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# Some Inclusion Relations of Certain Subclasses of Strongly Starlike, Convex and Close-to-Convex Functions Associated with a Pascal Operator

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**Abstract:** This paper studies some inclusion properties of some new subclasses of analytic functions in the open symmetric unit disc $U$ that are associated with the Pascal operator. Furthermore, the integral-preserving properties in a sector for these subclasses are also investigated.

**Keywords:** Pascal operator; analytic functions; strongly starlike functions; strongly convex functions; strongly close-to-convex functions; argument estimates

**MSC:** 30C45; 30C55



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## 1. Introduction

Let $\mathcal{A}$ be the class consisting of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k , \tag{1}$$

which are analytic in the open symmetric unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\delta$ and type $\gamma$, denoted by $\mathcal{S}^*(\delta, \gamma)$, if it satisfies

$$\left| \arg\left( \frac{zf'(z)}{f(z)} - \gamma \right) \right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma < 1 \text{ and } 0 < \delta \le 1. \tag{2}$$

On the other hand, a function $f \in \mathcal{A}$ is said to be strongly convex of order $\delta$ and type $\gamma$, denoted by $\mathcal{K}(\delta, \gamma)$,if it satisfies

$$\left| \arg\left( 1 + \frac{zf''(z)}{f'(z)} - \gamma \right) \right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma < 1 \text{ and } 0 < \delta \le 1. \tag{3}$$

In (2) and (3) , if $\gamma = 0$, then $f$ belongs to the class of strongly starlike and convex functions of order $\delta$, respectively, which have been studied by Mocanu [1] and Nunokawa [2], while if $\delta = 1$, then $f \in \mathcal{S}^*(\gamma)$ and $f \in \mathcal{K}(\gamma)$, where $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ are the classes of starlike and convex functions of order $\gamma$, respectively, which were introduced by Robertson [3]. In particular, if $\gamma = 0$ and $\delta = 1$, then the functions $f \in \mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $f \in \mathcal{K}(0) \equiv \mathcal{K}$, where $\mathcal{S}^*$ and $\mathcal{K}$ are the classes of starlike and convex functions, respectively. For $0 \leq \gamma, \alpha < 1$ and $0 < \delta, \beta \leq 1$, let $\mathcal{B}(\delta, \gamma, \beta, \alpha)$ be the class of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| \arg\left( \frac{zf'(z)}{k(z)} - \gamma \right) \right| < \frac{\delta\pi}{2}, \tag{4}$$

for some $k \in \mathcal{S}^*(\beta, \alpha)$. In (4), if $\delta = \beta = 1$, then the function $f \in \mathcal{B}(\gamma, \alpha)$, where $\mathcal{B}(\gamma, \alpha)$ is the class of close-to-convex functions of order $\gamma$ and type $\alpha$, which has been studied by Libera [4], while if $\gamma = \alpha = 0$ and $\beta = 1$, then $f$ belongs to the class of strongly close-to-convex functions of order $\delta$, which has been studied by Reade [5].

If $f, g \in \mathcal{A}$ such that $f$ is given by (1) and $g$ is given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, then the Hadamard product $(f * g)(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

It is well -known that

$$z(f * g)' = f * zg' = g * zf'. \tag{5}$$

Many real-life phenomena can be described and modelled using distributions of random variables, which have an important role in statistics and probability. Some of these distributions are commonly used and have been specified by special names to emphasize their significance, such as the Binomial, Poisson, and Pascal (or Negative Binomial) distribution. The Pascal distribution has been widely used in many fields such as communications, health, climatology, demographics, and engineering (see [6]). Recently, in geometric function theory, there has been a growing interest in studying the geometric properties of analytic functions associated with the Pascal distribution (see [7–13]).

A variable $x$ is said to be a Pascal (or Negative Binomial) distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$$(1-q)^m, \frac{qm(1-q)^m}{1!}, \frac{q^2 m(m+1)(1-q)^m}{2!}, \frac{q^3 m(m+1)(m+2)(1-q)^m}{3!}, \ldots,$$

respectively, where $m$ and $q$ are called the parameters, and thus

$$P(x = n) = \binom{n+m-1}{m-1} q^n (1-q)^m, n = 0, 1, 2, \ldots.$$

This distribution is based on the binomial theorem with a negative exponent and it describes the probability of $m$ success and $n$ failure in $(n+m-1)$ trials, and success on $(n+m)th$ trials where $(1-q)$ is the probability of success.

Recently, a power series whose coefficients are probabilities of the Pascal distribution was introduced by El-Deeb et al. [14] as follows

$$\Theta_q^m(z) = (1-q)^m z + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} q^{k-1} (1-q)^m z^k \quad (z \in U),$$

where $m \in \mathbb{Z}^+, 0 \leq q \leq 1$. By the ratio test, we can note that the radius of convergence of the series above is infinity. For $m \in \mathbb{Z}^+, 0 \leq q < 1$,we consider the Pascal operator

$$\Lambda_q^m : \mathcal{A} \to \mathcal{A},$$

which is defined as follows

$$\begin{aligned}\Lambda_q^m f(z) &= f_{q,m}(z) * f(z),\\ &= z + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} q^{k-1} a_k z^k \quad (z \in U),\end{aligned}$$

where

$$f_{q,m}(z) = \frac{\Theta_q^m(z)}{(1-q)^m} = z + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} q^{k-1} z^k \quad (z \in U).$$

Now, we define the operator $\mathcal{L}_q^m : \mathcal{A} \to \mathcal{A}$ which is analogous to the Pascal operator $\Lambda_q^m$, as follows

$$\mathcal{L}_q^m f(z) = f_{q,m}^{(-1)}(z) * f(z) \qquad (z \in U), \tag{6}$$

where

$$f_{q,m}(z) * f_{q,m}^{(-1)}(z) = \frac{z}{1-z} \qquad (z \in U). \tag{7}$$

We define and investigate the properties of the following new classes of analytic functions by using the two operators $\Lambda_q^m$ and $\mathcal{L}_q^m$. Let

$$\mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma) = \left\{ f \in \mathcal{A} : \Lambda_q^m(f) \in \mathcal{S}^*(\delta,\gamma), 0 \le \gamma < 1, 0 < \delta \le 1, z \in U \right\}, \tag{8}$$

$$\mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma) = \left\{ f \in \mathcal{A} : \mathcal{L}_q^m(f) \in \mathcal{S}^*(\delta,\gamma), 0 \le \gamma < 1, 0 < \delta \le 1, z \in U \right\}, \tag{9}$$

