symmetry

MDPI

Article

Multiplicity of Solutions for Quasilinear Differential Models Generated by Instantaneous and Non-Instantaneous Impulses

Tengfei Shen [1,*], Wenbin Liu [1] and Wei Zhang [2]

[1] School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China; liuwenbin-xz@163.com or wbliu@cumt.edu.cn

[2] School of Mathematics and Big Data, Anhui University of Science and Technology, Huainan 232001, China; zhangwei_azyw@163.com

* Correspondence: stfcool@126.com or shentengfei@cumt.edu.cn

Abstract: This paper aims to deal with the multiplicity of weak solutions for quasilinear differential models generated by instantaneous and non-instantaneous impulses. By establishing the new variational structure and overcoming the influence of impulsive effects brought by the quasilinear term, some new results are acquired via the gene property, which extends and enriches some previous results. Moreover, an example is given to illustrate the conclusion of the main results.

Keywords: multiplicity; non-instantaneous impulse; instantaneous impulse; gene property

MSC: 34A37; 34B37

check for updates

Citation: Shen, T.; Liu, W.; Zhang, W. Multiplicity of Solutions for Quasilinear Differential Models Generated by Instantaneous and Non-Instantaneous Impulses. *Symmetry* **2022**, *14*, 1141. https://doi.org/10.3390/sym14061141

Academic Editor: Mariano Torrisi

Received: 19 April 2022

Accepted: 28 May 2022

Published: 1 June 2022

1. Introduction

In this paper, we are concerned with the following one-dimensional second-order quasilinear differential equation with instantaneous and non-instantaneous impulses as follows.

$$\begin{cases} -u''(t) + b(t)u(t) - (u^2(t))''u(t) = f_i(t, u(t)), \ a.e.\ t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta(u'(t_i)) = I_i(u(t_i)), \ i = 1, 2, ..., N, \\ u'(t) = \theta_i(t_i^+), t \in (t_i, s_i], \ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), \ i = 1, 2, ..., N, \\ u(0) = u(T) = 0, \end{cases} \tag{1}$$

where $f_i(t, u) = g_i(t, u) + \lambda h_i(t)|u|^{\nu-2}u$, $g_i \in C((s_i, t_{i+1}] \times \mathbb{R}, \mathbb{R})$, $b \in L^\infty([0, T], \mathbb{R})$, $h \in L^\infty((s_i, t_{i+1}], \mathbb{R})$, $\theta_i \in L^\infty((t_i, s_i], \mathbb{R}^+)$, $\lambda \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, +\infty)$, $v \in [1, 2)$, $s_0 = 0 < t_1 < s_1 < t_2 < \cdots < s_N < t_{N+1} = T$, $I_i \in C(\mathbb{R}, \mathbb{R})$, $\Delta(u'(t_i)) = u'(t_i^+) - u'(t_i^-)$ and $u'(t_i^\pm) = \lim_{t \to t_i^\pm} u'(t)$, $\theta_i(t_i^+) = \lim_{t \to t_i^+} \theta_i(t)$.

This problem has a practical background that arises from the standing wave solutions ($\phi(t, x) = e^{-iwt}u(x)$, $w \in \mathbb{R}$) of a kind of quasilinear Schrödinger equation as follows.

$$i\partial_t\phi = -\partial_{xx}\phi + V(x)\phi - \partial_{xx}(|\phi|^2)\phi - |\phi|^{q-1}\phi, \ x \in \mathbb{R}, q > 1. \tag{2}$$

For the theme of existence and multiplicity of standing wave solutions for (2), one can refer to [1–4] and the references therein. Naturally, an interesting question is whether there is a standing wave solution to (2) with suitable boundary conditions when impulsive effects happen. The multiplicity of solutions of boundary value problems (BVPs for short) to differential equations is an important research topic in the qualitative theory of differential equations. It originated from the practical application in the fields of physics and engineering, etc., and can ensure that appropriate solutions may be found in practical nonlinear problems. Therefore, it has important theoretical significance. By establishing the new variational structure and overcoming the influence of impulsive effects, the multiplicity of weak solutions for Dirichlet BVP (1) is considered via the gene property.

As is known to all that the BVPs of impulsive differential equations are an effective means to describe the discontinuous change of things. It has many practical applications in the scientific and technological fields, such as SIR epidemic models, aerospace technology, controllability, optimization, signal communication, and economic regulation, etc. (see [5–8] and references therein). Thus, BVPs of differential equations with impulses have attracted the attention of many scholars. For example, Nieto and O'Regan [9] dealt with the instantaneous impulsive Dirichlet BVP

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), \text{ a.e. } t \in J, \\ \Delta(u'(t_j)) = I_j(u(t_j)), j = 1, 2, ..., m, \\ u(0) = u(T) = 0, \end{cases} \tag{3}$$

and achieved some existing results via employing some critical point theorems. Zhou and Li [10] extended the results of [9] to the case of the variable coefficient. For more articles concerning the second-order Dirichlet BVP with instantaneous impulsive effects, see Zhang and Yuan [11], Sun and Chen [12], etc. It should be mentioned that Shen and Liu [13] investigated the multiplicity of solutions for the Dirichlet BVP (1) by the symmetry mountain pass theorem with instantaneous impulsive effects.

