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A New Result in Form of Finite Triple Sums for a Series from Ramanujan's Notebooks

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Abstract: We consider a function $g(r, x, u)$ with $x, u \in \mathbb{C}$ and $r \in \mathbb{N}$, which, over a symmetric domain, equals the sum of an infinite series as noted in the 16th Entry of Chapter 3 in Ramanujan's second notebook. The function attracted new attention since it was established to be closely connected to the theory of labelled trees. However, to the best of our knowledge, a closed-form solution allowing, e.g., the rapid computation of $g(r, x, u)$ in Mathematica without explicit use of recursions has been lacking until now. Our proposed formula transforms the part depending on the variable u into a more symmetric form, which then appears inside a finite triple sum consisting of binomials and Stirling numbers of the second kind.

Keywords: Ramanujan; Stirling numbers; combinatorics

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1. Introduction

The documented ideas of Srinivasa Ramanujan (1887–1920) keep being an inspiration more than 100 years after they were written down. In [1], problems are presented and discussed that were contributed by him to the Journal of the Indian Mathematical Society. Equation (9) of [1] caught our attention, and the subsequent discussions lead us to [2], the summary/discussion of Chapter 3 of Ramanujan's Second Notebook written by Berndt, Evans and Wilson (1983). There, it is stated in the 16th Entry that over domains of convergence (c.f., Equation (11), Section 4):

$$\sum_{k=0}^{\infty} \frac{(k+x)^{k+r} e^{-u(x+k)} u^k}{k!} = \sum_{t=1}^{r+1} \frac{\psi_t(r,x)}{(1-u)^{r+t}} \equiv g(r,x,u), \tag{1}$$

where $x, u \in \mathbb{C}$ and $r \in \mathbb{N}$ and where the polynomial $\psi_t(r, x)$ is recursively defined by

$$\psi_t(r,x) = (x-r-t+1)\psi_t(r-1,x) + (r+t-2)\psi_{t-1}(r-1,x), \tag{2}$$

with $\psi_t(r, x) = 0$ if $t \notin [1, r+1]$ and $\psi_1(0, x) = 1$. Equation (2) is to be credited to Berndt, Evans and Wilson, while an equivalent recursion with three terms was provided in the original work of Ramanujan. We have not encountered in the literature any explicit formula allowing for the fast computation of $g(r, x, u)$ without the need of setting up recursions. The objectives of the present work are (i) to propose such a formula (key result given by Equation (5)), (ii) to back it up by computational means and (iii) to encourage the wider community to rigorously prove the conjectured identity.

In the introductory text of [2], the following statement is made: "Entries 16 and 17 do not seem to have been expanded upon in the literature and would appear to be a basis for further fruitful research". The statement is also quoted in [3], a work by Zeng from 1999, who further notes that, since the publication of [2], "several authors have made arithmetical and combinatorial studies of a sequence in Entry 17", while "nothing seems to have been do[ne] yet regarding Entry 16". It was noticed by Zeng that the study of labelled trees by

Shor [4] involves a recursion formula for polynomials $Q_{r,t}(x)$ similar to that of $\psi_t(r,x)$. As such, it was presented in [3] that

$$Q_{r,t}(x) \;=\; \psi_{t+1}(r-1, x+r), \tag{3}$$

meaning that from, e.g., $\psi_3(3,x) = 15x - 35$, follows $Q_{4,2}(x) = \psi_3(3, x+4) = 15x + 25$. Additionally, knowledge of $Q_{r+1,t-1}(x)$ allows us, through Equation (3) via an equation system, to work out $\psi_t(r,x)$. We should note that [3,5]