$$\mathcal{K}_{\Lambda,m,q}(\delta,\gamma) = \left\{ f \in \mathcal{A} : \Lambda_q^m(f) \in \mathcal{K}(\delta,\gamma), 0 \le \gamma < 1, 0 < \delta \le 1, z \in U \right\}, \tag{10}$$

$$\mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma) = \left\{ f \in \mathcal{A} : \mathcal{L}_q^m(f) \in \mathcal{K}(\delta,\gamma), 0 \le \gamma < 1, 0 < \delta \le 1, z \in U \right\}, \tag{11}$$

$$\begin{aligned}\mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha) \quad &= \quad \Big\{ f \in \mathcal{A} : \Lambda_q^m(f) \in \mathcal{B}(\delta,\gamma,\beta,\alpha), 0 \le \gamma, \alpha < 1\\ &\quad\quad \text{and } 0 < \delta, \beta \le 1, z \in U \}, \end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha) \quad &= \quad \Big\{ f \in \mathcal{A} : \mathcal{L}_q^m(f) \in \mathcal{B}(\delta,\gamma,\beta,\alpha), 0 \le \gamma, \alpha < 1\\ &\quad\quad \text{and } 0 < \delta, \beta \le 1, z \in U \}. \end{aligned} \tag{13}$$

In 1975, Ruscheweyh [15] introduced his famous differential operator of normalized analytic functions in the open symmetric unit disc $U$. This operator has an important role in geometric function theory. In this paper, motivated by the significant work of Ruschewey, we obtained some argument properties and inclusion relations of the classes $\mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$, $\mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma)$, $\mathcal{K}_{\Lambda,m,q}(\delta,\gamma)$, $\mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma)$, $\mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha)$, and $\mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha)$. Additionally, we derive the integral preserving properties of these classes.

## 2. Inclusion Relations

In proving our main results, we need the following lemmas.

**Lemma 1.** *[16] (Alexander's Theorem). Let $f \in \mathcal{A}$. Then $f \in \mathcal{K} \Longleftrightarrow zf' \in \mathcal{S}^*$.*

**Lemma 2.** *[2] Let $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$ be analytic function in $U$ and suppose that there exists a point $z_0 \in U$ such that*

$$|\arg l(z)| < \frac{\delta\pi}{2} \qquad (|z| < |z_o|),$$

*and*

$$|\arg l(z_o)| = \frac{\delta\pi}{2},$$

*where* $0 < \delta \leq 1$. *Then we have*

$$\frac{z_o l'(z_o)}{l(z_o)} = ib\delta,$$

*where*

$$b \geq \frac{1}{2}\left(d + \frac{1}{d}\right) \text{ when } \arg l(z_o) = \frac{\delta\pi}{2},$$

*and*

$$b \leq -\frac{1}{2}\left(d + \frac{1}{d}\right) \text{ when } \arg l(z_o) = -\frac{\delta\pi}{2},$$

*where* $(l(z_o))^{1/\delta} = \pm id$, $d > 0$.

**Proposition 1.** $z\left(\Lambda_q^m f(z)\right)' = m\Lambda_q^{m+1} f(z) - (m-1)\Lambda_q^m f(z)$.

**Proof.** Since

$$\Lambda_q^{m+1} f(z) = z + \sum_{k=2}^{\infty} \binom{k+m-1}{m} q^{k-1} a_k z^k,$$

then

$$\begin{aligned}
\Lambda_q^{m+1} f(z) &= z + \sum_{k=2}^{\infty} \frac{(k+m-1)(k+m-2)!}{m(m-1)!(k-1)!} q^{k-1} a_k z^k, \\
&= z + \frac{1}{m} \sum_{k=2}^{\infty} k \binom{k+m-2}{m-1} q^{k-1} a_k z^k + \frac{m-1}{m} \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} q^{k-1} a_k z^k, \\
&= \frac{1}{m}\left\{ z + \sum_{k=2}^{\infty} k \binom{k+m-2}{m-1} q^{k-1} a_k z^k \right\} \\
&\quad + \frac{m-1}{m}\left\{ z + \sum_{k=2}^{\infty} \binom{k+m-2}{m-1} q^{k-1} a_k z^k \right\}, \\
&= \frac{1}{m} z\left(\Lambda_q^m f(z)\right)' + \frac{m-1}{m} \Lambda_q^m f(z),
\end{aligned}$$

which is equivalent to

$$m\Lambda_q^{m+1} f(z) = z\left(\Lambda_q^m f(z)\right)' + (m-1)\Lambda_q^m f(z).$$

This completes the proof of Proposition 1. □

By using (5)–(7) and Proposition 1, we get the following identity

$$z\left(\mathcal{L}_q^{m+1} f(z)\right)' = m\left(\mathcal{L}_q^m f(z)\right) - (m-1)\left(\mathcal{L}_q^{m+1} f(z)\right). \tag{14}$$

In the following theorems, we will prove several inclusion relationships for analytic function classes, which are associated with $\Lambda_q^m$ and $\mathcal{L}_q^m$.

**Theorem 1.** $\mathcal{S}^*_{\Lambda,m+1,q}(\delta,\gamma) \subset \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$, $\quad 0 \leq \gamma < 1$ *and* $0 < \delta \leq 1$.

**Proof.** Let $f \in \mathcal{S}^*_{\Lambda,m+1,q}(\delta,\gamma)$. We need to show that

$$\left|\arg\left(\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m f(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma < 1, 0 < \delta \le 1.$$

Set

$$\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m f(z)} = \gamma + (1-\gamma)l(z), \tag{15}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$. By using Proposition 1 and (15), we get

$$\frac{\Lambda_q^{m+1} f(z)}{\Lambda_q^m f(z)} = \frac{1}{m}[\gamma + m - 1 + (1-\gamma)l(z)]. \tag{16}$$

