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Effects of a Geometrically Realized Early Dark Energy Era on the Spectrum of Primordial Gravitational Waves

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Abstract: In this work, we investigate the effects of a geometrically generated early dark energy era on the energy spectrum of the primordial gravitational waves. The early dark energy era, which we choose to have a constant equation of state parameter w , is synergistically generated by an appropriate $f(R)$ gravity in the presence of matter and radiation perfect fluids. As we demonstrate, the predicted signal for the energy spectrum of the $f(R)$ primordial gravitational waves is amplified and can be detectable, for various reheating temperatures, especially for large reheating temperatures. The signal amplitude depends on the duration of the early dark energy era and on the value of the dark energy equation of state parameter, with the latter affecting more crucially the amplification. Specifically, the amplification occurs when the equation of state parameter approaches the de Sitter value $w = -1$. Regarding the duration of the early dark energy era, we find that the largest amplification occurs when the early dark energy era commences at temperature $T = 0.85$ eV until $T = 7.8$ eV. Moreover, we study a similar scenario in which amplification occurs, where the early dark energy era commences at $T = 0.29$ eV and lasts until the temperature is increased by $\Delta T \sim 1.7$ eV. The discovery of primordial gravitational waves will reveal if several symmetries in the Universe exist or not so this work is important toward revealing the primordial gravitational waves.

Keywords: neutron stars; scalar-tensor gravity; Higgs inflationary model



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1. Introduction

One of the solutions proposed in the literature for solving the H_0 -tension problem is the existence of an early dark energy era [1–5]. If such an era is observed, then this could eliminate the H_0 -tension problem. In this work, the focus is on the effects of an early dark energy era on the energy spectrum of the primordial gravitational waves generated by an $f(R)$ gravity. Specifically we shall assume that an $f(R)$ gravity is the underlying geometric source that drives inflation, the early dark energy era, and the late-time acceleration era, the commonly known dark energy era. In our approach, inflation will be controlled by a vacuum R^2 gravity, while the early and ordinary dark energy era will synergistically be controlled by appropriate $f(R)$ gravity terms and radiation and cold dark matter fluids. Our aim is to see, in a quantitative way, the direct effect of a geometrically generated early dark energy era on the primordial gravitational waves energy spectrum. For our analysis, we shall use the theoretically predicted energy spectrum which appears in many studies, see, for example, Refs. [6–61] and references therein, and we shall numerically evaluate the effects of the $f(R)$ gravity on the general relativistic waveform. We shall use a WKB approach developed in [29] and we shall investigate the amount of amplification caused by an early dark energy era, which is generated by an $f(R)$ gravity in the presence of perfect matter and radiation fluids. The early dark energy era is described by a constant equation of state (EoS) parameter w , and we shall find which $f(R)$ gravity can generate such an evolution in the presence of matter and radiation perfect fluids. Then, we shall calculate the overall amplification factor of the general relativistic energy spectrum of the primordial gravitational waves. Our results will be confronted with the sensitivity

curves of future interferometric experiments, such as the LISA laser interferometer space antenna [62,63], the DECIGO [64,65], the Einstein Telescope [66], the future BBO (Big Bang Observatory) [67,68], and also the non-interferometric experiments Square Kilometer Array (SKA) [69] and the NANOGrav collaboration [70,71]. As we shall demonstrate, the general relativistic energy spectrum is amplified by the presence of an $f(R)$ gravity-generated early dark energy era, only if several conditions hold true. Specifically, the duration of the early dark energy era plays an important role, and also the value of the EoS parameter w also crucially affects the results. As we show, in most of the cases we studied, the energy spectrum signal is amplified due to this $f(R)$ gravity-generated early dark energy era. As a conclusion, we point out that a detection of a stochastic gravitational wave signal in future interferometric experiments may have many possible explanations, and the early dark energy realized by an $f(R)$ gravity is one of these explanations. Also such a discovery might highlight several unrevealed symmetries of the Universe, such as supersymmetry.

