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On the Aggregation of Comonotone or Countermonotone Fuzzy Relations

Yuanyuan Liu and Fan Jia *

School of Management Science and Engineering, Shangdong University of Finance and Economics, Jinan 250014, China; liuyuanyuan@shu.edu.cn
* Correspondence: fanta07@126.com

Abstract: The properties of fuzzy relations have been extensively studied, and the preservation of their properties plays a fundamental role in the various applications. However, either sufficient or necessity conditions for the preservation requires the aggregated functions of fuzzy relations to dominate or to be dominated by the corresponding operations, which constructs a significant limitation on applicable functions. This work concentrates on the preservation of transitivities and Ferrers property for the aggregation of comonotone or countermonotone fuzzy relations. Firstly, definitions of comonotonicity and countermonotonicity for binary functions are initially proposed. On the foundation of that, the relations of commuting and bisymmetry between min/max and commonly used increasing/decreasing functions are found. Afterwards, with the condition that underlying fuzzy relations are pair-wisely comonotone or countermonotone, theorems on the aggregation functions which can preserve the transitivities and the Ferrers property are proposed. Moreover, an interesting conclusion that the equivalent condition for the min-Ferrers property of fuzzy relations is clarified.

Keywords: fuzzy relation; aggregation; comonotonicity; countermonotonicity; transitivity; ferrers property

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1. Introduction

The use of fuzzy relations, a natural generalization of crisp relations, has become widespread due to their expressive power. Both in theoretical developments and practical applications, their properties are widely studied and play a fundamental role. These properties include reflexivity, (a)symmetry, transitivity, the Ferrers property, and so on [1–5]. In fields such as preference modelling, decision making and approximate reasoning, the aggregation of fuzzy relations is a common theme and, in particular, the preservation of said properties by such aggregation process.

Many researchers have contributed to the study of this preservation of properties [6–11]. More specifically, an n-ary function $F : [0,1]^n \to [0,1]$ is said to preserve a given property, if for any fuzzy relations $R_1, R_2, \ldots, R_n$ having this property, the aggregated result $R_F = F(R_1, R_2, \ldots, R_n)$ also has this property [12]. Under this overall consideration for underlying fuzzy relations, many remarkable studies made efforts on providing weak conditions on the functions of fuzzy relations to achieve the preservation.

In this paper, we focus in particular on the preservation of transitivity properties and the Ferrers property of fuzzy relations. As for the preservation of other properties, we refer to [13]. Recall that an n-ary ($n \geq 2$) aggregation function is a mapping $A : [0,1]^n \to [0,1]$ that is increasing (i.e. $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$) and satisfies the boundary conditions $A(0, \ldots, 0) = 0$, $A(1, \ldots, 1) = 1$. Inspired by the work of De Baets and Mesiar on the refinement of fuzzy partitions [14], Saminger et al. [12] showed that a necessary and sufficient condition for the preservation of T-transitivity of fuzzy relations is that the aggregation function A involved dominates the t-norm T, denoted as $A \gg T$, with the notion of dominance borrowed from the field of probabilistic metric spaces [15,16]. Continuing along this line, Drewniak [17] gave an extension on the applicable transitivities and the aggregated function of fuzzy relations for the preservation, and separately proposed the sufficient condition for any increasing n-ary functions $F : [0,1]^n \to [0,1]$ preserving B-transitivity of fuzzy relations where B can be any binary operation: $[0,1]^2 \to [0,1]$, and the necessary condition for any n-ary functions $F : [0,1]^n \to [0,1]$ preserving B-transitivity where B has the zero element $z = 0$. Both of the above sufficient and necessary condition presented in [17] required $F \gg B$.

Furthermore, in [13], the authors extended their study in [17] to negative transitivity, semitransitivity and the Ferrer property. With respect to increasing n-ary operation F: $[0,1]^n \to [0,1]$, they proposed the sufficient and necessary condition for the preservation of negative B-transitivity, i.e., $B \gg F$, where B has the zero element $z = 1$, and sufficient conditions for preservation of B_1-B_2-semitransitivity and the B_1-B_2-Ferrers property, i.e., $F \gg B_1$ and $B_2 \gg F$, where B_1 and B_2 are any binary operations: $[0,1]^2 \to [0,1]$.

Regarding the above advanced results for the preservation of tansitivities or the Ferrers property, we can notice that there are several restrictions. First of all is their definition for the preservation of the properties from an overall aspect. All the fuzzy relations having that property are taken into account to constraint the aggregated function to preserve the property. In fact, it is this restriction that results in the strong necessary condition on the aggregated function for the preservation. The second restriction is that either the sufficient condition or the necessary condition for the preservation asks the aggregated functions to dominate or to be dominated by the corresponding t-norms, t-conorms, or the general binary operations B. According to the definition of dominance, F dominates B if for an arbitrary matrix $[x_{ij}] = X \in [0,1]^{n\times 2}$, $F(B(x_{11}, x_{12}), \ldots, B(x_{n1}, x_{n2})) \geq B(F(x_{11}, \ldots, x_{n1}), F(x_{12}, \ldots, x_{n2}))$. No doubt that this is a very strict constraint for the function F, especially w.r.t. transitivities and the Ferrers properties involving min or max since any increasing function dominates max and is dominated by min, and there are a few functions could dominate min or be dominated by max, thus the applications are badly limited. Furthermore, based on the connections between dominance and commuting, the constraint on the aggregated function F actually requires that F commutes with min or max. The third limitation of existing results is the presumption for the functions of fuzzy relations, i.e., increasing functions, or the aggregation operators. We expect they could be generalized as much as possible to expand their applicable problems. Therefore, we can conclude that all these advanced results are meant to guarantee their methods adaptable for any fuzzy relations with the property, however, in order to achieve this, some redundant conditions have to impose on the aggregated function, which finally result that a few functions are available for the preservation.

Out of the above analysis, different from the investigates trying to weaken the restriction on the aggregated function F to preserve the property for all fuzzy relations with that property, we turn to focus on the fuzzy relations with some reasonable characteristics thereby accordingly give practical suggestions on aggregated functions for the preservation. Firstly, we initially give the definitions of comonotonicty and countermonotonicty for binary functions, then w.r.t. pair-wisely comonotone or countermonotone functions, we prove the commuting or bisymmetric relation for min or max with the general functions which are increasing or decreasing in each argument. Secondly, regarding the fuzzy relations with the characteristic of pair-wisely comonotone or countermonotone, some theorems are proposed for the preservation of the properties that are tough to handle with the existing results, like min-transitivity, negative max-transitivity, min-semitransitivity, and min-Ferrers property under the aggregation with general functions. Besides the more effective methods for users to preserve the fuzzy relations property, the transforming of min-transitivity and negative max-transitivity during the aggregated process with any decreasing function is detected. Furthermore, another interesting finding is the equivalent relation between the min-Ferrers property of fuzzy relations and their self-comonotonicity in one argument at any.

The paper is organized as follows. Section 2 introduces some basic concepts, e.g., fuzzy relations, t-norm (t-conorm), aggregation operator, dominance, etc., and briefly recalls the advanced results for the preservation of tansitivities and the Ferrers property. Section 3 presents the new definitions of comonotoncity and countermonotonicity and proves the relations of min and max with general functions which are the foundations for the preservation. Section 4 puts forward the results on preservations and transformations of fuzzy relations properties.

2. Preliminaries

In this section, the relevant concepts including fuzzy relation, the commonly studied properties of fuzzy relations, aggregated fuzzy relation, etc., are presented. In addition, advanced theorems proposed in [12,13,17] for the preservation of fuzzy relations properties are also recalled, which are the foundations of our work.

