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Fredholm Type Integral Equation in Controlled Rectangular Metric-like Spaces

Salma Haque, Fatima Azmi and Nabil Mlaiki *

Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia; shaque@psu.edu.sa (S.H.); fazmi@psu.edu.sa (F.A.)
* Correspondence: nmlaiki@psu.edu.sa or nmlaiki2012@gmail.com

Abstract: In this article, we present an extension of the controlled rectangular b-metric spaces, so-called controlled rectangular metric-like spaces, where we keep the symmetry condition and we only change the condition $[D(s,r) = 0 \Leftrightarrow s = r]$ to $[D(s,r) = 0 \Rightarrow s = r]$, which means we may have a non-zero self distance; also, $D(s,s)$ is not necessarily less than $D(s,r)$. This new type of metric space is a generalization of controlled rectangular b-metric spaces and partial rectangular metric spaces.

Keywords: controlled rectangular metric-like spaces; fixed point; integral equations

MSC: 47H10; 54H25

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Citation: Haque, S.; Azmi, F.; Mlaiki, N. Fredholm Type Integral Equation in Controlled Rectangular Metric-like Spaces. *Symmetry* **2022**, *14*, 991. https://doi.org/10.3390/sym14050991

Academic Editor: Boris Malomed

Received: 12 April 2022
Accepted: 10 May 2022
Published: 12 May 2022

1. Introduction

The uniqueness of a fixed-point theory for self-contractive mapping, which was introduced by Banach in 1922 [1], opened a new area of research in various fields. It has become an interesting domain and an exciting field of mathematical research see [2–4]; in fact, it has become an important tool now in many fields of mathematics , such as variational inequalities, approximation theory, linear inequalities nonlinear analysis, differential, and integral equations; for more details on these type of applications, see [5–7]. Its applications appear in mathematical sciences, super fractals, and more recently, in discrete dynamics. Kamran et al. [8] introduced extended b-metric spaces, which is a generalization of metric spaces and b-metric spaces [9]. Then, the generalization of these metrics appeared in the form of a controlled metric [10] and double controlled metric spaces [11]. Further, Branciari [12], in 2000, introduced rectangular metric spaces. Then, in 2015, George et al. in [13], generalized rectangular metric spaces to rectangular b-metric spaces. In 2020, Mlaiki et al., in [14], generalized the rectangular b-metric spaces by introducing the controlled rectangular metric spaces. Inspired by the work of Matthews in [15], where he introduced the notion of partial metric spaces, which is basically assuming that the self distance is not necessarily zero; however, we have $D(s,s) \leq D(s,r)$. Shukla, in [16], introduced the concept of partial rectangular metric spaces, which is basically the exact same work as Matthews, except it is in rectangular metric spaces. In this article, we present a generalization of controlled rectangular b-metric spaces and partial rectangular metric spaces, so-called controlled rectangular metric-like spaces. In the next section, we present some preliminaries and concepts needed later; while in the next section, we prove our main result. In the last section, we present an application of our results.

2. Preliminaries

We present some preliminary definitions of rectangular b-metric spaces, and controlled rectangular metric spaces, before presenting our new notion of a controlled rectangular metric-like space.

Definition 1 ([12]). *(Branciari metric spaces) Let $X \neq \phi$. A mapping $D : X^2 \to [0, \infty)$ is called a rectangular metric on X if for any $x, y \in X$ and all $u, \neq v \in X \setminus \{x, y\}$, having the following conditions:*
(R_1) $x = y \leftrightarrow D(x, y) = 0$;
(R_2) $D(x, y) = D(y, x)$;
(R_3) $D(x, y) \leq D(x, u) + D(u, v) + D(v, y)$.
In this case, then (X, D) is a rectangular metric space.

As a generalization of rectangular metric spaces, rectangular b-metric spaces were introduced in [13], where the triangle inequality has a constant $a > 1$.