2. $f(R)$ Gravity Realization of Inflation and Subsequent Eras

The standard approach for realizing various cosmological eras in Einstein–Hilbert cosmology is usually performed by using perfect matter fluids. The latter dominate the evolution at a certain point, and the corresponding era is controlled by those fluids, for example, the radiation domination era is controlled by the radiation fluid, the energy density of which redshifts as $\rho_r \sim a^{-4}$, where a is the scale factor of a flat Friedmann–Robertson–Walker (FRW) metric. Additionally for the matter domination era, the matter perfect fluid dominates the evolution, which describes non-baryonic non-relativistic matter and its energy density redshifts as $\rho_m \sim a^{-3}$. Apart from the three standard evolution eras that we usually assume that the Universe underwent, that is, the inflationary, radiation and matter domination, and dark energy era, we basically do not know the behavior of our Universe post-inflationary. We have hints for the post-inflationary era, but no proofs. This era is a mysterious era, and it commences with the reheating era, which is the beginning of the radiation era, and it is believed that the reheating era smoothly deforms to the radiation era. Remarkably, we also lack knowledge of what happened from the reheating era up to the matter domination era, and before the recombination era. Thus, the question is: What do we know? We know very well the physics beyond the recombination era up to the present day. The recombination era is basically where the last scattering surface of the CMB photons was formed, and from this era until present day, we understand the physics relatively well.

Thus, for the epoch before the recombination era, we have no measured data, only at last scattering and beyond, so let us assume that an early dark energy era is realized before the recombination and specifically from the matter–radiation equality redshift $z \sim 3400$ until some final redshift in the past, deeply in the matter domination era z_f . In this work, we shall assume that this final redshift is a free variable and we shall investigate what would be the effect of an early dark energy era on the spectrum of the primordial gravitational waves, focusing on modes which were subhorizon modes immediately after the inflationary era and during the first stages of the reheating era. Additionally, regarding the early dark energy era itself, we shall assume that it is described by a constant EoS parameter, and, more importantly, it is not realized by some perfect matter fluid, but it is realized geometrically, by some dominant form of $f(R)$ gravity for the whole early dark energy era. It is also natural to assume that $f(R)$ controls the inflationary and the late dark energy era, as follows:

$$f(R) = \begin{cases} R + \frac{R^2}{6M^2} & R \sim R_I, \\ F_w(R) & R_f \leq R \ll R_{eq}, \\ R + F_{DE}(R) & R \sim R_0 \ll R_{rec}, \end{cases}$$

with R_I being the curvature scale of inflation, which is calculated primordially when the modes exit the horizon at the first time, so at the beginning of inflation. Additionally,

the curvature scale R_{rec} is the curvature scale at the recombination, R_f is the curvature scale when the early dark energy era commences, R_{eq} is the curvature at matter radiation equality, and R_0 is the curvature scale at present day, so it is basically identical to the cosmological constant. Note that according to our scenario, the era between the curvature scales $R_I < R < R_f$ is not described by modified gravity, so modified gravity affects, post-inflationary, the evolution after $R > R_f$.

The exact forms of $F_w(R)$ and $F_{DE}(R)$ will be specified shortly on the basis of phenomenological viability. Regarding the function $F_w(R)$, this in conjunction with the matter and radiation perfect fluids will synergistically generate the early dark energy era with a constant EoS parameter w , and we shall find that shortly. The function $F_{DE}(R)$ will realize the dark energy era, at late-times, so some appropriate form of this will be used in order to provide a viable dark energy era phenomenology. In order to find the exact forms of the $f(R)$ gravity, we consider the gravitational action with perfect matter fluids present:

$$\mathcal{S} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \mathcal{S}_m, \quad (1)$$

with $\kappa^2 = 8\pi G = \frac{1}{M_p^2}$, with G being Newton's constant, and M_p stands for the reduced Planck mass. In the metric formalism, the field equations are

$$f_R(R)R_{\mu\nu}(g) - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R(R) + g_{\mu\nu} \square f_R(R) = \kappa^2 T_{\mu\nu}^m, \quad (2)$$

with $T_{\mu\nu}^m$ being the energy momentum tensor of the matter and radiation perfect fluids, and furthermore $f_R = \frac{df}{dR}$. For a flat FRW spacetime, in which case the line element reads

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^3 (dx^i)^2, \quad (3)$$

the field equations become

$$-18 \left(4H(t)^2 \dot{H}(t) + H(t) \ddot{H}(t) \right) f_{RR}(R) + 3 \left(H^2(t) + \dot{H}(t) \right) f_R(R) - \frac{f(R)}{2} + \kappa^2 (\rho_m + \rho_r) = 0, \quad (4)$$

with ρ_m, ρ_r standing for the cold dark matter energy density and radiation energy density, respectively.

