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Laplace Transform for Solving System of Integro-Fractional Differential Equations of Volterra Type with Variable Coefficients and Multi-Time Delay

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Abstract: This study is the first to use Laplace transform methods to solve a system of Caputo fractional Volterra integro-differential equations with variable coefficients and a constant multi-time delay. This technique is based on different types of kernels, which we will explain in this paper. Symmetry kernels, which have properties of difference kernels or simple degenerate kernels, are able to compute analytical work. These are demonstrated by solving certain examples and analyzing the effectiveness and precision of cause techniques.

Keywords: system fractional-integro differential equation; Laplace transform; Caputo fractional derivative; delay differential equations; difference and simple degenerate kernels

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1. Introduction

The purpose of this paper is to solve linear system integro-fractional differential equations of the Volterra type (LS-VIFDEs) with variable coefficients and multi-time delay of the retarded type (RD):

$$\begin{aligned} {}_a^C D_t^{\alpha} u_r(t) + \sum_{i=1}^{n-1} P_{ri}(t)\, {}_a^C D_t^{\alpha_{r(n-i)}} u_r(t) + P_{rn}(t) u_r(g(t,\tau_r)) \\ = f_r(t) + \sum_{j=0}^{m} \lambda_{rj} \int_a^t \mathcal{K}_{rj}\,(t,x) u_j\big(g\big(x,\tau_{rj}\big)\big)dx,\ a \le t \le b. \end{aligned} \tag{1}$$

All $r = 0,1,2,\cdots,m$, as well as the fractional orders, have the basic ordering property $\alpha_{rn} > \alpha_{r(n-1)} > \alpha_{r(n-2)} > \ldots > \alpha_{r1} > \alpha_{r0} = 0$, and are given together with the initial conditions. For all $r = 0,1,\ldots,m$; $\left[u_r^{(k_r)}(t)\right]_{t=a} = u_{r,k_r}$ and historical functions, $u_r^{(k_r)}(t) = \varphi_r^{(k_r)}(t)$ for all $t \in [\bar{a}, a]$, as well as $\bar{a} = a - \max\{\tau_r, \tau_{rj} : j = 0,1,\ldots,m\}$, $k_r = 0,1,\ldots,\mu_r - 1$, $\mu_r = \max\{d_{r\ell} | \ell = 0,1,2,\ldots,n\}$, $d_{r\ell} = \lceil \alpha_{r\ell} \rceil$, where $u_r(t)$ are $(m+1)$. This function is unknown and is the solution of LS-VIFDE's multi-time RD, Equation (1), with conditions and functions: $\mathcal{K}_{rj} : S \times \mathbb{R} \to \mathbb{R}$. ($S = \{(t,x) : a \le x \le t \le b\}$), $r, j = 0,1,2,\ldots,m$ and $f_r; P_{ri} : [a,b] \to \mathbb{R}$ for all $i = 1,2,\ldots,n$; $r = 0,1,\ldots,m$ for all real bounded continuous functions. In addition, for all $r = 0,1,\ldots,m$, where $u_r(t) \in \mathbb{R}$, ${}_a^C D_t^{\alpha_{r\ell}} u_r(t)$ is the $\alpha_{r\ell}$-fractional Caputo-derivative order of the functions u_r on $[a,b]$ and all $\alpha_{r\ell}, \in \mathbb{R}^+$, $d_{r\ell} - 1 > \alpha_{r\ell} \le d_{r\ell}$, $d_{r\ell} = \lceil \alpha_{r\ell} \rceil$ for all $r = 0,1,\ldots,m$; $\ell = 1,2,\ldots,n$. Moreover, the value of $\tau_{rj}, \tau_r \in \mathbb{R}^+$ for all $j = 0,1,\ldots,m$ are called positive constant time lags or time delays. Because the problem of LS-delayed VIFDE's time delay is a relatively new topic in mathematics, there are only one or two ways of solving it, and since the specific analytic solution no longer exists, an approximation method must be used. In this paper, we use the Laplace transform to provide an explanation for how to solve Equation (1) with conditions.

The Laplace transform is a very useful method for solving various types of equations, such as integro-differential equations, integral equations, fractional equations, and delay differential equations. It can also be used to solve initial and boundary value problems related to differential equations and partial differentials with constant coefficients [1–6]. This transform method is also used for solving linear Caputo fractional-integro differential equations with multi-time retarded delays [7] and for solving linear system integro-fractional differential equation of Volterra-type equations [8]. When using this technique, it is important and necessary to explain and define several properties of the Laplace transform that are important for driving this transformation of delay functions and the Caputo fractional derivative, which is expressed in Equation (1).

This work is classified into these sections as follows: some definitions and important properties are shown in Section 2. In Section 3, a system of integro-fractional differential equations of the Volterra type with variable coefficients and multi-time delay technique is presented. In Section 4, the results are illustrated with all of the examples. Finally, a discussion of this method is included in Section 5.

2. Definitions with Important Properties

2.1. Fractional Calculus

In this subsection, we recall the most common definitions and results of fractional calculus that will be useful for this research. First, we start from the definition of function space $C_\gamma, \gamma \in \mathbb{R}$, which is the basic definition that operational calculus needs for the differential operator:

Definition 1. [4,7]. *A real valued function u defined on$[a,b]$ is in the space of γ-functions $C_\gamma[a,b]$, $\gamma \in \mathbb{R}$ if there exists a real number $r > \gamma$, such that $u(t) = (t-a)^r \hat{u}(t)$, where $\hat{u} \in C[a,b]$, and it is said to be in the space$C_\gamma^n[a,b]$ if and only if $u^{(n)} \in C_\gamma[a,b],\ n \in \mathbb{N}_0$.*

Definition 2. [4,8]. *For a function $u \in C_\gamma[a,b],\ \delta \geq -1,$ the Reimann–Liouville fractional integral operator ${}_aJ_t^\alpha$ of fractional order $\alpha \in \mathbb{R}^+$ and origin point a is defined as:*

$$
\begin{gathered}
{}_aJ_t^\alpha u(t) = \tfrac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} u(x) dx. \\
{}_aJ_t^0 u(t) = u(t),\ a \leq t \leq b.
\end{gathered}
$$

where Γ is the gamma function. ${}_aJ_t^\alpha$ has an important (or semigroup) property, that is:${}_aJ_t^\alpha\, {}_aJ_t^\beta u(t) = {}_aJ_t^\beta\, {}_aJ_t^\alpha u(t) = {}_aJ_t^{\alpha+\beta} u(t)$ for arbitrary $\alpha > 0$ and $\beta > 0$. Additionally, it has the following properties

$$
{}_aJ_t^\alpha (t-a)^{\ \delta} = \frac{\Gamma(\delta)}{\Gamma(\delta+\alpha+1)} (t-a)^{\ \delta+\alpha},\ \delta > -1.
$$