Definition 1 (See [18])**.** *A fuzzy relation on a set $X \neq \varnothing$ is an arbitrary function $R : X^2 \to [0,1]$. The family of all fuzzy relations on X is denoted by $FR(X)$.*

Along with the proposition of fuzzy relations, their properties are studied. In our work, we mainly focus on the tansitivities and the Ferrers property.

Definition 2 (See [3])**.** *Let T be a t-norm and S a t-conorm. A fuzzy relation R on X is called*
(i) T-transitive if $\forall x, y, z \in X$,

$$T(R(x,y), R(y,z)) \leq R(x,z);$$

(ii) negatively S-transitive if $\forall x, y, z \in X$,

$$R(x,z) \leq S(R(x,y), R(y,z));$$

(iii) T-S-semitransitive if $\forall x, y, z, w \in X$,

$$T(R(x,y), R(y,z)) \leq S(R(x,w), R(w,z));$$

(iv) T-S-Ferrers if $\forall x, y, z, w \in X$,

$$T(R(x,y), R(z,w)) \leq S(R(x,w), R(z,y)).$$

Definition 3 (See [15])**.** *A binary operation T: $[0,1]^2 \to [0,1]$ is called a t-norm if it satisfies:*
(i) Neutral element 1: $\forall x \in [0,1], T(x,1) = T(1,x) = x$;
(ii) Monotonicity: T is increasing in each variable.
(iii) Commutativity: $\forall (x,y) \in [0,1]^2, T(x,y) = T(y,x)$;
(iv) Associativity: $\forall (x,y,z) \in [0,1]^3, T(x, T(y,z)) = T(T(x,y), z)$;
Any t-norm T corresponds a dual t-conorm S defined by $S(x,y) = 1 - T(1-x, 1-y)$.

Three popular t-norms are given by:
(i) the Lukasiewicz t-norm $T_L(x,y) = \max(0, x+y-1)$;
(ii) the algebraic product $T_P(x,y) = xy$;
(iii) the minimum operator $T_M(x,y) = \min(x,,y)$.

The corresponding dual t-conorms are as follows.
(i) the bounded sum t-conorm $S_L(x,y) = \min(x+y, 1)$;
(ii) the probabilistic sum $S_P = x + y - xy$;
(iii) the maximum operator $S_M(x,y) = \max(x,y)$.

It is well-known that t-norms and t-conorms are special cases of fuzzy conjunctions and fuzzy disjunctions, respectively.

Definition 4 (See [19]). *An operation* $C : [0,1]^2 \to [0,1]$ *is called a fuzzy conjunction if it is increasing and*

$$C(1,1) = 1, \ \ C(0,0) = C(0,1) = C(1,0) = 0.$$

An operation $D : [0,1]^2 \to [0,1]$ *is called a fuzzy disjunction if it is increasing and*

$$D(0,0) = 0, \ \ D(1,1) = C(0,1) = C(1,0) = 1.$$

Definition 5 (See [12]). *Let* $F : [0,1]^n \to [0,1]$, $R_1, \dots, R_n \in FR(X)$. *An aggregated fuzzy relation* $R_F \in FR(X)$ *is described by the formula*

$$R_F(x,y) = F(R_1(x,y), \dots, R_n(x,y)), \ \ x, y \in X.$$

The paper [12] shows the close connection between the preservation of T-transitivity for an aggregated fuzzy relation and the dominance of that aggregation operator over the corresponding t-norm T. The definitions of aggregation operator and dominance are as follows.

Definition 6 (See [20]). *A function* $A : \bigcup\limits_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ *is called an n-ary aggregation operator if it fulfills* $A(x_1, \dots, x_n) \leq A(y_1, \dots, y_n)$ *whenever* $x_i \leq y_i$ *for all* $i \in 1, \dots, n$ *and* $A(0, \dots, 0) = 0$, $A(1, \dots, 1) = 1$.

Definition 7 (See [12]). *Consider an n-ary aggregation operator* $A_{(n)}$ *and an m-ary aggregation operator* $B_{(m)}$. *We say that* $A_{(n)}$ *dominates* $B_{(m)}$ ($A_{(n)} \gg B_{(m)}$) *if for an arbitrary matrix* $[x_{ij}] = X \in [0,1]^{m \times n}$,

$$B_{(m)}(A_{(n)}(x_{11}, \dots, x_{1n}), \dots, A_{(n)}(x_{m1}, \dots, x_{mn})) \leq A_{(n)}(B_{(m)}(x_{11}, \dots, x_{m1}), \dots, B_{(m)}(x_{1n}, \dots, x_{mn})).$$

Theorem 1 (See [12]). *Let* T *be an arbitrary t-norm. An aggregated fuzzy relation* R_F *preserves the* T*-transitivity if and only if the aggregation operator* F *dominates* T, *i.e.,* $F \gg T$.

In [13,17], authors devote their work to generalize the conclusion for the preservation in Theorem 1 by weakening restrictions on the aggregated functions of fuzzy relations and extending the T-transitivity to other properties. In the following theorems, unless otherwise specified, B, B_1, B_2 are any binary operations: $[0,1]^2 \to [0,1]$, and F is an n-ary function: $[0,1]^n \to [0,1]$.

Theorem 2 (See [17]). *Let* B *have the zero element* $z = 0$. *If* $R_1, \dots, R_n$ *have the* B*-transitivity, and* R_F *preserves* B*-transitivity, then* $F \gg B$.

Theorem 3 (See [17]). *Let* F *be an increasing function. If* $R_1, \dots, R_n$ *have the* B*-transitivity, and* $F \gg B$, *then* R_F *preserves* B*-transitivity.*

Theorem 4 (See [13]). *Let* B *have the zero element* $z = 1$, *and* F *be an increasing function. Assume that* $R_1, \dots, R_n$ *have negative* B*-transitivity, then* R_F *preserves the negative* B*-transitivity if and only if* $B \gg F$.

Theorem 5 (See [13]). *Let* F *be an increasing function. If* $R_1, \dots, R_n$ *have the* B_1*-*B_2*-semitransitivity, and* $F \gg B_1$, $B_2 \gg F$, *then* R_F *preserves the* B_1*-*B_2*-semitransitivity.*

Theorem 6 (See [13]). *Let* F *be an increasing function. If* $R_1, \dots, R_n$ *have the* B_1*-*B_2*-Ferrers property, and* $F \gg B_1$, $B_2 \gg F$, *then* R_F *preserves the* B_1*-*B_2*-Ferrers property.*

As we can seen that no matter the preservation of T-transitivity in [12] or the preservations of B-transitivity, negative B-transitivity, B_1-B_2-semitransitivity and B_1-B_2-Ferrers property in [13,17], the dominance of the aggregation operators over the corresponding t-norms or the binary operations is included in the theorems. However, this condition is not easy to be achieved, especially for the transitivities and the Ferrers properties related with min or max. As is well known, the min dominates any increasing function, while the max is dominated by any increasing function. A few increasing functions could dominate the min and be dominated by the max.

Theorem 7. *An n-ary increasing function* $F : [0,1]^n \to [0,1]$ *dominates* min *if and only if for each* $x_1, \ldots, x_n \in [0,1]$, $F(x_1, \ldots, x_n) = \min(f_1(x_1), \ldots, f_n(x_n))$, *where* $f_i : [0,1] \to [0,1]$ *is increasing with* $i = 1, \ldots, n$.