Definition 2 ([13]). *(Rectangular b-metric spaces) Let $X \neq \phi$. A mapping $D : X^2 \to [0, \infty)$ is known as rectangular b-metric on X for a constant $a \geq 1$ such that any $xy \in X$ and points $u \neq, v \in X \setminus \{x, y\}$, which has the following conditions:*
(R_{b1}) $x = y \leftrightarrow D(x, y) = 0$;
(R_{b2}) $D(x, y) = D(y, x)$;
(R_{b3}) $D(x, y) \leq a[D(x, u) + D(u, v) + D(v, y)]$.
In this case, the pair (X, D) is called a rectangular b-metric space.

In 2020, a new extension to the rectangular metric spaces was defined as follows.

Definition 3 ([14]). *Let X be a non empty set, a function $\theta : X^4 \to [1, \infty)$ and $D : X^2 \to [0, \infty)$. We say that (X, D) is a controlled rectangular b-metric space if all distinct $x, y, u, v \in X$ we have:*

1. $D(x, y) = 0$ *if and only if* $x = y$;
2. $D(x, y) = D(y, x)$;
3. $D(x, y) \leq \theta(x, y, u, v)[D(x, u) + D(u, v) + D(v, y)]$.

In this manuscript, we define controlled rectangular metric-like spaces as follows;

Definition 4. *Let X be a non empty set, a function $\theta : X^4 \to [1, \infty)$ and $D : X^2 \to [0, \infty)$. We say that (X, D) is a controlled rectangular metric-like space if $x \neq y \neq u \neq v \in X$ having the functions:*

1. $D(x, y) = 0 \Rightarrow x = y$;
2. $D(x, y) = D(y, x)$; *(symmetric condition)*
3. $D(x, y) \leq \theta(x, y, u, v)[D(x, u) + D(u, v) + D(v, y)]$.

Remark 1. *Note that, in Definition 4, we are assuming that the space is symmetric. However, in the case where the symmetric condition is not satisfied, we will have a different space with a totally different topology.*

Next, we present two examples of controlled rectangular metric-like spaces that are not controlled rectangular b-metric spaces.

Example 1. *Let $X = [0, \infty)$ and $p : [0, \infty) \times [0, \infty) \to (1, \infty)$. Define $D : X^2 \to [0, \infty)$ by*

$$D(x, y) = (x + y)^{p(x,y)} \text{ for all } x, y \in X$$

Note that (X, D) is a controlled rectangular metric-like space with

$$\theta(x, y, u, v) = 2^{p(\max\{x,y\}, \max\{u,v\}) - 1}.$$

For all $0 < y < x$ we have

$$D(x, x) = (x + x)^{p(x,x)} > (x + y)^{p(x,y)} = D(x, y).$$

Thus, (X, D) is not a controlled rectangular b-metric space nor a partial rectangular metric space.

Example 2. *Let $X = Y \cup Z$ where $Y = \{\frac{1}{m} \mid m$ is a natural number$\}$ and $Z \subset \mathbb{R}^+$. We define $D : X^2 \to [0, \infty)$ by*

$$D(x,y) = \begin{cases} 0, & \text{if and only if } x = y \\ 2\beta, & \text{if } x, y \in Y \\ \frac{1}{2}, & x = y = 1 \\ \frac{\beta}{2}, & \text{otherwise,} \end{cases}$$

where β is a constant bigger than 0. Now, define $\theta : X^4 \to [1, \infty)$ by $\theta(x, y, u, v) = \max\{x, y, u, v\} + 2\beta$. It is quite easy to check that (X, D) is a controlled rectangular metric-like space. However, (X, D) is not a controlled rectangular metric type space nor a partial rectangular metric space, for example $D(1, 1) = \frac{1}{2} \neq 0$.

Remark 2. *Notice that by Example 1, not every controlled rectangular metric-like space is a controlled rectangular b-metric space. On the other hand, every controlled rectangular b-metric space and every partial rectangular metric space is a controlled rectangular metric-like space.*

Next, we present the topology of controlled rectangular metric-like spaces.

