\mu(xy^{-1}) \nleqslant b, \mu(x-y) \nleqslant b$. This shows that

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Therefore

$$\begin{aligned}\mathcal{F}(\mu) &= \bigvee \Big\{ a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F \Big\} \\ &\geqslant \bigvee \Big\{ a \in L \mid \forall b \in P(L), b \ngeqslant a, \mu^{(b)} \text{is a subfield of } F \Big\}.\end{aligned}$$

So (1) $\Longleftrightarrow$ (4) is clearly established;

(1) $\Longleftrightarrow$ (5) Suppose that

$$a \in \Big\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\Big\}.$$

Then for any $b \in \beta(a)$ and for any $x, y \in \mu_{(b)}$, it holds that

$$b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a) = \beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(xy)),$$
$$b \in \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a) = \beta(\mu(x) \wedge \mu(y) \wedge a) \subseteq \beta(\mu(x-y)),$$

i.e., $xy^{-1}, x-y \in \mu_{(b)}$. This shows that $\mu_{(b)}$ is a subfield of F. This means that

$$\begin{aligned}\mathcal{F}(\mu) =& \bigvee\Big\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\Big\} \\ \leqslant& \bigvee\Big\{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{is a subfield of } F\Big\}.\end{aligned}$$

Conversely, assume that

$$a \in \{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{is a subfield of } F\}.$$

Now we prove that for any $x, y \in F$,

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Let $b \in \beta(\mu(x) \wedge \mu(y) \wedge a)$. By

$$\beta(\mu(x) \wedge \mu(y) \wedge a) = \beta(\mu(x)) \cap \beta(\mu(y)) \cap \beta(a),$$

we know that $x, y \in \mu_{(b)}$ and $b \in \beta(a)$. Since $\mu_{(b)}$ is a sufield of F, it holds that xy^{-1}, $x-y \in \mu_{(b)}$, i.e., $b \in \beta(\mu(xy^{-1}))$ and $b \in \beta(\mu(x-y))$. This shows that

$$\mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y).$$

Therefore

$$\begin{aligned}\mathcal{F}(\mu) =& \bigvee\Big\{a \in L \mid \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(xy^{-1}), \mu(x) \wedge \mu(y) \wedge a \leqslant \mu(x-y), \forall x, y \in F\Big\} \\ \geqslant& \{a \in L \mid \forall b \in \beta(a), \mu_{(b)} \text{is a subfield of } F\}.\end{aligned}$$

So (1) $\Longleftrightarrow$ (5) is clearly established.
In conclusion, (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (5). $\square$

By Definition 1 and Definition 6 , we obtain the following Theorem.

Theorem 5. *Let μ be an L-fuzzy subset in a field F. Then $\mathcal{F}(\mu) \leqslant \mathcal{R}(\mu)$.*

4. The Relation between L-Fuzzy Subfield Measure and L-Fuzzy Convexity

In this section, we will investigate the relation between L-fuzzy subfield measure and L-fuzzy convexity. We will prove that a field homomorphism is exactly an L-fuzzy convex preserving mapping and an L-fuzzy convex-to-convex mapping.

For each $\mu \in L^F$, $\mathcal{F}(\mu)$ can be naturally considered as a mapping $\mathcal{F} : L^F \longrightarrow L$ defined by $\mu \mapsto \mathcal{F}(\mu)$.

The following Theorem shows that $\mathcal{F}$ is an L-fuzzy convexity on F.

Theorem 6. *Let F be a field. Then the mapping $\mathcal{F} : L^F \longrightarrow L$ defined by $\mu \mapsto \mathcal{F}(\mu)$ is an L-fuzzy convexity on F, which is called the L-fuzzy convexity induced by L-fuzzy subfield measure on F.*

Proof. (LMC1) It is straightforward that

$$\mathcal{F}(\chi_{\emptyset}) = \mathcal{F}(\chi_F) = \top.$$

(LMC2) Let $\{\mu_i \mid i \in \Omega\} \subseteq L^F$ be nonempty. To prove

$$\bigwedge_{i\in\Omega} \mathcal{F}(\mu_i) \leqslant \mathcal{F}\left(\bigwedge_{i\in\Omega} \mu_i\right).$$