Now let us proceed to the core of our analysis and we assume that the Universe goes through an intermediate early dark energy era, with a constant EoS parameter w , thus $p_{eff} = w\rho_{eff}$, where p_{eff} and ρ_{eff} are the Universe's total pressure and energy density. The early dark energy era will be assumed to last from $z = 3400$, so from the matter radiation equality, until a redshift z_f , which will be a free parameter in our analysis. During the early dark energy era, lasting from $z = 3400$ up to $z = z_f$, the scale factor of the Universe is

$$a(t) = a_{end} t^{\frac{2}{3(1+w)}}, \quad (5)$$

with a_{end} being the scale factor at the redshift $z = z_f$. It is easy to find which geometrical $f(R)$ theory can realize the cosmology (5) synergistically in the presence of matter and radiation perfect fluids. In order to do so, we shall apply the formalism of Ref. [72], so we use the e -foldings number as a dynamical variable; thus, we have

$$e^{-N} = \frac{a_i}{a}, \quad (6)$$

where a_i is some initial value of the scale factor. Using the e -foldings number N , the Friedmann equation takes the form

$$-18 \left[4H^3(N)H'(N) + H^2(N)(H')^2 + H^3(N)H''(N) \right] f_{RR}(R) + 3 \left[H^2(N) + H(N)H'(N) \right] f_R(R) - \frac{f(R)}{2} + \kappa^2 \rho = 0, \quad (7)$$

where $\rho = \rho_m + \rho_r$. Introducing the auxiliary function, $G(N) = H^2(N)$, the Ricci scalar is written as follows:

$$R = 3G'(N) + 12G(N). \quad (8)$$

Eventually, the Friedman equation becomes

$$-9G(N(R)) [4G'(N(R)) + G''(N(R))] f_{RR}(R) + \left[3G(N) + \frac{3}{2}G'(N(R)) \right] f_R(R) - \frac{f(R)}{2} + \kappa^2(\rho_m + \rho_r) = 0, \quad (9)$$

where $G'(N) = dG(N)/dN$ and $G''(N) = d^2G(N)/dN^2$. By solving the above equation, one obtains the $f(R)$ gravity which realizes the given scale factor of interest, which in our case is that of Equation (5). Let us now proceed to find the explicit form of the $f(R)$ gravity which realizes the scale factor (5). For the scale factor (5), $G(N)$ becomes

$$G(N) = \frac{4e^{-3N(w+1)}}{9(w+1)^2}, \quad (10)$$

and we took $a_r = 1$ for simplicity. Upon combining Equations (8) and (10), we obtain

$$N = \frac{\log\left(\frac{4(1-3w)}{3R(w+1)^2}\right)}{3(w+1)}. \quad (11)$$

The above, in conjunction with the following,

$$\rho_{tot} = \sum_i \rho_{i0} a_0^{-3(1+w_i)} e^{-3N(R)(1+w_i)}, \quad (12)$$

affect the Friedmann Equation (9), which becomes

$$a_1 R^2 \frac{d^2 f(R)}{dR^2} + a_2 R \frac{df(R)}{dR} - \frac{f(R)}{2} + \sum_i S_i R^{\frac{3(1+w_i)}{3(1+w)}} = 0, \quad (13)$$

where the index “ i ” takes values $i = (r, m)$, with $i = r$ indicating radiation and $i = m$ indicating cold dark matter perfect fluids. In addition, the parameters a_1 and a_2 , S_i , and A are defined in the following way:

$$a_1 = \frac{3(1+w)}{4-3(1+w)}, \quad a_2 = \frac{2-3(1+w)}{2(4-3(1+w))}, \quad S_i = \frac{\kappa^2 \rho_{i0} a_0^{-3(1+w_i)}}{[3A(4-3(1+w))]^{\frac{3(1+w_i)}{3(1+w)}}}, \quad A = \frac{4}{3(w+1)} \quad (14)$$

By solving (13) we will obtain the $f(R)$ gravity which generates the early dark energy era, which was denoted $F_w(R)$ previously, so the solution is

$$F_w(R) = \left[\frac{c_2 \rho_1}{\rho_2} - \frac{c_1 \rho_1}{\rho_2(\rho_2 - \rho_1 + 1)} \right] R^{\rho_2+1} + \sum_i \left[\frac{c_1 S_i}{\rho_2(\delta_i + 2 + \rho_2 - \rho_1)} \right] R^{\delta_i+2+\rho_2} - \sum_i B_i c_2 R^{\delta_i+\rho_2} + c_1 R^{\rho_1} + c_2 R^{\rho_2}, \quad (15)$$