In 2013, Hernández and O'Regan [14] firstly introduced the non-instantaneous impulsive problem, whose impulsive effects keep active on a finite time interval. Since then, more and more scholars have paid attention to this interesting problem (see [15,16] and references therein). Recently, Bai and Nieto [17] made use of the classical Lax–Milgram Theorem to construct the variational structure of the second-order Dirichlet BVP with non-instantaneous impulsive effects and obtained the existence and uniqueness of weak solutions. Khaliq and ur Rehman [18] extended the results of [17] to the case of the fractional Dirichlet BVP with non-instantaneous impulsive effects. Based on Ekeland's variational principle, Tian and Zhang [19] created a further study on the existence of solutions for the second-order Dirichlet BVP with non-instantaneous and instantaneous impulses as follows.

$$\begin{cases} -u''(t) = g_i(t, u(t)), t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N \\ \Delta(u'(t_i)) = I_i(u(t_i)), \ i = 1, 2, ..., N, \\ u'(t) = u(t_i^+), t \in (t_i, s_i], \ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-), \ i = 1, 2, ..., N, \\ u(0) = u(T) = 0. \end{cases} \tag{4}$$

Zhang and Liu [20] extended the results of [19] to the case of the fractional Dirichlet BVP with non-instantaneous and instantaneous impulsive effects. Moreover, for the topic of the existence of multiple weak solutions for impulsive equations, one can read [21,22] and references therein.

Motivated by the works mentioned above, we are concerned with the multiplicity of weak solutions for the Dirichlet BVP (1). Let us present the characteristics of this paper: First, under the influence of non-instantaneous and instantaneous impulsive effects, a new energy functional is established for the second-order Dirichlet BVP of quasilinear differential equations, which implies that the variational methods can be used to investigate the existence and multiplicity of weak solutions for this problem. Second, the non-instantaneous and instantaneous impulsive effects generated by the quasilinear term $(u^2)''u$ are more complicated than the case of u'', which makes this problem more interesting and difficult.

2. Preliminaries

To begin with, we introduce some necessary basic knowledge and signs. Let $C := C([0,T],\mathbb{R})$ with norm $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$ and $L^p := L^p([0,T],\mathbb{R})$ with norm $\|u\|_{L^p} = (\int_0^T |u(t)|^p dt)^{\frac{1}{p}}, 1 \le p \le \infty$. In the Sobolev space $H_0^1(0,T)$, define the inner product

$$< u, v >= \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt,\ \forall u,\ v \in H_0^1(0,T),$$

inducing the norm

$$\|u\| = \left(\int_0^T |u(t)|^2 + |u'(t)|^2 dt\right)^{\frac{1}{2}}. \tag{5}$$

By Poincaré's inequality $\|u\|_{L^2} \le \frac{1}{\sqrt{\mu}}\|u'\|_{L^2}$, where $\mu = \frac{\pi^2}{T^2}$ means the first eigenvalue relating to $-u'' = \mu u$ with Dirichlet boundary conditions, it follows that $\|u'\|_{L^2}^2 \le \|u\|^2 \le (1+\frac{1}{\mu})\|u'\|_{L^2}^2$. Therefore, the norm $\|u'\|_{L^2}$ is equivalent to $\|u\|$. In this paper, assume that $\operatorname{ess\,inf}_{t\in[0,T]} b(t) > q$ and $\rho = \min\{2\theta_{min}^2, q\} > -\mu$, where $\theta_{min} = \min_{i=1,2,\dots,N}\theta_i(t_i^+)$, q is a constant. If we consider the following inner product

$$< u, v >_\rho= \int_0^T \rho u(t)v(t)dt + \int_0^T u'(t)v'(t)dt,\ \forall u,\ v \in H_0^1(0,T),$$

inducing the norm

$$\|u\|_\rho = \left(\int_0^T \rho|u(t)|^2 + |u'(t)|^2 dt\right)^{\frac{1}{2}}, \tag{6}$$

by the Lemma 2.1 in [10] and Poincaré's inequality, there exists a constant $\vartheta \in (0,1)$ such that

$$\vartheta\|u'\|_{L^2}^2 \le \|u\|_\rho^2 \le (1+\frac{\rho}{\mu})\|u'\|_{L^2}^2. \tag{7}$$

Thus, the norms $\|u'\|_{L^2}$, $\|u\|_\rho$ and $\|u\|$ are equivalent. Moreover, in view of the Sobolev imbedding theorem, we can find a constant $S > 0$ such that $\|u\|_\infty \le S\|u\|$. It should be mentioned that for each $u \in H_0^1(0,T)$, u is absolutely continuous and $u' \in L^2$. Thus, impulsive effects may occur. Therefore, the following lemma can be established.

Lemma 1. *If a function $u \in H_0^1(0,T)$ is a solution of problem (1), then the following identity*

$$\begin{aligned}&\int_0^T u'(t)v'(t)dt + \int_0^T (2u'^2(t)u(t)v(t) + 2u^2(t)u'(t)v'(t))dt + \sum_{i=1}^N \int_{t_i}^{s_i} 2\theta_i^2(t_i^+)u(t)v(t)dt \\ &+ \sum_{i=0}^N \int_{s_i}^{t_{i+1}} b(t)u(t)v(t)dt + \sum_{i=1}^N (2u^2(t_i)+1)I_i(u(t_i))v(t_i) = \sum_{i=0}^N \int_{s_i}^{t_{i+1}} f_i(t,u(t))v(t)dt\end{aligned} \tag{8}$$

holds for any $v \in H_0^1(0,T)$.