Differentiating both sides of (16) logarithmically, we obtain

$$\frac{z\left(\Lambda_q^{m+1} f(z)\right)'}{\Lambda_q^{m+1} f(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{\gamma + m - 1 + (1-\gamma)l(z)}\right]. \tag{17}$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg l(z)| < \frac{\delta\pi}{2} \quad (|z| < |z_o|),$$

and

$$|\arg l(z_o)| = \frac{\delta\pi}{2},$$

where $0 < \delta \le 1$. By applying Lemma 2, we get

$$\frac{z_o l'(z_o)}{l(z_o)} = ib\delta,$$

where

$$b \ge \frac{1}{2}\left(d + \frac{1}{d}\right) \ when \ \arg l(z_o) = \frac{\delta\pi}{2},$$

and

$$b \le -\frac{1}{2}\left(d + \frac{1}{d}\right) when \ \arg l(z_o) = -\frac{\delta\pi}{2},$$

where

$$(l(z_o))^{1/\delta} = \pm id, d > 0.$$

At first, if $\arg l(z_o) = \frac{\delta\pi}{2}$, then

$$\arg\left(\frac{z_0\left(\Lambda_q^{m+1} f(z_0)\right)'}{\Lambda_q^{m+1} f(z_0)} - \gamma\right) = \arg\left[(1-\gamma)l(z_o)\left(1 + \frac{\frac{z_0 l'(z_0)}{l(z_o)}}{\gamma + m - 1 + (1-\gamma)l(z_0)}\right)\right],$$

$$= \arg\left[(1-\gamma)d^\delta e^{i\delta\pi/2}\left(1 + \frac{ib\delta}{\gamma + m - 1 + (1-\gamma)d^\delta e^{i\delta\pi/2}}\right)\right],$$

$$= \frac{\delta\pi}{2} + \arg\left[1 + \frac{ib\delta}{\gamma + m - 1 + (1-\gamma)d^\delta e^{i\delta\pi/2}}\right],$$

$$= \frac{\delta\pi}{2}$$

$$+ \tan^{-1}\left(\frac{b\delta\left[s + td^\delta \cos\left(\frac{\delta\pi}{2}\right)\right]}{s^2 + 2std^\delta \cos\left(\frac{\delta\pi}{2}\right) + t^2 d^{2\delta} + b\delta t d^\delta \sin\left(\frac{\delta\pi}{2}\right)}\right),$$

where $s = \gamma + m - 1$ and $t = 1 - \gamma$. Then

$$\arg\left(\frac{z_0\left(\Lambda_q^{m+1} f(z_0)\right)'}{\Lambda_q^{m+1} f(z_0)} - \gamma\right) \geq \frac{\delta\pi}{2},$$

which obviously contradicts the assumption $f \in \mathcal{S}^*_{\Lambda,m+1,q}(\delta,\gamma)$. Similarly, if $\arg l(z_o) = -\frac{\delta\pi}{2}$, then we get that

$$\arg\left(\frac{z_0\left(\Lambda_q^{m+1} f(z_0)\right)'}{\Lambda_q^{m+1} f(z_0)} - \gamma\right) \leq -\frac{\delta\pi}{2},$$

which also contradicts the same assumption $f \in \mathcal{S}^*_{\Lambda,m+1,q}(\delta,\gamma)$. Therefore, the function $l(z)$ should satisfy that $|\arg l(z)| < \frac{\delta\pi}{2} \quad (z \in U)$. This shows that

$$\left|\arg\left(\frac{z\left(\Lambda_q^{m} f(z)\right)'}{\Lambda_q^{m} f(z)} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow f \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma).$$

Hence, the proof is completed. □

**Theorem 2.** $\mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma) \subset \mathcal{S}^*_{\mathcal{L},m+1,q}(\delta,\gamma), \quad 0 \leq \gamma < 1$ *and* $0 < \delta \leq 1$.

**Proof.** Let $f \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma)$. We need to show that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^{m+1} f(z)\right)'}{\mathcal{L}_q^{m+1} f(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \leq \gamma < 1, 0 < \delta \leq 1.$$

Set

$$\frac{z\left(\mathcal{L}_q^{m+1} f(z)\right)'}{\mathcal{L}_q^{m+1} f(z)} = \gamma + (1-\gamma)l(z), \tag{18}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$. By using (14) and (18), we get

$$\frac{\mathcal{L}_q^{m} f(z)}{\mathcal{L}_q^{m+1} f(z)} = \frac{1}{m}[\gamma + m - 1 + (1-\gamma)l(z)]. \tag{19}$$

Using logarithmic differentiation for (19), we obtain

$$\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m f(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{\gamma + m - 1 + (1-\gamma)l(z)}\right]. \tag{20}$$

The proof is completed similarly to Theorem 1. □

**Theorem 3.** $\mathcal{K}_{\Lambda,m+1,q}(\delta,\gamma) \subset \mathcal{K}_{\Lambda,m,q}(\delta,\gamma), \quad 0 \leq \gamma < 1$ *and* $0 < \delta \leq 1$.

**Proof.** Let $f \in \mathcal{K}_{\Lambda,m+1,q}(\delta,\gamma)$. From (10), we have

$$\Lambda_q^{m+1}(f) \in \mathcal{K}(\delta,\gamma).$$

Applying Lemma 1, we obtain

$$z\left(\Lambda_q^{m+1}(f)\right)' \in \mathcal{S}^*(\delta,\gamma).$$

From (5), we have

$$\Lambda_q^{m+1}\left(zf'\right) \in \mathcal{S}^*(\delta,\gamma),$$

which is equivalent to

$$zf' \in \mathcal{S}^*_{\Lambda,m+1,q}(\delta,\gamma).$$

By using Theorem 1, we get

$$zf' \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma),$$

which is equivelant to

$$\Lambda_q^{m}\left(zf'\right) \in \mathcal{S}^*(\delta,\gamma).$$

From (5) and Lemma 1, we obtain

$$z\left(\Lambda_q^{m}(f)\right)' \in \mathcal{S}^*(\delta,\gamma) \Leftrightarrow \Lambda_q^m(f) \in \mathcal{K}(\delta,\gamma),$$

which means $f \in \mathcal{K}_{\Lambda,m,q}(\delta,\gamma)$. Hence, the proof is completed. □

**Theorem 4.** $\mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma) \subset \mathcal{K}_{\mathcal{L},m+1,q}(\delta,\gamma), \quad 0 \leq \gamma < 1$ *and* $0 < \delta \leq 1$.