$$Q_{r,t}(x) \;=\; \sum_{T\in\mathcal{T}_{r+1,t}} x^{\deg_T(1)-1}, \tag{4}$$

where $\mathcal{T}_{r+1,t}$ is the set of labelled trees on $\{1,2,\ldots,r+1\}$ with t improper edges, and where $\deg_T(1)$ is the number of children of the smallest node in the tree T. We complement here the detailed definitions given in Section 2 of [5] with an illustration for the case with $Q_{3,1}(x)$. All trees are considered that have four labelled nodes and one improper edge. An edge (a,b) is said to be proper if the minimal node of the subtree rooted in b (including b itself) has a value exceeding a; it is otherwise classified as improper. We find seven ways to achieve trees with four labelled nodes and one improper edge, as depicted in Figure 1. Node 1 has in three cases two children and in four cases only one. Equation (4) thus suggests that $Q_{3,1}(x) = 3x + 4$, agreeing with what may be read, e.g., in Table 2 of [5].

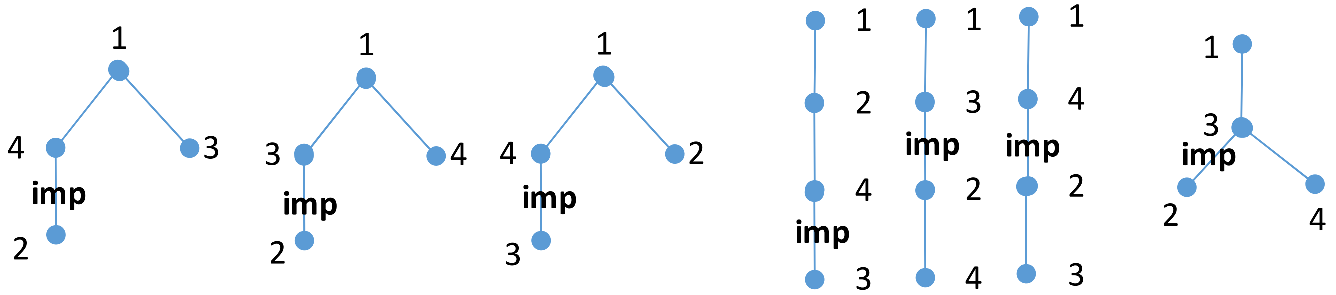

Figure 1. The seven possibilities to construct trees with four labelled nodes and one improper edge.

The cited works offer a combinatorial interpretation of recurrence relations given in [2]. It appears to us that most subsequent works dealing in one way or another with Entry 16 of [2] do so in the theoretical framework of labelled trees. As noted above, an explicit formula allowing for the fast computation of $g(r,x,u)$ without the need of setting up recursions has been lacking. The key result of this work is the conjectured identity (for $r \geq 0$):

$$\sum_{t=1}^{r+1} \frac{\psi_t(r,x)}{(1-u)^{r+t}} \;=\; \sum_{j=0}^{r}\sum_{k=0}^{j}\sum_{s=0}^{r-j}(-1)^s \binom{2r-j}{k}\binom{2r-j-k}{s}\left\{ {2r-j-k-s \atop r-j-s} \right\}\frac{x^k u^{r-j}}{(1-u)^{2r+1-j}} \tag{5}$$

where the curly brackets on the right-hand side are used to denote Stirling numbers of the second kind. The right-hand side of Equation (5) offers an efficient computation of $g(r,x,u)$ in Mathematica [6] through the built-in functions Binomial[] and StirlingS2[]. We describe in Section 2 how we arrived at the conjectured identity, while in Section 3, we describe efforts made to strengthen the conjecture. Concluding remarks are given in Section 4. It is stressed that, without proof, Equation (5) remains conjectured.