Proof. In view of (1), we have

$$
\begin{aligned}
&\int_0^T (2u'^2(t)u(t)v(t) + 2u^2(t)u'(t)v'(t))dt \\
&= \int_0^T (u(t)v(t))'(u^2(t))'dt \\
&= \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} ((u(t)v(t))'(u^2(t))'dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} ((u(t)v(t))'(u^2(t))'dt \\
&= \sum_{i=0}^{N} (2u^2(t)u'(t)v(t)) \mid_{s_i^+}^{t_{i+1}^-} - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (u^2(t))''u(t)v(t)dt \\
&\quad + \sum_{i=1}^{N} (2u^2(t)u'(t)v(t)) \mid_{t_i^+}^{s_i^-} - \sum_{i=1}^{N} \int_{t_i}^{s_i} (2u(t)u'(t))'u(t)v(t)dt \\
&= \sum_{i=1}^{N} (2u^2(t_i)u'(t_i^-)v(t_i) - 2u^2(t_i)u'(t_i^+)v(t_i)) + 2u^2(T)u'(T)v(T) - 2u^2(0)u'(0)v(0) \\
&\quad + \sum_{i=1}^{N} (2u^2(s_i)u'(s_i^-)v(s_i) - 2u^2(s_i)u'(s_i^+)v(s_i)) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (u^2(t))''u(t)v(t)dt \\
&\quad - \sum_{i=1}^{N} \int_{t_i}^{s_i} 2u'^2(t)u(t)v(t)dt \\
&= -\sum_{i=1}^{N} 2I_i(u(t_i))u^2(t_i)v(t_i) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} (u^2(t))''u(t)v(t)dt \\
&\quad - \sum_{i=1}^{N} 2\theta_i^2(t_i^+) \int_{t_i}^{s_i} u(t)v(t)dt.
\end{aligned} \tag{9}
$$

Similarly, it follows that

$$
\int_0^T u'(t)v'(t)dt = -\sum_{i=1}^{N} I_i(u(t_i))v(t_i) - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} u''(t)v(t)dt. \tag{10}
$$

Moreover, we can obtain

$$
\int_0^T b(t)u(t)v(t)dt = \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} b(t)u(t)v(t)dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} b(t)u(t)v(t)dt, \tag{11}
$$

which together with the eqution

$$
-u''(t) + b(t)u(t) - (u^2(t))''u(t) = f_i(t, u(t)),\ a.e.\ t \in (s_i, t_{i+1}],
$$

(9) and (10) yield (8). $\square$

Definition 1. *A function $u \in H_0^1(0,T)$ is labeled as a weak solution of problem (1), if (8) is satisfied for any $v \in H_0^1(0,T)$.*

Define the functional $\Phi : H_0^1(0,T) \to \mathbb{R}$ by

$$
\begin{aligned}
\Phi(u) \;=\; & \frac{1}{2}\int_0^T u'^2(t)dt + \frac{1}{2}\sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} b(t)u^2(t)dt + \sum_{i=1}^{N} \int_{t_i}^{s_i} \theta_i^2(t_i^+)u^2(t)dt + \int_0^T u'^2(t)u^2(t)dt \\
+ \; & \sum_{i=1}^{N} \int_0^{u(t_i)} (2t^2+1)I_i(t)dt - \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} G_i(t,u(t))dt - \frac{\lambda}{\nu} \sum_{i=0}^{N} \int_{s_i}^{t_{i+1}} h_i(t)|u(t)|^{\nu}dt.
\end{aligned}
$$

where $G_i(t,u) = \int_0^u g_i(t,s)ds$. In view of the continuity of g_i and I_i, by employing the standard approaches, we can get that $\Phi(u) \in C^1(H_0^1(0,T),\mathbb{R})$ and the critical points of $\Phi(u)$ are weak solutions of the problem (1).

Next, for obtaining our main results, some knowledge on "genus" will be presented. Let E be a Banach space, $\Phi \in C^1(E,\mathbb{R})$,

$$\Sigma = \{A \subset E \setminus \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$
$$K_\sigma = \{u \in E : \Phi(u) = \sigma, \Phi'(u) = 0\},\ \Phi^\sigma = \{u \in E : \Phi(u) \leq \sigma\},$$

where $\sigma \in \mathbb{R}$.

Definition 2 ([23]). *For $A \in \Sigma$, if there is an odd map $\phi \in C(A,\mathbb{R}^n \setminus \{0\})$ such that n is the smallest integer with this property, then the genus of A is n defined by $\gamma(A) = n$.*

Lemma 2 ([23]). *Assume that $\Phi \in C^1(E,\mathbb{R})$ meets the (PS)-condition. Moreover, Φ is an even functional. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \geq n\},\ \sigma_n = \inf_{A \in \Sigma_n} \sup_{u \in A} \Phi(u).$$

(i) If $\Sigma_n \neq 0$ and $\sigma_n \in \mathbb{R}$, then σ_n is a critical value of Φ;
(ii) If there exists $\kappa \in \mathbb{N}$ such that $\sigma_n = \sigma_{n+1} = \cdots = \sigma_{n+\kappa} = \sigma \in \mathbb{R}$, and $\sigma \neq \Phi(0)$, then $\gamma(K_\sigma) \geq \kappa + 1$.

3. Main Results

In order to describe our main results, the following assumptions are given.