**Proof.** Let $f \in \mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma)$. From (11), we have

$$\mathcal{L}_q^m(f) \in \mathcal{K}(\delta,\gamma).$$

Applying Lemma 1, we obtain

$$z\left(\mathcal{L}_q^m(f)\right)' \in \mathcal{S}^*(\delta,\gamma).$$

From (5), we have

$$\mathcal{L}_q^m\left(zf'\right) \in \mathcal{S}^*(\delta,\gamma),$$

which is equivalent to

$$zf' \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma).$$

By using Theorem 2, we get

$$zf' \in \mathcal{S}^*_{\mathcal{L},m+1,q}(\delta,\gamma),$$

which is equivelant to
$$\mathcal{L}_q^{m+1}\left(zf'\right) \in \mathcal{S}^*(\delta,\gamma).$$

From (5) and Lemma 1, we obtain
$$z\left(\mathcal{L}_q^{m+1}(f)\right)' \in \mathcal{S}^*(\delta,\gamma) \Leftrightarrow \mathcal{L}_q^{m+1}(f) \in \mathcal{K}(\delta,\gamma),$$
which means $f \in \mathcal{K}_{\mathcal{L},m+1,q}(\delta,\gamma)$. Hence, the proof is completed. □

**Theorem 5.** $\mathcal{B}_{\Lambda,m+1,q}(\delta,\gamma,\beta,\alpha) \subset \mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha),\ \ 0 \le \gamma,\alpha < 1$ *and* $0 < \delta,\beta \le 1.$

**Proof.** Let $f \in \mathcal{B}_{\Lambda,m+1,q}(\delta,\gamma,\beta,\alpha)$ which is equivalent to
$$\Lambda_q^{m+1}(f) \in \mathcal{B}(\delta,\gamma,\beta,\alpha).$$

Then there exists a function $k \in \mathcal{S}^*(\beta,\alpha)$ such that
$$\left|\arg\left(\frac{z\left(\Lambda_q^{m+1}f(z)\right)'}{k(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma,\alpha < 1, 0 < \delta,\beta \le 1.$$

Letting $k(z) = \Lambda_q^{m+1}g(z)$, we have $g \in \mathcal{S}^*_{\Lambda,m+1,q}(\beta,\alpha)$ and
$$\left|\arg\left(\frac{z\left(\Lambda_q^{m+1}f(z)\right)'}{\Lambda_q^{m+1}g(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}.$$

Now, we set
$$\frac{z\left(\Lambda_q^{m}f(z)\right)'}{\Lambda_q^{m}g(z)} = \gamma + (1-\gamma)l(z), \tag{21}$$
where $l(z) = 1 + c_1z + c_2z^2 + ....$ By using Proposition 1 and (21), we get
$$m\Lambda_q^{m+1}f(z) - (m-1)\Lambda_q^{m}f(z) = \Lambda_q^{m}g(z)[\gamma + (1-\gamma)l(z)]. \tag{22}$$

Now, differentiating (22), we obtain
$$\begin{aligned} mz\left(\Lambda_q^{m+1}f(z)\right)' &= z\left(\Lambda_q^{m}g(z)\right)'[\gamma + (1-\gamma)l(z)] + \left(\Lambda_q^{m}g(z)\right)(1-\gamma)zl'(z) \\ &\quad +(m-1)z\left(\Lambda_q^{m}f(z)\right)'. \end{aligned} \tag{23}$$

If we apply Proposition 1 for the function $g(z)$, then (23) gives
$$\begin{aligned} \frac{z\left(\Lambda_q^{m+1}f(z)\right)'}{\Lambda_q^{m+1}g(z)} &= \left[1 - \frac{m-1}{m}\frac{\Lambda_q^{m}g(z)}{\Lambda_q^{m+1}g(z)}\right][\gamma + (1-\gamma)l(z)] \\ &\quad + \frac{\Lambda_q^{m}g(z)}{\Lambda_q^{m+1}g(z)}\frac{(1-\gamma)zl'(z)}{m} \\ &\quad + \frac{m-1}{m}\frac{z\left(\Lambda_q^{m}f(z)\right)'}{\Lambda_q^{m+1}g(z)}. \end{aligned}$$

By using Proposition 1 and (22), we have

$$\frac{z\left(\Lambda_q^{m+1}f(z)\right)'}{\Lambda_q^{m+1}g(z)} = \gamma + (1-\gamma)l(z) + \frac{\Lambda_q^{m}g(z)}{\Lambda_q^{m+1}g(z)}\,\frac{(1-\gamma)zl'(z)}{m}. \tag{24}$$

Since $k = \Lambda_q^{m+1}(g) \in \mathcal{S}^*(\beta,\alpha)$, an application of Theorem 1, we have

$$\left|\arg\left(\frac{z\left(\Lambda_q^{m}g(z)\right)'}{\Lambda_q^{m}g(z)} - \alpha\right)\right| < \frac{\beta\pi}{2}.$$

Now, let

$$\frac{z\left(\Lambda_q^{m}g(z)\right)'}{\Lambda_q^{m}g(z)} = \alpha + (1-\alpha)L(z), \tag{25}$$

where $L(z) = |L(z)|e^{i\arg L(z)}$, $|\arg(L(z))| < \frac{\beta\pi}{2}$. Therefore, we can rewrite (24) as

$$\frac{z\left(\Lambda_q^{m+1}f(z)\right)'}{\Lambda_q^{m+1}g(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{\alpha+m-1+(1-\alpha)L(z)}\right]. \tag{26}$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg l(z)| < \frac{\delta\pi}{2} \quad (|z| < |z_o|),$$

and

$$|\arg l(z_o)| = \frac{\delta\pi}{2},$$

where $0 < \delta \leq 1$. By applying Lemma 2, we get

$$\frac{z_o l'(z_o)}{l(z_o)} = ib\delta,$$

where

$$b \geq \frac{1}{2}\left(d + \frac{1}{d}\right) \text{ when } \arg l(z_o) = \frac{\delta\pi}{2},$$

and

$$b \leq -\frac{1}{2}\left(d + \frac{1}{d}\right) \text{ when } \arg l(z_o) = -\frac{\delta\pi}{2},$$

where

$$(l(z_o))^{1/\delta} = \pm id, d > 0.$$

Let $\alpha+m-1+(1-\alpha)L(z_0)=\rho e^{i\frac{\theta\pi}{2}}$ where $\alpha+m-1<\rho<\infty$ and $-\beta\leq\theta\leq\beta$. At first, if $\arg l(z_o)=\frac{\delta\pi}{2}$, then

$$\begin{aligned}
\arg\left(\frac{z_0\left(\Lambda_q^{m+1}f(z_0)\right)'}{\Lambda_q^{m+1}g(z_0)}-\gamma\right) &= \arg\left[(1-\gamma)l(z_o)\left(1+\frac{\frac{z_0 l'(z_0)}{l(z_o)}}{\alpha+m-1+(1-\alpha)L(z_0)}\right)\right],\\
&= \arg\left[(1-\gamma)d^{\delta}e^{i\delta\pi/2}\left(1+\frac{ib\delta}{\rho e^{i\theta\pi/2}}\right)\right],\\
&= \frac{\delta\pi}{2}+\arg\left[1+\frac{b\delta}{\rho}e^{i\frac{\pi}{2}(1-\theta)}\right],\\
&= \frac{\delta\pi}{2}+\tan^{-1}H(\theta),
\end{aligned}$$

where

$$\begin{aligned}
H(\theta) &= \frac{b\delta\cos\frac{\theta\pi}{2}}{\rho+b\delta\sin\frac{\theta\pi}{2}},\\
&= \frac{n_1\cos\frac{\theta\pi}{2}}{1+n_1\sin\frac{\theta\pi}{2}},
\end{aligned}$$