Definition 5. *Let (X, D) be controlled rectangular metric-like space,*

1. *A sequence $\{x_l\}$ in a controlled rectangular metric-like space (X, D) is called $D-$convergent, if there exists $x \in X$ such that $\lim_{l\to\infty} D(x_l, \nu) = D(\nu, \nu)$.*
2. *A sequence $\{x_l\}$ is called $D-$Cauchy if and only if $\lim_{l,m\to\infty} D(x_l, x_m)$ exists and finite.*
3. *A controlled rectangular metric-like space (X, D) is called D-complete if for every $D-$Cauchy sequence $\{x_n\}$ in X, if there exists $\nu \in X$, such that*

$$\lim_{l\to\infty} D(x_l, \nu) = \lim_{l,m\to\infty} D(x_l, x_m) = D(\nu, \nu).$$

4. *For $a \in X$, an open ball in a controlled rectangular metric-like space (X, D) define by*

$$B_D(a, \eta) = \{b \in X \mid |D(a, b) - D(a, a)| < \eta\}.$$

Next, we define continuity in controlled rectangular metric-like spaces.

Definition 6. *A self-mapping function ζ in F is said continuous at $x \in F$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\zeta(B(s, \delta)) \subseteq B(\zeta(s, \varepsilon))$, that is $\lim_{n\to\infty} \zeta(x_n) = \zeta(\lim_{n\to\infty} x_n)$.*

In the next section, we present our main results by proving the existence of a fixed point for mappings that satisfies different types of contractions in controlled rectangular metric-like spaces.

3. Main Results

Theorem 1. *Let (X, D) be a complete controlled rectangular metric-like space, and T is continuous and maps to itself on X. If there exists $0 < k < 1$, such that $D(Tx, Ty) \leq kD(x, y)$ and*

$$\sup_{m>1} \lim_{l\to\infty} \theta(x_l, x_{l+1}, x_{l+2}, x_m) \leq \frac{1}{k},$$

then in X there is a unique fixed point of T.

Proof. Let $x_0 \in X$ and it is a sequence $\{x_l\}$ as follows $x_1 = Tx_0, x_2 = T^2x_0, \cdots, x_l = T^l x_0, \cdots$ Now, by the hypothesis of the theorem, we have

$$D(x_l, x_{l+1}) \leq kD(x_{l-1}, x_l) \leq k^2 D(x_{l-2}, x_{l-1}) \leq \cdots \leq k^l D(x_0, x_1).$$

Note that taking the limit of the above inequality as $n \to \infty$ we deduce that $D(x_l, x_{l+1}) \to 0$ as $l \to \infty$. Denote by $D_i = D(x_{l+i}, x_{l+i+1})$. For all $l \geq 1$, we have two cases.

Case 1: Let $x_l = x_m$ for some integers $l \neq m$. Therefore, if for $m > l$ we have $T^{m-l}(x_l) = x_l$. Choose $y = x_l$ and $p = m - l$. Then $T^p y = y$, and that is, y is a periodic point of T. Thus, $D(y, Ty) = D(T^p y, T^{p+1} y) \leq k^p D(y, Ty)$. Since $k \in (0,1)$, we obtain $D(y, Ty) = 0$, so $y = Ty$, therefore, T has a fixed point y.

Case 2: Suppose $T^l x \neq T^m x$ for all integers $l \neq m$. Let $l < m \in N$, and to show that $\{x_l\}$ is a $D-$Cauchy sequence, we considered two subcases:
Subcase 1: Assume that $m = l + 2p + 1$. By property (3) of the controlled rectangular-like metric spaces, we have,