(I1) For any $u \in \mathbb{R}$, $I_i(u)$ are odd in u and $I_i(u)u \geq 0$, $i = 1,2...,N$.

(I2) There exist constants $\alpha_i > 0$, $d_1 > 0$ and $\gamma_i \in [0,1)$ such that

$$|I_i(u)| \leq \alpha_i |u|^{\gamma_i} \text{for any } |u| \leq d_1.$$

(G1) There exist constants $\beta_i > 0$, $d_2 > 0$ and $l \in [0,1)$ such that $g_i(t,u)$ are odd in u, $\forall (t,u) \in (s_i,t_{i+1}] \times [-d_2,d_2]$ and

$$g_i(t,u) \leq \beta_i |u|^l, \forall (t,u) \in (s_i,t_{i+1}] \times \mathbb{R}, i = 1,2...,N.$$

(G2) There exist constants $\xi_i > 0$, $d_3 > 0$, $\tau \in [\nu,\gamma_* + 1)$ and the open sets $\Omega_i \subset (s_i,t_{i+1}]$ such that

$$G_i(t,u) \geq \xi_i |u|^\tau,\ \forall\ (t,u) \in \Omega_i \times [-d_3,d_3], i = 1,2...,N,$$

where $\gamma_* = \min_{i=1,2,...,N} \gamma_i$. Moreover, $\int_{s_i}^{t_{i+1}} G_i(t,u)dt > 0$.

Let $\xi_{min} = \min_{i=1,2,...,N} \xi_i$. Now, we state our main results.

Theorem 1. *Assuming that the conditions (I1), (I2), (G1) and (G2) are fulfilled, there exist positive constants ξ_*, λ_* such that if $\xi_{min} \in (0,\xi_*)$ and $\lambda \in [0,\lambda_*)$, the Dirichlet BVP (1) has infinitely many nontrivial weak solutions $\{u_k\}$ satisfying $\Phi(u_k) \to 0$ as $k \to +\infty$.*

Remark 1. *In (G1), the oddness of $g_i(t,u)$ in u are local.*

For obtaining our main results, inspired by [24], by constructing the following truncated functional, the following Lemma 3 can be established.

$$
\begin{aligned}
J(u) \;=\;& \frac{1}{2}\int_0^T u'^2(t)dt + \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b(t)u^2(t)dt + \sum_{i=1}^{N}\int_{t_i}^{s_i}\theta_i^2(t_i^+)u^2(t)dt + \int_0^T u'^2(t)u^2(t)dt \\
&- \frac{\lambda}{\nu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} h_i(t)|u(t)|^\nu dt + \sum_{i=1}^{N}\int_0^{u(t_i)}(2t^2+1)I_i(t)dt - \Upsilon(\|u\|)\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} G_i(t,u(t))dt,
\end{aligned}
$$

where $\Upsilon \in C^1(\mathbb{R}^+, [0,1])$ satisfying

$$
\begin{cases}
\Upsilon'(t) \le 0, \ \forall t \in [0,T]; \\
\Upsilon(t) = 0, \forall t \ge \frac{\eta}{S}; \\
\Upsilon(t) = 1, \forall t \le \frac{\eta}{2S},
\end{cases} \tag{12}
$$

where $\eta = \min\{d_1, d_2, d_3\}$. Assume that $\beta_{\max} < \frac{\vartheta\mu\eta^{1-l}}{4TS^2(1+\mu)}$, where $\beta_{max} = \max_{i=1,2,\ldots,N}\beta_i$. Thus, the critical points $\{u_n\}$ of J satisfying $\|u_n\| \le \frac{\eta}{2S}$ are the critical points of Φ. Next, we show that the functional J satisfies the (PS)-condition.

Lemma 3. *Assume that the conditions of Theorem 1 hold, then there exists a positive constant λ^* such that if $\lambda \in [0, \lambda^*)$, $J(u)$ satisfies the (PS)-condition, i.e., for any $\{u_n\} \in H_0^1(0,T)$, if*

$$\{J(u_n)\} \text{ is bounded and } J'(u_n) \to 0 \text{ as } n \to +\infty,$$

then $\{u_n\}$ has a convergent subsequence in $H_0^1(0,T)$.

Proof. Based on the definition of $J(u)$, if $\|u\| \ge \frac{\eta}{S}$, by (I1), we can obtain

$$
\begin{aligned}
J(u) \;=\;& \frac{1}{2}\int_0^T u'^2(t)dt + \frac{1}{2}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b(t)u^2(t)dt + \sum_{i=1}^{N}\int_{t_i}^{s_i}\theta_i^2(t_i^+)u^2(t)dt \\
&+ \int_0^T u'^2(t)u^2(t)dt + \sum_{i=1}^{N}\int_0^{u(t_i)}(2t^2+1)I_i(t)dt - \frac{\lambda}{\nu}\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} h_i(t)|u(t)|^\nu dt \\
\ge\;& \frac{1}{2}\int_0^T u'^2(t)dt + \frac{1}{2}\int_0^T \rho u^2(t)dt - \frac{\lambda h_{max}}{\nu}\int_0^T |u(t)|^\nu dt \\
\ge\;& \frac{\vartheta\mu}{2(1+\mu)}\|u\|^2 - \frac{\lambda h_{max} T S^\nu}{\nu}\|u\|^\nu,
\end{aligned}
$$