where c_1, c_2 are simple integration constants, and also δ_i and B_i are

$$\delta_i = \frac{3(1+w_i) - 23(1+w)}{3(1+w)} - \rho_2 + 2, \quad B_i = \frac{S_i}{\rho_2 \delta_i}, \quad (16)$$

with $i = (r, m)$. Let us now specify the late time era, and we shall choose a convenient $f(R)$ gravity which is known to provide a viable late-time phenomenology. We shall choose the one used in Ref. [51], which is known to be a viable dark energy $f(R)$ gravity model, which is

$$F_{DE}(R) = -\gamma \Lambda \left(\frac{R}{3m_s^2} \right)^\delta, \quad (17)$$

where m_s is $m_s^2 = \frac{\kappa^2 \rho_m^{(0)}}{3}$, and also $\rho_m^{(0)}$ denotes the cold dark matter energy density at present day. Furthermore, the parameter δ is chosen, $0 < \delta < 1$, γ is an arbitrary dimensionless parameter, and Λ is the cosmological constant. Hence, the Universe's evolution is controlled during inflation by an R^2 model, while from z_f and up to $z = 3400$ by $F_w(R)$ given in Equation (15), and at late times by $F_{DE}(R)$ given in Equation (17). In principle, it is not hard to find such a phenomenological model, for example, such a phenomenological model would look similar to

$$f(R) = e^{-\frac{\Lambda}{R}} \left(R + \frac{R^2}{6M^2} \right) + e^{-\frac{R}{\Lambda}} \tanh\left(\frac{R - R_f}{\Lambda}\right) \left(\frac{R - R_{eq}}{\Lambda} \right) F_{DE}(R) + e^{-\frac{R}{\Lambda}} \tanh\left(\frac{R - R_0}{\Lambda}\right) \left(F_w(R) - R - \frac{R^2}{6M^2} \right). \quad (18)$$

It should be noted that the above model is not the only one that can reproduce the phenomenology we want to describe; it is one example, but certainly not the only one. We just quote one example for completeness. With regard to the early dark energy era, we shall consider three cases for values of the EoS parameter, described below:

$$\text{Scenario I: EoS } w = -0.99, \quad (19)$$

$$\text{Scenario II: EoS } w = -\frac{1}{3} - 0.001,$$

$$\text{Scenario III: EoS } w = -0.7.$$

Scenarios I and II basically describe the limiting cases of accelerating expansion, nearly a de Sitter one (Scenario I) and nearly accelerating (Scenario II, slightly smaller EoS parameter compared to the value of the EoS parameter for which non-accelerating nor decelerating cosmology occurs, namely $w = -\frac{1}{3}$). Finally, for Scenario III, the EoS parameter takes an intermediate value for the sake of completeness.

Let us now proceed to the evaluation of the observational indices relevant to the calculation of the energy spectrum of the primordial gravitational waves. Specifically, we shall calculate the tensor-to-scalar ratio and the tensor spectral index. We shall be interested in modes with, which is the pivot scale used in Planck. For $f(R)$ gravity, the tensor spectral index is [51,73,74]

$$n_T \simeq -2\epsilon_1^2, \quad (20)$$

and the tensor-to-scalar ratio is [51,73,74]

$$r = 48\epsilon_1^2, \quad (21)$$

with ϵ_1 being the first slow-roll index $\epsilon_1 = -\dot{H}/H^2$. For the R^2 gravity we have $\epsilon_1 \simeq \frac{1}{2N}$; hence

$$n_T \simeq -\frac{1}{2N^2}, \quad (22)$$

and the corresponding tensor-to-scalar ratio is

$$r = \frac{12}{N^2}. \quad (23)$$

In the next section we shall evaluate numerically the energy spectrum of the primordial gravitational waves at present day, for all the modes that became subhorizon during the early stages of the reheating era. We are interested in short wavelength modes with $\lambda \gg 10$ Mpc, or equivalently with wavenumbers $k > 10^7 \text{ Mpc}^{-1}$ up to $k > 10^{18} \text{ Mpc}^{-1}$, which corresponds to the frequency range $10^{-7} < f < 10^3 \text{ Hz}$.

3. Primordial Gravitational Wave Energy Spectrum: The Effects of an Early Dark Energy Phase

In the next ten years, several space interferometers will provide observational data on whether the theoretically predicted stochastic gravitational wave background exists or not.

Already in the literature, the theoretical predictions for the stochastic background of primordial gravitational waves are intensively studied, see Refs. [6–61] and references therein.