Theorem 8. *An n-ary increasing function* $F : [0,1]^n \to [0,1]$ *is dominated by* max *if and only if for each* $x_1, \ldots, x_n \in [0,1]$, $F(x_1, \ldots, x_n) = \max(f_1(x_1), \ldots, f_n(x_n))$, *where* $f_i : [0,1] \to [0,1]$ *is increasing with* $i = 1, \ldots, n$.

Therefore, for min-tansitivity, negative max-transitivity, min-semitransitivity and min-Ferrers property, a few increasing functions can satisfy the constraints in Theorems 3–6. Removing the condition of aggregation operator in the definition of dominance, n-ary functions satisfying Theorem 2 are still very limited. Meanwhile, note that the sufficient and necessary conditions in Theorems 3–6 actually imply a requirement for the relation of commuting or bisymmetry between the function F and min or max. The definition of commuting is originally defined for aggregation operators, and an extension of that for any functions is given in next section.

Definition 8. *Consider an n-ary aggregation operator* $A_{(n)}$ *and an m-ary aggregation operator* $B_{(m)}$. *We say that* $A_{(n)}$ *and* $B_{(m)}$ *commute with each other if for all* $x_{i,j} \in [0,1]$ *with* $i \in \{1, \ldots, m\}$ *and* $j \in \{1, \ldots, n\}$, *the following equation holds:*

$$B_{(m)}(A_{(n)}(x_{11}, \ldots, x_{1n}), \ldots, A_{(n)}(x_{m1}, \ldots, x_{mn})) = A_{(n)}(B_{(m)}(x_{11}, \ldots, x_{m1}), \ldots, B_{(m)}(x_{1n}, \ldots, x_{mn})).$$

The conditions for any increasing function that dominates min or is dominated by max are adaptable for the commuting with min or max (cf. [21], Proposition 21). What is more, two operators commuting with each other is a special case of generalized bisymmetry.

Definition 9 (See [22])**.** *The functional equation of* $m \times n$ *generalized bisymmetry, or the GEB for short, is*

$$G(F_1(x_{11}, \ldots, x_{1n}), \ldots, F_m(x_{m1}, \ldots, x_{mn})) = F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots, x_{mn})).$$

3. Comonotonicity and Countermonotonicity

In this section, definitions of comonotonicity and countermonotonicity for binary functions are introduced firstly. Then, the commuting and bisymmetric relations between min or max and some commonly used functions are proposed.

3.1. Definitions of Comonotonicity and Countermonotonicity

The paper [23] gives the definition of comonotonicity for functions with one argument. We extend that work into binary functions.

Definition 10 (See [23])**.** *Let* Ω *be a non-empty set. Two functions* $f, g : \Omega \to \mathbb{R}$ *are said to be comonotone, if for all* $x, y \in \Omega$,

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq 0.$$

Definition 11. *(Comonotonicity) Let Ω be a non-empty set. Two functions $F, G : \Omega^2 \to \mathbb{R}$ are said to be comonotone, if for all $x_1, x_2, y_1, y_2 \in \Omega$,*

$$\big(F(x_1, y_1) - F(x_2, y_2)\big)\big(G(x_1, y_1) - G(x_2, y_2)\big) \geq 0.$$

Example 1. *PD and NSD are comonotone for any fuzzy intervals.*

Proof. The indices PD and NSD were proposed in [24] for comparing fuzzy intervals. Let M and N be two fuzzy intervals with membership functions π_M and π_N, then

$$PD(M, N) = \sup_{u \geq v} \min\big(\pi_M(u), \pi_N(v)\big),$$

$$NSD(M, N) = 1 - \sup_{u \leq v} \min\big(\pi_M(u), \pi_N(v)\big).$$

In addition, it is known that (i) $PD(M, N) + NSD(N, M) = 1$ (ii) $\max(PD(M, N), PD(N, M)) = 1$. From (i), we can get $(PD(X, Y) - PD(Z, W))(NSD(X, Y) - NSD(Z, W)) = (PD(X, Y) - PD(Z, W))(PD(W, Z) - PD(Y, X))$.

Based on (ii), there are four cases in total: $PD(X, Y) = PD(Z, W) = 1$, $PD(X, Y) = PD(W, Z) = 1$, $PD(Y, X) = PD(Z, W) = 1$, and $PD(Y, X) = PD(W, Z) = 1$. It can always be deduced that $(PD(X, Y) - PD(Z, W))(NSD(X, Y) - NSD(Z, W)) \geq 0$ for each case, which implies that PD and NSD are comonotone. □

Furthermore, the Definition 11 of comonotonicity can be reformulated as follows.

Lemma 1. *Functions F, $G : \Omega^2 \to \mathbb{R}$ are comonotone if and only if for all $x_1, x_2, y_1, y_2 \in \Omega$,*

$$F(x_1, y_1) > F(x_2, y_2) \;\; \textit{implies} \;\; G(x_1, y_1) \geq G(x_2, y_2).$$

Lemma 2. *For functions $F, G, H : \Omega^2 \to \mathbb{R}$:*
(i) if both F and G are comonotone with H, then $F + G$ and H are comonotone;
(ii) if F, G and H are pair-wisely comonotone, then $\max(F, G)$ and H are comonotone, as are $\min(F, G)$ and H.

Remark 1. *If both F and G are comonotone with H, there is no conclusion that F and G are comonotone.*

For example, suppose F, G, H are functions: $\Omega^2 \to \mathbb{R}$. H is a constant function, F is strictly increasing in the first argument, while G is strictly decreasing in the first argument. We can know that both F and G are comonotone with H, however, F and G are not comonotone since for all $y \in \Omega$, if $x_1 \neq x_2$, $(F(x_1, y) - F(x_2, y))(G(x_1, y) - G(x_2, y)) < 0$.

Definition 12. *(Countermonotonicity) Let Ω be a non-empty set. Two functions $F, G : \Omega^2 \to \mathbb{R}$ are said to be countermonotone, if for all $x_1, x_2, y_1, y_2 \in \Omega$,*

$$\big(F(x_1, y_1) - F(x_2, y_2)\big)\big(G(x_1, y_1) - G(x_2, y_2)\big) \leq 0.$$

This definition can be reformulated as follows.

Lemma 3. *Functions F and $G : \Omega^2 \to \mathbb{R}$ are countermonotone, if and only if for all $x_1, x_2, y_1, y_2 \in \Omega$,*

$$F(x_1, y_1) > F(x_2, y_2) \;\; \textit{implies} \;\; G(x_1, y_1) \leq G(x_2, y_2).$$

Lemma 4. *For functions $F, G, H : \Omega^2 \to \mathbb{R}$:*
(i) if F and G are countermonotone with H, respectively, then $F + G$ and H are countermonotone;
(ii) if F, G and H are pair-wisely countermonotone, then $\max(F, G)$ and H are countermonotone, as are $\min(F, G)$ and H.

Similarly with the situation of comonotonicity, given both functions F and G are countermonotone with H, there's no conclusion that F and G are countermonotone or comonotone.

3.2. Relations with Commuting and Bisymmetry

Ahead of proposing the theorems on the preservation of fuzzy relations properties, some employed results are presented firstly in this part. The aggregated functions for fuzzy relations in our work are not restricted to be aggregation operators or increasing functions any more, and some corresponding extensions are given as follows.

