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Thermodynamics for a Rotating Chiral Fermion System in the Uniform Magnetic Field

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Abstract: We study the thermodynamics for a uniformly rotating system of chiral fermions under the uniform magnetic field. Then, we obtain the mathematical expressions of some thermodynamic quantities in terms of the series with respect to the external magnetic field B, the angular velocity Ω and the chemical potential μ, expanded around $B = 0$, $\Omega = 0$ and $\mu = 0$. Our results given by such series are a generalization of the expressions available in the references simply corresponding to the lower-order terms of our findings. The zero-temperature limit of our results is also discussed.

Keywords: magnetic field; rotating; series expansion; chiral magnetic effect; chiral vortical effect

Citation: Fang, R.-H. Thermodynamics for a Rotating Chiral Fermion System in the Uniform Magnetic Field. *Symmetry* **2022**, *14*, 1106. https://doi.org/10.3390/sym14061106

Academic Editor: Shi Pu

Received: 19 April 2022
Accepted: 26 May 2022
Published: 27 May 2022

1. Introduction

The properties of the Dirac fermion system have been investigated from many aspects for a long time. For a hydrodynamic system consisting of Dirac fermions under the background of electromagnetic fields, the Wigner function is an appropriate tool, which can provide a covariant and gauge invariant formalism [1,2]. It is worth pointing out that, although the Wigner function defined in 8-dimensional phase space is not always non-negative, one can always obtain non-negative probability density when the 4-dimensional momentum is integrated out. For a massless (or chiral) fermion system with uniform vorticity and electromagnetic fields, the charge current and the energy-momentum tensor up to the second order have been obtained from the Wigner function approach, including the chiral anomaly equation, chiral magnetic and vortical effects [3,4]. The pair production in parallel electric and magnetic fields with finite temperature and chemical potential from the Wigner function approach has also been investigated recently [5]. Without external electromagnetic fields, the energy-momentum tensor and charge current of the massless fermion system up to second order in vorticity have been obtained from thermal field theory [6–8]. For a uniformly rotating massless fermion system, the analytic expressions of the charge current and the energy-momentum tensor are obtained [9]. For the massive and massless fermion systems under the background of a uniform magnetic field, the general expansions with respect to fermion mass, magnetic field and chemical potential are derived by the approaches of proper-time and grand partition function [10–12]. There are also some investigations on the system of free fermion gas, quark matter or hadronic matter, with pure rotation [13–15], or with the coexistence of rotation and magnetic field [16,17], or with specific boundary conditions [18–22]. The quantum superfluid phenomena of Dirac fermions in the background of magnetic field and rotation have been discussed recently [23,24].

In this article, we consider a uniformly rotating chiral fermion system in a uniform magnetic field, where we ignore the interaction among the fermions and the directions of the angular velocity and the magnetic field are chosen to be parallel. In this article, we will adopt the approach of normal ordering and ensemble average to calculate the thermodynamic quantities of the system. First, we briefly derive the Dirac equation in a rotating frame under the background of a uniform magnetic field from the Dirac equation in curved space. Then, through solving the eigenvalue equation of the Hamiltonian in cylindrical coordinates, we can obtain a series of Landau levels, from which one can calculate the expectation value of corresponding thermodynamic quantities for each eigenstate. From the approach of ensemble average used in [25–27], the macroscopic thermodynamic quantities can be expressed as the summation over the product of the particle number (Fermi–Dirac distribution) and the expectation value in each eigenstate. We expand all thermodynamic quantities as threefold series at $B = 0$, $\Omega = 0$ and $\mu = 0$, where the lower orders are consistent with that from the approaches of thermal field theory and the Wigner function, respectively [4,6,7], and to our knowledge, the general orders have not been obtained before. We also calculate all quantities in a zero temperature limit, and obtain the equality of particle/energy density and corresponding currents along the z-axis, which can provide a qualitative reference for the thermodynamics of compact stars in astrophysics, such as the neutron star and magnetic star, since the magnitudes of the magnetic field and rotational speed are huge compared to the temperature of the compact stars [28,29]. In this article, all thermodynamic quantities will be calculated at the rotating axis ($r = 0$), so the boundary condition at the speed-of-light surface will not affect our results.

From the point of view of hydrodynamics, it has been pointed out that the relativistic hydrodynamical equations with only the first-order term does not obey the causality [30–32], i.e., the group speed of some transport coefficients, such as heat conductivity, would exceed the speed of light [33]. Therefore, the high-order terms in hydrodynamics are necessary, which indeed repair the issue of causality. There have been some earlier works to study the second-order terms of transport phenomena, such as the Kubo formula from quantum field theory [34,35], thermal field theory [6,7], Wigner function [4,36], etc. All of these works are essentially perturbation theory, from which the general order terms have not been obtained. In this article, we consider a special configuration for the electromagnetic field and vorticity field in hydrodynamics, i.e., with a pure homogeneous magnetic field parallel to a homogeneous vorticity field, and obtain the general order terms of all thermodynamic quantities, which is important to study the analytic behavior of hydrodynamics in mathematics.

The rest of this article is organized as follows. In Sections 2 and 3, we briefly derive the Dirac equation in a uniformly rotating frame and list the Landau levels and corresponding eigenfunctions of a single right-handed fermion, which are just reference reviews. In Sections 4 and 5, we obtain the expressions of some thermodynamic quantities in terms of the series with respect to the external magnetic field B, the angular velocity Ω and the chemical potential μ, expanded around $B = 0$, $\Omega = 0$ and $\mu = 0$, which are our main results. In Section 6, the zero temperature limit of the thermodynamic quantities is discussed. This article is summarized in Section 7.

Throughout this article we adopt natural units where $\hbar = c = k_B = 1$. We use the Heaviside–Lorentz convention for electromagnetism and the chiral representation for gamma matrices where $\gamma^5 = \mathrm{diag}\,(-1,-1,+1,+1)$, which is the same as Peskin and Schroeder [37].

2. Dirac Equation in a Uniformly Rotating Frame

In this section, we briefly introduce the Dirac equation in curved spacetime [38], which is applied to a uniformly rotating frame [16].