$$
\begin{aligned}
D(x_l, x_{l+2p+1}) &\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})[D(x_l, x_{l+1}) + D(x_{l+1}, x_{l+2}) + D(x_{l+2}, x_{l+2p+1})] \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D(x_l, x_{l+1}) + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D(x_{l+1}, x_{l+2}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})[D(x_{l+2}, x_{l+3}) \\
&+ D(x_{l+3}, x_{l+4}) + D(x_{l+4}, x_{l+2p+1})] \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D(x_l, x_{l+1}) + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D(x_{l+1}, x_{l+2}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D(x_{l+2}, x_{l+3}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D(x_{l+3}, x_{l+4}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D(x_{l+4}, x_{l+2p+1}) \\
&\leq \cdots \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D(x_l, x_{l+1}) + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D(x_{l+1}, x_{l+2}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D(x_{l+2}, x_{l+3}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D(x_{l+3}, x_{l+4}) \\
&+ \cdots + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1}) \\
&\cdots \theta(x_{l+2p-2}, x_{l+2p-1}, x_{l+2p}, x_{l+2p+1})D(x_{l+2p}, x_{l+2p+1}) \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D_0 + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})D_1 \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D_2 \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})D_3 \\
&+ \cdots \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1}) \times \cdots \\
&\times \cdots \theta(x_{l+2p-2}, x_{l+2p-1}, x_{l+2p}, x_{l+2p+1})D_{2p}
\end{aligned}
$$

$$
\begin{aligned}
&= \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})[D_0 + D_1] \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})[D_2 + D_3] \\
&+ \cdots + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1}) \times \cdots \\
&\times \cdots \theta(x_{l+2p-2}, x_{l+2p-1}, x_{l+2p}, x_{l+2p+1})[D_{2p-1} + D_{2p}] \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})[(k^l + k^{l+1})D(x_0, x_1)] \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})[(k^{l+2} + k^{l+3})D(x_0, x_1)] \\
&+ \cdots + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1}) \times \cdots \\
&\times \cdots \theta(x_{l+2p-2}, x_{l+2p-1}, x_{l+2p}, x_{l+2p+1})[(k^{l+2p-2} + k^{l+2p-1})D(x_0, x_1)] \\
&\leq [\theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})(k^l + k^{l+1}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1})(k^{l+2} + k^{l+3}) + \\
&\cdots + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p+1})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p+1}) \times \cdots \\
&\times \cdots \theta(x_{l+2p-2}, x_{l+2p-1}, x_{l+2p}, x_{l+2p+1})(k^{l+2p-2} + k^{l+2p-1})]D(x_0, x_1) \\
&= \sum_{r=0}^{p-1} \prod_{i=0}^{r} \theta(x_{l+2i}, x_{l+2i+1}, x_{l+2i+2}, x_{l+2p+1})[k^{l+2l} + k^{l+2l+1}]D(x_0, x_1) \\
&= \sum_{r=0}^{p-1} \prod_{i=0}^{r} \theta(x_{l+2i}, x_{l+2i+1}, x_{l+2i+2}, x_{l+2p+1})[1 + k]k^{l+2r}D(x_0, x_1)
\end{aligned}
$$

As$k < 1$, the above inequalities imply the following:

$$
D(x_l, x_{l+2p+1}) < \sum_{r=0}^{p-1} \prod_{i=0}^{r} \theta(x_{l+2i}, x_{l+2i+1}, x_{l+2i+2}, x_{l+2p+1}) 2k^{l+2r} D(x_0, x_1).
$$

Since $\sup_{m>1} \lim_{l\to\infty} \theta(x_l, x_{l+1}, x_{l+2}, x_m) \leq \frac{1}{k}$ we deduce that,

$$
\begin{aligned}
\lim_{l,p\to\infty} D(x_l, x_{l+2p+1}) &< \sum_{r=0}^{\infty} \prod_{i=0}^{r} \theta(x_{l+2i}, x_{l+2i+1}, x_{l+2i+2}, x_{l+2p+1}) 2k^{l+2r} D(x_0, x_1) \\
&\leq \sum_{r=0}^{\infty} \frac{1}{k^{r+1}} 2k^{l+2r} D(x_0, x_1) \\
&\leq \sum_{r=0}^{\infty} 2k^{l+r-1} D(x_0, x_1).
\end{aligned}
$$