The notions of dominance and commuting w.r.t. aggregation operators can be generalized for any functions, i.e., we can say that an n-ary function $F:\Omega^n \to \mathbb{R}$ dominates an m-ary function $G:\Omega^m \to \mathbb{R}$ ($F \gg G$) if for any matrix $[x_{ij}] = X \in \Omega^{m\times n}$,

$$G(F(x_{11},\ldots,x_{1n}),\ldots,F(x_{m1},\ldots,x_{mn})) \leq F(G(x_{11},\ldots,x_{m1}),\ldots,G(x_{1n},\ldots,x_{mn})),$$

and F, G commute with each other if

$$G(F(x_{11},\ldots,x_{1n}),\ldots,F(x_{m1},\ldots,x_{mn})) = F(G(x_{11},\ldots,x_{m1}),\ldots,G(x_{1n},\ldots,x_{mn})).$$

Proposition 1. *Any n-ary increasing function* commutes *with* min *for inputs generated by pair-wisely comonotone functions $F_1,\ldots,F_n$: $\Omega^2 \to \mathbb{R}$.*

Proof. An interpretation of Proposition 1: let A be an arbitrary n-ary increasing function, for all $x_1, x_2, y_1, y_2 \in \Omega$, $A\big(\min(F_1(x_1,y_1),F_1(x_2,y_2)),\ldots,\min(F_n(x_1,y_1),F_n(x_2,y_2))\big) = \min\big(A(F_1(x_1,y_1),\ldots,F_n(x_1,y_1)), A(F_1(x_2,y_2),\ldots,F_n(x_2,y_2))\big)$.

Case 1: Suppose there exists $F_i(x_1,y_1) > F_i(x_2,y_2), 1 \leq i \leq n$. Since $F_1,\ldots,F_n$ are pair-wisely comonotone, we have

$$F_i(x_1,y_1) \geq F_i(x_2,y_2), i = 1,\ldots,n.$$

A is an increasing function, so

$$A\big(F_1(x_1,y_1),\ldots,F_n(x_1,y_1)\big) \geq A\big(F_1(x_2,y_2),\ldots,F_n(x_2,y_2)\big).$$

Then, we obtain

$$\begin{aligned} &A\big(\min(F_1(x_1,y_1),F_1(x_2,y_2)),\ldots,\min(F_n(x_1,y_1),F_n(x_2,y_2))\big) \\ &= A\big(F_1(x_2,y_2),\ldots,F_n(x_2,y_2)\big) \\ &= \min\big(A(F_1(x_1,y_1),\ldots,F_n(x_1,y_1)), A(F_1(x_2,y_2),\ldots,F_n(x_2,y_2))\big). \end{aligned}$$

Case 2: Suppose there exists $F_i(x_1,y_1) < F_i(x_2,y_2), 1 \leq i \leq n$. It is similar with case 1, and is omitted.

Case 3: Suppose for $i = 1,\ldots,n$, $F_i(x_1,y_1) = F_i(x_2,y_2)$. The equality for the proposition can be derived immediately. □

Corollary 1. *Any n-ary increasing function* commutes *with* max *for inputs generated by pair-wisely comonotone functions $F_1,\ldots,F_n$: $\Omega^2 \to \mathbb{R}$.*

This can be obtained through similar analysis with the three cases in Proposition 1, and is omitted.

Proposition 2. *For pair-wisely comonotone functions $F_1,\ldots,F_n$: $\Omega^2 \to \mathbb{R}$, and an aribitrary n-ary decreasing function A, the* "generalized equation of bisymmetry"*: $A\big(\max(F_1(x_1,y_1),F_1(x_2,y_2)),\ldots,\max(F_n(x_1,y_1),F_n(x_2,y_2))\big) = \min\big(A(F_1(x_1,y_1),\ldots,F_n(x_1,y_1)), A(F_1(x_2,y_2),\ldots,F_n(x_2,y_2))\big)$ holds for all $x_1,x_2,y_1,y_2 \in \Omega$.*

Proof. Case 1: Suppose there exists $F_i(x_1, y_1) > F_i(x_2, y_2), 1 \leq i \leq n$. Since $F_1, \ldots, F_n$ are pair-wisely comonotone, we have

$$F_i(x_1, y_1) \geq F_i(x_2, y_2), i = 1, \ldots, n.$$

A is a decreasing function, so

$$A\big(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)\big) \leq A\big(F_1(x_2, y_2), \ldots, F_n(x_2, y_2)\big).$$

Then, we obtain

$$\begin{aligned}
&A\big(\max(F_1(x_1, y_1), F_1(x_2, y_2)), \ldots, \max(F_n(x_1, y_1), F_n(x_2, y_2))\big) \\
&= A\big(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)\big) \\
&= \min\big(A(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)), A(F_1(x_2, y_2), \ldots, F_n(x_2, y_2))\big).
\end{aligned}$$

Case 2: Suppose there exists $F_i(x_1, y_1) < F_i(x_2, y_2), 1 \leq i \leq n$. It is similar with case 1, and is omitted.

Case 3: Suppose for $i = 1, \ldots, n$, $F_i(x_1, y_1) = F_i(x_2, y_2)$. The equality in the proposition can be derived immediately. □

Corollary 2. *For pair-wisely comonotone functions $F_1, \ldots, F_n$: $\Omega^2 \to \mathbb{R}$, and an arbitrary n-ary decreasing function A, the* "generalized equation of bisymmetry"*:* $A\big(\min(F_1(x_1, y_1), F_1(x_2, y_2)), \ldots, \min(F_n(x_1, y_1), F_n(x_2, y_2))\big) = \max\big(A(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)), A(F_1(x_2, y_2), \ldots, F_n(x_2, y_2))\big)$ *holds for all $x_1, x_2, y_1, y_2 \in \Omega$.*

The proof is omitted.

Proposition 3. *Given that functions $F_1, \ldots, F_n : \Omega^2 \to \mathbb{R}$, $F_i, i = 1, \ldots, m$ and $F_j, j = m + 1, \ldots, n$ are pair-wisely comonotone, respectively, and F_i and F_j are pair-wisely countermonotone. If an n-ary function $A(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ is increasing w.r.t. $x_1, \ldots, x_m$ and decreasing w.r.t. $x_{m+1}, \ldots, x_n$, the* "generalized equation of bisymmetry"*:* $A\big(\min(F_1(x_1, y_1), F_1(x_2, y_2)), \ldots, \min(F_m(x_1, y_1), F_n(x_2, y_2)), \max(F_{m+1}(x_1, y_1), F_{m+1}(x_2, y_2)), \ldots, \max(F_n(x_1, y_1), F_n(x_2, y_2))\big) = \min\big(A(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)), A(F_1(x_2, y_2), \ldots, F_n(x_2, y_2))\big)$ *holds for all $x_1, x_2, y_1, y_2 \in \Omega$.*

Proof. Case 1: Suppose there exists $F_i(x_1, y_1) > F_i(x_2, y_2), 1 \leq i \leq m$. According to the assumption, we have

$$F_i(x_1, y_1) \geq F_i(x_2, y_2), i = 1, \ldots, m,$$

and

$$F_j(x_1, y_1) \leq F_j(x_2, y_2), i = m + 1, \ldots, n.$$

The function A is increasing w.r.t. $x_1, \ldots, x_m$ and decreasing w.r.t. $x_{m+1}, \ldots, x_n$, so

$$A\big(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)\big) \geq A\big(F_1(x_2, y_2), \ldots, F_n(x_2, y_2)\big).$$

Then, we obtain

$$\begin{aligned}
A\big(&\min(F_1(x_1, y_1), F_1(x_2, y_2)), \ldots, \min(F_m(x_1, y_1), F_m(x_2, y_2)), \\
&\max(F_{m+1}(x_1, y_1), F_{m+1}(x_2, y_2)), \ldots, \max(F_n(x_1, y_1), F_n(x_2, y_2))\big) \\
&= A\big(F_1(x_2, y_2), \ldots, F_n(x_2, y_2)\big) \\
&= \min\big(A(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)), A(F_1(x_2, y_2), \ldots, F_n(x_2, y_2))\big).
\end{aligned}$$

Case 2: Suppose there exists $F_i(x_1, y_1) < F_i(x_2, y_2)$, $F_j(x_1, y_1) < F_j(x_2, y_2)$, or $F_j(x_1, y_1) > F_j(x_2, y_2), 1 \le i \le m, m+1 \le j \le n$. All these situations are similar with case 1, and omitted.