In curved spacetime, under the background of the electromagnetic field, the Dirac equation for a single chiral fermion is

$$i\underline{\gamma}^\mu D_\mu \psi(x) = 0, \tag{1}$$

where the covariant derivative D_μ and gamma matrices $\underline{\gamma}^\mu$ are defined as

$$D_\mu = \partial_\mu + ieA_\mu + \Gamma_\mu, \quad \underline{\gamma}^\mu = \gamma^a e_a^{\ \mu}. \tag{2}$$

The underline in $\underline{\gamma}^\mu$ is used to distinguish the spacetime-dependent gamma matrices $\underline{\gamma}^\mu$ from the constant gamma matrices γ^a, and $\Gamma_\mu = \frac{1}{8}\omega_{\mu ab}[\gamma^a, \gamma^b]$ is the affine connection. The definitions of vierbein $e_a^{\ \mu}$, metric tensor $g_{\mu\nu}$, and spin connection $\omega_{\mu ab}$ are listed as follows,

$$e_a^{\ \mu} = \frac{\partial x^\mu}{\partial X^a}, \quad e^a_{\ \mu} = \frac{\partial X^a}{\partial x^\mu}, \quad g_{\mu\nu} = \eta_{ab} e^a_{\ \mu} e^b_{\ \nu}, \tag{3}$$

$$\omega_{\mu ab} = g_{\alpha\beta} e_a^{\ \alpha}(\partial_\mu e_b^{\ \beta} + \Gamma^\beta_{\mu\nu} e_b^{\ \nu}), \tag{4}$$

$$\Gamma^\beta_{\mu\nu} = \frac{1}{2} g^{\beta\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}), \tag{5}$$

where $\eta_{ab} = \mathrm{diag}\,(+1, -1, -1, -1)$ is the metric tensor in Minkowski space, X^a and x^μ are the coordinates in a local Lorentz frame and in a general frame, respectively.

In curved spacetime, the vector J^μ_V, axial vector J^μ_A and symmetric energy-momentum tensor $T^{\mu\nu}$ become

$$J^\mu_V = \overline{\psi}\underline{\gamma}^\mu \psi, \quad J^\mu_A = \overline{\psi}\underline{\gamma}^\mu \gamma^5 \psi, \tag{6}$$

$$T^{\mu\nu} = \frac{1}{4}\Big(\overline{\psi} i \underline{\gamma}^\mu D^\nu \psi + \overline{\psi} i \underline{\gamma}^\nu D^\mu \psi + \mathrm{H.C.}\Big), \tag{7}$$

where D^μ, $\underline{\gamma}^\mu$ in curved spacetime have replaced ∂^a, γ^a in flat spacetime.

Now we consider a frame $\mathcal{K}$ rotating uniformly with angular velocity $\boldsymbol{\Omega} = \Omega \mathbf{e}_z$ relative to an inertial frame K. The coordinates in $\mathcal{K}$ and K are denoted as $x^\mu = (t, x, y, z)$ and $X^a = (T, X, Y, Z)$, respectively, which are related to each other by following transformations,

$$\begin{cases} T &= & t \\ X &= & x\cos\Omega t - y \sin\Omega t \\ Y &= & x \sin\Omega t + y\cos\Omega t \\ Z &= & z \end{cases}. \tag{8}$$

It should be pointed out that the rotational angular velocity Ω cannot be too large, otherwise the synchronous condition in Equation (8) cannot be satisfied. According to Equation (3), the metric tensor $g_{\mu\nu}$ and its inverse are

$$g_{\mu\nu} = \begin{pmatrix} 1-(x^2+y^2)\Omega^2 & y\Omega & -x\Omega & 0 \\ y\Omega & -1 & 0 & 0 \\ -x\Omega & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{9}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & y\Omega & -x\Omega & 0 \\ y\Omega & y^2\Omega^2 - 1 & -xy\Omega^2 & 0 \\ -x\Omega & -xy\Omega^2 & x^2\Omega^2 - 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{10}$$

Keeping $g_{\mu\nu}$ unchanged, the vierbein $e^a_{\ \mu}$ still has a freedom degree of an arbitrary local Lorentz transformation. We can choose $e^a_{\ \mu}$ as

$$e^0_{\ 0} = e^1_{\ 1} = e^2_{\ 2} = e^3_{\ 3} = 1, \quad e^1_{\ 0} = -y\Omega, \quad e^2_{\ 0} = x\Omega, \tag{11}$$

with zeros for other components.

Now we consider a single chiral fermion in a uniformly rotating frame under the background of a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$, and we choose the gauge potential in the inertial frame as $A^a = (0, \mathbf{A})$ with $\mathbf{B} = \nabla \times \mathbf{A}$. The covariant derivative D_μ and gamma matrices $\underline{\gamma}^\mu$ become

$$D_\mu = \Big(\partial_t - \frac{i}{2}\Omega\Sigma_3 + \Omega(yA_x - xA_y), \partial_x - ieA_x, \partial_y - ieA_y, \partial_z - ieA_z\Big), \tag{12}$$

$$\underline{\gamma}^{\mu} = \left(\gamma^0, y\Omega\gamma^0 + \gamma^1, -x\Omega\gamma^0 + \gamma^2, \gamma^3\right), \tag{13}$$

and in this case, the Dirac equation for a single chiral fermion can be written as

$$i\frac{\partial}{\partial t}\psi(x) = \left[-i\gamma^0\gamma\cdot(\nabla - ie\mathbf{A}) - \Omega J_z\right]\psi(x), \tag{14}$$

where e is the charge of the chiral fermion, $J_z = \frac{1}{2}\Sigma_3 - i(x\partial_y - y\partial_x)$ is the z-component of the total angular momentum $\mathbf{J}$, and the term $-\Omega J_z$ can be naturally explained as the coupling of the angular momentum $\mathbf{J}$ and the angular velocity $\boldsymbol{\Omega}$.

3. Landau Levels for a Single Right-Handed Fermion in a Rotating Frame

In the chiral representation of gamma matrices, where $\gamma^5 = \mathrm{diag}\,(-1, -1, +1, +1)$, we can divide the chiral fermion field into left- and right-handed fermion fields, i.e., $\psi = (\psi_L, \psi_R)^T$. Since the equations of motion for ψ_L and ψ_R decouple, we only discuss right-handed fermion field in this article. All results can be directly generalized to the left-handed case. In the following, we set $eB > 0$ for simplicity.