which yields that

$$J(u) \to +\infty \text{ as } \|u\| \to +\infty, \tag{13}$$

where $h_{max} = \max_{i=1,2,\ldots,N}\|h_i\|_{L^\infty}$. Thus, $J(u)$ is coercive and bounded from below. Moreover, for any $\{u_n\} \in H_0^1(0,T)$, if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$, it follows that $\{u_n\}$ is bounded in $H_0^1(0,T)$ by (13). Based on the fact that $H_0^1(0,T)$ is a reflexive Banach space, $\{u_n\}$ has a convergent subsequence (called again $\{u_n\}$). Since $H_0^1(0,T)$ is compactly embedded into C, so $u_n \rightharpoonup u$ in $H_0^1(0,T)$, $u_n \to u$ uniformly in C. If $\frac{\eta}{2S} < \|u_n\| \le \frac{\eta}{S}$, by (I1), (G1) and (G2), we have

$$
\begin{aligned}
J'(u_n)u_n &= \int_0^T u'^2_n(t)dt + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b(t)u_n^2(t)dt + 2\sum_{i=1}^{N}\int_{t_i}^{s_i}\theta_i^2(t_i^+)u_n^2(t)dt + 4\int_0^T u'^2_n(t)u_n^2(t)dt \\
&\quad + \sum_{i=1}^{N}(2u^2(t_i)+1)I_i(u(t_i))u_n(t_i) - \lambda\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} h_i(t)|u_n(t)|^\nu dt \\
&\quad - \Upsilon'(\|u_n\|)\|u_n\|\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} G_i(t,u_n(t))dt - \Upsilon(\|u_n\|)\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} g_i(t,u_n(t))u_n(t)dt \\
&\geq \int_0^T u'^2_n(t)dt + \int_0^T \rho u_n^2(t)dt - \lambda h_{max}\int_0^T |u_n(t)|^\nu dt - \beta_{max}\int_0^T |u_n(t)|^{l+1}dt \\
&\geq \frac{\vartheta\mu}{1+\mu}\|u_n\|^2 - \lambda h_{max}TS^\nu\|u_n\|^\nu - \beta_{max}TS^{l+1}\|u_n\|^{l+1} \\
&\geq \frac{\vartheta\mu}{1+\mu}(\frac{\eta}{2S})^2 - \lambda h_{max}T\eta^\nu - \beta_{max}T\eta^{l+1},
\end{aligned}
$$

which together with $\beta_{\max} < \frac{\vartheta\mu\eta^{1-l}}{4TS^2(1+\mu)}$ yield that there exists a positive constant λ^{**} such that if $\lambda \in [0,\lambda^{**})$, $J'(u_n)u_n > 0$. If $\|u_n\| > \frac{\eta}{S}$, $J'(u_n)u_n \geq \|u_n\|^\upsilon(\frac{\vartheta\mu}{1+\mu}(\frac{\eta}{s})^{2-\upsilon} - \lambda h_{max}TS^\nu)$. Hence, there exists a positive constant λ^{***} such that if $\lambda \in [0,\lambda^{***})$, $J'(u_n)u_n > 0$. Therefore, $J'(u_n) \nrightarrow 0$ for$\lambda \in [0,\lambda^*)$, where $\lambda^* = \min\{\lambda^{**},\lambda^{***}\}$. Thus, we just need to deal with the case of $\|u_n\| \leq \frac{\eta}{2S}$. It follows that $|u_n| \leq \|u_n\|_\infty \leq S\|u_n\| \leq \frac{\eta}{2}$, which together with (I2), (G1), $u_n \rightharpoonup u$ in $H_0^1(0,T)$, $u_n \to u$ uniformly in C, $J'(u_n) \to 0$ as $n \to +\infty$ and

$$
\begin{aligned}
&\int_0^T (u_n^2(t)u_n'(t) - u^2(t)u'(t))(u_n'(t) - u'(t))dt \\
&= \int_0^T u_n'(t)(u_n'(t) - u'(t))(u_n^2(t) - u^2(t))dt + \int_0^T u^2(t)|u_n'(t) - u'(t)|^2dt
\end{aligned}
$$

yields that

$$
\begin{aligned}
o(1) &= < J'(u_n) - J'(u), u_n - u >= \int_0^T |u_n'(t) - u'(t)|^2dt + \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} b(t)|u_n(t) - u(t)|^2dt \\
&\quad + \sum_{i=1}^{N}\int_{t_i}^{s_i} 2\theta_i^2(t_i^+)|u_n(t) - u(t)|^2dt + \int_0^T (u_n'^2(t)u_n(t) - u'^2(t)u(t))(u_n(t) - u(t))dt \\
&\quad + \int_0^T (u_n^2(t)u_n'(t) - u^2(t)u'(t))(u_n'(t) - u'(t))dt \\
&\quad - \lambda\sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} h_i(t)(|u_n(t)|^{\nu-2}u_n(t) - |u(t)|^{\nu-2}u(t))(u_n(t) - u(t))dt \\
&\quad + \sum_{i=1}^{N}(2u_n^2(t_i)+1)I_i(u_n(t_i))(u_n(t_i) - u(t_i)) - \sum_{i=1}^{N}(2u^2(t_i)+1)I_i(u(t_i))(u_n(t_i) - u(t_i)) \\
&\quad - \sum_{i=0}^{N}\int_{s_i}^{t_{i+1}} (g_i(t,u_n(t)) - g_i(t,u(t)))(u_n(t) - u(t))dt \\
&= \int_0^T |u_n'(t) - u'(t)|^2dt + \int_0^T u^2(t)|u_n'(t) - u'(t)|^2dt + o(1),
\end{aligned}
$$

which leads to that $u_n \to u$ in $H_0^1(0,T)$, which means that $J(u)$ satisfies the (PS)-condition. □