$-1\leq\theta\leq 1$ and $0<n_1=\frac{b\delta}{\rho}<1$. We note that $H(\theta)$ is a decreasing function in $\left[\frac{2}{\pi}\sin^{-1}(-n_1),1\right]$ and an increasing function in $\left[-1,\frac{2}{\pi}\sin^{-1}(-n_1)\right]$. Therefore, $H(\theta)\geq 0$ on $[-1,1]$ and

$$\arg\left(\frac{z_0\left(\Lambda_q^{m+1}f(z_0)\right)'}{\Lambda_q^{m+1}g(z_0)}-\gamma\right)\geq\frac{\delta\pi}{2},$$

which obviously contradicts the assumption $f\in\mathcal{B}_{\Lambda,m+1,q}(\delta,\gamma,\beta,\alpha)$. Similarly, if $\arg l(z_o)=-\frac{\delta\pi}{2}$, we get

$$\arg\left(\frac{z_0\left(\Lambda_q^{m+1}f(z_0)\right)'}{\Lambda_q^{m+1}g(z_0)}-\gamma\right)=-\frac{\delta\pi}{2}+\tan^{-1}H(\theta),$$

where

$$\begin{aligned}
H(\theta) &= \frac{b\delta\cos\frac{\theta\pi}{2}}{\rho+b\delta\sin\frac{\theta\pi}{2}},\\
&= \frac{n_2\cos\frac{\theta\pi}{2}}{1+n_2\sin\frac{\theta\pi}{2}},
\end{aligned}$$

$-1\leq\theta\leq 1$ and $-1<n_2=\frac{b\delta}{\rho}<0$. We note that $H(\theta)$ is an increasing function in $\left[\frac{2}{\pi}\sin^{-1}(-n_2),1\right]$ and a decreasing function in $\left[-1,\frac{2}{\pi}\sin^{-1}(-n_2)\right]$. Therefore, $H(\theta)\leq 0$ on $[-1,1]$ and

$$\arg\left(\frac{z_0\left(\Lambda_q^{m+1}f(z_0)\right)'}{\Lambda_q^{m+1}g(z_0)}-\gamma\right)\leq-\frac{\delta\pi}{2},$$

which also contradicts the same assumption $f \in \mathcal{B}_{\Lambda,m+1,q}(\delta,\gamma,\beta,\alpha)$. Therefore, the function $l(z)$ should satisfy that $|\arg l(z)| < \frac{\delta\pi}{2}$ $(z \in U)$. This shows that

$$\left|\arg\left(\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m g(z)} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow f \in \mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha).$$

Hence, the proof is completed. □

**Theorem 6.** *$\mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha) \subset \mathcal{B}_{\mathcal{L},m+1,q}(\delta,\gamma,\beta,\alpha)$, $0 \le \gamma,\alpha < 1$ and $0 < \delta,\beta \le 1$.*

**Proof.** Let $f \in \mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha)$ which is equivalent to

$$\mathcal{L}_q^m(f) \in \mathcal{B}(\delta,\gamma,\beta,\alpha).$$

Then there exists a function $k \in \mathcal{S}^*(\beta,\alpha)$ such that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{k(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma,\alpha < 1, 0 < \delta,\beta \le 1.$$

Letting $k(z) = \mathcal{L}_q^m g(z)$, we have $g \in \mathcal{S}^*_{\mathcal{L},m,q}(\beta,\alpha)$ and

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}.$$

Now,we set

$$\frac{z\left(\mathcal{L}_q^{m+1} f(z)\right)'}{\mathcal{L}_q^{m+1} g(z)} = \gamma + (1-\gamma)l(z), \tag{27}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + ....$ By using (14) and (27), we get

$$m\mathcal{L}_q^m f(z) - (m-1)\mathcal{L}_q^{m+1} f(z) = \mathcal{L}_q^{m+1} g(z)[\gamma + (1-\gamma)l(z)]. \tag{28}$$

Now, differentiating (28), we obtain

$$\begin{aligned} mz\left(\mathcal{L}_q^m f(z)\right)' &= z\left(\mathcal{L}_q^{m+1} g(z)\right)'[\gamma + (1-\gamma)l(z)] + \left(\mathcal{L}_q^{m+1} g(z)\right)(1-\beta)zl'(z) \\ &\quad +(m-1)z\left(\mathcal{L}_q^{m+1} f(z)\right)'. \end{aligned} \tag{29}$$

If we apply (14) for the function $g(z)$, then (29) gives

$$\begin{aligned} \frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} &= \left[1 - \frac{(m-1)}{m}\frac{\mathcal{L}_q^{m+1} g(z)}{\mathcal{L}_q^m g(z)}\right][\gamma + (1-\gamma)l(z)] \\ &\quad + \frac{\mathcal{L}_q^{m+1} g(z)}{\mathcal{L}_q^m g(z)}\frac{(1-\gamma)zl'(z)}{m} \\ &\quad + \frac{m-1}{m}\frac{z\left(\mathcal{L}_q^{m+1} f(z)\right)'}{\mathcal{L}_q^m g(z)}. \end{aligned}$$

By using (14) and (28), we have

$$\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} = \gamma + (1-\gamma)l(z) + \frac{(1-\gamma)zl'(z)}{m}\frac{\mathcal{L}_q^{m+1}g(z)}{\mathcal{L}_q^m g(z)}. \tag{30}$$

Since $k = \mathcal{L}_q^m(g) \in \mathcal{S}^*(\beta,\alpha)$ and by Theorem 2, we get

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^{m+1}g(z)\right)'}{\mathcal{L}_q^{m+1}g(z)} - \alpha\right)\right| < \frac{\beta\pi}{2}.$$

Now, let

$$\frac{z\left(\mathcal{L}_q^{m+1}g(z)\right)'}{\mathcal{L}_q^{m+1}g(z)} = \alpha + (1-\alpha)L(z), \tag{31}$$

where $L(z) = |L(z)|e^{i\arg L(z)}$, $|\arg(L(z))| < \frac{\beta\pi}{2}$. Therefore, (30) can be written as

$$\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{\alpha + m - 1 + (1-\alpha)L(z)}\right]. \tag{32}$$

The rest of the proof is the same as in Theorem 5. Then we obtain that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^{m+1}f(z)\right)'}{\mathcal{L}_q^{m+1}g(z)} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow f \in \mathcal{B}_{\mathcal{L},m+1,q}(\delta,\gamma,\beta,\alpha).$$

Hence, the proof is completed. □

## 3. Integral Operator

In this section, we will prove several integral-preserving properties of analytic function classes which are introduced above.