2. Methods

By targeting simple special cases of the left-hand side of Equation (1), for instance, with x set to small integers and u set to simple fractions (such as $1/2$, $1/3$, etc.), we were able to come up with functions that seemingly offered closed-form solutions to the infinite sum for, e.g., $r = 2$ and $r = 3$. After familiarizing ourselves with [2] and Equations (1) and (2), we noticed a discrepancy comparing the forms of our solutions. Whereas the use of the $\psi_t(r,x)$-polynomials render explicit solutions of the form

$$g(r,x,u) \;=\; \sum_{j\,=\,0}^{r}\sum_{k\,=\,0}^{j} \frac{\beta_{r,j,k}x^k}{(1-u)^{2r+1-j}}\,, \tag{6}$$

with $\beta_{r,j,k}$ taking integer values (not exclusively non-negative), our few experimentally found solutions were of the form

$$g(r,x,u) \;=\; \sum_{j\,=\,0}^{r}\sum_{k\,=\,0}^{j} \frac{\rho_{r,j,k}x^k u^{r-j}}{(1-u)^{2r+1-j}}\,, \tag{7}$$

where the integer coefficients $\rho_{r,j,k} = 0$ if $k < j = r$ and are strictly positive otherwise. As an example, in the case with $r = 3$, we have (check via Equation (4) or see, e.g., [5]): $\psi_1(3,x) = x^3 - 6x^2 + 11x - 6$; $\psi_2(3,x) = 6x^2 - 26x + 26$; $\psi_3(3,x) = 15x - 35$; $\psi_4(3,x) = 15$. This gives

$$g(3,x,u) \;=\; x^3\left(\frac{1}{u_*^4}\right) + x^2\left(-\frac{6}{u_*^4}+\frac{6}{u_*^5}\right) + x\left(\frac{11}{u_*^4}-\frac{26}{u_*^5}+\frac{15}{u_*^6}\right) + \left(-\frac{6}{u_*^4}+\frac{26}{u_*^5}-\frac{35}{u_*^6}+\frac{15}{u_*^7}\right), \tag{8}$$

where $u_* = 1-u$. Expressing this in the form of Equation (7) while setting $\rho_{3,3,k} = 0$ for $k = 0, 1$ and 2, is a straightforward algebraic exercise yielding $\rho_{3,3,3} = 1$, $\rho_{3,2,2} = 6$, $\rho_{3,2,1} = 4$, $\rho_{3,1,1} = 15$, $\rho_{3,2,0} = 1$, $\rho_{3,1,0} = 10$ and $\rho_{3,0,0} = 15$. In line with this example, we constructed arrays of $\rho_{r,j,k}$ coefficients for $r = 3$, 4 and 5, as shown in Figure 2 (top arrays).

$r = 3$

$j \setminus k$	0	1	2	3
0	15			
1	10	15		
2	1	4	6	
3	0	0	0	1

$r = 4$

$j \setminus k$	0	1	2	3	4
0	105				
1	105	105			
2	25	60	45		
3	1	5	10	10	
4	0	0	0	0	1

$r = 5$

$j \setminus k$	0	1	2	3	4	5
0	945					
1	1260	945				
2	490	840	420			
3	56	175	210	105		
4	1	6	15	20	15	
5	0	0	0	0	0	1

Divide entries in top arrays by $\binom{2r-j}{k}$ to get entries in bottom arrays

$j \setminus k$	0	1	2	3
0	15			
1	10	3		
2	1	1	1	
3	0	0	0	1

$j \setminus k$	0	1	2	3	4
0	105				
1	105	15			
2	25	10	3		
3	1	1	1	1	
4	0	0	0	0	1

$j \setminus k$	0	1	2	3	4	5
0	945					
1	1260	105				
2	490	105	15			
3	56	25	10	3		
4	1	1	1	1	1	
5	0	0	0	0	0	1

Figure 2. The top arrays display values of $\rho_{r,j,k}$ for the cases with $r = 3$ (**left**), $r = 4$ (**middle**) and $r = 5$ (**right**). The bottom arrays show the resulting quotients upon division by binomials on the form "$2r - j$ over k". The colored numbers are referenced in the text.