The series $\sum_{r=0}^{\infty} 2k^{l+r-1} D(x_0, x_1)$ is convergent by the ratio test, which implies that $D(x_l, x_{l+2p+1})$ converges as $l, p \to \infty$.
Subcase 2: $m = l + 2p$ Fist of all, note that

$$
D(x_l, x_{l+2}) \leq kD(x_{l-1}, x_{l+1}) \leq k^2 D(x_{l-2}, x_l) \leq \cdots \leq k^l D(x_0, x_2),
$$

which leads us to conclude that $D(x_l, x_{l+2}) \to 0 \text{ as } l \to \infty$. Similarly to Subcase 1, we have:

$$
\begin{aligned}
D(x_l, x_{l+2p}) &\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p})[D(x_l, x_{l+1}) + D(x_{l+1}, x_{l+2}) + D(x_{l+2}, x_{l+2p})] \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p}) D(x_l, x_{l+1}) + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p}) D(x_{l+1}, x_{l+2}) \\
&+ \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p})\theta(x_{l+2}, x_{l+3}, x_{l+4}, x_{l+2p})[D(x_{l+2}, x_{l+3}) \\
&+ D(x_{l+3}, x_{l+4}) + D(x_{l+4}, x_{l+2p})] \\
&\leq \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p}) D(x_l, x_{l+1}) + \theta(x_l, x_{l+1}, x_{l+2}, x_{l+2p}) D(x_{l+1}, x_{l+2})
\end{aligned}
$$

$$+\,\theta(x_l,x_{l+1},x_{l+2},x_{l+2p})\theta(x_{l+2},x_{l+3},x_{l+4},x_{l+2p})D(x_{l+2},x_{l+3})$$
$$+\,\theta(x_l,x_{l+1},x_{l+2},x_{l+2})\theta(x_{l+2},x_{l+3},x_{l+4},x_{l+2p})D(x_{l+3},x_{l+4})$$
$$+\,\theta(x_l,x_{l+1},x_{l+2},x_{l+2p})\theta(x_{l+2},x_{l+3},x_{l+4},x_{l+2p})D(x_{l+4},x_{l+2p})$$

$$\begin{aligned}
&\leq \theta(x_l,x_{l+1},x_{l+2},x_{l+2p})D_0+\theta(x_l,x_{l+1},x_{l+2},x_{l+2p})D_1\\
&+\theta(x_l,x_{l+1},x_{l+2},x_{l+2p})\theta(x_{l+2},x_{l+3},x_{l+4},x_{l+2p})D_2\\
&+\theta(x_l,x_{l+1},x_{l+2},x_{l+2p})\theta(x_{l+2},x_{l+3},x_{l+4},x_{l+2p})D_3\\
&+\cdots\\
&+\theta(x_l,x_{l+1},x_{l+2},x_{l+2p})\theta(x_{l+2},x_{l+3},x_{l+4},x_{l+2p})\times\cdots\\
&\times\cdots\theta(x_{l+2p-3},x_{l+2p-2},x_{l+2p-1},x_{l+2p})D_{2p}\\
&+\prod_{i=0}^{2p-2}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+1},x_{l+2p})D(x_{l+2p-2},x_{l+2p})
\end{aligned}$$

$$\begin{aligned}
&=\sum_{r=0}^{p-1}\prod_{i=0}^{r}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+2},x_{l+2p+1})[k^{l+2r}+k^{l+2r+1}]D(x_0,x_1)\\
&+\prod_{i=0}^{2p-2}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+1},x_{l+2p})D(x_{l+2p-2},x_{l+2p})
\end{aligned}$$

$$\begin{aligned}
&=\sum_{r=0}^{p-1}\prod_{i=0}^{r}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+2},x_{l+2p+1})[1+k]k^{l+2r}D(x_0,x_1)\\
&+\prod_{i=0}^{2p-2}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+1},x_{l+2p})D(x_{l+2p-2},x_{l+2p})\\
&\leq\sum_{r=0}^{p-1}\prod_{i=0}^{r}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+2},x_{l+2p+1})[1+k]k^{l+2r}D(x_0,x_1)\\
&+\prod_{i=0}^{2p-2}\theta(x_{l+2i},x_{l+2i+1},x_{l+2i+1},x_{l+2p})k^{l+2p-2}D(x_0,x_2)
\end{aligned}$$