Definition 3. [7,8]. *Let $\alpha > 0,\ m = \alpha$ and $a \in \mathbb{R}$. The Reimann–Liouville fractional derivative of order α and starting pointa of a function $u(t) \in C_{-1}^m[a,b]$ is given as:*

$$
\begin{gathered}
{}_a^RD_t^\alpha u(t) = D^m \left[{}_aJ_t^{m-\alpha} u(t)\right]. \\
{}_a^RD_t^0 u(t) = u(t),\ a \leq t \leq b.
\end{gathered}
$$

If $\alpha = m(\in \mathbb{Z}^+)$ and $u \in C^m[a,\ b]$, thus ${}_a^RD_t^m u(t) = \frac{d^m}{dt^m} u(t)$.

Definition 4. [8,9]. *Let $\alpha > 0,\ m = \alpha$,* then the *Caputo fractional derivative of order α and starting pointa of a function $u(t) \in C_{-1}^m[a,b]$ is given as:*

$$
\begin{gathered}
{}_a^CD_t^\alpha u(t) = {}_aJ_t^{m-\alpha} u^{(m)}(t) = \tfrac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} u^{(m)}(s) ds. \\
{}_a^CD_t^0 u(t) = u(t),\ a \leq t \leq b.
\end{gathered}
$$

Additionally, if $\alpha = m(\in \mathbb{Z}^+)$ and $u \in C^m[a,\ b]$, *thus ${}_a^CD_t^m u(t) = \frac{d^m}{dt^m} u(t)$.*

2.2. Some Important Properties

In this subsection, we are interested some important properties which are used later on this paper [4,7–9].

i. ${}_{a}^{R}D_{t}^{\alpha}\mathcal{A} = \mathcal{A}\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$; where $\mathcal{A}$ is any constant; ($\alpha \geq 0$, $\alpha \notin \mathbb{N}$).

ii. If the Caputo fractional derivative of a constant function is equal to zero, it means ${}_{a}^{C}D_{t}^{\alpha}\mathcal{A} = 0$, for any constant $\mathcal{A}$ and all $\alpha > 0$.

iii. The relationship between the R-L integral and Caputo derivatives are shown here: Let $\alpha \geq 0,\ m = \alpha$ and $u \in C^{m}[a,b]$, then:

$$ {}_{a}^{C}D_{t}^{\alpha}[{}_{a}J_{t}^{\alpha}u(t)] = u(t)\ ;\ a \leq t \leq b $$

$$ {}_{a}J_{t}^{\alpha}\left[{}_{a}^{C}D_{t}^{\alpha}u(t)\right] = u(t) - \sum_{k=0}^{m-1}\frac{u^{(k)}(a)}{k!}(t-a)^{k}. $$

iv. Let $T_{m-1}[\gamma;a]$ be the Taylor polynomial of degree $(m-1)$ for the function γ, then:

$$ {}_{a}^{C}D_{t}^{\alpha}\gamma(t) = {}_{a}^{R}D_{t}^{\alpha}[\gamma(t) - T_{m-1}[\gamma;a]], $$

where $(m-1 < \alpha \leq m)$.

v. Let $u(t) = (t-a)^{\beta}$ and $\alpha > 0\ ; m = \alpha$ for some $\beta \geq 0$, then:

$$ {}_{a}^{C}D_{t}^{\alpha}u(t) = \begin{cases} 0 & \text{if } \beta \in \{0,1,2,\cdots,m-1\} \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}\,(t-a)^{\beta-\alpha} & \text{if } \beta \in \mathbb{N} \text{ and } \beta \geq m \\ & \text{or } \beta \notin \mathbb{N} \text{ and } \beta > m-1 \end{cases} $$

Definition 5. [1,10]. *The Laplace transform (LT) for the suitable function, $u(t)$, of real variable $t \geq 0$, is the function $U(s)$, which is defined by the integral form:*

$$ U(s) = \mathcal{L}\{u(t)\} = \int_{0}^{\infty} e^{-st}u(t)dt \tag{2} $$

with $U(s)$ the LT of $u(t)$, and inverse Laplace transform of $U(s)$, denoted by $\mathcal{L}^{-1}\ \{U(s);t\}$, being the function u defined on $[0,\infty)$, which has the fewest number of discontinuities and satisfies $\mathcal{L}\{u(t);s\} = U(s)$. Laplace transform has various properties with some lemmas, which are the key for our work, as shown below [1,6,11–13]*:*

i. *If $u(t)$ and $q(t)$ have well-defined Laplace transforms, then $U(s) = \mathcal{L}\{u(t)\}$ and $Q(s) = \mathcal{L}\{q(t)\}$, respectively. Now, the Laplace transform of the convolution integral is defined by the form:*

$$ \mathcal{L}\{(u*q)(t)\} = \mathcal{L}\left\{\int_{0}^{t} u(t-x)\ q(x)dx\right\} = U(s)Q(s) \tag{3} $$

If $u = 1$, then:

$$ \mathcal{L}\left\{\int_{0}^{t} q(x)dx;s\right\} = \frac{1}{s}\ Q(s) \tag{4} $$

ii. *Put the power function t^{m} of order $m \in \mathbb{Z}^{+}$, then:*

$$ \mathcal{L}\{t^{m}u(t)\} = (-1)^{m}\ \frac{d^{m}}{ds^{m}}\ \mathcal{L}\{u(t)\} = (-1)^{m}\ \frac{d^{m}}{ds^{m}}\ U(s) \tag{5} $$

iii. *From (ii and iii), we obtain:*

$$ \mathcal{L}\left\{\int_{0}^{t}\ t\ u(x)dx;s\right\} = -\frac{d}{ds}\left(\frac{1}{s}\ U(s)\right) \quad \text{and} \quad \mathcal{L}\left\{\int_{0}^{t} x\ u(x)dx\ ;s\right\} = -\frac{1}{s}\frac{d}{ds}\ U(s)\ . \tag{6} $$