The arrays were contemplated for some time until noticing that the second row from the bottom stood out as familiar binomial coefficients. It was then found that binomials "$2r - j$ over k" not only captured the sequence at row $j = r - 1$, but also provided a proper divisor for each element of the arrays. The resulting quotients are displayed as the bottom arrays in Figure 2. The breakthrough step came when consulting the Online Encyclopedia of Integer Sequences (OEIS) [7] with the query of finding successive appearances of 3, 10, 25 and 56 (marked red in Figure 2). This led us to sequence A000247, where cross-references were given also to sequences A000478 starting with 15, 105, 490 (as the green marked numbers in Figure 2) and A058844 starting with 105, 1260 (as the blue marked numbers in Figure 2). This suggested, through descriptions on OEIS, that the entries of our constructed arrays corresponded to the number of ways of placing $2r - j - k$ labelled items into $r - j$ indistinguishable boxes, such that at least 2 items are placed in each box. Letting $f(a,\ b)$ denote the number of ways of placing a labelled items into b indistinguishable boxes, such that at least 2 items are placed in each box, we further learned from [7] (through comments to the sequence A008299) that

$$f(a,b) \;=\; \sum_{s\,=\,0}^{b} (-1)^s \binom{a}{s} \left\{ \begin{matrix} a-s \\ b-s \end{matrix} \right\}, \tag{9}$$

where the curly brackets denote Stirling numbers of the second kind. Altogether, this led us to conjecture the identity expressed by Equation (5), where Equation (9) is used with the settings $a \;=\; 2r - j - k$ and $b \;=\; r - j$.

The recursively defined function $g(r, x, u)$ contains the variable u as powers of $1/(1-u)$. This gives it a quite short and elegant appearance at the cost of time intensive recursions at deeper levels of r. Now, $g(r, x, u)$ has been transformed via an expansion of the recursion up to certain levels leading to the identification of patterns in the coefficients of powers of a variable $(1-u)/u$, which looks algebraically more symmetric, because both nominator and denominator vary with u. The revealed connection to the problem of distributing balls into boxes is quite unexpected.