Proof. Note that $J \in C^1(H_0^1(0,T),\mathbb{R})$ and $J(0) = 0$. If $\|u\| \geq \frac{\eta}{S}$, based on the definition of Υ, we have $J(-u) = J(u)$. If $\|u\| \leq \frac{\eta}{S}$, we have $|u| \leq \|u\|_\infty \leq S\|u\| \leq \eta$, which together with (I1) and (G1) yield that $J(-u) = J(u)$. Thus, $J(-u) = J(u), \forall u \in H_0^1(0,T)$. Next, in view of Lemma 2, we aim to present that there exists $m > 0$ such that

$$\gamma(J^{-m}) \geq k,\ k \in \mathbb{Z}^+.$$

Let k disjoint open sets Λ_i satisfy $\bigcup_{i=1}^k \Lambda_i \subset \bigcup_{i=1}^N \Omega_i$, where $k \geq N$. Moreover, there has at least one $\iota \in \{1,2,...,k\}$ such that

$$\Lambda_\iota \subset \Omega_i \subset (s_i, t_{i+1}].$$

Let $u_i \in H_0^1(\Lambda_i) \setminus \{0\}$, $\|u_i\| = 1$ and

$$\Delta_k = \mathrm{span}\{u_1, u_2, ..., u_k\},\ \Pi_k = \{u \in \Delta_k : \|u\| = 1\}.$$

For any $u \in \Delta_k$, in view of the equivalence of the norms on the finite-dimensional space, there exist $c_1 > 0, c_2 > 0$ such that $c_1\|u\| \leq \|u\|_{L^\tau}, \|u\|_{L^\nu} \leq c_2\|u\|$. Moreover, it follows that there exists $\omega_i \in \mathbb{R}, i = 1,2,...,k$ such that $u(t) = \sum_{i=1}^k \omega_i u_i(t)$,

$$\|u\|_{L^\tau}^\tau = \sum_{i=1}^k |\omega_i|^\tau \int_{\Lambda_i} |u_i(t)|^\tau dt,\ \|u\|_{L^\nu}^\nu = \sum_{i=1}^k |\omega_i|^\nu \int_{\Lambda_i} |u_i(t)|^\nu dt$$

and

$$\|u\|^2 = \sum_{i=1}^k \omega_i^2 \int_{\Lambda_i} (|u_i(t)|^2 + |u_i'(t)|^2) dt = \sum_{i=1}^k \omega_i^2. \tag{14}$$

Let us reorder γ_i $(i = 1,2,...,N)$ as follows.

$$0 \leq \widetilde{\gamma}_1 \leq \widetilde{\gamma}_2 \leq \cdots \leq \widetilde{\gamma}_N < 1,$$

where $\widetilde{\gamma}_1 = \gamma_{min}$. Now, we need to consider two cases:

(i) h_i is a negative or sign-changing function; (ii) h_i is a positive function.

For case (i), from (I2), (G2) and (14), for any $u \in \Pi_k$ and $0 < \zeta \leq r$ that $r = \min\{1, \frac{\eta}{2S}\}$, we have

$$\begin{aligned}
J(\zeta u) \leq\ & \frac{(\|b\|_{L^\infty} + 2\theta_{max}^2 + 1)\zeta^2}{2}\|u\|^2 + \sum_{i=1}^N \alpha_i S^{\gamma_i+1}\zeta^{\gamma_i+1}\|u\|^{\gamma_i+1} + \sum_{i=1}^N 2\alpha_i S^{\gamma_i+3}\zeta^{\gamma_i+3}\|u\|^{\gamma_i+3} \\
& + S^2\zeta^4\|u\|^4 - \zeta^\tau \xi_{min} \sum_{i=1}^k |\omega_i|^\tau \int_{\Lambda_i} |u_i(t)|^\tau dt + \lambda\zeta^\nu h_{max} \sum_{i=1}^k |\omega_i|^\nu \int_{\Lambda_i} |u_i(t)|^\nu dt \\
\leq\ & \frac{(\|b\|_{L^\infty} + 2\theta_{max}^2 + 1)\zeta^2}{2} + \sum_{i=1}^N \alpha_i S^{\gamma_i+1}\zeta^{\gamma_i+1} + \sum_{i=1}^N 2\alpha_i S^{\gamma_i+3}\zeta^{\gamma_i+3} \\
& + S^2\zeta^4 - \zeta^\tau \xi_{min} c_1^\tau + \lambda\zeta^\nu h_{max} c_2^\nu \\
\leq\ & M\zeta^{\widetilde{\gamma}_1+1} - \zeta^\tau \xi_{min} c_1^\tau + \lambda\zeta^\nu h_{max} c_2^\nu \\
=\ & \zeta^\nu (M\zeta^{\widetilde{\gamma}_1+1-\nu} - \zeta^{\tau-\nu}\xi_{min} c_1^\tau + \lambda h_{max} c_2^\nu),
\end{aligned}$$

where $M = \frac{\|b\|_{L^\infty} + 2\theta_{max}^2 + 1}{2} + \sum_{i=1}^N \alpha_i S^{\gamma_i+1} + \sum_{i=1}^N 2\alpha_i S^{\gamma_i+3} + S^2$, $\theta_{max} = \max_{i=1,2,...,N} \theta_i(t_i^+)$. Define

$$\psi(t) = Mt^{\widetilde{\gamma}_1+1-\nu} - t^{\tau-\nu}\xi_{min} c_1^\tau,\ 0 < t \leq r.$$