In this paper, the focus is to investigate the effect of an $f(R)$ gravity-realized early dark energy era, with constant EoS parameter w , which commences at the matter–radiation equality and stretches up to a final redshift z_f , which shall be a free variable for the moment. Our main assumption is that $z_f \sim \mathcal{O}(10^4)$, but we shall allow for other higher values for completeness in order to see the effect on the energy spectrum of the primordial gravitational waves. Let us note that the choice of z_f is arbitrary, it just indicates the end of the early dark energy era. This is the maximum redshift which we will allow the upper redshift limit of the early dark energy era to be.

Let us discuss more the duration of the early dark energy era, and as we stated we shall assume that it starts around the matter–radiation equality at redshift $z = 3400$, so at a temperature $T = 0.85$ eV [75], and it ends at z_f which will be assumed in the range $z_f = [10^4, 3 \times 10^4]$, which corresponds to a temperature range $T_f = [0.89, 7.8]$ eV [75]. In terms of the temperature, the early dark energy era does not go deeply in the radiation domination era, and specifically we shall assume that it lasts from $T = 0.89$ eV up to $T_f = 2.6$ eV; however, we shall extend the duration up to $T_f = 7.8$ eV.

The procedure to extract the overall effect of the $f(R)$ gravity on the primordial gravitational waves is based on a WKB method which applies on the modes that became subhorizon just after the inflationary era ended, so during the reheating era. For a pictorial representation of the subhorizon modes, see Figure 1. The WKB method was developed in Refs. [29,30] and relies on the calculation of the parameter a_M , defined as

$$a_M = \frac{f_{RR}\dot{R}}{f_R H}, \quad (24)$$

and the modified gravity effect on the waveform is

$$h = e^{-\mathcal{D}} h_{GR}, \quad (25)$$

where h_{GR} is the waveform with $a_M = 0$ which corresponds to the general relativity case, and also the quantity $\mathcal{D}$ is defined as

$$\mathcal{D} = \frac{1}{2} \int^\tau a_M \mathcal{H} d\tau_1 = \frac{1}{2} \int_0^z \frac{a_M}{1+z'} dz'. \quad (26)$$

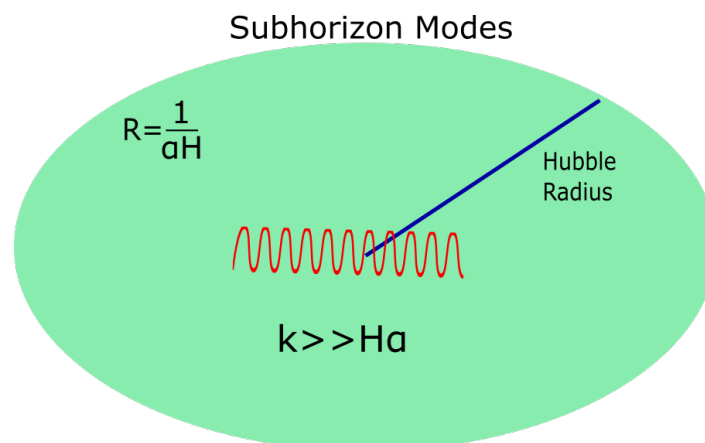


Figure 1. Subhorizon inflationary modes which, during the inflationary era, were at subhorizon scales, and have the smallest wavelength. These are the first that reenter the horizon after the end of the inflationary era and during the early stages of the reheating era. The subhorizon modes will be probed by future space interferometers.

The primordial gravity waves energy density including the $f(R)$ gravity effects is [9,21,22,29,30,32,51],

$$\Omega_{\text{gw}}(f) = e^{-2\mathcal{D}} \times \frac{k^2}{12H_0^2} r \mathcal{P}_\zeta(k_{\text{ref}}) \left(\frac{k}{k_{\text{ref}}} \right)^{n_T} \left(\frac{\Omega_m}{\Omega_\Lambda} \right)^2 \left(\frac{g_*(T_{\text{in}})}{g_{*0}} \right) \left(\frac{g_{*0}}{g_{*s}(T_{\text{in}})} \right)^{4/3} \left(\frac{3j_1(k\tau_0)}{k\tau_0} \right)^2 T_1^2(x_{\text{eq}}) T_2^2(x_R), \quad (27)$$

with $k_{\text{ref}} = 0.002 \text{ Mpc}^{-1}$ being CMB pivot scale. Additionally, n_T denotes the tensor spectral index and r denotes the tensor-to-scalar ratio. The calculation of the quantity $\mathcal{D}$ is the main aim hereafter, for the redshift ranges $z = [0, 3400]$, and $z = [3400, 3 \times 10^4]$, and to see the overall amplification or damping on the energy spectrum. Thus, the quantity that needs to be calculated is