The right-handed part of Equation (14) is

$$i\frac{\partial}{\partial t}\psi_R(x) = H\psi_R(x), \tag{15}$$

$$H = -i\boldsymbol{\sigma}\cdot(\nabla - ie\mathbf{A}) - \Omega J_{R,z}, \tag{16}$$

where H, $J_{R,z} = \frac{1}{2}\sigma_3 - i(x\partial_y - y\partial_x)$ is Hamiltonian and the z-component of the total angular momentum of the right-handed fermion. In this article, we shall choose the symmetric gauge for $\mathbf{A}$, i.e., $\mathbf{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$. Then, the explicit form of the Hamiltonian is

$$H = -i\boldsymbol{\sigma}\cdot\nabla + \frac{1}{2}eB(y\sigma_1 - x\sigma_2) - \Omega J_{R,z}. \tag{17}$$

It can be proved that these three Hermitian operators, $H, \hat{p}_z = -i\partial_z, J_{R,z}$, are commutative with each other, then we can construct the common eigenfunctions of them. According to the calculations for Landau levels in Appendix A, we list the common eigenfunctions and corresponding energy in the cylindrical coordinate system (where the three coordinate variables are z, r, ϕ) as follows:

When $m = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$,

$$\psi_{\lambda n m p_z} = \sqrt{\frac{n!}{(n+m-\frac{1}{2})!}}\begin{pmatrix} \sqrt{\frac{eB(E+p_z+m\Omega)}{2(E+m\Omega)}}e^{-\frac{\rho}{2}}\rho^{\frac{m}{2}-\frac{1}{4}}L_n^{m-\frac{1}{2}}(\rho)e^{i(m-\frac{1}{2})\phi} \\ \frac{i\lambda eB}{\sqrt{(E+m\Omega)(E+p_z+m\Omega)}}e^{-\frac{\rho}{2}}\rho^{\frac{m}{2}+\frac{1}{4}}L_{n-1}^{m+\frac{1}{2}}(\rho)e^{i(m+\frac{1}{2})\phi} \end{pmatrix}\frac{e^{-iEt+izp_z}}{2\pi}, \tag{18}$$

$$E = \begin{cases} \lambda\sqrt{p_z^2 + 2eBn} - m\Omega, & n > 0 \\ p_z - m\Omega, & n = 0 \end{cases}. \tag{19}$$

When $m = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$,

$$\psi_{\lambda n m p_z} = \sqrt{\frac{n!}{(n-m+\frac{1}{2})!}}\begin{pmatrix} \sqrt{\frac{eB(E+p_z+m\Omega)}{2(E+m\Omega)}}e^{-\frac{\rho}{2}}\rho^{\frac{1}{4}-\frac{m}{2}}L_n^{\frac{1}{2}-m}(\rho)e^{i(m-\frac{1}{2})\phi} \\ -\frac{i\lambda eB(n-m+\frac{1}{2})}{\sqrt{(E+m\Omega)(E+p_z+m\Omega)}}e^{-\frac{\rho}{2}}\rho^{-\frac{1}{4}-\frac{m}{2}}L_n^{-\frac{1}{2}-m}(\rho)e^{i(m+\frac{1}{2})\phi} \end{pmatrix}\frac{e^{-iEt+izp_z}}{2\pi}, \tag{20}$$

$$E = \lambda\sqrt{p_z^2 + 2eB\left(n - m + \frac{1}{2}\right)} - m\Omega, \tag{21}$$

where $\rho = \frac{1}{2}eBr^2$, $L_n^{\mu}(\rho)$ is the general Laguerre polynomial as introduced in Appendix B, m is the magnetic quantum number, $\lambda = \pm 1$ represent the states with positive and negative energy, respectively, and $n = 0, 1, 2, \cdots$ represent different Landau levels. The eigenfunctions $\psi_{\lambda n m p_z}$ are denoted by the group of good quantum numbers (λ, n, m, p_z), which are normalized according to

$$\int dV \psi^{\dagger}_{\lambda' n' m' p'_z} \psi_{\lambda n m p_z} = \delta_{\lambda'\lambda}\delta_{n'n}\delta_{m'm}\delta(p'_z - p_z). \tag{22}$$

4. Particle Current

In this section, we consider a right-handed fermion system under the background of a uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$, and the system is rotating uniformly with angular velocity $\mathbf{\Omega} = \Omega\mathbf{e}_z$. The interaction among the fermions in this system is ignored. We assume that this rotating system is in equilibrium with a reservoir, which keeps constant temperature $T = 1/\beta$ and constant chemical potential μ.

4.1. Ensemble Average

We will calculate the macroscopic particle current of the system at the rotation axis (i.e., at $r = 0$) through the ensemble average approach, in which all macroscopic thermodynamic quantities are the ensemble average of the normal ordering of the corresponding field operators.

The forms of the eigenfunctions in Equations (18) and (20) at $r = 0$ or $\rho = 0$ are simplified to

$$\psi_{\lambda n m p_z} = \frac{e^{-iEt+izp_z}}{2\pi}\begin{pmatrix} \sqrt{\frac{eB(E+p_z+\Omega/2)}{2(E+\Omega/2)}}\delta_{m,1/2} \\ 0 \end{pmatrix} + \frac{e^{-iEt+izp_z}}{2\pi}\begin{pmatrix} 0 \\ -\frac{i\lambda eB\sqrt{n+1}}{\sqrt{(E-\Omega/2)(E+p_z-\Omega/2)}}\delta_{m,-1/2} \end{pmatrix}, \tag{23}$$

which are to be used in the following calculations of ensemble average. We find that the z-component m of the total angular momentum can only take values $\pm 1/2$ due to the absence of the orbital angular momentum at $r = 0$.

For the right-handed fermion system, the field operator of the particle current at $r = 0$ is

$$J^{\mu} = \psi^{\dagger}_R \sigma^{\mu} \psi_R, \tag{24}$$

with $\sigma^{\mu} = (1, \boldsymbol{\sigma})$. From the approach of ensemble average used in [25–27], the macroscopic particle current $\mathcal{J}^{\mu}$ can be calculated from J^{μ} as follows,

$$\begin{aligned} \mathcal{J}^{\mu} &= \langle : J^{\mu} : \rangle \\ &= \sum_{n=1}^{\infty}\sum_{\lambda}\int_{-\infty}^{\infty} dp_z \frac{\lambda}{e^{\beta(E_n - \lambda\Omega/2 - \lambda\mu)}+1}\psi^{\dagger}_{\lambda,n,1/2,p_z}\sigma^{\mu}\psi_{\lambda,n,1/2,p_z} \\ &\quad + \sum_{n=0}^{\infty}\sum_{\lambda}\int_{-\infty}^{\infty} dp_z \frac{\lambda}{e^{\beta(E_{n+1} + \lambda\Omega/2 - \lambda\mu)}+1}\psi^{\dagger}_{\lambda,n,-1/2,p_z}\sigma^{\mu}\psi_{\lambda,n,-1/2,p_z} \\ &\quad + \sum_{\lambda}\int_{-\infty}^{\infty} dp_z \frac{\lambda\theta(\lambda p_z)}{e^{\beta(|p_z| - \lambda\Omega/2 - \lambda\mu)}+1}\psi^{\dagger}_{\lambda,0,1/2,p_z}\sigma^{\mu}\psi_{\lambda,0,1/2,p_z}, \end{aligned} \tag{25}$$

where $\langle : \cdots : \rangle$ means normal ordering and ensemble average of corresponding field operator [2,39], $\theta(x)$ is the step function, and we have defined $E_n = \sqrt{p_z^2 + 2eBn}$. The second, third, and fourth lines of Equation (25) represent the contributions of high Landau levels with $m = 1/2$, all Landau levels with $m = -1/2$, and the lowest Landau level with $m = 1/2$, respectively. We can see that the macroscopic particle current $\mathcal{J}^{\mu}$ consists of the summation over the product of the particle number (Fermi–Dirac distribution) and the expectation value in each mode described by the quantum numbers (λ, n, m, p_z).