The following shows the important lemma for the Laplace transform of a constant delay function:

Lemma 1. [7]. *Let $u(t)$be a continuous differentiable function on a closed bounded interval* $[0, b]$, $b \in \mathbb{R}^+$*, and let* τ *be a constant delay such that:*

$$u(t) = \varphi(t), \text{ for } -\tau \le t < 0. \tag{7}$$

Then, the Laplace transform of a $\tau - \text{delay}$ *function is given by:*

$$\mathcal{L}\{u(t-\tau); s\} = e^{-s\tau}\,[U(s) + Q(s,\tau)]. \tag{8}$$

where $Q(s,\tau) = \int_{-\tau}^{0} e^{-st}\varphi(t)dt$ *and* $\mathcal{L}\{u(t)\} = U(s)$*. If the historical function* $\varphi(t)$ *is defined by power function* $t^n (n \in \mathbb{Z}^+)$*, we obtain:*

$$\mathcal{L}\{u(t-\tau); s\} = e^{-s\tau}\,U(s) + \sum_{p=0}^{n} (-1)^{n-p}\,p!\begin{pmatrix} n \\ p \end{pmatrix} \frac{\tau^{n-p}}{s^{p+1}} - \frac{n!}{s^{n+1}}\,e^{-s\tau} \tag{9}$$

Lemma 2. [4,9]. *Laplace transform of Caputo fractional of order* α $(m-1 < \alpha \le m)$, $m = \alpha$ *can be obtained as:*

$$\begin{aligned}&\mathcal{L}\{{}_a^cD_t^{\alpha}\,u(t); s\} = \mathcal{L}\{J_t^{m-\alpha}D_t^m u(t); s\} = s^{-(m-\alpha)}\,\mathcal{L}\Big\{u^{(m)}(t); s\Big\} \\ &= s^{-(m-\alpha)}\left[s^m\,U(s) - \sum_{k=0}^{m-1} s^{m-k-1}\,u^{(k)}(0)\right] = s^{\alpha}\,U(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1}\,u^{(k)}(0)\,.\end{aligned} \tag{10}$$

3. Solving LS-VIFDE's Multi-Time RD Using the Laplace Transform Technique

In this section, we try to find a general analytical solution to a linear system of integro-differential equations of the arbitrary orders with variable coefficients and multi-time delays using the Laplace transform method in various types of kernels.

3.1. First Type (Difference Kernel)

We use Equation (1) with different kernels and $a = 0$ as the starting point. Furthermore, we consider $P_{ri}(t)$ as a power function, with difference kernels form $\mathcal{K}_{rj}\,(t,x) = \mathcal{K}_{rj}\,(t-x)$, where $C_{ri}t^{\ell_{ri}}$, $C_{ri} \in \mathbb{R}$ are constants and ℓ_{ri} are arbitrary non-negative integers for all r and i, and the Laplace transformation is taken for all $r = 0, 1, \ldots, m$, which is:

$$\begin{aligned}&\mathcal{L}\{{}_a^CD_t^{\alpha_{rn}}u_r(t); s\} + \mathcal{L}\left\{\sum_{i=1}^{n-1} P_{ri}(t)\,{}_a^CD_t^{\alpha_{r(n-i)}}u_r(t); s\right\} + \mathcal{L}\{P_{rn}(t)u_r(g(t,\tau_r)); s\} \\ &\qquad = \mathcal{L}\{f_r(t); s\} + \sum_{j=0}^{m} \lambda_{rj}\mathcal{L}\Big\{\int_0^t \mathcal{K}_{rj}\,(t-x)u_j\big(g\big(x,\tau_{rj}\big)\big)dx; s\Big\}.\end{aligned} \tag{11}$$

After applying the Laplace transformation in Equation (11), using Lemma 2 with the initial condition for the first part, where $m_{\alpha_{rn}} - 1 < \alpha_{rn} \le m_{\alpha_{rn}}$ for all $r = 0, 1, \ldots, m$, and also using Definition (5; part (ii)) and Lemma 2 for second parts, where $m_{\alpha_{r(n-i)}} - 1 < \alpha_{r(n-i)} \le m_{\alpha_{r(n-i)}}$, for all $r = 0, 1, \ldots, m$, we obtain:

$$\begin{aligned}\mathcal{L}\{{}_a^CD_t^{\alpha_{rn}}u_r(t); s\} &= s^{\alpha_{rn}}\,U_r(s) - \sum_{k_r=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k_r-1}\,u_r^{(k_r)}(0) \\ &= s^{\alpha_{rn}}\,U_r(s) - \sum_{k_r=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k_r-1}\,u_{r,k_r}.\end{aligned} \tag{12}$$

where u_{r,k_r} are given for all r from the conditions. For all $r = 0, 1, \ldots, m$ using Equations (5) and (10) and conditions, for each $i = 1, 2, \ldots, n-1$, we obtain:

$$\begin{aligned}&\mathcal{L}\Big\{P_{ri}(t)\,{}_a^C D_t^{\alpha_{r(n-i)}} u_r(t); s\Big\}\\ &= C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[s^{\alpha_{r(n-i)}}\, U_r(s)\right]\\ &- C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[\sum_{k_r=0}^{m_{\alpha_{r(n-i)}}-1} s^{\alpha_{r(n-i)}-k_r-1} u_{r,k_r}\right].\end{aligned} \tag{13}$$

where ℓ_{ri} is the order of $P_{ri}(t)$ for each $i = 1, 2, \ldots, n-1$ and $r = 0, 1, \ldots, m$. Consequently, we use Equation (5) and then apply the Lemma (1, Equations (8) and (9)), respectively, with the defined $g(t, \tau_r) = t - \tau_r$, thus obtaining for each r:

$$\mathcal{L}\{P_{rn}(t) u_r(g(t, \tau_r)); s\} = C_{rn}(-1)^{\ell_{rn}} \frac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[e^{-s\tau_r}(U_r(s) + Q_r(s, \tau_r))\right].$$

where:

$$Q_r(s, \tau_r) = \int_{-\tau_r}^{0} e^{-st} \varphi_r(t) dt.$$

If the historical function $\varphi_r(t)$ is t^{q_r}, $q_r \in \mathbb{Z}^+$ for all $r = 0, 1, \ldots, m$, in this special case, we obtain:

$$\begin{aligned}&\mathcal{L}\{P_{rn}(t) u_r(g(t, \tau_r)); s\}\\ &= C_{rn}\,(-1)^{\ell_{rn}}\left\{\tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[e^{-s\tau_r} U_r(s)\right] + \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\sum_{p=0}^{q_r}(-1)^{q_r-p}\, p!\begin{pmatrix} q_r \\ p \end{pmatrix}\tfrac{\tau_r{}^{q_r-p}}{s^{p+1}}\right] - \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\tfrac{q_r!}{s^{q_r+1}} e^{-s\tau_r}\right]\right\}.\end{aligned} \tag{14}$$

the Laplace transform of the homogenous part is simply written as:

$$\mathcal{L}\{f_r(t); s\} = F_r(s),\ r = 0, 1, \ldots, m. \tag{15}$$

By applying Equation (3) from Definition 4 with Lemma (1, Equations (8) and (11)) with defined $g(x, \tau_{rj}) = x - \tau_{rj}$ for all $r; j = 0, 1, \ldots, m$, the last part of Equation (11) will become:

$$\mathcal{L}\left\{\int_0^t \mathcal{K}_{rj}\,(t-x) u_j(g(x, \tau_{rj})) dx; s\right\} = \mathcal{K}_{rj}(s)\, e^{-s\tau_{rj}}\left[U_j(s) + Q_j(s, \tau_{rj})\right].$$

where:

$$Q_j(s, \tau_{rj}) = \int_{-\tau_{rj}}^{0} e^{-st} \varphi_j(t) dt.$$

The symbolic $\mathcal{K}_{rj}(s)$ is the Laplace transform of the difference kernel $\mathcal{K}_{rj}\,(t-x)$ for each r and j. If the historical function $\varphi_j(t)$ is t^{q_r}, $q_r \in \mathbb{Z}^+$ for all $r = 0, 1, \ldots, m$, in this special case, we obtain:

$$\begin{aligned}&\mathcal{L}\left\{\int_0^t \mathcal{K}_{rj}\,(t-x) u_j(g(x, \tau_{rj})) dx; s\right\}\\ &= \mathcal{K}_{rj}(s)\left[e^{-s\tau_{rj}}\, U_j(s) + \sum_{p=0}^{q_r}(-1)^{q_r-p}\, p!\begin{pmatrix} q_r \\ p \end{pmatrix}\tfrac{\tau_{rj}{}^{q_r-p}}{s^{p+1}} - \tfrac{q_r!}{s^{q_r+1}} e^{-s\tau_{rj}}\right].\end{aligned} \tag{16}$$

After putting Equations (12)–(16) into Equation (11), they become:

$$
\begin{aligned}
& s^{\alpha_{rn}} U_r(s) - \sum_{k_r=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k_r-1} u_{r,k_r} + \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[s^{\alpha_{r(n-i)}} U_r(s)\right] \\
& - \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}} \left[\sum_{k_r=0}^{m_{\alpha_{r(n-i)}}-1} s^{\alpha_{r(n-i)}-k_r-1} u_{r,k_r}\right] + C_{rn}\ (-1)^{\ell_{rn}} \left\{ \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}} [e^{-s\tau_r}(U_r(s) + Q_r(s,\tau_r))]\right] \\
& = F_r(s) + \sum_{j=0}^{m} \lambda_{rj} \mathcal{K}_{rj}(s)\ e^{-s\tau_{rj}}\left[U_j(s) + Q_j(s,\tau_{rj})\right].
\end{aligned}
$$

If $t^{q_r},\ q_r \in \mathbb{Z}^+$ for each $r = 0, 1, \ldots, m$ is a power function, which is also a historical function, using part two of Lemma 1 above the equation means it becomes:

$$
\begin{aligned}
& s^{\alpha_{rn}} U_r(s) - \sum_{k_r=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k_r-1} u_{r,k_r} \\
& + \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[s^{\alpha_{r(n-i)}} U_r(s)\right] - \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}} \left[\sum_{k_r=0}^{m_{\alpha_{r(n-i)}}-1} s^{\alpha_{r(n-i)}-k_r-1} u_{r,k_r}\right] \\
& + C_{rn}\ (-1)^{\ell_{rn}} \left\{ \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}[e^{-s\tau_r} U_r(s)] + \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\sum_{p=0}^{q_r} (-1)^{q_r-p}\ p! \binom{q_r}{p} \tfrac{\tau_r{}^{q_r-p}}{s^{p+1}}\right] - \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\tfrac{q_r!}{s^{q_r+1}} e^{-s\tau_r}\right]\right\} \\
& = F_r(s) + \sum_{j=0}^{m} \lambda_{rj}\ \mathcal{K}_{rj}(s)\left[e^{-s\tau_{rj}}\ U_j(s) + \sum_{p=0}^{q_r}(-1)^{q_r-p}\ p!\binom{q_r}{p}\tfrac{\tau_{rj}{}^{q_r-p}}{s^{p+1}} - \tfrac{q_r!}{s^{q_r+1}} e^{-s\tau_{rj}}\right].
\end{aligned}
\tag{17}
$$