Since $\sup_{m>1}\lim_{l\to\infty}\theta(x_l,x_{l+1},x_{l+2},x_m)\leq\frac{1}{k}$ we deduce that,

$$\begin{aligned}
\lim_{l,p\to\infty}D(x_l,x_{l+2p})&\leq\lim_{l,p\to\infty}\sum_{r=0}^{p-1}\frac{1}{k^{l+1}}[1+k]k^{l+2r}D(x_0,x_1)+k^{2p-1}k^{l+2p-2}D(x_0,x_2)\\
&=\lim_{l,p\to\infty}\sum_{r=0}^{p-1}[1+k]k^{l+r-1}D(x_0,x_1)+k^{l-1}D(x_0,x_2)\\
&\leq\sum_{m=0}^{\infty}[1+k]k^mD(x_0,x_1)+k^mD(x_0,x_2)
\end{aligned}$$

From the ratio test, it is easy to show that the series

$$\sum_{m=0}^{\infty}[1+k]k^mD(x_0,x_1)+k^mD(x_0,x_2)$$

converges. Hence, $D(x_l,x_{l+2p})$ converges as l,p going toward ∞. Thus, by subcase 1 and subcase 2, it is proved that the sequence $\{x_l\}$ is a $D-$Cauchy sequence. Since (X,D) is a D-complete extended rectangular metric-like space, we deduce that $\{x_l\}$ converges to some $\nu \in X$. Now, we show that ν is fixed point of T.

$$D(x_l, x_{l+1}) = D(Tx_{l-1}, Tx_l) \leq kD(x_{l-1}, x_l) = kD(x_{l-1}, Tx_{l-1}) < D(x_{l-1}, Tx_{l-1})$$

Now, taking the limit $l \to \infty$, and as T is continuous, we deduce that

$$D(\nu, T\nu) < D(\nu, T\nu),$$

which leads us to a contradiction. Hence, $D(\nu, T\nu) = 0$ and that is $T\nu = \nu$ and ν is a fixed point of T.

Finally, for uniqueness, let us assume two fixed points of T say ν and μ such that $\nu \neq \mu$. By the contractive property of T we have:

$$D(\nu, \mu) = D(T\nu, T\mu) \leq kD(\nu, \mu) < D(\nu, \mu)$$

which leads us to contradiction.Thus, T has a unique fixed point as required. □

Theorem 2. *Let (X, D) be a complete controlled rectangular metric-like space, and T a continuous self-mapping on X satisfying the following condition; for all $x, y \in X$ there exists $0 < k < \frac{1}{2}$ such that*

$$D(Tx, Ty) \leq k[D(x, Tx) + D(y, Ty)]$$

Furthermore, if

$$\sup_{m>1} \lim_{l\to\infty} \theta(x_l, x_{l+1}, x_{l+2}, x_m) \leq \frac{1}{k},$$

and for all $u, v \in X$, we have:

$$\lim_{l\to\infty} \theta(u, v, x_l, x_{l+1}) \leq 1,$$

then T has a fixed point in X. Moreover, if for every fixed point ν of T we have $D(\nu, \nu) = 0$, then the fixed point of T is unique.

Proof. Let $x_0 \in X$ and define the sequence $\{x_l\}$ as follows

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_l = Tx_{l-1} = T^n x_0, \cdots$$

First of all, note that for all $l \geq 1$, we have

$$\begin{aligned}
&D(x_l, x_{l+1}) \leq k[D(x_{l-1}, x_l) + D(x_l, x_{l+1})] \\
&\Rightarrow (1-k)D(x_l, x_{l+1}) \leq kD(x_{l-1}, x_l) \\
&\Rightarrow D(x_l, x_{l+1}) \leq \frac{k}{1-k} D(x_{l-1}, x_l).
\end{aligned}$$

Since $0 < k < \frac{1}{2}$, one can easily deduce that $0 < \frac{k}{1-k} < 1$. Therefore, let $\mu = \frac{k}{1-k}$.