Thus, from $\xi_{min} \in (0, \xi^*)$ that $\xi^* = \frac{Mr^{\widetilde{\gamma}_1+1-\nu}(\widetilde{\gamma}_1+1-\nu)}{c_1^{\tau}(\tau-\nu)}$, there exists

$$t_0 = (\frac{\xi_{min} c_1^{\tau}(\tau - \nu)}{M(\widetilde{\gamma}_1 + 1 - \nu)})^{\frac{1}{\widetilde{\gamma}_1+1-\tau}} \in (0, r],$$

such that for $\tau - \widetilde{\gamma}_1 - 1 < 0$,

$$\psi(t_0) = \min_{0<t\leq r} \psi(t) = \frac{\xi_{min} c_1^{\tau}(\tau - \widetilde{\gamma}_1 - 1)}{\widetilde{\gamma}_1 + 1 - \nu} (\frac{\xi_{min} c_1^{\tau}(\tau - \nu)}{M(\widetilde{\gamma}_1 + 1 - \nu)})^{\frac{\tau-\nu}{\widetilde{\gamma}_1+1-\tau}} < 0.$$

Hence, if $\lambda \in [0, \lambda^{****})$ and

$$\lambda^{****} = \frac{\xi_{min} \nu c_1^{\tau}(\widetilde{\gamma}_1 + 1 - \tau)}{c_2^{\nu} h_{max}(\widetilde{\gamma}_1 + 1 - \nu)} (\frac{\xi_{min} c_1^{\tau}(\tau - \nu)}{M(\widetilde{\gamma}_1 + 1 - \nu)})^{\frac{\tau-\nu}{\widetilde{\gamma}_1+1-\tau}} > 0,$$

we can find constats $m > 0$, $\varpi > 0$ such that

$$J(\varpi u) < -m \text{ for } u \in \Pi_k. \tag{15}$$

Moreover, (15) holds for case (ii), provided that $\xi_{min} \in \mathbb{R}^+ \setminus \{0\}, \lambda \in \mathbb{R}^+$. Choose $\lambda \in [0, \lambda_*)$, where $\lambda_* = \min\{\lambda^{****}, \lambda^*\}$. It implies that J satisfies the (PS)-condition and (15). Let

$$\Pi_k^{\varpi} = \{\varpi u : u \in \Pi_k\},\ \Xi = \{(\omega_1, \omega_2, ..., \omega_k) \in \mathbb{R}^k : \sum_{i=1}^{k} \omega_i^2 < \varpi^2\}.$$

From (15), we have

$$J(u) < -m \text{ for } u \in \Pi_k^{\varpi},$$

which implies that $\Pi_k^{\varpi} \subset J^{-m} \in \Sigma$. Moreover, based on (15), there exists an odd homeomorphism mapping $\phi \in C(\Pi_k^{\varpi}, \partial\Xi)$, which together with the properties of genus (see [23]) yield that

$$\gamma(J^{-m}) \geq \gamma(\Pi_k^{\varpi}) = \gamma(\partial\Xi) = k. \tag{16}$$

Let

$$\sigma_k = \inf_{A\in\Sigma_k} \sup_{u\in A} J(u).$$

Since $J(u)$ is coercive and bounded from below, by (16), one has $-\infty < \sigma_k \leq -m < 0$. Based on the above facts, by Lemma 2, the Dirichlet BVP (1) has infinitely many nontrivial weak solutions $\{u_k\}$ satisfying $\|u_k\| \leq \frac{\eta}{2S}$ and $\Phi(u_k) \to 0$ as $k \to +\infty$. $\square$

Next, an example is given to illustrate the conclusion of the main results.

4. Example

Consider the following problem:

$$\begin{cases} -u''(t) + (t+1)u(t) - (u^2(t))''u(t) = u^{\frac{1}{9}}(t) + \lambda|u(t)|^{-\frac{17}{18}}u(t),\ a.e.\ t \in (s_i, t_{i+1}], i = 0, 1, 2, ..., N, \\ \Delta(u'(t_i)) = u^{\frac{1}{3}}(t_i),\ i = 1, 2, ..., N, \\ u'(t) = \sin^2(t_i^+), t \in (t_i, s_i],\ i = 1, 2, ..., N, \\ u'(s_i^+) = u'(s_i^-),\ i = 1, 2, ..., N, \\ u(0) = u(T) = 0. \end{cases} \tag{17}$$

It is not difficult to verify that the conditions (I1), (I2), (G1), (G2) are satisfied.

5. Conclusions

Under the influence of non-instantaneous and instantaneous impulsive effects, by constructing a new energy functional, which makes the variational methods applicable, and overcoming the difficulties brought by the quasilinear term $(u^2(t))''u$, the multiplicity of weak solutions for quasilinear differential models generated by instantaneous and non-instantaneous impulses are obtained, which extend and enrich some previous results. In the future, we will develop a further study of the multiplicity of weak solutions to BVPs of differential equations with non-instantaneous and instantaneous impulsive effects when the nonlinear term g satisfies the satisfies superlinear growth.