Case 3: Suppose for $k = 1, \ldots, n$, $F_k(x_1, y_1) = F_k(x_2, y_2)$. The equality in the proposition can be derived immediately. □

Corollary 3. *Given that functions $F_1, \ldots, F_n : \Omega^2 \to \mathbb{R}$, $F_i, i = 1, \ldots, m$ and $F_j, j = m+1, \ldots, n$ are pair-wisely comonotone, respectively, and F_i and F_j are pair-wisely countermonotone. If an n-ary function $A(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$ is increasing w.r.t. $x_1, \ldots, x_m$ and decreasing w.r.t. $x_{m+1}, \ldots, x_n$, the* "generalized equation of bisymmetry"*: $A\big(\max(F_1(x_1, y_1), F_1(x_2, y_2)), \ldots, \max(F_m(x_1, y_1), F_n(x_2, y_2)), \min(F_{m+1}(x_1, y_1), F_{m+1}(x_2, y_2)), \ldots, \min(F_n(x_1, y_1), F_n(x_2, y_2))\big) = \max\big(A(F_1(x_1, y_1), \ldots, F_n(x_1, y_1)), A(F_1(x_2, y_2), \ldots, F_n(x_2, y_2))\big)$ holds for all $x_1, x_2, y_1, y_2 \in \Omega$.*

Remark 2. *Binary operations* min *and* max *in Propositions 1–3 and Corollaries 1–3 can be extended to minimum and maximum for any n arguments $(n \ge 2)$, respectively, and they all hold.*

4. Preservation of Transitivities and the Ferrers Properties

In this section, based on the conclusions in Section 3, theorems on the preservations of min-transitivity, negative max-transitivity, min-semitransitivity and min-Ferrers property for the aggregation of pair-wisely comonotone or countermonotone fuzzy relations are presented.

4.1. Preservation of Transitivities

Theorem 9. *Given that F: $[0,1]^n \to [0,1]$ is an increasing function. If all fuzzy relations $R_1, \ldots, R_n$ are* min*-transitive and pair-wisely comonotone, the aggregated relation R_F is* min*-transitive.*

Proof. Fuzzy relations R_i, $i = 1, \ldots, n$ are min-transitive, then $\min(R_i(x, y), R_i(y, z)) \le R_i(x, z)$. Based on Proposition 1, any increasing function commutes with min for pair-wisely comonotone fuzzy relations. We obtain

$$\begin{aligned}\min(R_F(x,y), R_F(y,z)) &= \min\big(F(R_1(x,y), \ldots, R_n(x,y)), F(R_1(y,z), \ldots, R_n(y,z))\big)\\ &= F\big(\min(R_1(x,y), R_1(y,z)), \ldots, \min(R_n(x,y), R_n(y,z))\big)\\ &\le F(R_1(x,z), \ldots, R_n(x,z)) = R_F(x,z).\end{aligned}$$

As a result, the min-transitivity of R_F is proved. □

Example 2. *Given that fuzzy relations R_1, R_2 are* min*-transitive and comonotone, the new relation R_C and R_D aggregated with any fuzzy conjunction C and disjunction D also* min*-transitive.*

This can be directly derived from the property of increasing in two arguments for any conjunction C or disjunction D. For example, let $D = \max$, for any $x, y, z \in X$, there is always $\min\big(\max(R_1(x,y), R_2(x,y)), \max(R_1(y,z), R_2(y,z))\big) \le \max(R_1(x,z), R_2(x,z))$.

Theorem 10. *Given that F: $[0,1]^n \to [0,1]$ is an increasing function. If all fuzzy relations $R_1, \ldots, R_n$ are negatively* max*-transitive and pair-wisely comonotone, the aggregated relation R_F is negatively* max*-transitive.*

Proof. Fuzzy relations R_i, $i = 1, \ldots, n$ are negatively max-transitive, then $\max(R_i(x, y), R_i(y, z)) \ge R_i(x, z)$. Based on Corollary 1, any increasing function commutes with max for pair-wisely comonotone fuzzy relations. We obtain

$$\begin{aligned}\max(R_F(x,y), R_F(y,z)) &= \max\big(F(R_1(x,y), \ldots, R_n(x,y)), F(R_1(y,z), \ldots, R_n(y,z))\big)\\ &= F\big(\max(R_1(x,y), R_1(y,z)), \ldots, \max(R_n(x,y), R_n(y,z))\big)\\ &\ge F(R_1(x,z), \ldots, R_n(x,z)) = R_F(x,z).\end{aligned}$$

As a result, the negative max-transitivity of R_F is proved. □

Example 3. *Given that fuzzy relations R_1, R_2 are negatively* max*-transitive and comonotone, the new relation R_C and R_D aggregated with any fuzzy conjunction C and disjunction D also are negatively* max*-transitive.*

Corollary 4. *Given that F: $[0,1]^n \to [0,1]$ is a decreasing function. If all fuzzy relations $R_1, \ldots, R_n$ are* min*-transitive and pair-wisely comonotone, the aggregated relation R_F is negatively* max*-transitive.*

Proof. Based on Proposition 2, we obtain

$$\begin{aligned}\max(R_F(x,y), R_F(y,z)) &= \max\big(F(R_1(x,y),\ldots,R_n(x,y)), F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\min(R_1(x,y),R_1(y,z)),\ldots,\min(R_n(x,y),R_n(y,z))\big)\\ &\geq F(R_1(x,z),\ldots,R_n(x,z)) = R_F(x,z).\end{aligned}$$

As a result, the negative max-transitivity of R_F is proved. □

Corollary 5. *Given that F: $[0,1]^n \to [0,1]$ is a decreasing function. If all fuzzy relations $R_1, \ldots, R_n$ are negatively* max*-transitive and pair-wisely comonotone, the aggregated relation R_F is* min*-transitive.*

Proof. Based on Corollary 2, we obtain

$$\begin{aligned}\min(R_F(x,y), R_F(y,z)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)), F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\max(R_1(x,y),R_1(y,z)),\ldots,\max(R_n(x,y),R_n(y,z))\big)\\ &\leq F(R_1(x,z),\ldots,R_n(x,z)) = R_F(x,z).\end{aligned}$$

As a result, the min-transitivity of R_F is proved. □

Corollary 6. *Given that F: $[0,1]^n \to [0,1]$ is an increasing (or a decreasing) function. If all fuzzy relations $R_1, \ldots, R_n$ are* min*-transitive and negatively* max*-transitive and pair-wisely comonotone, the aggregated relation R_F is* min*-transitive and negatively* max*-transitive.*

Proof. Case 1: If F is an increasing function, the preservations of min-transitivity and negative max-transitivity for R_F can be immediately got through Theorem 9 and Theorem 10, respectively.