For any $a \leqslant \bigwedge_{i\in\Omega} \mathcal{F}(\mu_i)$, by Theorem 4, for all $i \in \Omega$, we have

$$\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(xy^{-1}), \mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(x-y),$$

for all $x, y \in F$. This implies

$$\left(\bigwedge_{i\in\Omega} \mu_i(x)\right) \wedge \left(\bigwedge_{j\in\Omega} \mu_j(y)\right) \wedge a \leqslant \bigwedge_{i\in\Omega} (\mu_i(x) \wedge \mu_i(y) \wedge a) \leqslant \bigwedge_{i\in\Omega} \mu_i(xy^{-1}),$$
$$\left(\bigwedge_{i\in\Omega} \mu_i(x)\right) \wedge \left(\bigwedge_{j\in\Omega} \mu_j(y)\right) \wedge a \leqslant \bigwedge_{i\in\Omega} (\mu_i(x) \wedge \mu_i(y) \wedge a) \leqslant \bigwedge_{i\in\Omega} \mu_i(x-y).$$

By Lemma 3, we have $a \leqslant \mathcal{F}\left(\bigwedge_{i\in\Omega} \mu_i\right)$. From the arbitrarin ess of a, we obtain

$$\bigwedge_{i\in\Omega} \mathcal{F}(\mu_i) \leqslant \mathcal{F}\left(\bigwedge_{i\in\Omega} \mu_i\right).$$

(LMC3) Let $\{\mu_i \mid i \in \Omega\} \subseteq L^F$ be nonempty and totally ordered. To prove

$$\bigwedge_{i\in\Omega} \mathcal{F}(\mu_i) \leqslant \mathcal{F}\left(\bigvee_{i\in\Omega} \mu_i\right).$$

For any $a \leqslant \bigwedge_{i\in\Omega} \mathcal{F}(\mu_i)$. By Lemma 3, for all $i \in \Omega$, we have

$$\mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i\left(xy^{-1}\right), \mu_i(x) \wedge \mu_i(y) \wedge a \leqslant \mu_i(x-y),$$

for all $x, y, \in F$. Let $b \in J(L)$ such that

$$b \prec \left(\bigvee_{i\in\Omega} \mu_i(x)\right) \wedge \left(\bigvee_{i\in\Omega} \mu_i(y)\right) \wedge a,$$

then we have

$$b \prec \bigvee_{i\in\Omega} \mu_i(x), b \prec \bigvee_{i\in\Omega} \mu_i(y), b \leqslant a.$$

Hence there exists some $i, j \in \Omega$ such that $b \leqslant \mu_i(s), b \leqslant \mu_j(t), b \leqslant a$. Since $\{\mu_i \mid i \in \Omega\}$ is totally ordered, we assume $\mu_j \leqslant \mu_i$, it follows that $b \leqslant \mu_i \wedge \mu_i(y) \wedge a$.

By

$$\mu_i \wedge \mu_i(y) \wedge a \leqslant \mu_i(xy^{-1}), \mu_i \wedge \mu_i(y) \wedge a \leqslant \mu_i(x-y),$$

we obtain $b \leqslant \mu_i(xy^{-1})$ and $b \leqslant \mu_i(x-y)$.

Hence $b \leqslant \bigwedge_{i\in\Omega} \mu_i(xy^{-1})$ and $b \leqslant \bigwedge_{i\in\Omega} \mu(x-y)$.

From the arbitrariness of b, we have

$$\left(\bigvee_{i\in\Omega}\mu_i(x)\right)\wedge\left(\bigvee_{i\in\Omega}\mu_i(y)\right)\wedge a\leqslant\bigvee_{i\in\Omega}\mu_i(xy^{-1}),$$
$$\left(\bigvee_{i\in\Omega}\mu_i(x)\right)\wedge\left(\bigvee_{i\in\Omega}\mu_i(y)\right)\wedge a\leqslant\bigvee_{i\in\Omega}\mu_i(x-y).$$

Combining Lemma 3, we have $a\leqslant\mathcal{F}\left(\bigvee_{i\in\Omega}\mu_i\right)$. By the arbitrariness of a, we obtain $\bigwedge_{i\in\Omega}\mathcal{F}(\mu_i)\leqslant\mathcal{F}\left(\bigvee_{i\in\Omega}\mu_i\right)$. Therefore $\mathcal{F}$ is an L-fuzzy convexity on F. □

Now we consider the L-fuzzy subfield measures of homomorphic image and preimage of L-fuzzy sets.