$$\mathcal{D} = \frac{1}{2} \left(\int_0^{3400} \frac{a_{M_1}}{1+z'} dz' + \int_{3400}^{z_f} \frac{a_{M_2}}{1+z'} dz' \right), \quad (28)$$

with a_{M_1} and a_{M_2} being calculated for $f(R) \sim R + f_{DE}(R)$ and $f(R) \sim F_w(R)$, respectively, and with z_f varying in the range $z_f = [10^4, 3 \times 10^4]$. Let us now present our analysis for the three different scenarios we defined in the previous section. For the scenarios II and III, the overall amplification factors $\mathcal{D}$ are nearly zero; hence, the predicted energy spectrum of the primordial gravitational waves is undetectable. Just for the sake of being precise, for scenario II, the parameter $\mathcal{D}$ is $\mathcal{D} \sim 10^{-68}$ and these results apply for both the redshift ranges $z = [3400, 10^4]$ and $z = [3400, 3 \times 10^4]$. In addition, the first integral $\frac{1}{2} \left(\int_0^{3400} \frac{a_{M_1}}{1+z'} dz' \right)$ for all scenarios is of the order ~ -0.05 for all the scenarios. The only non-trivial result occurs only for the Scenario I, in which case we obtain $\int_{3400}^{10^4} \frac{a_{M_2}}{1+z'} dz' = -4.6353$ and $\int_{3400}^{3 \times 10^4} \frac{a_{M_2}}{1+z'} dz' = -9.35798$, so in both cases, amplification occurs for the energy spectrum. In Figures 2 and 3, we plot the $f(R)$ gravity h^2 -scaled energy spectrum as a function of the frequency, and specifically Figure 2 corresponds to the case for which the early dark energy era lasts up to $z_f = 10^4$ and Figure 3 for $z_f = 3 \times 10^4$. In both the plots, we also included the sensitivity curves of most of the future interferometer experiments, and the predicted energy spectrum is presented for three distinct reheating temperatures, for $T_R = 10^{12} \text{ GeV}$ (purple curve), for $T_R = 10^7 \text{ GeV}$ (red curve), and $T_R = 10^2 \text{ GeV}$ (blue curve).

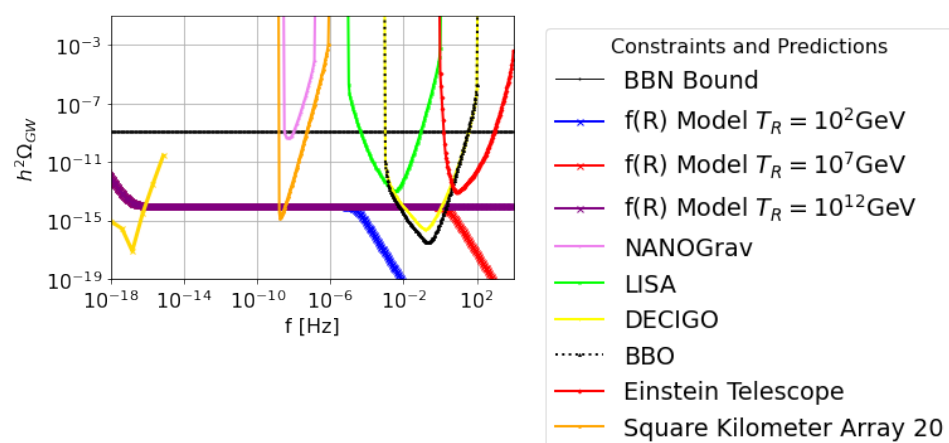


Figure 2. The h^2 -scaled gravitational wave energy spectrum for the early dark energy w -era generated by $f(R)$ gravity, for the Scenario I for $z_f = 10^5$, with reheating temperatures $T_R = 10^{12} \text{ GeV}$ (purple), to $T_R = 10^7 \text{ GeV}$ (red) and to (blue) $T_R = 10^2 \text{ GeV}$.