4.2. Particle Number Density

Firstly we calculate the particle number density $\rho \equiv \mathcal{J}^0$ of the system. Making use of

$$\psi^{\dagger}_{\lambda n m p_z}\psi_{\lambda n m p_z} = \begin{cases} \frac{eB}{4\pi^2}\frac{E+\Omega/2+p_z}{E+\Omega/2}, & m = \frac{1}{2} \\ \frac{eB}{4\pi^2}\frac{E-\Omega/2-p_z}{E-\Omega/2}, & m = -\frac{1}{2} \end{cases} \tag{26}$$

and from Equation (25) one can obtain

$$\rho\beta^3 = \frac{b\omega}{16\pi^2} + \frac{1}{2}\sum_{s=\pm1}\frac{\partial}{\partial a}g\left(a+\frac{1}{2}s\omega, b\right), \tag{27}$$

where we have defined three dimensionless quantities $a = \beta\mu$, $b = 2eB\beta^2$, $\omega = \beta\Omega$, and have defined $g(x,b)$ as

$$g(x,b) = \frac{b}{4\pi^2}\int_0^\infty dy \sum_{n=0}^\infty \sum_{s=\pm1}\left(1-\frac{1}{2}\delta_{n,0}\right)\ln\left(1+e^{sx-\sqrt{nb+y^2}}\right). \tag{28}$$

In a recent article [11], making use of the Abel–Plana formula, the authors obtained the asymptotic expansion of $g(x,b)$ at $b=0$ as follows

$$\begin{aligned} g(x,b) &= \left(\frac{7\pi^2}{360}+\frac{x^2}{12}+\frac{x^4}{24\pi^2}\right) - \frac{b^2\ln b^2}{384\pi^2} - \frac{b^2}{96\pi^2}\ln\left(\frac{e}{2G^6}\right) \\ &\quad -\frac{1}{2\pi^2}\sum_{n=0}^\infty \frac{(4n+1)!!}{(4n+4)!!}\mathscr{B}_{2n+2}C_{2n+1}(x)b^{2n+2}, \end{aligned} \tag{29}$$

where $G = 1.28242...$ is the Glaisher number, $\mathscr{B}_n$ are Bernoulli numbers, and $C_{2n+1}(x)$ is defined and expanded at $x=0$ in the following,

$$\begin{aligned} C_{2n+1}(x) &= -\delta_{n,0} + \frac{1}{(4n+1)!}\int_0^\infty dy \ln y \frac{d^{4n+1}}{dy^{4n+1}}\left(\frac{1}{e^{y+x}+1}+\frac{1}{e^{y-x}+1}\right) \\ &= (\ln 4+\gamma-1)\delta_{n,0} + \frac{2}{(4n+1)!}\sum_{k=0}^\infty\left(2^{4n+2k+1}-1\right)\zeta'(-4n-2k)\frac{x^{2k}}{(2k)!}. \end{aligned} \tag{30}$$

Plugging Equations (29) and (30) into Equation (27), one can obtain the threefold series expansion of the particle number density at $a=0$, $b=0$, $\omega=0$ or $\mu=0$, $B=0$, $\Omega=0$ as follows,

$$\begin{aligned} \rho\beta^3 &= \frac{a}{6}+\frac{a^3}{6\pi^2}+\frac{a\omega^2}{8\pi^2}+\frac{b\omega}{16\pi^2} \\ &\quad -\frac{1}{\pi^2}\sum_{n=0}^\infty\frac{\mathscr{B}_{2n+2}b^{2n+2}}{(4n+4)!!(4n)!!}\sum_{j=0}^\infty\frac{\omega^{2j}}{(2j)!2^{2j}} \\ &\quad \times\sum_{k=0}^\infty\left(2^{4n+2k+2j+3}-1\right)\zeta'(-4n-2k-2j-2)\frac{a^{2k+1}}{(2k+1)!}. \end{aligned} \tag{31}$$

The lower orders $O(b^2,\omega^2,b\omega)$ in Equation (31) are consistent with the perturbative results in [4,6,7], where the authors used the approaches of thermal field theory and the Wigner function, respectively.

4.3. Particle Current along Z-Axis

Next we calculate the space components of the particle current $\mathcal{J}^\mu$. According to the rotation symmetry along z-axis of the system, the x- and y-components of $\mathcal{J}^\mu$ vanish. The unique nonzero component is $\mathcal{J}^z$. Making use of

$$\psi^\dagger_{\lambda nmp_z}\sigma_3\psi_{\lambda nmp_z} = \begin{cases} \frac{eB}{4\pi^2}\frac{E+\Omega/2+p_z}{E+\Omega/2}, & m=\frac{1}{2} \\ -\frac{eB}{4\pi^2}\frac{E-\Omega/2-p_z}{E-\Omega/2}, & m=-\frac{1}{2} \end{cases} \tag{32}$$

and from Equation (25) one can obtain

$$\mathcal{J}^z\beta^3 = \frac{ab}{8\pi^2} + \frac{1}{2}\sum_{s=\pm 1} s\frac{\partial}{\partial a} g\left(a + \frac{1}{2}s\omega, b\right), \tag{33}$$

which can be expanded as the threefold series at $a = 0, b = 0, \omega = 0$ or $\mu = 0, B = 0, \Omega = 0$ as follows,

$$\begin{aligned}
\mathcal{J}^z\beta^3 \quad = \quad & \frac{ab}{8\pi^2} + \frac{\omega}{12} + \frac{\omega^3}{48\pi^2} + \frac{\omega a^2}{4\pi^2} \\
& -\frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{\mathscr{B}_{2n+2}b^{2n+2}}{(4n+4)!!(4n)!!}\sum_{j=0}^{\infty}\frac{\omega^{2j+1}}{(2j+1)!2^{2j+1}} \\
& \times\sum_{k=0}^{\infty}\left(2^{4n+2k+2j+3}-1\right)\zeta'(-4n-2k-2j-2)\frac{a^{2k}}{(2k)!}.
\end{aligned} \tag{34}$$

when $\omega = 0$ or $\Omega = 0$ in Equation (34), one can obtain $\mathcal{J}^z\beta^3 = \frac{ab}{8\pi^2}$, which is the chiral magnetic effect [40–44]; when $b = 0$ or $B = 0$ and keeping the leading order of ω in Equation (34), one can obtain $\mathcal{J}^z\beta^3 = \frac{\omega}{12}\left(1 + \frac{3a^2}{\pi^2}\right)$, which is the chiral vortical effect [45–50].