Hence,

$$\begin{aligned}
D(x_l, x_{l+1}) &\leq \mu D(x_{l-1}, x_l) \\
&\leq \mu^2 D(x_{l-2}, x_{l-1}) \\
&\leq \cdots \\
&\leq \mu^l D(x_0, x_1).
\end{aligned}$$

Therefore,

$$D(x_l, x_{l+1}) \to 0 \quad \text{as} \quad l \to \infty.$$

Furthermore, for all $l \geq 1$, we have

$$D(x_l, x_{l+2}) \leq k[D(x_{l-1}, x_l) + D(x_{l+1}, x_{l+2})]$$

Thus, by using the fact that $D(x_l, x_{l+1}) \to 0$ as $l \to \infty$, we deduce that

$$D(x_l, x_{l+2}) \to 0 \quad \text{as} \quad l \to \infty.$$

Now, similarly to the proof of case 1 and case 2 of Theorem 1, we deduce that the sequence $\{x_l\}$ is a $D-$Cauchy sequence. Since (X, D) is a D-complete controlled rectangular metric-like space, we conclude that $\{x_l\}$ converges to some $\nu \in X$. Now, we show that ν is a fixed point of T.

$$D(x_l, x_{l+1}) = D(Tx_{l-1}, Tx_l) \leq k[D(x_{l-1}, Tx_{l-1}) + D(x_l, Tx_l)]$$

Now, taking the limit as $l \to \infty$ and using the fact that T is continuous, we deduce that

$$D(\nu, T\nu) < k[D(\nu, T\nu) + D(\nu, T\nu)] = 2kD(\nu, T\nu) < D(\nu, T\nu) \text{ since } k < \frac{1}{2},$$

which leads us to a contradiction. Hence, $D(\nu, T\nu) = 0$, and that is, $T\nu = \nu$, and ν is a fixed point of T. To show uniqueness, we assume two fixed points of T, say ν and μ such that $\nu \neq \mu$. By the contractive property of T, we have:

$$D(\nu, \mu) = D(T\nu, T\mu) \leq k[D(\nu, T\nu) + D(\mu, T\mu)] = k[D(\nu, \nu) + D(\mu, T\mu)] = 0.$$

Thus, $D(\nu, \mu) = 0$ and, that is, $\nu = \mu$. Therefore, T has a unique fixed point as required. □

In the next section, we present an application of our result.

4. Application

Let X be the set $C([0, 1], \mathbb{R})$ and consider the following Fredholm type integral equation:

$$\zeta'(t) = \int_0^1 F(\psi, \omega, \zeta'(t))ds, \quad \text{for} \quad \psi, \omega \in [0, 1] \tag{1}$$

where $F(\psi, \omega, \zeta'(t))$ is continuous from $[0, 1]^2 \to \mathbb{R}$. Next, let

$$\begin{aligned} D : X \times X &\longrightarrow \mathbb{R} \\ (\zeta, \varrho) &\mapsto \sup_{t \in [0,1]} \left(\frac{|\zeta'(t)| + |\varrho(t)|}{2} \right) \end{aligned}$$

Notice that (X, D) is a complete controlled rectangular metric-like space, where

$$\theta(x, y, u, v) = \max\{x, y, u, v\}.$$

Theorem 3. *If $\zeta, \varrho \in X$ satisfies the following conditions*
(1) $|F(\psi, \omega, \zeta'(t))| + |F(\psi, \omega, \varrho(t))| \leq k(|\zeta'(t)| + |\varrho(t)|)$, *for some* $k \in (0, 1)$;
(2) $F(\psi, \omega, \int_0^1 F(\psi, \omega, \zeta'(t))ds) < F(\psi, \omega, \zeta'(t))$ *for all* ψ, ω,
then Equation (1) has a unique solution.