Author Contributions: Conceptualization, T.S.; writing—original draft, T.S.; writing—review and editing, W.L.; formal analysis, W.Z. All authors have read and approved the final manuscript.

Funding: Supported by the Natural Science Foundation of Jiangsu Province (No. BK20190620) and the Fundamental Research Funds for the Central Universities (No. 2019QNA05).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Alves, C.; Miyagaki, O.; Soares, S. On the existence and concentration of positive solutions to a class of quasilinear elliptic problems on $\mathbb{R}$. *Math. Nachr.* **2011**, *284*, 1784–1795. [CrossRef]
2. Ambrosetti, A.; Wang, Z. Positive solutions to a class of quasilinear elliptic equation on $\mathbb{R}$. *Discrete Contin. Dyn. Syst.* **2003**, *9*, 55–68. [CrossRef]
3. Poppenberg, M.; Schmitt, K.; Wang, Z. On the existence of soliton solutions to a quasilinear Shrodinger equations. *Calc. Var. Part. Differ. Equ.* **2002**, *14*, 329–344. [CrossRef]
4. Shen, Z.; Han, Z. Existence of solutions to quasilinear Schrödinger equations with indefinite potential. *Electron. J. Diff. Equ.* **2015**, *91*, 1–9.
5. Choisy, M.; Guégan, J.; Rohani, P. Dynamics of infectious diseases and pulse vaccination: Teasing apart the embedded resonance effects. *Physics D* **2006**, *223*, 26–35. [CrossRef]
6. D'Onofrio, A. On pulse vaccination strategy in the SIR epidemic model with vertical transmission. *Appl. Math. Lett.* **2005**, *18*, 729–732. [CrossRef]
7. Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S. Theory of Impulsive Differential Equations. In *Series Modern Applied Mathematics*; World Scientific: Teaneck, NJ, USA, 1989; Volume 6.
8. Nenov, S. Impulsive controllability and optimization problems in population dynamics. *Nonlinear Anal.* **1999**, *36*, 881–890. [CrossRef]
9. Nieto, J.; O'Regan, D. Variational approach to impulsive differential equations. *Nonlinear Anal.* **2009**, *10*, 680–690. [CrossRef]
10. Zhou, J.; Li, Y. Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects. *Nonlinear Anal.* **2009**, *71*, 2856–2865. [CrossRef]
11. Zhang, Z.; Yuan, R. An application of variational methods to Dirichlet boundary value problem with impulses. *Nonlinear Anal.* **2010**, *11*, 155–162. [CrossRef]
12. Sun, J.; Chen, H. Multiplicity of solutions for a class of impulsive differential equations with Dirichlet boundary conditions via variant fountain theorems. *Nonlinear-Anal.-Real World Appl.* **2010**, *11*, 4062–4071. [CrossRef]
13. Shen, T.; Liu, W. Multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects. *Electron. J. Qual. Theory Differ. Equ.* **2015**, *97*, 1–10. [CrossRef]
14. Hernández, E.; O'Regan, D. On a new class of abstract impulsive differential equations. *Proc. Am. Math. Soc.* **2013**, *141*, 1641–1649. [CrossRef]
15. Wang, J. Stability of noninstantaneous impulsive evolution equations. *Appl. Math. Lett.* **2017**, *73*, 157–162. [CrossRef]
16. Agarwal, R.; O'Regan, D.; Hristova, S. Stability by lyapunov like functions of nonlinear differential equations with non-instantaneous impulses. *J. Appl. Math. Comput.* **2017**, *53*, 147–168. [CrossRef]
17. Bai, L.; Nieto, J. Variational approach to differential equations with not instantaneous impulses. *Appl. Math. Lett.* **2017**, *73*, 44–48. [CrossRef]
18. Khaliq, A.; ur Rehman, M. On variational methods to non-instantaneous impulsive fractional differential equation. *Appl. Math. Lett.* **2018**, *83*, 95–102. [CrossRef]
19. Tian, Y.; Zhang, M. Variational method to differential equations with instantaneous and non-instantaneous impulses. *Appl. Math. Lett.* **2019**, *94*, 160–165. [CrossRef]
20. Zhang, W.; Liu, W. Variational approach to fractional Dirichlet problem with instantaneous and non-instantaneous impulses. *Appl. Math. Lett.* **2020**, *99*, 105993. [CrossRef]

21. Yan, Q.; Chen, F.; An, Y. Variational method for p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses. *Math. Methods Appl. Sci.* **2021**, *44*, 8543–8553.
22. Min, D.; Chen, F. Variational methods to the p-Laplacian type nonlinear fractional order impulsive differential equations with Sturm-Liouville boundary-value problem. *Fract. Calc. Appl. Anal.* **2021**, *24*, 1069–1093. [CrossRef]
23. Rabinowitz, P. *Minimax Methods in Critical Point Theory with Applications to Differential Equations*; CBMS Regional Conference Series in Mathematics; American Mathematical Society: Washington, DC, USA, 1986; Volume 65.
24. Chen, G.; Schechter, M. Multiple periodic solutions for damped vibration systems with general nonlinearities at infinity. *Appl. Math. Lett.* **2019**, *90*, 69–74. [CrossRef]