5. Energy-Momentum Tensor

In this section, we will calculate the energy-momentum tensor $\mathcal{T}^{\mu\nu}$ (at $r = 0$) of the right-handed fermion system as described in Section 4. According to the rotation symmetry along the z-axis, the energy-momentum tensor at $r = 0$ are unchanged under the rotation along the z-axis, which leads to following constraints on $\mathcal{T}^{\mu\nu}$:

$$\mathcal{T}^{01} = \mathcal{T}^{02} = \mathcal{T}^{12} = \mathcal{T}^{13} = \mathcal{T}^{23} = 0, \quad \mathcal{T}^{11} = \mathcal{T}^{22}. \tag{35}$$

The possible nonzero components of $\mathcal{T}^{\mu\nu}$ are $\mathcal{T}^{00}$, $\mathcal{T}^{11} = \mathcal{T}^{22}$, $\mathcal{T}^{33}$, and $\mathcal{T}^{03}$.

For the right-handed fermion system, the field operator of the symmetric energy-momentum tensor at $r = 0$ is

$$T^{\mu\nu} = \frac{1}{4}\left(\psi_R^\dagger i\sigma^\mu D_R^\nu \psi_R + \psi_R^\dagger i\sigma^\nu D_R^\mu \psi_R + \text{H.C.}\right), \tag{36}$$

with $\sigma^\mu = (1, \boldsymbol{\sigma})$ and the right-handed covariant derivative D_R^μ defined as

$$D_R^\mu = \left(\partial_t - \frac{i}{2}\Omega\sigma_3, -\partial_x, -\partial_y, -\partial_z\right). \tag{37}$$

The macroscopic energy-momentum tensor $\mathcal{T}^{\mu\nu}$ can be calculated from $T^{\mu\nu}$ as follows,

$$\begin{aligned}
\mathcal{T}^{\mu\nu} \quad = \quad & \langle : T^{\mu\nu} : \rangle \\
= \quad & \frac{1}{4}\sum_{n=1}^{\infty}\sum_{\lambda}\int_{-\infty}^{\infty} dp_z \frac{\lambda}{e^{\beta(E_n - \lambda\Omega/2 - \lambda\mu)} + 1}\psi^\dagger_{\lambda,n,1/2,p_z}\left(i\sigma^\mu D_R^\nu + i\sigma^\nu D_R^\mu\right)\psi_{\lambda,n,1/2,p_z} \\
& + \frac{1}{4}\sum_{n=0}^{\infty}\sum_{\lambda}\int_{-\infty}^{\infty} dp_z \frac{\lambda}{e^{\beta(E_{n+1} + \lambda\Omega/2 - \lambda\mu)} + 1}\psi^\dagger_{\lambda,n,-1/2,p_z}\left(i\sigma^\mu D_R^\nu + i\sigma^\nu D_R^\mu\right)\psi_{\lambda,n,-1/2,p_z} \\
& + \frac{1}{4}\sum_{\lambda}\int_{-\infty}^{\infty} dp_z \frac{\lambda\theta(\lambda p_z)}{e^{\beta(|p_z| - \lambda\Omega/2 - \lambda\mu)} + 1}\psi^\dagger_{\lambda,0,1/2,p_z}\left(i\sigma^\mu D_R^\nu + i\sigma^\nu D_R^\mu\right)\psi_{\lambda,0,1/2,p_z} + \text{H.C.}
\end{aligned} \tag{38}$$

5.1. Energy Density

Firstly, we calculate the energy density $\varepsilon \equiv \mathcal{T}^{00}$ of the system. Making use of

$$\psi^\dagger_{\lambda nmp_z}\left(i\partial_t + \frac{1}{2}\Omega\sigma_3\right)\psi_{\lambda nmp_z} = \begin{cases} \frac{eB}{8\pi^2}(E + p_z + \Omega/2), & m = \frac{1}{2} \\ \frac{eB}{8\pi^2}(E - p_z - \Omega/2), & m = -\frac{1}{2} \end{cases} \tag{39}$$

and from Equation (25) one can obtain

$$\varepsilon\beta^4 = \frac{ab\omega}{16\pi^2} + \sum_{s=\pm 1}\left(\frac{3}{2} - b\frac{\partial}{\partial b}\right)g\left(a + \frac{1}{2}s\omega, b\right), \tag{40}$$

which can be expanded as the threefold series at $a = 0, b = 0, \omega = 0$ or $\mu = 0, B = 0, \Omega = 0$ as follows,

$$\begin{aligned}\varepsilon\beta^4 \quad = \quad & \frac{7\pi^2}{120} + \frac{a^2}{4} + \frac{\omega^2}{16} + \frac{a^4}{8\pi^2} + \frac{3a^2\omega^2}{16\pi^2} + \frac{\omega^4}{128\pi^2} \\ & + \frac{ab\omega}{16\pi^2} + \frac{b^2\ln b^2}{384\pi^2} + \frac{b^2}{96\pi^2}\ln\left(\frac{2e^{\gamma+1}}{G^6}\right) \\ & + \frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{(4n+1)\mathscr{B}_{2n+2}b^{2n+2}}{(4n+4)!!(4n)!!}\sum_{j=0}^{\infty}\frac{\omega^{2j}}{(2j)!2^{2j}} \\ & \times\sum_{k=0}^{\infty}\left(2^{4n+2k+2j+1} - 1\right)\zeta'(-4n-2k-2j)\frac{a^{2k}}{(2k)!},\end{aligned} \tag{41}$$

where the logarithmic term $b^2 \ln b^2$ has been discussed in detail in [11], and its coefficient is independent of ω in this work. It is worth noting that there would be no such logarithmic term if the un-normal ordering description of field operators was adopted [4,51].

5.2. Pressure

The pressure P of the system is $\mathcal{T}^{33}$. Making use of

$$\psi^\dagger_{\lambda nmp_z}\sigma_3(-i\partial_z)\psi_{\lambda nmp_z} = \begin{cases} \frac{eB}{8\pi^2}\frac{(E+p_z+\Omega/2)p_z}{E+\Omega/2}, & m = \frac{1}{2} \\ -\frac{eB}{8\pi^2}\frac{(E-p_z-\Omega/2)p_z}{E-\Omega/2}, & m = -\frac{1}{2}\end{cases} \tag{42}$$

and from Equation (25) one can obtain

$$P\beta^4 = \frac{ab\omega}{16\pi^2} + \frac{1}{2}\sum_{s=\pm 1}g\left(a + \frac{1}{2}s\omega, b\right), \tag{43}$$

which can be expanded as the threefold series at $a = 0, b = 0, \omega = 0$ or $\mu = 0, B = 0, \Omega = 0$ as follows,

$$\begin{aligned}P\beta^4 \quad = \quad & \frac{7\pi^2}{360} + \frac{a^2}{12} + \frac{\omega^2}{48} + \frac{a^4}{24\pi^2} + \frac{a^2\omega^2}{16\pi^2} + \frac{\omega^4}{384\pi^2} \\ & + \frac{ab\omega}{16\pi^2} - \frac{b^2\ln b^2}{384\pi^2} - \frac{b^2}{96\pi^2}\ln\left(\frac{2e^{\gamma}}{G^6}\right) \\ & - \frac{1}{\pi^2}\sum_{n=0}^{\infty}\frac{\mathscr{B}_{2n+2}b^{2n+2}}{(4n+4)!!(4n)!!}\sum_{j=0}^{\infty}\frac{\omega^{2j}}{(2j)!2^{2j}} \\ & \times\sum_{k=0}^{\infty}\left(2^{4n+2k+2j+1} - 1\right)\zeta'(-4n-2k-2j)\frac{a^{2k}}{(2k)!}.\end{aligned} \tag{44}$$