Proof. Let $T : X \longrightarrow X$ be defined by $T\zeta'(t) = \int_0^1 F(\psi, \omega, \zeta'(t))ds$, then $D(T\zeta, T\varrho) = \sup_{t\in[0,1]}(\frac{|T\zeta'(t)|+|T\varrho(t)|}{2})$. Hence,

$$\begin{aligned}\frac{|T\zeta'(t)| + |T\varrho(t)|}{2} &= \frac{|\int_0^1 F(\psi,\omega,\zeta'(t))ds| + |\int_0^1 F(\psi,\omega,\varrho(t))ds|}{2} \\ &\le \frac{\int_0^1 |F(\psi,\omega,\zeta'(t))|ds + \int_0^1 |F(\psi,\omega,\varrho(t))|ds}{2} \\ &= \frac{\int_0^1 (|F(\psi,\omega,\zeta'(t))| + |F(\psi,\omega,\varrho(t))|)ds}{2} \\ &\le \frac{\int_0^1 k(|\zeta'(t)| + |\varrho(t)|)ds}{2} \\ &\le kD(\zeta', \varrho).\end{aligned}$$

Thus, $D(T\zeta, T\varrho) \le kD(\zeta, \varrho)$. Now, let $n \in \mathbb{N}^\star$ and $\zeta \in X$;

$$\begin{aligned}(T^n\zeta)(t) = T(T^{n-1}\zeta'(t)) &= \int_0^1 F(\psi,\omega,T^{n-1}\zeta'(t))ds \\ &= \int_0^1 F(\psi,\omega,T(T^{n-2}\zeta)(t))ds \\ &= \int_0^1 F(\psi,\omega,\int_0^1 F(\psi,\omega,(T^{n-2}\zeta'(t))))ds \\ &< \int_0^1 F(\psi,\omega,(T^{n-2}\zeta'(t)))ds = (T^{n-1}\zeta'(t))\end{aligned}$$

Therefore, for all $t \in [0,1]$ we have $(T^n\zeta'(t))_n$, which is a strictly decreasing and bounded-below sequence and, hence, converges to some l. Since $(T_n)_n$ is a monotone sequence, it follows from Dini Theorem that $\sup_t |T^n\zeta'(t)|$ converges to some $l' \le \sup_{\psi,\omega} |F(\psi,\omega,\zeta'(t))|$. Now, it is not difficult to see that all the hypotheses of Theorem 1 are satisfied, and therefore, Equation (1) has a unique solution as required. □

5. Conclusions

In this manuscript, we have introduced a new type of metric space, which is a generalization of rectangular metric spaces, rectangular b-metric spaces, and controlled rectangular metric spaces. We have proved the existence and uniqueness of a fixed point for self-mapping on controlled rectangular metric-like spaces. Our results are a generalization of many theorems in the literature. Finally, we gave an application of our result to the Fredholm-type integral equation.
In closing, we would like to present the following two questions;

Question 1. *Let (X, D) be a controlled rectangular metric like space, and T a map from $X \to X$. Assume that for all $\zeta, \eta, T\zeta, T\eta \in X$ there exists $k \in (0,1)$, where*

$$D(T\zeta, T\eta) \le k\theta(\zeta, \eta, T\zeta, T\eta)D(\zeta, \eta)$$

under what other condition(s) does T have a unique fixed point in X?

Question 2. *Let (X, D) be a controlled rectangular metric-like space, and T a map from $X \to X$. Assume that for all $\zeta, \eta, T\zeta, T\eta \in X$ there exists $k \in (0,1)$, where*

$$D(T\zeta, T\eta) \le \theta(\zeta, \eta, T\zeta, T\eta)[D(\zeta, T\zeta) + D(\eta, T\eta)]$$

under what other condition(s) does T have a unique fixed point in X?

Author Contributions: Conceptualization, S.H., F.A. and N.M.; methodology, S.H., F.A. and N.M.; supervision, N.M.; validation, S.H., F.A. and N.M.; writing—original draft preparation, S.H., F.A. and N.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of Interest: The authors declare no conflict of interest.

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