Suppose that $f \in \mathcal{A}$ and $c > -1$. For $z \in U$, the Bernardi operator [17] is defined as

$$J_c(f(z)) = \frac{c+1}{z}\int_0^z t^{c-1}f(t)dt. \tag{33}$$

when $c = 1$; the integral operator $J_1$ was introduced by Libera [18]. From (33), we can easily get that

$$z\left(\Lambda_q^m J_c(f(z))\right)' = (c+1)\Lambda_q^m f(z) - c\Lambda_q^m J_c(f(z)), \tag{34}$$

and

$$z\left(\mathcal{L}_q^m J_c(f(z))\right)' = (c+1)\mathcal{L}_q^m f(z) - c\mathcal{L}_q^m J_c(f(z)). \tag{35}$$

**Theorem 7.** *For $c > \gamma$, let $0 \le \gamma < 1$ and $0 < \delta \le 1$. If $f \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$, then $J_c(f) \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$.*

**Proof.** Let $f \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$. We need to show that

$$\left|\arg\left(\frac{z\left(\Lambda_q^m J_c(f(z))\right)'}{\Lambda_q^m J_c(f(z))} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma < 1, 0 < \delta \le 1 \text{ and } c > \gamma.$$

Set

$$\frac{z\left(\Lambda_q^m J_c(f(z))\right)'}{\Lambda_q^m J_c(f(z))} = \gamma + (1-\gamma)l(z), \tag{36}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$. By using (34) and (36), we get

$$\frac{\Lambda_q^m f(z)}{\Lambda_q^m J_c(f(z))} = \frac{1}{c+1}[(c+\gamma) + (1-\gamma)l(z)]. \tag{37}$$

Differentiating both sides of (37) logarithmically, we obtain

$$\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m f(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{c+\gamma+(1-\gamma)l(z)}\right]. \tag{38}$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg l(z)| < \frac{\delta\pi}{2} \quad (|z| < |z_o|),$$

and

$$|\arg l(z_o)| = \frac{\delta\pi}{2},$$

where $0 < \delta \leq 1$. By applying Lemma 2, we get

$$\frac{z_o l'(z_o)}{l(z_o)} = ib\delta,$$

where

$$b \geq \frac{1}{2}\left(d + \frac{1}{d}\right) \ when \ \arg l(z_o) = \frac{\delta\pi}{2},$$

and

$$b \leq -\frac{1}{2}\left(d + \frac{1}{d}\right) when \ \arg l(z_o) = -\frac{\delta\pi}{2},$$

where

$$(l(z_o))^{1/\delta} = \pm id, d > 0.$$

At first, if $\arg l(z_o) = \frac{\delta\pi}{2}$, then

$$\begin{aligned}
\arg\left(\frac{z_0\left(\Lambda_q^m f(z_0)\right)'}{\Lambda_q^m f(z_0)} - \gamma\right) &= \arg\left[(1-\gamma)l(z_o)\left(1 + \frac{\frac{z_0 l'(z_0)}{l(z_o)}}{c+\gamma+(1-\gamma)l(z_0)}\right)\right], \\
&= \arg\left[(1-\gamma)d^\delta e^{i\delta\pi/2}\left(1 + \frac{ib\delta}{c+\gamma+(1-\gamma)d^\delta e^{i\delta\pi/2}}\right)\right], \\
&= \frac{\delta\pi}{2} + \arg\left[1 + \frac{ib\delta}{c+\gamma+(1-\gamma)d^\delta e^{i\delta\pi/2}}\right], \\
&= \frac{\delta\pi}{2} \\
&+ \tan^{-1}\left(\frac{b\delta\left[k + td^\delta\cos\left(\frac{\delta\pi}{2}\right)\right]}{k^2 + 2ktd^\delta\cos\left(\frac{\delta\pi}{2}\right) + t^2 d^{2\delta} + b\delta t d^\delta \sin\left(\frac{\delta\pi}{2}\right)}\right).
\end{aligned}$$

where $k = c + \gamma$ and $t = 1 - \gamma$. Then

$$\arg\left(\frac{z_0\left(\Lambda_q^m f(z_0)\right)'}{\Lambda_q^m f(z_0)} - \gamma\right) \geq \frac{\delta\pi}{2},$$

which obviously contradicts the assumption $f \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$. Similarly, if $\arg l(z_o) = -\frac{\delta\pi}{2}$, then we get that

$$\arg\left(\frac{z_0\left(\Lambda_q^m f(z_0)\right)'}{\Lambda_q^m f(z_0)} - \gamma\right) \leq -\frac{\delta\pi}{2},$$

which also contradicts the same assumption $f \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma)$. Therefore, the function $l(z)$ should satisfy that $|\arg l(z)| < \frac{\delta\pi}{2} \quad (z \in U)$. This shows that

$$\left|\arg\left(\frac{z\left(\Lambda_q^m J_c(f(z))\right)'}{\Lambda_q^m J_c(f(z))} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow J_c(f) \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma).$$

Hence, the proof is completed. □

**Theorem 8.** *For $c > \gamma$, let $0 \leq \gamma < 1$ and $0 < \delta \leq 1$. If $f \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma)$, then $J_c(f) \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma)$.*

**Proof.** Let $f \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma)$. We need to show that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m J_c(f(z))\right)'}{\mathcal{L}_q^m J_c(f(z))} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \leq \gamma < 1, 0 < \delta \leq 1 \text{ and } c > \gamma.$$

Set

$$\frac{z\left(\mathcal{L}_q^m J_c(f(z))\right)'}{\mathcal{L}_q^m J_c(f(z))} = \gamma + (1-\gamma)l(z), \tag{39}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$. By using (35) and (39), we get

$$\frac{\mathcal{L}_q^m f(z)}{\mathcal{L}_q^m J_c(f(z))} = \frac{1}{c+1}\{(c+\gamma) + (1-\gamma)l(z)\}. \tag{40}$$