Theorem 7. *Let $f: F\to F'$ be a field homomorphism, $\mu\in L^F$ and $\eta\in L^{F'}$. Then*
(1) *$\mathcal{F}(f_L^{\rightarrow}(\mu))\geqslant\mathcal{F}(\mu)$; if f is injective, then $\mathcal{F}(f_L^{\rightarrow}(\mu))=\mathcal{F}(\mu)$.*
(2) *$\mathcal{F}(f_L^{\leftarrow}(\eta))\geqslant\mathcal{F}(\eta)$; if f is suijective, then $\mathcal{F}(f_L^{\leftarrow}(\eta))=\mathcal{F}(\eta)$.*

Proof. (1) can be proved from Theorem 4 and the following fact.

$$\begin{aligned}
\mathcal{F}(f_L^{\rightarrow}(\mu)) &= \bigvee\left\{a\in L\,\middle|\,\begin{array}{c} f_L^{\rightarrow}(\mu)(x')\wedge f_L^{\rightarrow}(\mu)(y')\wedge a\leqslant f_L'(\mu)\left(x'y'^{-1}\right),\\ f_L^{\rightarrow}(\mu)(x')\wedge f_L^{\rightarrow}(\mu)(y')\wedge a\leqslant f_L'(\mu)(x'-y'),\forall x',y'\in F'\end{array}\right\}\\
&= \bigvee\left\{a\in L\,\middle|\,\begin{array}{c} \bigvee_{f(x)=x'}\mu(x)\wedge\bigvee_{f(y)=y'}\mu(y)\wedge a\leqslant\bigvee_{f(z)=x'y'^{-1}}\mu(z),\\ \bigvee_{f(x)=x'}\mu(x)\wedge\bigvee_{f(y)=y'}\mu(y)\wedge a\leqslant\bigvee_{f(z)=x'-y'}\mu(z),\forall x',y'\in F'\end{array}\right\}\\
&\geqslant\bigvee\left\{a\in L\,\middle|\,\mu(x)\wedge\mu(y)\wedge a\leqslant\mu(xy^{-1}),\mu(x)\wedge\mu(y)\wedge a\leqslant\mu(x-y),\forall x,y\in F\right\}\\
&=\mathcal{F}(\mu).
\end{aligned}$$

If f is injective, the above $\geqslant$ can be replaced by $=$. Hence $\mathcal{F}(\mu)=\mathcal{F}(f_L^{\rightarrow}(\mu))$.
(2) can be proved from the following fact.

$$\begin{aligned}
\mathcal{F}(f_L^{\leftarrow}(\eta)) = &\bigwedge_{x,y\in F}\left((f_L^{\leftarrow}(\eta)(x)\wedge f_L^{\leftarrow}(\eta)(y))\mapsto f_L^{\leftarrow}(\eta)\left(xy^{-1}\right)\right)\\
&\wedge\bigwedge_{x,y\in F}((f_L^{\leftarrow}(\eta)(x)\wedge f_L^{\leftarrow}(\eta)(y))\mapsto f_L^{\leftarrow}(\eta)(x-y))\\
= &\bigwedge_{x,y\in F}\left((\eta(f(x))\wedge\eta(f(y)))\mapsto\eta\left(f(x)f(y^{-1})\right)\right)\\
&\wedge\bigwedge_{x,y\in F}((\eta(f(x))\wedge\eta(f(y)))\mapsto\eta(f(x)-f(y)))\\
\geqslant &\bigwedge_{x',y'\in F'}\left((\eta(x')\wedge\eta(y'))\mapsto\eta\left(x'y'^{-1}\right)\right)\\
&\wedge\bigwedge_{x',y'\in F'}((\eta(x')\wedge\eta(y'))\mapsto\eta(x'-y'))\\
= &\mathcal{F}(\eta).
\end{aligned}$$

If f is surjective, the above $\geqslant$ can be replaced by $=$. Thus we can obtain that $\mathcal{F}(\eta)=\mathcal{F}(f_L^{\leftarrow}(\eta))$. □

By Theorem 7 (1) and (2) , we obtain the following Theorem.

Theorem 8. *Let $\mathcal{F}_F$ and $\mathcal{F}_{F'}$ be the L-fuzzy convexities induced by L-fuzzy subfield measures on F and F'. If $f : F \longrightarrow F'$ is a field homomorphism, then $f : (F, \mathcal{F}_F) \longrightarrow (F', \mathcal{F}_{F'})$ is an L-fuzzy convexity preserving and L-fuzzy convex-to-convex mapping.*

The following corollary is obvious.

Corollary 1. *Let $\mathcal{F}_F$ and $\mathcal{F}_{F'}$ be the L-fuzzy convexities induced by L-fuzzy subfield measures on F and F'. If $f : F \longrightarrow F'$ is a field isomorphism, then $f : (F, \mathcal{F}_F) \longrightarrow (F', \mathcal{F}_{F'})$ is an L-fuzzy isomorphism.*

5. The Operations of *L*-Fuzzy Subfields

In this section, we shall discuss some operation properties of L-fuzzy subfields measures.