As it is obvious from both Figures 2 and 3, the signals corresponding to the $f(R)$ gravity models are detectable from most future experiments. Specifically, for the case with $z_f = 10^4$, the $f(R)$ gravity signal will be detected by the SKA experiment for all

reheating temperatures, and by the DECIGO and BBO experiments for reheating temperatures $T_R = 10^7$ GeV and $T_R = 10^{12}$ GeV. As for the case with $z_f = 3 \times 10^4$, in which case the early dark energy era lasts slightly longer compared to the standard scenarios in the literature, the signal will be detected by all the experiments, except for the case with reheating temperature $T_R = 10^2$ GeV, which will be detected only by the SKA and NANOGrav experiments. Let us also note that if the early dark energy era commences earlier, for example at the recombination with $z \sim 1100$, and lasts until $z = 10^4$, similar results are obtained. Indeed, in this case we obtain $\int_{1100}^{10^4} \frac{a M_2}{1+z} dz' = -10$, and in Figure 4 we plot the h^2 -scaled energy spectrum for this case. As it can be seen, the signal will be detected by all the future experiments, except for the low reheating temperature scenario. As an overall comment for all the cases we studied in this section, it seems that the signal of $f(R)$ gravity gravitational waves depends on the duration of the early dark energy era, and for a good possibility of detection, the eras between which the early dark energy era occurs must have a temperature difference approximately of the order $\Delta T \sim 1.7$ eV. We also need to comment on an important issue: since our WKB approach affects the subhorizon modes during reheating, the figures we presented must be looked at with caution for small frequencies, because these modes were superhorizon during reheating. Hence, our results are valid for frequencies starting from the NANOGrav until the Einstein Telescope.

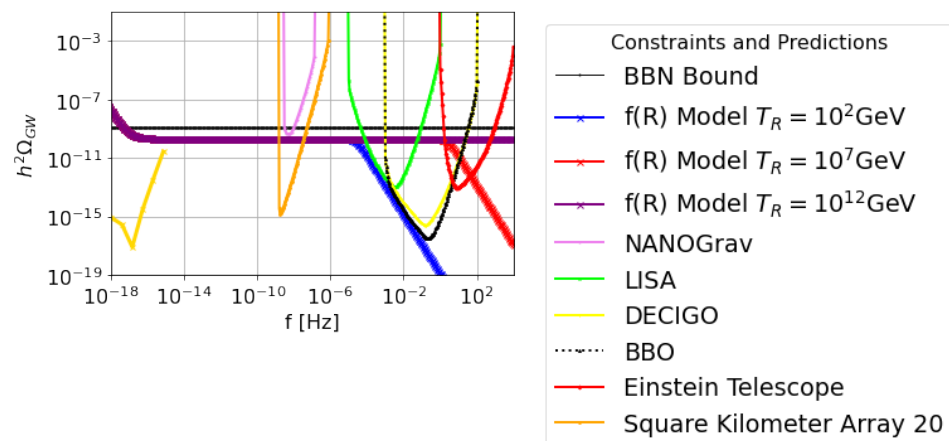


Figure 3. The h^2 -scaled gravitational wave energy spectrum for the early dark energy w -era generated by $f(R)$ gravity, for Scenario I for $z_f = 3 \times 10^4$, with reheating temperatures $T_R = 10^{12}$ GeV (purple), to $T_R = 10^7$ GeV (red) and to (blue) $T_R = 10^2$ GeV.

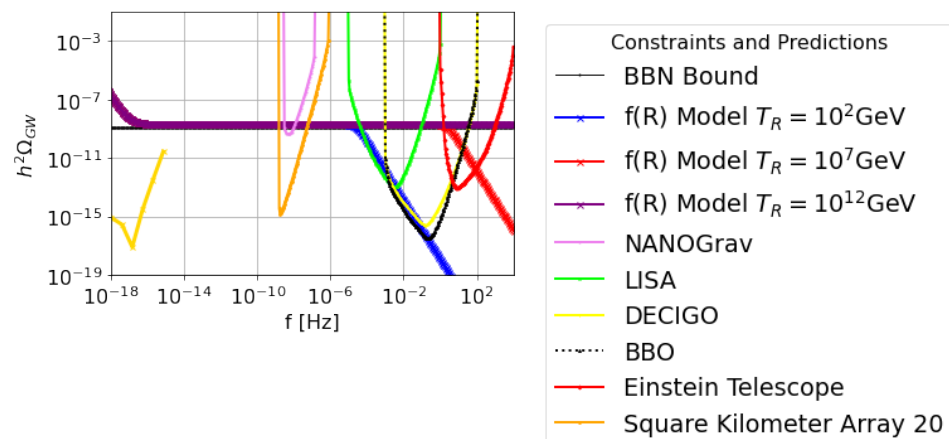


Figure 4. The h^2 -scaled gravitational wave energy spectrum for the early dark energy w -era generated by $f(R)$ gravity, for Scenario I for $z_f = 1 \times 10^4$ and for starting redshift of the early dark energy era $z = 1100$, with reheating temperatures $T_R = 10^{12}$ GeV (purple), to $T_R = 10^7$ GeV (red) and to (blue) $T_R = 10^2$ GeV.