Consequently, the system of ordinary differential equation of components $\{U_r(s) : r = 0, 1, \ldots, m\}$ is solved to find $U_r(s)$. In the end, the inverse of the Laplace transform on $U_r(s)$ is used to obtain the solution $u_r(t)$ of LS-VIFDEs for multi-time RD (1). After some simple manipulations, from Equation (17), we obtain:

$$
\begin{aligned}
& \left[s^{\alpha_{rn}} + \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}} s^{\alpha_{r(n-i)}} + C_{rn}\ (-1)^{\ell_{rn}} \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}} e^{-s\tau_r} - \lambda_{rr}\ \mathcal{K}_{rr}(s) e^{-s\tau_{rr}}\right] U_r(s) \\
& - \sum_{\substack{j=0 \\ j \neq r}}^{m} \lambda_{rj}\ \mathcal{K}_{rj}(s) e^{-s\tau_{rj}} U_j(s) + \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} s^{\alpha_{r(n-i)}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}} U_r(s) \\
& + C_{rn}\ (-1)^{\ell_{rn}} e^{-s\tau_r} \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}} U_r(s) = \overline{F}_r(s).\ r = 0, 1, \ldots, m
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{F}_r(s) = {} & F_r(s) + \sum_{k_r=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k_r-1} u_{r,k_r} + \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}} \left[\sum_{k_r=0}^{m_{\alpha_{r(n-i)}}-1} s^{\alpha_{r(n-i)}-k_r-1} u_{r,k_r}\right] \\
& - C_{rn}(-1)^{\ell_{rn}} \left\{ \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\sum_{p=0}^{q_r}(-1)^{q_r-p}\ p!\binom{q_r}{p}\tfrac{\tau_r{}^{q_r-p}}{s^{p+1}}\right] - \tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\tfrac{q_r!}{s^{q_r+1}} e^{-s\tau_r}\right]\right\} \\
& + \sum_{j=0}^{m} \lambda_{rj}\ \mathcal{K}_{rj}(s)\left[\sum_{p=0}^{q_r}(-1)^{q_r-p}\ p!\binom{q_r}{p}\tfrac{\tau_{rj}{}^{q_r-p}}{s^{p+1}} - \tfrac{q_r!}{s^{q_r+1}} e^{-s\tau_{rj}}\right].
\end{aligned}
$$

As a special case, if the $P_{ri}(t)$ and $P_{rn}(t)$ are only constants, this means that ℓ_{ri} and ℓ_{rn} are equal to zero. Thus, after some simple manipulations, from Equation (17), we obtain the following system for all $r = 0, 1, \ldots, m$:

$$
H_r(s) U_r(s) - \sum_{\substack{j=0 \\ j \neq r}}^{m} \lambda_{rj}\ \mathcal{K}_{rj}(s) e^{-s\tau_{rj}} U_j(s) = \overline{F}_r(s). \tag{18}
$$

where:

$$H_r(s) = s^{\alpha_{rn}} + \sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}} \frac{d^{\ell_{ri}}}{ds^{\ell_{ri}}} s^{\alpha_{r(n-i)}} + C_{rn}\,(-1)^{\ell_{rn}} \frac{d^{\ell_{rn}}}{ds^{\ell_{rn}}} e^{-s\tau_r} - \lambda_{rr}\,\mathcal{K}_{rr}(s)e^{-s\tau_{rr}}.$$

Finally, the system of ordinary differential equation of components $\{U_r(s) : r = 0, 1, \ldots, m\}$ is solved to find $U_r(s)$. In the end, the inverse of the Laplace transform on $U_r(s)$ is used to obtain the solution $u_r(t)$ of LS-VIFDEs for multi-time RD (1).

3.2. Second Type (Simple Degenerate Kernel)

Some types of linear-system VIFDEs of consistent multi-time can be solved using the Laplace transform approach. We take the same conditions as Equation (12) with all conditions by changing the kernel from difference kernel to a simple degenerate kernel. Define the kernel: $\mathcal{K}_{rj}\,(t,x) = c_{rj}t^{k^1_{rj}} + d_{rj}x^{k^2_{rj}}$, where $c_{rj},\ d_{rj} \in \mathbb{R}$ for all $r, j = 0, 1, \ldots, m$ and $k^1_{rj}, k^2_{rj} \in \mathbb{Z}^+$; then:

$$\begin{aligned}\mathcal{L}\{{}^C_aD^{\alpha_{rn}}_t u_r(t);s\} + \mathcal{L}\left\{\sum_{i=1}^{n-1} P_{ri}(t)\ {}^C_aD^{\alpha_{r(n-i)}}_t u_r(t);s\right\} + \mathcal{L}\{P_{rn}(t)u_r(g(t,\tau_r));s\} \\ = \mathcal{L}\{f_r(t);s\} + \sum_{j=0}^{m} \lambda_{rj}\mathcal{L}\Big\{ \int_0^t \Big[c_{rj}t^{k^1_{rj}} + d_{rj}x^{k^2_{rj}}\Big] u_j\big(g\big(x,\tau_{rj}\big)\big)dx;s\Big\}.\end{aligned} \tag{19}$$

The left hands in all parts of Equation (19) are the same as Equation (11) in Section 3.1, while for the integral part, it is different. We apply the important property of Equation (6) part (iii) in Section 2.2 using Equations (8) and (9), respectively, and for higher derivative of multiplication functions using Leibniz's formula [7,14], with the property $g\big(x,\tau_{rj}\big) = x - \tau_{rj}$; then, after some manipulating, we obtain:

$$\begin{aligned}&\mathcal{L}\Big\{ \int_0^t \Big[c_{rj}t^{k^1_{rj}} + d_{rj}x^{k^2_{rj}}\Big] u_j\big(g\big(x,\tau_{rj}\big)\big)dx;s\Big\} \\ &= \tfrac{e^{-s\tau_{rj}}}{s}\left\{\left[c_{rj}\left(\sum_{b=0}^{k^1_{rj}} b!\begin{pmatrix} k^1_{rj} \\ b\end{pmatrix}\tfrac{1}{s^b}{\tau_{rj}}^{k^1_{rj}-b}\right) + d_{rj}{\tau_{rj}}^{k^2_{rj}}\right] + \left[d_{rj}\sum_{b=0}^{k^2_{rj}-1}(-1)^{b+k^2_{rj}}{\tau_{rj}}^{b}\begin{pmatrix} k^2_{rj} \\ b\end{pmatrix}\tfrac{d^{k^2_{rj}-b}}{ds^{k^2_{rj}-b}}\right]\right. \\ &\quad + \left[c_{rj}\sum_{b=0}^{k^1_{rj}-1}(-1)^{b+k^1_{rj}}b!\begin{pmatrix} k^1_{rj} \\ b\end{pmatrix}\tfrac{1}{s^b}\left(\sum_{p=0}^{k^1_{rj}-b-1}(-1)^p{\tau_{rj}}^p\begin{pmatrix} k^1_{rj}-b \\ p\end{pmatrix}\tfrac{d^{k^1_{rj}-b-p}}{ds^{k^1_{rj}-b-p}}\right)\right]\Bigg\}U_j(s) \\ &\quad + \tfrac{1}{s}\left\{c_{rj}\left[\sum_{b=0}^{k^1_{rj}}(-1)^{b+k^1_{rj}}b!\begin{pmatrix} k^1_{rj} \\ b\end{pmatrix}\tfrac{1}{s^b}\tfrac{d^{k^1_{rj}-b}}{ds^{k^1_{rj}-b}}\right] + d_{rj}\left[(-1)^{k^2_{rj}}\tfrac{d^{k^2_{rj}}}{ds^{k^2_{rj}}}\right]\right\}H^q_{rj}(s).\end{aligned} \tag{20}$$