One can obtain $\mathcal{T}^{11}$ from the traceless condition for energy-momentum tensor, $\mathcal{T}^{00} = 2\mathcal{T}^{11} + \mathcal{T}^{33}$.

5.3. Energy Current

The energy current along the z-axis is $\mathcal{T}^{03}$. Making use of

$$\psi^\dagger_{\lambda nmp_z}\left(-i\partial_z + \sigma_3 i\partial_t + \frac{1}{2}\Omega\right)\psi_{\lambda nmp_z} = \begin{cases} \frac{eB}{8\pi^2}\frac{(E+p_z+\Omega/2)^2}{E+\Omega/2}, & m = \frac{1}{2} \\ -\frac{eB}{8\pi^2}\frac{(E-p_z-\Omega/2)^2}{E-\Omega/2}, & m = -\frac{1}{2}\end{cases} \tag{45}$$

and from Equation (25) one can obtain

$$\mathcal{T}^{03}\beta^4 = \frac{b}{8\pi^2}\left(\frac{\pi^2}{6} + \frac{\omega^2}{8} + \frac{a^2}{2}\right) + \sum_{s=\pm 1} s\left(1 - \frac{b}{2}\frac{\partial}{\partial b}\right)g\left(a + \frac{1}{2}s\omega, b\right), \tag{46}$$

which can be expanded as the threefold series at $a = 0, b = 0, \omega = 0$ or $\mu = 0, B = 0, \Omega = 0$ as follows,

$$\begin{aligned}\mathcal{T}^{03}\beta^4 \quad = \quad & \frac{b}{8\pi^2}\left(\frac{\pi^2}{6} + \frac{\omega^2}{8} + \frac{a^2}{2}\right) + \left(\frac{a\omega}{6} + \frac{a^3\omega}{6\pi^2} + \frac{a\omega^3}{24\pi^2}\right) \\ & + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{n\mathscr{B}_{2n+2}b^{2n+2}}{(4n+4)!!(4n)!!}\sum_{j=0}^{\infty}\frac{\omega^{2j+1}}{(2j+1)!2^{2j+1}} \\ & \times \sum_{k=0}^{\infty}\left(2^{4n+2k+2j+3} - 1\right)\zeta'(-4n - 2k - 2j - 2)\frac{a^{2k+1}}{(2k+1)!}. \end{aligned} \tag{47}$$

Up to now, we have obtained all thermodynamic quantities of the right-handed fermion system. For the left-handed fermion system, one can derive corresponding quantities from the right-handed case through space inversion: $\rho_R \to \rho_L$, $\mathcal{J}_R^z \to -\mathcal{J}_L^z$, $\varepsilon_R \to \varepsilon_L$, $P_R \to P_L$, $\mathcal{T}_R^{03} \to -\mathcal{T}_L^{03}$, $\mu_R \to \mu_L$, $B \to B$, $\Omega \to \Omega$, where the subscripts R, L are used to distinguish the quantities in the right-handed case from that in the left-handed case.

6. Zero Temperature Limit

Now we turn to the thermodynamics of the system at zero temperature limit. When the temperature tends to be zero, with chemical potential μ, magnetic field B, and angular velocity Ω fixed, then the three dimensionless quantities $a = \beta\mu$, $b = 2eB\beta^2$, $\omega = \beta\Omega$ all tend to be infinity. The asymptotic behavior of $g(x, b)$ as $x \to \infty$ and $b \to \infty$ has been obtained in [11],

$$\lim_{x,b\to\infty} g(x, b) = \frac{x^2 b}{16\pi^2}. \tag{48}$$

From Equations (27) and (33), one can derive the expressions of the particle density ρ and the current $\mathcal{J}^z$ at the zero temperature limit as follows,

$$\rho = \mathcal{J}^z = \frac{eB}{4\pi^2}\left(\mu + \frac{\Omega}{2}\right). \tag{49}$$

At the zero temperature limit, due to the coupling of the spin with the magnetic field and the angular velocity, the spin alignment of all particles and antiparticles will be along the z-axis of the system. Since these particles are right-handed, they will move along the z-axis with the speed of light c ($c = 1$ in natural unit), so it is reasonable that the particle density ρ equals to the z-component current $\mathcal{J}^z$ at the zero temperature limit.

From Equations (40), (43) and (46), the expressions of energy density ε, pressure P and energy current $\mathcal{T}^{03}$ at the zero temperature limit are

$$\varepsilon = P = \mathcal{T}^{03} = \frac{eB}{8\pi^2}\left(\mu + \frac{\Omega}{2}\right)^2. \tag{50}$$

The movements of the particles and antiparticles with the speed of light along the z-axis leads to the equality of the energy density ε and the energy current $\mathcal{T}^{03}$. Since there is no energy current along the direction of the x- and y-axis, then $\mathcal{T}^{11}$ and $\mathcal{T}^{22}$ vanish in this system, which results in the equality of the energy density ε and the pressure P.

7. Summary

In this article, we have investigated the thermodynamics of the uniformly rotating right-handed fermion system under the background of a uniform magnetic field through the approach of normal ordering and ensemble average, where all thermodynamic quantities are expanded as threefold series at $B = 0$, $\Omega = 0$ and $\mu = 0$. For these threefold series, our results at lower orders are consistent with previous ones by other authors. It is worth pointing out that the general orders of B and Ω in the expressions of the thermodynamic quantities are obtained for the first time and can provide a useful reference for the high-order calculations from several different approaches, such as thermal field theory and the Wigner function. We also calculate all quantities in the zero temperature limit, and obtain the equality of particle/energy density and corresponding currents along the z-axis. Since for the chiral fermion the right-handed part decouples from the left-handed part, in this article, we only considered the case of the right-handed fermion system, which can be directly generalized to the left-handed case through space inversion. In this article, the currents and energy-momentum tensor are calculated at the rotating axis ($r = 0$), so the boundary condition at the speed-of-light surface will not affect our results. The calculations for these quantities off or far from the rotating axis ($r \neq 0$), as well as with the boundary condition at the speed-of-light surface may be investigated in the future.