Using logarithmic differentiation for (40), we get

$$\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m f(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{c+\gamma+(1-\gamma)l(z)}\right]. \tag{41}$$

The rest of the proof is the same as in Theorem 7. Then we obtain that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m J_c(f(z))\right)'}{\mathcal{L}_q^m J_c(f(z))} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow J_c(f) \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma).$$

Hence, the proof is completed. □

**Theorem 9.** *For $c > \gamma$, let $0 \leq \gamma < 1$ and $0 < \delta \leq 1$. If $f \in \mathcal{K}_{\Lambda,m,q}(\delta,\gamma)$, then $J_c(f) \in \mathcal{K}_{\Lambda,m,q}(\delta,\gamma)$.*

**Proof.** Let $f \in \mathcal{K}_{\Lambda,m,q}(\delta,\gamma)$. From (10), we have

$$\Lambda_q^m(f) \in \mathcal{K}(\delta,\gamma).$$

Applying Lemma 1, we obtain

$$z\left(\Lambda_q^m(f)\right)' \in \mathcal{S}^*(\delta,\gamma).$$

(5) gives

$$\Lambda_q^m\left(zf'\right) \in \mathcal{S}^*(\delta,\gamma),$$

which is equivalent to

$$zf' \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma).$$

An application of Theorem 7 yields

$$J_c\left(zf'\right) \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma),$$

or

$$z(J_c(f))' \in \mathcal{S}^*_{\Lambda,m,q}(\delta,\gamma).$$

Applying again Lemma 1, we obtain

$$J_c(f) \in \mathcal{K}_{\Lambda,m,q}(\delta,\gamma).$$

Hence, the proof is completed. □

**Theorem 10.** *For $c > \gamma$, let $0 \leq \gamma < 1$ and $0 < \delta \leq 1$. If $f \in \mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma)$, then $J_c(f) \in \mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma)$.*

**Proof.** Let $f \in \mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma)$. From (11), we have

$$\mathcal{L}_q^m(f) \in \mathcal{K}(\delta,\gamma).$$

Applying Lemma 1, we obtain

$$z\left(\mathcal{L}_q^m(f)\right)' \in \mathcal{S}^*(\delta,\gamma).$$

(5) gives

$$\mathcal{L}_q^m\left(zf'\right) \in \mathcal{S}^*(\delta,\gamma),$$

which is equivalent to

$$zf' \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma).$$

An application of Theorem 8 yields

$$J_c\left(zf'\right) \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma),$$

or

$$z(J_c(f))' \in \mathcal{S}^*_{\mathcal{L},m,q}(\delta,\gamma).$$

Applying again Lemma 1, we obtain

$$J_c(f) \epsilon \mathcal{K}_{\mathcal{L},m,q}(\delta,\gamma).$$

Hence, the proof is completed. □

**Theorem 11.** *For $c > \alpha$, let $0 \le \gamma, \alpha < 1$ and $0 < \delta, \beta \le 1$. If $f \in \mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha)$, then $J_c(f) \in \mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha)$.*

**Proof.** Let $f \in \mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha)$ which is equivalent to

$$\Lambda_q^m(f) \in \mathcal{B}(\delta,\gamma,\beta,\alpha).$$

Then there exists a function $k \in \mathcal{S}^*(\beta,\alpha)$ such that

$$\left|\arg\left(\frac{z\left(\Lambda_q^m f(z)\right)'}{k(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \le \gamma, \alpha < 1, 0 < \delta, \beta \le 1.$$

Letting $k(z) = \Lambda_q^m g(z)$ where the function $g \in \mathcal{S}^*_{\Lambda,m,q}(\beta,\alpha)$ and

$$\left|\arg\left(\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m g(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}.$$

Now, put

$$\frac{z\left(\Lambda_q^m J_c(f(z))\right)'}{\Lambda_q^m J_c(g(z))} = \gamma + (1-\gamma)l(z), \tag{42}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$. By using (34) and (42), we get

$$(c+1)\Lambda_q^m f(z) - c\Lambda_q^m J_c(f(z)) = \Lambda_q^m J_c(g(z))[\gamma + (1-\gamma)l(z)]. \tag{43}$$

By differentiating (43), we obtain

$$\begin{aligned}(c+1)z\left(\Lambda_q^m f(z)\right)' &= z\left(\Lambda_q^m J_c(g(z))\right)'[\gamma + (1-\gamma)l(z)] \\ &\quad + \left(\Lambda_q^m J_c(g(z))\right)(1-\gamma)zl'(z) \\ &\quad + cz\left(\Lambda_q^m J_c(f(z))\right)'.\end{aligned} \tag{44}$$

If we apply (34) for the function $g(z)$, then (44) gives

$$\begin{aligned}\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m g(z)} &= \left[1 - \frac{c}{c+1}\frac{\Lambda_q^m J_c(g(z))}{\Lambda_q^m g(z)}\right][\gamma + (1-\gamma)l(z)] \\ &\quad + \frac{\Lambda_q^m J_c(g(z))}{\Lambda_q^m g(z)}\frac{(1-\gamma)zl'(z)}{c+1} \\ &\quad + \frac{c}{c+1}\frac{z\left(\Lambda_q^m J_c(f(z))\right)'}{\Lambda_q^m g(z)}.\end{aligned}$$

By using (34) and (43), we have

$$\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m g(z)} = \gamma + (1-\gamma)l(z) + \frac{\Lambda_q^m J_c(g(z))}{\Lambda_q^m g(z)}\frac{(1-\gamma)zl'(z)}{c+1}. \tag{45}$$

Since $k = \Lambda_q^m(g) \in \mathcal{S}^*(\beta,\alpha)$, by applying Theorem 7, we have

$$J_c(g) \in \mathcal{S}^*_{\Lambda,m,q}(\beta,\alpha).$$

If we let

$$\frac{z\left(\Lambda_q^m J_c(g(z))\right)'}{\Lambda_q^m J_c(g(z))} = \alpha + (1-\alpha)L(z), \tag{46}$$

where $L(z) = |L(z)|e^{i\arg L(z)}$, $|\arg(L(z))| < \frac{\beta\pi}{2}$, then we can rewrite (45) as

$$\frac{z\left(\Lambda_q^m f(z)\right)'}{\Lambda_q^m g(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{c+\alpha+(1-\alpha)L(z)}\right]. \tag{47}$$