for all $r, j = 0, 1, \ldots, m$, where:

$$H^q_{rj}(s) = \begin{cases} e^{-s\tau_{rj}}Q_j\big(s,\tau_{rj}\big)\ ; \text{ if the historical function be any countinous differential function.} \\ \sum_{p=0}^{q_r}(-1)^{q_r-p}\ p!\begin{pmatrix} q_r \\ p\end{pmatrix}\tfrac{{\tau_{rj}}^{q_r-p}}{s^{p+1}} - \tfrac{q_r!}{s^{q_r+1}}e^{-s\tau_{rj}}\ ;\ if\ \varphi_r(t) = t^{q_r}.\end{cases}$$

and:

$$Q_j\big(s,\tau_{rj}\big) = \int_{-\tau_{rj}}^{0} e^{-st}\varphi_j(t)dt.$$

After some simple manipulations, and using Equation (20), we obtain the general solution for Equation (19):

$$
\begin{aligned}
&s^{\alpha_{rn}}\ U_r(s)-\sum_{k=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k-1}\ u_{r,k_r}\\
&+\sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}}\tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[s^{\alpha_{r(n-i)}}\ U_r(s)\right]+\sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}}\tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[\sum_{k=0}^{m_{\alpha_{r(n-i)}}-1} s^{\alpha_{r(n-i)}-k-1}u_{r,k_r}\right]\\
&+C_{rn}\ (-1)^{\ell_{rn}}\left\{\tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[e^{-s\tau_r}U_r(s)\right]+\tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\sum_{p=0}^{q_r}(-1)^{q_r-p}\ p!\begin{pmatrix} q_r\\ p\end{pmatrix}\tfrac{\tau_r{}^{q_r-p}}{s^{p+1}}\right]-\tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\tfrac{q_r!}{s^{q_r+1}}e^{-s\tau_r}\right]\right\}\\
&=F_r(s)\\
&+\sum_{j=0}^{m}\lambda_{rj}\tfrac{e^{-s\tau_{rj}}}{s}\left\{\begin{array}{l}
\left\{\begin{array}{l}\left[c_{rj}\left(\sum_{b=0}^{k_{rj}^1} b!\begin{pmatrix} k_{rj}^1\\ b\end{pmatrix}\tfrac{1}{s^b}\tau_{rj}{}^{k_{rj}^1-b}\right)+d_{rj}\tau_{rj}{}^{k_{rj}^2}\right]+\left[d_{rj}\sum_{b=0}^{k_{rj}^2-1}(-1)^{b+k_{rj}^2}\tau_{rj}{}^{b}\begin{pmatrix} k_{rj}^2\\ b\end{pmatrix}\tfrac{d^{k_{rj}^2-b}}{ds^{k_{rj}^2-b}}\right]\\
+\left[c_{rj}\sum_{b=0}^{k_{rj}^2-1}(-1)^{b+k_{rj}^1}b!\begin{pmatrix} k_{rj}^1\\ b\end{pmatrix}\tfrac{1}{s^b}\left(\sum_{p=0}^{k_{rj}^2-b-1}(-1)p_{\tau_{rj}}p\begin{pmatrix} k_{rj}^1-b\\ p\end{pmatrix}\tfrac{d^{k_{rj}^1-b-p}}{ds^{k_{rj}^1-b-p}}\right)\right]\end{array}\right\}U_j(s)\\
+\tfrac{1}{s}\left\{c_{rj}\left[\sum_{b=0}^{k_{rj}^1}(-1)^{b+k_{rj}^1}b!\begin{pmatrix} k_{rj}^1\\ b\end{pmatrix}\tfrac{1}{s^b}\tfrac{d^{k_{rj}^1-b}}{ds^{k_{rj}^1-b}}\right]+d_{rj}\left[(-1)^{k_{rj}^2}\tfrac{d^{k_{rj}^2}}{ds^{k_{rj}^2}}\right]\right\}H_{rj}^q(s)
\end{array}\right\}.
\end{aligned}
\tag{21}
$$

Equation (21) becomes:

$$
\begin{aligned}
\overline{F}_r(s)=F_r(s)&+\sum_{k_r=0}^{m_{\alpha_{rn}}-1} s^{\alpha_{rn}-k_r-1}\ u_{r,k_r}+\sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}}\tfrac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}\left[\sum_{k_r=0}^{m_{\alpha_{r(n-i)}}-1} s^{\alpha_{r(n-i)}-k_r-1}u_{r,k_r}\right]\\
&-C_{rn}(-1)^{\ell_{rn}}\left\{\tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\sum_{p=0}^{q_r}(-1)^{q_r-p}\ p!\begin{pmatrix} q_r\\ p\end{pmatrix}\tfrac{\tau_r{}^{q_r-p}}{s^{p+1}}\right]-\tfrac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}\left[\tfrac{q_r!}{s^{q_r+1}}e^{-s\tau_r}\right]\right\}\\
&+\sum_{j=0}^{m}\lambda_{rj}\tfrac{1}{s}\left\{c_{rj}\left[\sum_{b=0}^{k_{rj}^1}(-1)^{b+k_{rj}^1}b!\begin{pmatrix} k_{rj}^1\\ b\end{pmatrix}\tfrac{1}{s^b}\frac{d^{k_{rj}^1-b}}{ds^{k_{rj}^1-b}}\right]+d_{rj}\left[(-1)^{k_{rj}^2}\frac{d^{k_{rj}^2}}{ds^{k_{rj}^2}}\right]\right\}H_{rj}^q(s).
\end{aligned}
$$