In a field F, given two L-fuzzy sets μ, ν, $\mu \circ \nu$ is defined in Defintion 5. Now we present its representations by means of cut sets.

Theorem 9. *Let F be a field, $\mu, \nu \in L^F$. Then the following conditions are true.*

(1) $\forall a \in L, (\mu \circ \nu)_{(a)} \subseteq \mu_{(a)} \circ \nu_{(a)} \subseteq \mu_{[a]} \circ \nu_{[a]} \subseteq (\mu \circ \nu)_{[a]}$.

(2) $\forall a \in L, (\mu \circ \nu)^{(a)} \subseteq \mu^{(a)} \circ \nu^{(a)} \subseteq \mu^{[a]} \circ \nu^{[a]} \subseteq (\mu \circ \nu)^{[a]}$, *in particular, if $a \in P(L)$, then* $(\mu \circ \nu)^{(a)} = \mu^{(a)} \circ B^{(a)}$.

(3) $\mu \circ \nu = \bigwedge\limits_{a \in L} \{a \vee \left(\mu^{[a]} \circ \nu^{[a]}\right)\} = \bigwedge\limits_{a \in P(L)} \{a \vee \left(\mu^{[a]} \circ \nu^{[a]}\right)\}$.

(4) $\mu \circ \nu = \bigwedge\limits_{a \in L} \{a \vee \left(\mu^{(a)} \circ \nu^{(a)}\right)\} = \bigwedge\limits_{a \in P(L)} \{a \vee \left(\mu^{(a)} \circ \nu^{(a)}\right)\}$.

(5) $\mu \circ \nu = \bigvee\limits_{a \in L} \{a \wedge \left(\mu_{[a]} \circ \nu_{[a]}\right)\} = \bigvee\limits_{a \in M(L)} \{a \wedge \left(\mu_{[a]} \circ \nu_{[a]}\right)\}$.

(6) $\mu \circ \nu = \bigvee\limits_{a \in L} \{a \wedge \left(\mu_{(a)} \circ \nu_{(a)}\right)\} = \bigvee\limits_{a \in M(L)} \{a \wedge \left(\mu_{(a)} \circ \nu_{(a)}\right)\}$.

Proof. (1) $\forall a \in L$, first we prove that $(\mu \circ \nu)_{(a)} \subseteq \mu_{(a)} \circ \nu_{(a)}$. By

$$\begin{aligned} z \in (\mu \circ \nu)_{(a)} \quad \Rightarrow \quad a \in \beta((\mu \circ \nu)(x)) \quad &= \quad \beta\left(\bigvee_{z=xy} (\mu(x) \wedge \nu(y))\right) \\ = \quad \bigcup_{z=xy} \beta(\mu(x) \wedge \nu(y)) \quad &\subseteq \quad \bigcup_{z=xy} \beta(\mu(x) \cap \nu(y)) \end{aligned}$$

we can obtain

$$a \in \beta(\mu(x)) \text{ and } a \in \beta(\nu(y)) \Rightarrow x \in \mu_{(a)} \text{ and } y \in \nu_{(a)} \Rightarrow z = xy \in \mu_{(a)} \circ \nu_{(a)},$$

This shows $(\mu \circ \nu)_{(a)} \subseteq \mu_{(a)} \circ \nu_{(a)}$.

It is obvious that $\mu_{(a)} \circ \nu_{(a)} \subseteq \mu_{[a]} \circ \nu_{[a]}$.

$\mu_{[a]} \circ \nu_{[a]} \subseteq (\mu \circ \nu)_{[a]}$ is proved as follows.

Suppose that $z \in \mu_{[a]} \circ \nu_{[a]}$. Then by $\mu_{[a]} \circ \nu_{[a]} = \{z \in F \mid x \in \mu_{[a]}, y \in \nu_{[a]}, z = xy\}$, we have

$$(\mu \circ \nu)(z) = \bigvee_{z=xy} (\mu(x) \wedge \nu(y)) \geqslant \bigvee_{z=xy} (a \wedge a) = a,$$

that is, $z \in (\mu \circ \nu)_{[a]}$, so $\mu_{[a]} \circ \nu_{[a]} \subseteq (\mu \circ \nu)_{[a]}$.

So $\forall a \in L, (\mu \circ \nu)_{(a)} \subseteq \mu_{(a)} \circ \nu_{(a)} \subseteq \mu_{[a]} \circ \nu_{[a]} \subseteq (\mu \circ \nu)_{[a]}$.