Suppose that there exists a point $z_0 \in U$ such that

$$|\arg l(z)| < \frac{\delta\pi}{2} \quad (|z| < |z_o|),$$

and

$$|\arg l(z_o)| = \frac{\delta\pi}{2},$$

where $0 < \delta \leq 1$. By applying Lemma 2, we get

$$\frac{z_o l'(z_o)}{l(z_o)} = ib\delta,$$

where

$$b \geq \frac{1}{2}\left(d + \frac{1}{d}\right) \ \textit{when} \ \arg l(z_o) = \frac{\delta\pi}{2},$$

and

$$b \leq -\frac{1}{2}\left(d + \frac{1}{d}\right) \textit{when} \ \arg l(z_o) = -\frac{\delta\pi}{2},$$

where

$$(l(z_o))^{1/\delta} = \pm id, d > 0.$$

Let $c + \alpha + (1-\alpha)L(z_0) = \rho e^{i\theta\pi/2}$ where $c + \alpha < \rho < \infty$ and $-\beta \leq \theta \leq \beta$. The rest of the proof is the same as in Theorem 5. Then we obtain that

$$\left|\arg\left(\frac{z\left(\Lambda_q^m J_c(f(z))\right)'}{\Lambda_q^m J_c(g(z))} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow J_c(f) \in \mathcal{B}_{\Lambda,m,q}(\delta,\gamma,\beta,\alpha).$$

Hence, the proof is completed. □

**Theorem 12.** *For $c > \alpha$, let $0 \leq \gamma, \alpha < 1$ and $0 < \delta, \beta \leq 1$. If $f \in \mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha)$, then $J_c(f) \in \mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha)$.*

**Proof.** Let $f \in \mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha)$ which is equivalent to

$$\mathcal{L}_q^m(f) \in \mathcal{B}(\delta,\gamma,\beta,\alpha).$$

Then there exists a function $k \in \mathcal{S}^*(\beta,\alpha)$ such that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{k(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}, \quad 0 \leq \gamma, \alpha < 1, 0 < \delta, \beta \leq 1.$$

Letting $k(z) = \mathcal{L}_q^m g(z)$ where the function $g \in \mathcal{S}^*_{\mathcal{L},m,q}(\beta,\alpha)$ and

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} - \gamma\right)\right| < \frac{\delta\pi}{2}.$$

Now, put

$$\frac{z\left(\mathcal{L}_q^m J_c(f(z))\right)'}{\mathcal{L}_q^m J_c(g(z))} = \gamma + (1-\gamma)l(z), \tag{48}$$

where $l(z) = 1 + c_1 z + c_2 z^2 + \ldots$. By using (35) and (48), we get

$$(c+1)\mathcal{L}_q^m f(z) - c\mathcal{L}_q^m J_c(f(z)) = \mathcal{L}_q^m J_c(g(z))[\gamma + (1-\gamma)l(z)]. \tag{49}$$

By differentiating (49), we obtain

$$\begin{aligned}(c+1)z\left(\mathcal{L}_q^m f(z)\right)' &= z\left(\mathcal{L}_q^m J_c(g(z))\right)'[\gamma + (1-\gamma)l(z)] \\ &\quad + \left(\mathcal{L}_q^m J_c(g(z))\right)(1-\gamma)zl'(z) \\ &\quad + cz\left(\mathcal{L}_q^m J_c(f(z))\right)'. \end{aligned} \tag{50}$$

If we apply (35) for the function $g(z)$, then (50) gives

$$\begin{aligned}\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} &= \left[1 - \frac{c}{c+1}\frac{\mathcal{L}_q^m J_c(g(z))}{\mathcal{L}_q^m g(z)}\right][\gamma + (1-\gamma)l(z)] \\ &\quad + \frac{\mathcal{L}_q^m J_c(g(z))}{\mathcal{L}_q^m g(z)}\frac{(1-\gamma)zl'(z)}{c+1} \\ &\quad + \frac{c}{c+1}\frac{z\left(\mathcal{L}_q^m J_c(f(z))\right)'}{\mathcal{L}_q^m g(z)}. \end{aligned}$$

By using (35) and (49), we have

$$\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} = \gamma + (1-\gamma)l(z) + \frac{\mathcal{L}_q^m J_c(g(z))}{\mathcal{L}_q^m g(z)}\frac{(1-\gamma)zl'(z)}{c+1}. \tag{51}$$

Since $k = \mathcal{L}_q^m(g) \in \mathcal{S}^*(\beta,\alpha)$, by applying Theorem 7, we have

$$J_c(g) \in \mathcal{S}^*_{\mathcal{L},m,q}(\beta,\alpha).$$

If we let

$$\frac{z\left(\mathcal{L}_q^m J_c(g(z))\right)'}{\mathcal{L}_q^m J_c(g(z))} = \alpha + (1-\alpha)L(z), \tag{52}$$

where $L(z) = |L(z)|e^{i\arg L(z)}$, $|\arg(L(z))| < \frac{\beta\pi}{2}$, then we can rewrite (51) as

$$\frac{z\left(\mathcal{L}_q^m f(z)\right)'}{\mathcal{L}_q^m g(z)} - \gamma = (1-\gamma)\left[l(z) + \frac{zl'(z)}{c+\alpha+(1-\alpha)L(z)}\right]. \tag{53}$$

The rest of the proof is the same as in Theorem 5. Then we obtain that

$$\left|\arg\left(\frac{z\left(\mathcal{L}_q^m J_c(f(z))\right)'}{\mathcal{L}_q^m J_c(g(z))} - \gamma\right)\right| < \frac{\delta\pi}{2} \Leftrightarrow J_c(f) \in \mathcal{B}_{\mathcal{L},m,q}(\delta,\gamma,\beta,\alpha).$$

Hence, the proof is completed. □

## 4. Conclusions

Recently, the Pascal distribution has attracted the attention of many researchers in the field of geometric function theory. This distribution was used by various authors; see [8–13] to consider the properties of some famous subclasses of analytic functions. In the present paper, using the normalized Pascal operator $\Lambda_q^m$ and its dual $\mathcal{L}_q^m$, we introduced new subclasses of analytic functions. Due to the earlier works on different operators such as the Ruscheweyh diffrential operator [15] and Noor integral operator [19], we find inclusion relations of certain new subclasses of analytic functions in the open symmetric unit disc $U$ that are associated with the Pascal distribution. Furthermore, we studied the integral-preserving properties for these subclasses. Making use of the definition of Pascal operators could inspire researchers to create new different subclasses of analytic functions.

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