As a special case, if the $P_{ri}(t)$ and $P_{rn}(t)$ are the only constants, this means that ℓ_{ri} and ℓ_{rn} are equal to zero. Thus, after some simple manipulations, system Equation (21) was formed, and we obtained the following system, for all $r=0,1,\ldots,m$:

$$
H_r(s)U_r(s)-\sum_{\substack{j=0\\ j\neq r}}^{m}\lambda_{rj}\,\overline{K}_{rj}(s)e^{-s\tau_{rj}}U_j(s)=\overline{F}_r(s). \tag{22}
$$

where:

$$
H_r(s)=s^{\alpha_{rn}}+\sum_{i=1}^{n-1} C_{ri}(-1)^{\ell_{ri}}\frac{d^{\ell_{ri}}}{ds^{\ell_{ri}}}s^{\alpha_{r(n-i)}}+C_{rn}\ (-1)^{\ell_{rn}}\frac{d^{\ell_{rn}}}{ds^{\ell_{rn}}}e^{-s\tau_r}-\lambda_{rr}\,\overline{K}_{rr}(s)e^{-s\tau_{rr}}.
$$

and:

$$
\begin{aligned}
\overline{K}_{rj}(s)=&\left[c_{rj}\left(\sum_{b=0}^{k_{rj}^1} b!\begin{pmatrix} k_{rj}^1\\ b\end{pmatrix}\tfrac{1}{s^b}\tau_{rj}{}^{k_{rj}^1-b}\right)+d_{rj}\tau_{rj}{}^{k_{rj}^2}\right]+\left[d_{rj}\sum_{b=0}^{k_{rj}^2-1}(-1)^{b+k_{rj}^2}\tau_{rj}{}^{b}\begin{pmatrix} k_{rj}^2\\ b\end{pmatrix}\tfrac{d^{k_{rj}^2-b}}{ds^{k_{rj}^2-b}}\right]\\
&+\left[c_{rj}\sum_{b=0}^{k_{rj}^1-1}(-1)^{b+k_{rj}^1}b!\begin{pmatrix} k_{rj}^1\\ b\end{pmatrix}\tfrac{1}{s^b}\left(\sum_{p=0}^{k_{rj}^1-b-1}(-1)^p\tau_{rj}{}^p\begin{pmatrix} k_{rj}^1-b\\ p\end{pmatrix}\tfrac{d^{k_{rj}^1-b-p}}{ds^{k_{rj}^1-b-p}}\right)\right].
\end{aligned}
$$

If the (HF) is any continuously differentiable function $\varphi_r(t)$. Consequently, there is an ordinary differential equation in $U_r(s), U_j(s)$, which is solved to find $U_r(s), U_j(s)$. Finally, the inverse of the Laplace transform is used on $U_r(s), U_j(s)$ to obtain the solution $u_r(t), u_j(t)$ for the system of integro-fractional differential equations with variable coefficients and multi-delays.

4. Analytic Examples

Here are some examples of the system of integro-fractional differential equations with variable coefficients and multi-delays, which were solved by Laplace transform method:

Example 1. *Consider the linear SIFDEs of the Volterra type with the constant multi-time delay and variable coefficients of retarded delay on* $[0,\ 1]$:

$$\begin{aligned} &{}_0^C D_t^{1.5} u_0(t) - t\, {}_0^C D_t^{0.5} u_0(t) - 3t\ u_0(t-1) \\ &\qquad = f_0(t) + \int_0^t \left[(t-x) u_0(x-2) - e^{t-x} u_1(x-1) \right] dx. \end{aligned} \tag{23}$$

$$\begin{aligned} &{}_0^C D_t^{0.9} u_1(t) - \tfrac{1}{2}\, {}_0^C D_t^{0.5} u_1(t) + \tfrac{1}{2}\ u_1(t-0.2) \\ &\qquad = f_1(t) + \int_0^t \left[(t-x) u_0(x-0.3) + (t-x)^2 u_1(x-0.5) \right] dx. \end{aligned} \tag{24}$$

where:

$$f_0(t) = \frac{2}{\Gamma(1.5)} t^{0.5} + \frac{2}{\Gamma(2.5)} t^{2.5} + e^t - \frac{1}{12} t^4 - \frac{7}{3} t^3 + 4t^2 - 5t - 1.$$

$$f_1(t) = \frac{2}{\Gamma(1.1)} t^{0.1} - \frac{1}{\Gamma(1.5)} t^{0.5} - \frac{1}{4} t^4 + 0.1\ t^3 - 0.045\ t^2 + t + 0.3.$$

with historical function (HF) and initial condition $u_0(0) = 0; u_0'(0) = 0; \varphi_0(t) = t^2; u_1(0) = 1;\ \ u_1'(0) = 2; \varphi_1(t) = 2t+1$, *so we have:*$\mathcal{K}_{0,1}(t,x) = (t-x)\ ; \mathcal{K}_{0,2}(t,x) = e^{t-x}, \mathcal{K}_{1,1}(t,x) = (t-x); \mathcal{K}_{1,2}(t,x) = (t-x)^2$ *and* $\tau_0 = 1, \tau_{0,1} = 2, \tau_{0,2} = 1, \tau_1 = 0.2,\ \tau_{1,1} = 0.3\ ; \tau_{1,2} = 0.5$, *which are constant different time delays, and* $P_{0,1}(t) = t,\ P_{0,2}(t) = -3t,\ P_{1,1}(t) = \frac{-1}{2}, P_{1,2}(t) = \frac{1}{2}$ *are variable coefficients.*

The Laplace transform is taken to the above equation and Equations (17) and (18) are used to obtain:

$$H_0(s) U_0(s) + \frac{e^{-s}}{(s-1)} U_1(s) = \overline{F}_0(s). \tag{25}$$

$$H_1(s) U_1(s) - \frac{e^{-0.3s}}{s^2} U_0(s) = \overline{F}_1(s). \tag{26}$$

where:

$$H_0(s) = s^{1.5} - \frac{d}{ds} s^{0.5} + 3 \frac{d}{ds} e^{-s} - \frac{1}{s^2} e^{-2s}.$$

$$H_1(s) = s^{0.9} - \frac{1}{2} s^{0.5} + \frac{1}{2} e^{-0.2s} - \frac{2}{s^3} e^{-0.5s}.$$

and:

$$\overline{F}_0(s) = \frac{2}{s^{1.5}} + \frac{5}{s^{3.5}} - \frac{6\ e^{-s}}{s^3} - \frac{18\ e^{-s}}{s^4} - \frac{2\ e^{-2s}}{s^5} + \frac{2e^{-s}}{s^2(s-1)} + \frac{e^{-s}}{s(s-1)}.$$

$$\overline{F}_1(s) = \frac{2}{s^{1.1}} - \frac{1}{s^{1.5}} + \frac{1}{s^{0.1}} - \frac{1}{2s^{0.5}} + \frac{e^{-0.2s}}{s^2} + \frac{e^{-0.2s}}{2s} - \frac{2e^{-0.3s}}{s^5} - \frac{4e^{-0.5s}}{s^5} - \frac{2e^{-0.5s}}{s^4}.$$