(2) $(\mu \circ \nu)^{(a)} \subseteq \mu^{(a)} \circ \nu^{(a)}$ can be proved from the following implications.

$$
\begin{aligned}
z \in (\mu \circ \nu)^{(a)} \quad &\Rightarrow (\mu \circ \nu)(z) \nleqslant a \\
&\Rightarrow \bigvee_{z=xy} (\mu(x) \wedge \nu(y)) \nleqslant a \\
&\Rightarrow \exists x, y \in F \text{ such that } z = xy,\ \mu(x) \nleqslant a,\ \nu(y) \nleqslant a \\
&\Rightarrow \exists x, y \in F \text{ such that } z = xy,\ x \in \mu^{(a)},\ y \in \nu^{(a)} \\
&\Rightarrow z = xy \in \mu^{(a)} \circ \nu^{(a)},
\end{aligned}
$$

we can obtain $(\mu \circ \nu)^{(a)} \subseteq \mu^{(a)} \circ \nu^{(a)}$.

In particular, if $a \in P(L)$, then the inverse of the above implications are true. In this case, $(\mu \circ \nu)^{(a)} = \mu^{(a)} \circ \nu^{(a)}$.

It is obvious that $\mu^{(a)} \circ \nu^{(a)} \subseteq \mu^{[a]} \circ \nu^{[a]}$, next we prove that $\mu^{[a]} \circ \nu^{[a]} \subseteq (\mu \circ \nu)^{[a]}$.

Suppose that $z \notin (\mu \circ \nu)^{[a]}$, then $a \in \alpha((\mu \circ \nu)(z))$. By

$$
\begin{aligned}
a \in \alpha((\mu \circ \nu)(z)) \quad &= \alpha\left(\bigvee_{z=xy} (\mu(x) \wedge \nu(y)) \right) \\
&\subseteq \bigcap_{z=xy} (\alpha(\mu(x)) \cup \alpha(\nu(y))) \\
&\Rightarrow \forall x, y \in F \text{ with } z = xy, \text{ it follows } a \in \alpha(\mu(x)) \text{ or } a \in \alpha(\nu(y)) \\
&\Rightarrow \forall x, y \in F \text{ with } z = xy, \text{ it follows } x \notin \mu^{[a]} \text{ or } y \notin \nu^{[a]} \\
&\Rightarrow z \notin \mu^{[a]} \circ \nu^{[a]}.
\end{aligned}
$$

We can obtain $\mu^{(a)} \circ \nu^{(a)} \subseteq \mu^{[a]} \circ \nu^{[a]}$.

From (1) and (2) and Theorem 2, we can obtain (3), (4), (5) and (6). □

Theorem 10. *Let F be a field , $\mu, \nu \in L^F$, then $\mathcal{F}(\mu \circ \nu) \geqslant \mathcal{F}(\mu) \wedge \mathcal{F}(\nu)$.*

Proof. By Theorem 4, we can obtain the following fact.

$$
\begin{aligned}
\mathcal{F}(\mu \circ \nu) &= \bigvee \{a \in L \mid \forall b \in P(L), b \ngeqslant a, (\mu \circ \nu)^{(b)} \text{ is a subfield of } F\} \\
&= \bigvee \{a \in L \mid \forall b \in P(L), b \ngeqslant a, \mu^{(b)} \circ \nu^{(b)} \text{ is a subfield of } F\} \\
&\geqslant \left(\bigvee \{a \in L \mid \forall b \in P(L), b \ngeqslant a, \mu^{(b)} \text{ is a subfield of } F\} \right) \\
&\quad \wedge \left(\bigvee \{a \in L \mid \forall b \in P(L), b \ngeqslant a, \nu^{(b)} \text{ is a subfield of } F\} \right) \\
&= \mathcal{F}(\mu) \wedge \mathcal{F}(\nu).
\end{aligned}
$$

□

6. Conclusions

We define a mapping $\mathcal{F} : L^X \to L$ from the family of all the L-fuzzy sets on a field X to L such that each L-fuzzy set is an L-fuzzy subfield to some extent. Some equivalent characterizations $\mathcal{F}(\mu)$ are given by means of cut sets. It is proved that $\mathcal{F}$ is L-fuzzy convex structure on X, hence $(X, \mathcal{F})$ forms an L-fuzzy convexity space. A homomorphism between fields is exactly an L-fuzzy convexity preserving mapping and an L-fuzzy convex-to-convex mapping. Finally we discuss some operations of L-fuzzy subsets. This method can be applied to other algebra systems, such as groups, ideals, and so on, that is, we can applied L-fuzzy convexity into L-fuzzy groups, L-fuzzy ideals, and so on. It may be further applied to the study of fuzzy topological vector space.

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