Funding: This work was supported by the National Natural Science Foundation of China under Grants No. 11890713, and No. 12073008.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: I thank De-Fu Hou for helpful discussion.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Landau Levels for a Single Right-Handed Fermion

The Hamiltonian for a right-handed fermion under the background of the uniform magnetic field $\mathbf{B} = B\mathbf{e}_z$ is

$$H = -i\boldsymbol{\sigma} \cdot (\nabla - ie\mathbf{A}) = -i\boldsymbol{\sigma} \cdot \nabla + \frac{1}{2}eB(y\sigma_1 - x\sigma_2), \tag{A1}$$

where we have chosen $\mathbf{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$ for the gauge potential. One can refer to [39,51,52] for other choices of the gauge potential.

In the following, we will solve the eigenvalue equation of H in cylindrical coordinates,

$$H\psi = E\psi. \tag{A2}$$

We can see that the three Hermitian operators, H, $\hat{p}_z = -i\partial_z$, $\hat{J}_z = \frac{1}{2}\sigma_3 + (x\hat{p}_y - y\hat{p}_x)$ are commutative with each other, so the eigenfunction ψ can be chosen as

$$\psi = \begin{pmatrix} f(r)e^{i(m-\frac{1}{2})\phi} \\ ig(r)e^{i(m+\frac{1}{2})\phi} \end{pmatrix} e^{izp_z}, \tag{A3}$$

where $-\infty < p_z < \infty$ and $m = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$ are the eigenvalues of $\hat{p}_z$ and $\hat{J}_z$, respectively. The explicit form of the Hamiltonian H in cylindrical coordinates is

$$H = \begin{pmatrix} -i\frac{\partial}{\partial z} & e^{-i\phi}\left(-i\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial}{\partial \phi} + \frac{i}{2}eBr\right) \\ e^{i\phi}\left(-i\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \phi} - \frac{i}{2}eBr\right) & i\frac{\partial}{\partial z} \end{pmatrix}, \tag{A4}$$

then from Equation (A2) we can obtain two differential equations for $f(r), g(r)$ as follows,

$$(p_z - E)f(r) + \left(\frac{\partial}{\partial r} + \frac{m+\frac{1}{2}}{r} - \frac{1}{2}eBr\right)g(r) = 0, \tag{A5}$$

$$\left(-\frac{\partial}{\partial r} + \frac{m-\frac{1}{2}}{r} - \frac{1}{2}eBr\right)f(r) + (-p_z - E)g(r) = 0, \tag{A6}$$

which are equivalent to

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\left(m-\frac{1}{2}\right)^2}{r^2} - \left[p_z^2 - E^2 - eB\left(m+\frac{1}{2}\right)\right] - \frac{1}{4}e^2B^2r^2\right\}f(r) = 0 \tag{A7}$$

$$g(r) = \frac{1}{p_z + E}\left(-\frac{\partial}{\partial r} + \frac{m-\frac{1}{2}}{r} - \frac{1}{2}eBr\right)f(r) \tag{A8}$$

We can define a dimensionless variable $\rho = \frac{1}{2}eBr^2$, then

$$\frac{d}{dr} = eBr\frac{d}{d\rho}, \quad \frac{d^2}{dr^2} = eB\frac{d}{d\rho} + 2eB\rho\frac{d^2}{d\rho^2}. \tag{A9}$$

Now Equation (A7) becomes

$$\left\{\rho\frac{d^2}{d\rho^2} + \frac{d}{d\rho} - \frac{\left(m-\frac{1}{2}\right)^2}{4\rho} - \frac{1}{2eB}\left[p_z^2 - E^2 - eB\left(m+\frac{1}{2}\right)\right] - \frac{1}{4}\rho\right\}f = 0 \tag{A10}$$

Next, we choose

$$f = e^{-\frac{\rho}{2}}\rho^{\frac{m}{2}-\frac{1}{4}}G(\rho), \tag{A11}$$

then Equations (A8) and (A10) become

$$g = -\frac{\sqrt{2eB}}{E+p_z}e^{-\frac{\rho}{2}}\rho^{\frac{m}{2}+\frac{1}{4}}G'(\rho), \tag{A12}$$

$$\rho G'' + \left[\left(m+\frac{1}{2}\right) - \rho\right]G' - \frac{1}{2eB}\left(p_z^2 - E^2\right)G = 0. \tag{A13}$$

Define following two quantities,

$$\gamma = m + \frac{1}{2}, \quad \alpha = \frac{1}{2eB}\left(p_z^2 - E^2\right), \tag{A14}$$

then Equation (A13) becomes

$$\rho G'' + (\gamma - \rho)G' - \alpha G = 0, \tag{A15}$$

which is the confluent hypergeometric equation [53]. With the boundary conditions, $|f(0)|, |f(\infty)| < \infty$, the solutions for $G(\rho), f(\rho), g(\rho)$ can be chosen as:

(1) When $\gamma = 0, -1, -2, \ldots$, i.e., $m = -1/2, -3/2, -5/2, \ldots$, the boundary condition $|f(0)| < \infty$ requires that

$$G(\rho) = \rho^{1-\gamma}F(\alpha - \gamma + 1, 2 - \gamma, \rho) = \rho^{\frac{1}{2}-m}F\left(\alpha - m + \frac{1}{2}, \frac{3}{2} - m, \rho\right), \tag{A16}$$

where $F(\alpha,\gamma,\rho)$ is the confluent hypergeometric function as discussed in Appendix B. In addition, the boundary condition $|f(\infty)|<\infty$ requires that

$$\alpha - m + \frac{1}{2} = -n \ \ (n \in \mathbb{N}), \quad E = \lambda\sqrt{p_z^2 + 2eB\left(n - m + \frac{1}{2}\right)} \ \ (\lambda = \pm 1), \tag{A17}$$

$$G(\rho) = \rho^{\frac{1}{2}-m} F\left(-n, \frac{3}{2} - m, \rho\right) \sim \rho^{\frac{1}{2}-m} L_n^{\frac{1}{2}-m}(\rho), \tag{A18}$$

where $L_n^k(\rho)$ is the general Laguerre polynomial as discussed in Appendix B. Then one obtains

$$f(\rho) \sim e^{-\frac{\rho}{2}} \rho^{\frac{1}{4}-\frac{m}{2}} L_n^{\frac{1}{2}-m}(\rho), \quad g(\rho) \sim -\frac{\sqrt{2eB}}{E+p_z}\left(n - m + \frac{1}{2}\right) e^{-\frac{\rho}{2}} \rho^{-\frac{1}{4}-\frac{m}{2}} L_n^{-\frac{1}{2}-m}(\rho). \tag{A19}$$