After substituting Equation (25) into Equation (26) and solving this with $U(\infty) = 0$, which is ODE of the first order, the following is obtained: $U_0(s) = \frac{2}{s^3}$.

By substituting $U_0(s)$ into one of either Equation (25) or Equation (26), we obtain: $U_1(s) = \frac{2}{s^2} + \frac{1}{s}$.

By taking the inverse of the Laplace transform of $U_0(s)$ and $U_1(s)$, the exact solutions, $u_0(t)$ and $u_1(t)$, are obtained from Equations (22) and (23): $u_0(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s^3} \right\} = t^2; u_1(t) = \mathcal{L}^{-1}\left\{ \frac{2}{s^2} + \frac{1}{s} \right\} = 2t + 1$, which is the exact solution for our given system.

Example 2. *Consider linear SIFDEs of a constant multi-time retarded delay with variable coefficients on* $[0,\ 1]$:

$$ {}_{0}^{C}D_{t}^{1.3}u_0(t) + 2\,u_0(t-0.4) = f_0(t) + \int_0^t (t+x)u_1(x-1)\,dx. \tag{27} $$

$$ {}_{0}^{C}D_{t}^{0.8}u_1(t) - \frac{1}{2}\,{}_{0}^{C}D_{t}^{0.5}u_1(t) = f_1(t) + \int_0^t \left(2t+2x^2\right)u_1(x-0.2)\,dx. \tag{28} $$

$$ \text{where}:\ f_0(t) = \frac{1}{\Gamma(1.7)}t^{0.7} - \frac{5t}{6}t^3 + t^2 - 0.8\,t + 0.16. $$

$$ f_1(t) = \frac{1}{\Gamma(1.2)}t^{0.2} - \frac{1}{2\,\Gamma(1.5)}t^{0.5} - \frac{1}{5}t^5 - \frac{0.7}{3}\,t^4 + \frac{0.64}{3}\,t^3 + 0.04\,\ t^2. $$

With initial conditions and historical functions:

$u_0(0) = 0; u_0'(0) = 0; \varphi_0(t) = \frac{1}{2}t^2$; $u_1(0) = 1; u_1'(0) = 2; \varphi_1(t) = t+1$, *since here we have:* $\mathcal{K}_{0,1}(t,x) = (t+x)$; $\mathcal{K}_{1,1}(t,x) = \left(2t+2x^2\right)$; $\tau_0 = 0.4,\ \tau_{0,1} = 1, \tau_{1,1} = 0.2$, *which are constant different time delays, and* $P_{0,2}(t) = 2,\ P_{1,1}(t) = \frac{-1}{2}$ *; are variable coefficients.*

Taking the Laplace transform for the above equation and using Equations (21) and (22), we obtain:

$$ H_0(s)U_0(s) + \sum_{\substack{j=0 \\ j\neq r}}^{m} \lambda_{01}\,\overline{K}_{01}(s)e^{-s\tau_{01}}U_1(s) = \overline{F}_0(s). \tag{29} $$

$$ H_1(s)U_1(s) + \sum_{\substack{j=0 \\ j\neq r}}^{m} \lambda_{10}\,\overline{K}_{10}(s)e^{-s\tau_{10}}U_0(s) = \overline{F}_1(s). \tag{30} $$

where:

$$ \overline{K}_{01}(s) = \frac{1}{s}\left\{2 + \frac{1}{s} - 2\frac{d}{ds}\right\}. $$

$$ \overline{K}_{10}(s) = \frac{1}{s}\left\{0.48 + \frac{2}{s} + 2\frac{d^2}{ds^2} - 2.8\frac{d}{ds}\right\}. $$

$$ H_0(s) = s^{1.3} + 2e^{-0.4s}. $$

$$ H_1(s) = s^{0.8} - \frac{1}{2}s^{0.5}. $$

and:

$$ \overline{F}_0(s) = \frac{1}{s^{1.7}} + \frac{2e^{-0.4s}}{s^3} - \frac{5\,e^{-s}}{s^3} - \frac{5\,e^{-s}}{s^4} - \frac{2\,e^{-s}}{s^2}. $$

$$ \overline{F}_1(s) = \frac{1}{s^{1.2}} - \frac{1}{2s^{1.5}} + \frac{1}{s^{0.2}} - \frac{1}{2s^{0.5}} - \frac{10.4\,e^{-0.2s}}{s^5} - \frac{0.48\,e^{-0.2s}}{s^4} - \frac{24\,e^{-0.2s}}{s^6}. $$

After substituting Equation (29) into Equation (30) and with $U(\infty) = 0$, which is ODE of the first order, after solving it, the following is obtained:$U_0(s) = \frac{1}{s^3}$. Next, by substituting $U_0(s)$ in either one of Equation (29) or Equation (30), we obtain: $U_1(s) = \frac{2}{s} + \frac{1}{s}$. By taking the inverse of the Laplace transform of $U_0(s)$ and $U_1(s)$, the exact solutions, $u_0(t)$ and $u_1(t)$, are obtained from Equations (27) and (28): $u_0(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$; $u_1(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2} + \frac{1}{s}\right\} = t+1$, which are the exact solutions for our given system.

5. Discussion

In this work, after using the Laplace transform to solve a linear system of integro-fractional differential equations of the Volterra type with variable coefficients and multi-time retarded delay using some illustrating examples, we found the following:

1. Generally, this method which was amended here, provided good results and validation.
2. Here: we successfully applied the Laplace transform method for two different types of kernels, which were difference and simple degenerate kernels.

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