Before closing this section, an important comment is in order. In standard contexts of the Starobinsky inflationary scenario, the signal of the primordial gravitational waves is undetectable. This has to do with the fact that the tensor spectral index for the standard Starobinsky inflation is negative, and thus, the energy spectrum of the primordial gravitational waves generated is significantly lower than the sensitivity curves of most future gravitational waves experiments. This result, however, is mainly based on a standard post-inflationary evolution, which includes the reheating, radiation domination, and matter domination eras. However, with this work, we aimed to show in a quantitative way that a geometrically generated early dark energy era which occurs just before recombination can amplify the energy spectrum significantly. By geometrically generated, we mean that the origin of this era is an $f(R)$ gravity in the presence of matter and radiation perfect fluids. The reason for this amplification is an overall amplifying factor $\sim e^{-2\mathcal{D}}$ appearing in front of Equation (27). This factor takes into account the WKB effects of the modified gravity beyond the inflationary era. The gravitational waves still redshift as radiation but their energy spectrum is amplified, and this amplification is a direct modified gravity effect, due to the fact that the evolution equation contains a WKB overall factor. In most studies known for the Starobinsky inflation predictions for primordial gravitational waves, this post-inflationary amplification is absent, due to the fact that the post-inflationary evolution comprises the reheating, radiation domination, and matter domination eras. In our case, post-inflationary there are non-trivial effects generated by the $f(R)$ gravity geometrically generated early dark energy era. To show in a simple and brief way this issue, in the presence of a non-trivial modified gravity, the evolution equation for tensor perturbations is described by the following equation of the tensor perturbation h_{ij} :

$$\ddot{h}(k) + (3 + \alpha_M)H\dot{h}(k) + \frac{k^2}{a^2}h(k) = 0, \quad (29)$$

where α_M is, in our case, given in Equation (24). If post-inflationary the evolution is the standard one, then the solution to the above equation is described by the one appearing in Equation (25) without the factor $\sim e^{-\mathcal{D}}$; thus, $h(k) \sim h_{GR}$. In our case though, the amplifying factor is non-trivial and as we showed numerically by evaluating $\mathcal{D}$ in Equation (26) for a constant EoS parameter early dark energy era generated by $f(R)$ gravity in the presence of matter and radiation perfect fluids, the amplification is significant.

Furthermore, we need to clarify that the epochs before the early dark energy era are not affected at all in our framework. The only effect of the early dark energy era is contained in the amplifying factor $\mathcal{D}$ in Equation (27). The post-inflationary epochs in between inflation and the early dark energy era are contained in Equation (27), and even the era of BBN is contained there. The BBN bounds are also taken into consideration in Figures 2–4, corresponding to a straight solid black line.

4. Conclusions

In this work, we investigated quantitatively the effect of a geometrically generated early dark energy era on the energy spectrum of the primordial gravitational waves. Specifically the early dark energy era is generated by an appropriate $f(R)$ gravity in the presence of matter and radiation perfect fluids. The main assumption we made is that $f(R)$ gravity generates the inflationary era, specifically, an R^2 gravity, the late-time acceleration era, and also the early dark energy era. As we showed, the energy spectrum of the primordial gravitational waves is significantly amplified in a detectable way, and the signal can be detected by most of the future gravitational waves experiments that will search for stochastic gravitational waves. Our analysis indicates that the amplification of the signal depends on two crucial parameters, the value of the dark energy EoS parameter w and the duration of the early dark energy era. As we showed, the amplification occurs only when the EoS parameter of the early dark energy era is close to the de Sitter value, and we specifically studied the case $w = -0.9$. In addition, the duration of the early dark energy era affects the amplification. We assumed that the early dark energy era started at $z = 3400$

up to $z = 3 \times 10^4$ and we also considered the case in which the early dark energy era started at $z = 1100$ up to $z = 10^4$. In both cases, the amplification is significant for $w = -0.9$ and for large reheating temperatures. In conclusion, the scenario we described in this paper indicates one certain thing: if a stochastic gravitational wave signal is detected in future gravitational waves experiments, the source of this signal is far from being certain, since several scenarios might lead to such a spectrum.

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