Case 2: If F is a decreasing function, the min-transitivity and negative max-transitivity for R_F can be obtained through the transformings from the negative max-transitivity and min-transitivity of $R_1, \ldots, R_n$ based on Corollarie 4 and Corollarie 5, respectively. □

Theorem 11. *Given that fuzzy relations $R_1, \ldots, R_n$ are* min*-transitive and negatively* max*-transitive, $R_i, i = 1, \ldots, m$ and $R_j, j = m+1, \ldots, n$ are pair-wisely comonotone, respectively, and R_i and R_j are pair-wisely countermonotone. If the function $F(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$: $[0,1]^n \to [0,1]$ is increasing w.r.t. $x_1, \ldots, x_m$ and decreasing w.r.t. $x_{m+1}, \ldots, x_n$, the aggregated relation R_F is* min*-transitive and negatively* max*-transitive.*

Proof. On the basis of the assumptions, combining Proposition 3 and Corollary 3, we have

$$\begin{aligned}\min(R_F(x,y), R_F(y,z)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)), F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\min(R_1(x,y),R_1(y,z)),\ldots,\min(R_m(x,y),R_m(y,z)),\\ &\qquad \max(R_{m+1}(x,y),R_{m+1}(y,z)),\ldots,\max(R_n(x,y),R_n(y,z))\big)\\ &\leq F\big(R_1(x,z),\ldots,R_m(x,z),R_{m+1}(x,z),\ldots,R_n(x,z)\big) = R_F(x,z)\end{aligned}$$

and

$$\begin{aligned}\max(R_F(x,y),R_F(y,z)) &= \max\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\max(R_1(x,y),R_1(y,z)),\ldots,\max(R_m(x,y),R_m(y,z)),\\ &\qquad \min(R_{m+1}(x,y),R_{m+1}(y,z)),\ldots,\min(R_n(x,y),R_n(y,z))\big)\\ &\geq F\big(R_1(x,z),\ldots,R_m(x,z),R_{m+1}(x,z),\ldots,R_n(x,z)\big)=R_F(x,z)\end{aligned}$$

As a result, the new aggregated relation R_F preserves min-transitivity and negative max-transitivity. □

Theorem 12. *Given that F: $[0,1]^n \to [0,1]$ is an increasing (or a decreasing) function, and fuzzy relations $R_1,\ldots,R_n$ are* min*-semitransitive and pair-wisely comonotone, the aggregated relation R_F is* min*-semitransitive .*

Proof. Case 1: F is an increasing function. Given that fuzzy relations $R_1,\ldots,R_n$ are pair-wisely comonotone, combining Proposition 1 and Corollary 1, we have

$$\begin{aligned}\min(R_F(x,y),R_F(y,z)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\min(R_1(x,y),R_1(y,z)),\ldots,\min(R_n(x,y),R_n(y,z))\big)\\ &\leq F\big(\max(R_1(x,w),R_1(w,z)),\ldots,\max(R_n(x,w),R_n(w,z))\big)\\ &= \max\big(F(R_1(x,w),\ldots,R_n(x,w)),F(R_1(w,z),\ldots,R_n(w,z))\big)\\ &= \max(R_F(x,w),R_F(w,z))\end{aligned}$$

Case 2: F is a decreasing function. Given that fuzzy relations $R_1,\ldots,R_n$ are pair-wisely comonotone, combining Proposition 2 and Corollary 2, we have

$$\begin{aligned}\min(R_F(x,y),R_F(y,z)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\max(R_1(x,y),R_1(y,z)),\ldots,\max(R_n(x,y),R_n(y,z))\big)\\ &\leq F\big(\min(R_1(x,w),R_1(w,z)),\ldots,\min(R_n(x,w),R_n(w,z))\big)\\ &= \max\big(F(R_1(x,w),\ldots,R_n(x,w)),F(R_1(w,z),\ldots,R_n(w,z))\big)\\ &= \max(R_F(x,w),R_F(w,z))\end{aligned}$$

Therefore, no matter the aggregated function F is an increasing function or a decreasing function, the min-semitransitivity can always been preserved for R_F w.r.t. pair-wisely comonotone fuzzy relations $R_1,\ldots,R_n$. □

Theorem 13. *Given that fuzzy relations $R_1,\ldots,R_n$ are* min*-semitransitive, $R_i, i=1,\ldots,m$ and $R_j, j=m+1,\ldots,n$ are pair-wisely comonotone, respectively, and R_i and R_j are pair-wisely countermonotone. If the function $F(x_1,\ldots,x_m,x_{m+1},\ldots,x_n):[0,1]^n \to [0,1]$ is increasing w.r.t. $x_1,\ldots,x_m$ and decreasing w.r.t. $x_{m+1},\ldots,x_n$, the aggregated relation R_F is* min*-semitransitive.*

Proof. On the basis of the assumptions, combining Proposition 3 and Corollary 3, we have

$$\begin{aligned}\min(R_F(x,y),R_F(y,z)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(y,z),\ldots,R_n(y,z))\big)\\ &= F\big(\min(R_1(x,y),R_1(y,z)),\ldots,\min(R_m(x,y),R_m(y,z)),\\ &\qquad \max(R_{m+1}(x,y),R_{m+1}(y,z)),\ldots,\max(R_n(x,y),R_n(y,z))\big)\\ &\leq F\big(\max(R_1(x,w),R_1(w,z)),\ldots,\max(R_m(x,w),R_m(w,z)),\\ &\qquad \min(R_{m+1}(x,w),R_{m+1}(w,z)),\ldots,\min(R_n(x,w),R_n(w,z))\big)\\ &= \max\big(F(R_1(x,w),\ldots,R_n(x,w)),F(R_1(w,z),\ldots,R_n(w,z))\big)\\ &= \max(R_F(x,w),R_F(w,z))\end{aligned}$$

As a result, the min-semitransitivity of R_F is proved. □

4.2. Preservation of the Ferrers Property

In this part, an exploration for the min-Ferrers property is primarily presented. Under the inspiration of the comonotonicity and countermonotonicity of binary functions, we find a similar property for fuzzy relations themselves, named as self-comonotonicity, and we bring out its equivalent relation with the min-Ferrers property. Then, thereoms for the preservation of min-Ferrers property are proposed, together with some illustrated examples.

Definition 13. *Let Ω be a non-empty sets. A function $F : \Omega^2 \to \mathbb{R}$ is said to be self-comonotone in the first argument at (x_1, x_2, y_1, y_2), if*

$$\big(F(x_1, y_1) - F(x_2, y_1)\big)\big(F(x_1, y_2) - F(x_2, y_2)\big) \geq 0.$$

Definition 14. *Let Ω be a non-empty sets. A function $F : \Omega^2 \to \mathbb{R}$ is said to be self-comonotone in the second argument at (x_1, x_2, y_1, y_2), if*

$$\big(F(x_1, y_1) - F(x_1, y_2)\big)\big(F(x_2, y_1) - F(x_2, y_2)\big) \geq 0.$$

The above definitions can be reformulated as follows.