(2) When $\gamma = 1,2,3,\ldots$, i.e., $m = 1/2, 3/2, 5/2, \ldots$, the boundary condition $|f(0)|<\infty$ requires that

$$G(\rho) = F(\alpha,\gamma,\rho) = F\left(\alpha, \frac{1}{2} + m, \rho\right). \tag{A20}$$

In addition, the boundary condition $|f(\infty)|<\infty$ requires that

$$\alpha = -n \ \ (n \in \mathbb{N}), \quad E = \lambda\sqrt{p_z^2 + 2eBn} \ \ (\lambda = \pm 1), \tag{A21}$$

$$G(\rho) = F\left(-n, \frac{1}{2} + m, \rho\right) \sim L_n^{m-\frac{1}{2}}(\rho). \tag{A22}$$

Then one obtains

$$f(\rho) \sim e^{-\frac{\rho}{2}} \rho^{\frac{m}{2}-\frac{1}{4}} L_n^{m-\frac{1}{2}}(\rho), \quad g(\rho) \sim \frac{\sqrt{2eB}}{E+p_z} e^{-\frac{\rho}{2}} \rho^{\frac{m}{2}+\frac{1}{4}} L_{n-1}^{m+\frac{1}{2}}(\rho). \tag{A23}$$

There is a special case we must point out here: When $m > 0$, $n = 0$, we must choose $E = p_z$, in which case we have $f(\rho) = e^{-\frac{\rho}{2}} \rho^{\frac{m}{2}-\frac{1}{4}}$, $g(\rho) = 0$. There is no physical solution for $m > 0$, $n = 0$, $E = -p_z$.

Making use of the orthonormal relation of the general Laguerre polynomials,

$$\int_0^\infty dx e^{-x} x^\gamma L_m^\gamma(x) L_n^\gamma(x) = \frac{\Gamma(n+\gamma+1)}{n!}\delta_{mn}, \tag{A24}$$

we can obtain the normalized eigenfunctions as follows:

When $m < 0$,

$$\psi_{\lambda n m p_z} = \sqrt{\frac{n!}{(n-m+\frac{1}{2})!}} \begin{pmatrix} \sqrt{\frac{eB(E+p_z)}{2E}} e^{-\frac{\rho}{2}} \rho^{\frac{1}{4}-\frac{m}{2}} L_n^{\frac{1}{2}-m} e^{i(m-\frac{1}{2})\phi} \\ -\frac{i\lambda eB(n-m+\frac{1}{2})}{\sqrt{E(E+p_z)}} e^{-\frac{\rho}{2}} \rho^{-\frac{1}{4}-\frac{m}{2}} L_n^{-\frac{1}{2}-m} e^{i(m+\frac{1}{2})\phi} \end{pmatrix} \frac{e^{izp_z}}{2\pi}, \tag{A25}$$

$$E = \lambda\sqrt{p_z^2 + 2eB\left(n - m + \frac{1}{2}\right)}. \tag{A26}$$

When $m > 0$,

$$\psi_{\lambda n m p_z} = \sqrt{\frac{n!}{(n+m-\frac{1}{2})!}} \begin{pmatrix} \sqrt{\frac{eB(E+p_z)}{2E}} e^{-\frac{\rho}{2}} \rho^{\frac{m}{2}-\frac{1}{4}} L_n^{m-\frac{1}{2}} e^{i(m-\frac{1}{2})\phi} \\ \frac{i\lambda eB}{\sqrt{E(E+p_z)}} e^{-\frac{\rho}{2}} \rho^{\frac{m}{2}+\frac{1}{4}} L_{n-1}^{m+\frac{1}{2}} e^{i(m+\frac{1}{2})\phi} \end{pmatrix} \frac{e^{izp_z}}{2\pi}, \tag{A27}$$

$$E = \lambda\sqrt{p_z^2 + 2eBn}. \tag{A28}$$

All normalized eigenfunctions are orthogonal with each other,

$$\int dV \psi^{\dagger}_{\lambda' n' m' p'_z} \psi_{\lambda n m p_z} = \delta_{\lambda'\lambda}\delta_{n'n}\delta_{m'm}\delta(p'_z - p_z). \tag{A29}$$

Appendix B. Confluent Hypergeometric Function and Laguerre Polynomial

The confluent hypergeometric equation is [53]

$$zy'' + (\gamma - z)y' - \alpha y = 0. \tag{A30}$$

when $\gamma \notin \mathbb{Z}$, there are two independent solutions as follows,

$$\begin{aligned} y_1 &= F(\alpha,\gamma,z), \\ y_2 &= z^{1-\gamma}F(\alpha-\gamma+1, 2-\gamma, z), \end{aligned} \tag{A31}$$

where $F(\alpha,\gamma,z)$ is the confluent hypergeometric function defined as

$$F(\alpha,\gamma,z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k}\frac{z^k}{k!} \equiv 1 + \frac{\alpha}{\gamma}z + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots. \tag{A32}$$

The asymptotic behavior of $F(\alpha,\gamma,z)$ as $z \to \infty$ is the same as e^z. When α is a non-positive integer, then $F(\alpha,\gamma,z)$ becomes a polynomial.

The general Laguerre polynomial $L_n^{\gamma}(z)$ is defined from $F(\alpha,\gamma,z)$ as follows [54],

$$L_n^{\gamma}(z) = \frac{\Gamma(\gamma+n+1)}{n!\Gamma(\gamma+1)}F(-n,\gamma+1,z) = \begin{pmatrix} \gamma+n \\ n \end{pmatrix} F(-n,\gamma+1,z), \tag{A33}$$

where $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$. Laguerre polynomial $L_n^{\gamma}(z)$ satisfies following differential equation

$$zy'' + (\gamma + 1 - z)y' + ny = 0 \tag{A34}$$

We can rewrite Equation (A34)) as a type of Sturm–Liouvelle equation,

$$\frac{d}{dz}\left(z^{\gamma+1}e^{-z}\frac{dy}{dz}\right) + nz^{\gamma}e^{-z}y = 0, \tag{A35}$$

which gives the orthogonality of $L_n^{\gamma}(z)$,

$$\int_0^{\infty} dz e^{-z} z^{\gamma} L_m^{\gamma}(z) L_n^{\gamma}(z) = \frac{\Gamma(n+\gamma+1)}{n!}\delta_{mn}. \tag{A36}$$

when $\gamma = 0$, then $L_n^{\gamma}(z)$ becomes the normal Laguerre polynomial $L_n(z)$,

$$L_n(z) = L_n^0(z) = F(-n,1,z). \tag{A37}$$

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