3. Test of Validity

A Mathematica code verifying that Equation (5) holds for integers $0 \le r \le 30$ is provided in Figure 3. The first two lines define $\psi_t(r,x) \;=\; 0$ if $t \notin [1, r+1]$ and $\psi_1(0,x) \;=\; 1$, respectively. The third line represents the recursion formula for $\psi_t(r,x)$ as given by Equation (4). The fourth line defines the function $f(a,\ b)$ similar to that in Equation (9). The fifth line involves the calculation of the difference between the left- and right-hand sides of Equation (5), stepping through r from 0 to 30 with increasing execution time needed for higher recursion levels. Note that the right-hand side here is expressed via the use of Equation (9). The output is as desired, an array of zeros, showing that, for each considered value of r, the conjectured identity holds true.

```
psi[t_][r_] := 0 /; t < 1 || t > r + 1;
psi[1][0] = 1;
psi[t_][r_] := psi[t][r] = (x - r - t + 1) * psi[t][r - 1] + (r + t - 2) psi[t - 1][r - 1];
f[a_, b_] := Sum[(-1) ^ s * Binomial[a, s] * StirlingS2[a - s, b - s], {s, 0, b}];

Table[Expand[Sum[psi[t][r] / (1 - u) ^ (r + t), {t, 1, r + 1}] -
    Sum[Binomial[2 * r - j, k] * f[2 * r - j - k, r - j] * x ^ k * u ^ (r - j) / (1 - u) ^ (2 * r + 1 - j),
     {j, 0, r}, {k, 0, j}]], {r, 0, 30}] // Simplify

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

Figure 3. A Mathematica code verifying Equation (5) for integers $0 \le r \le 30$.

4. Concluding Remarks

By the mere complexity of the expanded forms of either side of Equation (5), already at $r \;=\; 10$, there is little doubt that Equation (5) holds true in general for $r \ge 0$. With that said, we leave it as an open problem to rigorously prove Equation (5). Potentially useful references for approaching a proof include, e.g., [8,9], but in none of these are attempts made to specifically provide expressions for the $\beta_{r,j,k}$ coefficients of Equation (6) or the $\rho_{r,j,k}$

coefficients of Equation (7). Provided in [10], as well as in [2], are closed-form solutions to the infinite sum also in a few cases with negative values of r. From these results, we have not yet identified patterns that allow us to conjecture a general closed-form solution for the sum on the left-hand side of Equation (1) for integers $r < 0$.

We have also investigated the domain of u over which either side of Equation (5) serves as a solution to the infinite sum of Equation (1) for $r \geq 0$. The numerical evaluation of the series, i.e., of the LHS of Equation (1), presents some difficulties, because each summand contains a factor strongly increasing with k, the running index, and a damping component $u\exp(-u)$. Both taken together lead to a large amplitude in a certain range of k, until the exponential damping takes over. For negative u, the series terms alternate, and there is a cancellation of large numbers within the sum. Almost everywhere in the convergence region, high-precision arithmetic is needed. There is alternating behaviour of the summands at odd intervals of k at negative values of u. For the non-alternating case ($u > 0$), there is no cancellation of successive k summands, and consequently, the values of the sum become very large. The series contains a singularity at $u = 1$. To achieve a reliable numerical evaluation for higher values of r near the pole, where this becomes more difficult because of the large numbers involved, the following procedure is applied: as the value of the sum in this region is determined by the largest summands that occur at ever higher k values only the high k values are taken into account by using an asymptotic representation of the k summand as $k \to$ infinity. This also reveals details about the convergence region for complex u, which will be discussed later. As attested by computation in Mathematica, for u near the pole the value of the LHS of Equation (1), say I, is approximately calculated by replacing the summation over the index k through an integral:

$$I = \frac{e^x}{\sqrt{2\pi}} \int_1^\infty \frac{1}{\sqrt{k}} \left(\exp\left[\frac{2rx - x^2}{2k} + r\ln k + k\left(1 + \ln\left[ue^{-u}\right]\right) \right] \right) dk\,. \tag{10}$$

We may then inspect the whole range. The left panel of Figure 4 shows a comparison of logarithmic absolute values of Equations (1) and (5) as functions of u specifically for the case with $x = 1.9$ and $r = 10$. The dashed line in black represents the RHS of Equation (5), the line in orange displays the result through the LHS of Equation (5), and the blue dots indicate the value of the series. For the computation of the latter, integral approximation was used for $u > 0.3$. Good agreement is seen overall.

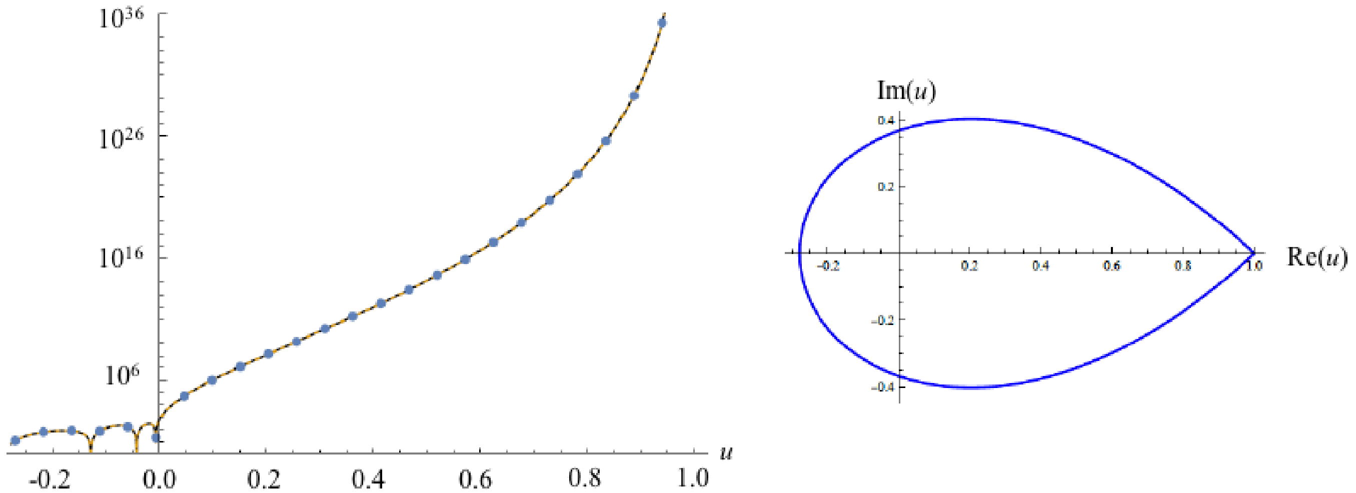

Figure 4. (**Left**) LHS of Equation (5) (orange), RHS of Equation (5) (black dashed) and computed LHS of Equation (1) (blue points) for $x = 1.9$, $r = 10$ and varying u. (**Right**) Region of complex plane, where the series on the LHS of Equation (1) converges and where Equation (1) holds true (see Equation (11)).

The behaviour of the asymptotic term for large values of k is determined by the factor $\exp[k(1+\ln[ue^{-u}])]$ that avoids explosion with increasing k only when $(1+\ln[ue^{-u}]) < 0$. For complex u, this yields a symmetric area of convergence, as illustrated by the right panel of Figure 4. For values of u outside this area, the series diverges. In short, our evaluation suggests that Equation (1) holds for non-negative integers r and while u satisfies both of the following relations:

$$\begin{cases} -0.27846454\ldots < \mathrm{Re}(u) < 1 \\ [\mathrm{Im}(u)]^2 < \exp[2(\mathrm{Re}(u)-1)] - [\mathrm{Re}(u)]^2 \end{cases}; \tag{11}$$

where $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$ denote the real and imaginary parts of u, respectively, and where using the Lambert W function the range for the real part of u is reaching in a somewhat symmetric way from $-W(1/e)$ to $W(e)$, c.f., Equation (16.6) in [2].

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References

1. Berndt, B.C.; Choi, Y.-S.; Kang, S.-Y. The problems submitted by Ramanujan to the Journal of the Indian Mathematical Society. In *Continued Fractions: From Analytic Number Theory to Constructive Approximation*; American Mathematical Society: Providence, RI, USA, 1999; pp. 15–56.
2. Berndt, B.C.; Evans, R.J.; Wilson, B.M. Chapter 3 of Ramanujan's second notebook. *Adv. Math.* **1983**, *49*, 123–169. [CrossRef]
3. Zeng, J. A Ramanujan Sequence that Refines the Cayley Formula for Trees. *Ramanujan J.* **1999**, *3*, 45–54. [CrossRef]
4. Shor, P. A New Proof of Cayley's Formula for Counting Labelled Trees. *J. Combin. Theory Ser. A* **1995**, *71*, 154–158. [CrossRef]
5. Chen, W.Y.C.; Guo, V.J.W. Bijections Behind the Ramanujan Polynomials. *Adv. Appl. Math.* **2001**, *27*, 336–356. [CrossRef]
6. Wolfram Research, Inc. *Mathematica, Version 12.1*; Wolfram Research, Inc.: Champaign, IL, USA, 2020.
7. Sloane, N.J.A. The Online Encyclopedia of Integer Sequences. Available online: https://oeis.org (accessed on 10 May 2022).
8. Guo, V.J.W.; Zeng, J. A Generalization of the Ramanujan Polynomials and Plane Trees. *Adv. Appl. Math.* **2007**, *39*, 96–115. [CrossRef]
9. Randazzo, A. Arboretum for a Generalization of Ramanujan Polynomials. *Ramanujan J.* **2021**, *54*, 591–604. [CrossRef]
10. Dieckmann, A. A Collection of Infinite Products and Series. Available online: http://www-elsa.physik.uni-bonn.de/~{}dieckman/InfProd/InfProd.html (accessed on 2 May 2022).