Lemma 5. *(i) A function $F : \Omega^2 \to \mathbb{R}$ is self-comonotone in the first argument at (x_1, x_2, y_1, y_2) if*

$$F(x_1, y_1) > F(x_2, y_1) \text{ implies } F(x_1, y_2) \geq F(x_2, y_2);$$

(ii) A function $F : \Omega^2 \to \mathbb{R}$ is self-comonotone in the second argument at (x_1, x_2, y_1, y_2) if

$$F(x_1, y_1) > F(x_1, y_2) \text{ implies } F(x_2, y_1) \geq F(x_2, y_2).$$

Corollary 7. *If a function $F : \Omega^2 \to \mathbb{R}$ is increasing (or decreasing) in the first (or the second) argument, then F is self-comonotone in the first (or the second) argument at any $(x_1, x_2, y_1, y_2) \in \Omega^4$.*

Example 4. *Any fuzzy conjunction C and disjunction D are self-comonotone in both arguments.*

This can be got from the property of increasing in two arguments for any conjunction C and any disjunction D.

Theorem 14. *A fuzzy relation $R : X^2 \to [0, 1]$ has the* min*-Ferrers property, if and only if it is self-comonotone in the first or the second argument for any $(x, z, y, w) \in X^4$.*

Proof. (i) If R is self-comonotone in the first or the second argument for any $(x, z, y, w) \in X^4$, then it has the min-Ferrers property.

Suppose R is self-comonotone in the first argument at (x, z, y, w), i.e.

$$\big(R(x, y) - R(z, y)\big)\big(R(x, w) - R(z, w)\big) \geq 0,$$

then, we have three cases:
Case 1: $R(x, y) = R(z, y)$;
Case 2: $R(x, y) > R(z, y)$ and $R(x, w) \geq R(z, w)$;
Case 3: $R(x, y) < R(z, y)$ and $R(x, w) \leq R(z, w)$.

For the above cases, we always have

$$\min\big(R(x, y), R(z, w)\big) \leq \max\big(R(x, w), R(z, y)\big).$$

Similarly, if R is self-comonotone in the second argument at (x, z, y, w), we can also derive that R has the min-Ferrers property, which is omitted.

(ii) If R has the min-Ferrers property, then it is self-comonotone in the first or the second argument for any $(x, z, y, w) \in X^4$.

Suppose $\min\big(R(x, y), R(z, w)\big) = R(x, y)$ and $\max\big(R(x, w), R(z, y)\big) = R(z, y)$, then we have $R(x, y) \leq R(z, y)$ and $R(z, w) \geq R(x, w)$. Furthermore, we can get $\big(R(x, y) -$

$R(z,y)\big)\big(R(x,w) - R(z,w)\big) \geq 0$, which means R is self-comonotone in the first argument at (x,z,y,w).

For the other similar cases w.r.t. $\min\big(R(x,y), R(z,w)\big) = R(z,w)$ or $\max\big(R(x,w), R(z,y)\big) = R(x,w)$, it always can be derived either $\big(R(x,y) - R(x,w)\big)\big(R(z,y) - R(z,w)\big) \geq 0$ or $\big(R(x,y) - R(z,y)\big)\big(R(x,w) - R(z,w)\big) \geq 0$. Finally we obtain if R has the min-Ferrers property, then it must be self-comonotone in the first or the second argument for any $(x,z,y,w) \in X^4$. □

Corollary 8. *If the fuzzy relation R is self-comonotone in the first (or the second) argument for any $(x,z,y,w) \in X^4$, then it has the* min*-Ferrers property.*

Example 5. *Any fuzzy conjunction C or disjunction D has the* min*-Ferrers property.*

Example 6. *Suppose two fuzzy relations $R_1, R_2 \in FR(X)$, and $R_2(x,y) = a + bR_1(x,y), x,y \in X, b \leq 1 - a$. If R_1 has the* min*-Ferrers property, then R_2 also has the* min*-Ferrers property.*

Example 7. *Suppose two fuzzy relations $R_1, R_2 \in FR(X)$ and R_1 has the* min*-Ferrers property, If there exists an increasing (or a decreasing) function F: $[0,1] \to [0,1]$ that $R_2(x,y) = F(R_1(x,y))$ for any $x,y \in X$, then R_2 also has the* min*-Ferrers property.*

Case 1: If at (x,z,y,w),$\big(R_1(x,y) - R_1(x,w)\big)\big(R_1(z,y) - R_1(z,w)\big) = 0$, or $\big(R_1(x,y) - R_1(z,y)\big)\big(R_1(x,w) - R_1(z,w)\big) = 0$, then $\big(R_2(x,y) - R_2(x,w)\big)\big(R_2(z,y) - R_2(z,w)\big) = 0$, or $\big(R_2(x,y) - R_2(z,y)\big)\big(R_2(x,w) - R_2(z,w)\big) = 0$ can be obtained because R_2 is a function of R_1.

Case 2: If at (x,z,y,w),

$$\big(R_1(x,y) - R_1(x,w)\big)\big(R_1(z,y) - R_1(z,w)\big) > 0,$$

or

$$\big(R_1(x,y) - R_1(z,y)\big)\big(R_1(x,w) - R_1(z,w)\big) > 0,$$

then

$$\big(R_2(x,y) - R_2(x,w)\big)\big(R_2(z,y) - R_2(z,w)\big) \geq 0,$$

or

$$\big(R_2(x,y) - R_2(z,y)\big)\big(R_2(x,w) - R_2(z,w)\big) \geq 0$$

can be obtained because for all $x,y \in X$, $R_2(x,y)$ increases or decreases with $R_1(x,y)$.

Finally, we can derive that R_2 is self-comonotone in the first or the second argument at any $(x,z,y,w) \in X^4$ and has the min-Ferrers property.

Theorem 15. *Given that F: $[0,1]^n \to [0,1]$ is an increasing (or a decreasing) function. If all fuzzy relations $R_1, \ldots, R_n$ are* min*-Ferrers and pair-wisely comonotone, the aggregated relation R_F is* min*-Ferrers.*

Proof. The proof for Theorem 15 is similar with that for Theorem 12.

Case 1: F is an increasing function.

$$\begin{aligned}
\min(R_F(x,y), R_F(z,w)) &= \min\big(F(R_1(x,y), \ldots, R_n(x,y)), F(R_1(z,w), \ldots, R_n(z,w))\big) \\
&= F\big(\min(R_1(x,y), R_1(z,w)), \ldots, \min(R_n(x,y), R_n(z,w))\big) \\
&\leq F\big(\max(R_1(x,w), R_1(z,y)), \ldots, \max(R_n(x,w), R_n(z,y))\big) \\
&= \max\big(F(R_1(x,w), \ldots, R_n(x,w)), F(R_1(z,y), \ldots, R_n(z,y))\big) \\
&= \max(R_F(x,w), R_F(z,y)).
\end{aligned}$$

Case 2: F is a decreasing function.

$$\begin{aligned}\min(R_F(x,y),R_F(z,w)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(z,w),\ldots,R_n(z,w))\big)\\ &= F\big(\max(R_1(x,y),R_1(z,w)),\ldots,\max(R_n(x,y),R_n(z,w))\big)\\ &\le F\big(\min(R_1(x,w),R_1(z,y)),\ldots,\min(R_n(x,w),R_n(z,y))\big)\\ &= \max\big(F(R_1(x,w),\ldots,R_n(x,w)),F(R_1(z,y),\ldots,R_n(z,y))\big)\\ &= \max(R_F(x,w),R_F(z,y)).\end{aligned}$$

□

Theorem 16. *Given that fuzzy relations $R_1,\ldots,R_n$ are* min*-Ferrers, $R_i, i=1,\ldots,m$ and $R_j, j=m+1,\ldots,n$ are pair-wisely comonotone, respectively, and R_i and R_j are pair-wisely countermonotone. If the function $F(x_1,\ldots,x_m,x_{m+1},\ldots,x_n):[0,1]^n\to[0,1]$ is increasing w.r.t. $x_1,\ldots,x_m$ and decreasing w.r.t. $x_{m+1},\ldots,x_n$, the aggregated relation R_F is* min*-Ferrers.*

Proof. The proof for Theorem 16 is similar with that for Theorem 13.

$$\begin{aligned}\min(R_F(x,y),R_F(z,w)) &= \min\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(z,w),\ldots,R_n(z,w))\big)\\ &= F\big(\min(R_1(x,y),R_1(z,w)),\ldots,\min(R_m(x,y),R_m(z,w)),\\ &\qquad \max(R_{m+1}(x,y),R_{m+1}(z,w)),\ldots,\max(R_n(x,y),R_n(z,w))\big)\\ &\le F\big(\max(R_1(x,y),R_1(z,w)),\ldots,\max(R_m(x,y),R_m(z,w)),\\ &\qquad \min(R_{m+1}(x,y),R_{m+1}(z,w)),\ldots,\min(R_n(x,y),R_n(z,w))\big)\\ &= \max\big(F(R_1(x,y),\ldots,R_n(x,y)),F(R_1(z,w),\ldots,R_n(z,w))\big)\\ &= \max(R_F(x,y),R_F(z,w)).\end{aligned}$$

As a result, the preservation of min-Ferrers property for R_F is proved. □

Example 8. *According to Example 1, the indices PD and NSD are comonotone for any fuzzy intervals. In addition, it is known that both PD and NSD have the* min*-Ferrers property. Then, according to Theorem 15, any increasing or decreasing function of PD and NSD preserves the* min*-Ferrers property. By utilizing this conclusion, a new aggregated fuzzy relation representing a combination of the possibility of $M \geq N$ and the necessity of $M > N$ can be constructed by the decision makers which preserves the* min*-Ferrers property.*

5. Conclusions

Instead of restricting the aggregated functions to maintain the preservation for any fuzzy relation that has the property, this work attended to give more general and sensible aggregation functions without the constraint of dominance in a global sense through attaching some reasonable conditions on the underling fuzzy relations. We gave the initial definitions of comonotonicity and countermonotonicity for binary functions, and proved the commuting and bisymmetric relations between min, max and some general functions. Based on that, the preservations of min-transitivity, negative max-transitivity, min-semitransitivity and min-Ferrers property for pair-wisely comonotone or countermonotone fuzzy relations are suggested. Furthermore, some interesting relevant findings were also presented including the definition of self-comonotonicity for binary functions, the equivalent conditions of the min-Ferrers property of fuzzy relations, the transforming of min-transitivity and negative max-transitivity, etc.

However, it also should be noted that (i) the theorems only provide the sufficient conditions for the preservation of tansitivities and the Ferrers property; (ii) the conclusions for the preservation are specially for the tansitivities and the Ferrers property involving min and max. For the transitivities and the Ferrers properties with t-norms, t-conorms or more general binary operations, they are not applicable. A further investigation on the limitations of this paper would be an interesting work.

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References

1. Díaz, S.; de Baets, B.; Montes, S. On the Ferrers property of valued interval orders. *Top* **2011**, *19*, 421–447. [CrossRef]
2. de Baets, B.; van de Walle, B.; Kerre, E. Fuzzy preference structures without incomparability. *Fuzzy Sets Syst.* **1995**, *76*, 333–348. [CrossRef]
3. Fodor, J.; Roubens, M. *Fuzzy Preference Modelling and Multicriteria Decision Support*; Kluwer Academic Publishers: Dordercht, The Netherlands; Boston, MA, USA; London, UK, 1994.
4. Herrera-Viedma, E.; Herrera, F.; Chiclana, F.; Luque, M. Some issues on consistency of fuzzy preference relations. *Eur. J. Oper. Res.* **2004**, *154*, 98–109. [CrossRef]
5. Wang, X.; Xue, Y. Traces and property indicators of fuzzy relations. *Fuzzy Sets Syst.* **2014**, *246*, 78–90. [CrossRef]
6. Chiclana, F.; Herrera-Viedma, E.; Herrera, F.; Alonso, S. Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations. *Eur. J. Oper. Res.* **2007**, *182*, 383–399. [CrossRef]
7. Juan-De-Dios, G.; Juan-Jose, M.; Oscar, V. Aggregation of Indistinguishability Fuzzy Relations Revisited. *Mathamtics* 2021, *in press*.
8. Peneva, V.; Ivan, P. Properties of the aggregation operators related with fuzzy relations. *Fuzzy Sets Syst.* **2003**, *139*, 615–633. [CrossRef]
9. Pradera, A.; Trillas, E.; Casti?eira, E. On the aggregation of some classes of fuzzy relations. In *Technologies for Constructing Intelligent Systems 2*; Physica-Verlag HD: Berlin/Heidelberg, Germany, 2002; pp. 125–136.
10. Urszula, B. New types of aggregation functions for interval-valued fuzzy setting and preservation of pos-B and nec-B-transitivity in decision making problems. *Inf. Sci.* **2015**, *424*, 385–399.
11. Xu, Z. Intuitionistic fuzzy aggregation operators. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1179–1187.
12. Saminger, S.; Mesiar, R.; Bodenhofer, U. Domination of aggregation operators and preservation of transitivity. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.* **2002**, *10*, 11–35. [CrossRef]
13. Urszula, B.; Król, A. Preservation of fuzzy relation properties based on fuzzy conjunctions and disjunctions during aggregation process. *Fuzzy Sets Syst.* **2015**, *291*, 98–113.
14. de Baets, B.; Mesiar, R. Metrics and T-equalities. *J. Math. Anal. Appl.* **2002**, *267*, 531–547. [CrossRef]
15. Schweizer, B.; Sklar, A. *Probabilistic Metric Spaces*; Elsevier: New York, NY, USA, 1983.
16. Tardiff, R. Topologies for probabilistic metric spaces. *Pacific J. Math.* **1976**, *65*, 233–251. [CrossRef]
17. Drewniak, J.; Dudziak, U. Preservation of properties of fuzzy relations during aggregation processes. *Kybernetika* **2007**, *43*, 115–132.
18. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
19. Drewniak, J.; Król, A. A survey of weak connectives and the preservation of their properties by aggregations. *Fuzzy Sets Syst.* **2010**, *161*, 202–215. [CrossRef]
20. Calvo, T.; Kolesárová, A.; Komorníková, M.; Mesiar, R. Aggregation operators: Properties, classes and construction methods. In *Aggregation Operators*; Studies in Fuzziness and Soft Computing; Calvo, T., Mayor, G., Mesiar, R., Eds.; Physica: Berlin/Heildelberg, Germany, 2002; Volume 97, pp. 3–104.
21. Saminger, S.; Mesiar, R.; Dubois, D. Aggregation operators and commuting. *IEEE Trans. Fuzzy Syst.* **2007**, *15*, 1032–1045. [CrossRef]
22. Aczél, J.; Gyula, M. Solution of the Rectangular m×n Generalized Bisymmetry Equation and of the Problem of Consistent Aggregation. *J. Math. Anal. Appl.* **1996**, *203*, 104–126. [CrossRef]
23. Schmeidler, D. Integral representation without additivity. *Proc. Amer. Math. Soc.* **1986**, *97*, 255–261. [CrossRef]
24. Dubois, D.; Henri, P. Ranking fuzzy numbers in the setting of possibility theory. *Inf. Sci.* **1983**, *30*, 183–224. [CrossRef]