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Constraining Non-Dissipative Transport Coefficients in Global Equilibrium

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Abstract: The fluid in global equilibrium must fulfill some constraints. These constraints can be derived from quantum statistical theory or kinetic theory. In this work, we show how these constraints can be applied to determine the non-dissipative transport coefficients for chiral systems along with the energy-momentum conservation, chiral anomaly for charge current and trace anomaly in the energy-momentum tensor.

Keywords: chiral anomaly; trace anomaly; anomalous fluid; non-dissipative transport

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1. Introduction

The charge currents associated withchiral anomaly exhibit peculiar properties which normal currents do not possess, such as the famous chiral magnetic effect [1,2] and chiral vortical effect [3–5]. These currents are all non-dissipative and may exist even in global equilibrium. These anomalous currents can be derived from various approaches, such as gauge/gravity duality [6–11], principle of entropy increase [12–15], Kubo formula from quantum field theory [16–21] or quantum kinetic equation [22–29]. Because these currents are non-dissipative, they may exist even in global equilibrium. However, in order to arrive at global equilibrium, the system must satisfy some specific constraints, especially when the electromagnetic field is present. In this work, we use these constraints along with the energy-momentum conservation law, trace anomaly for energy-momentum tensor and chiral anomaly for charge current to determine or constrain the non-dissipative transport coefficients up to the second order. A similar method was used to deal with first-order coefficients in References [30,31]. In Section 2, we first review how the constraint in global equilibrium can be derived from either quantum statistical theory or kinetic theory when the electromagnetic field is imposed. In Section 3, we show how to determine the energy-momentum tensor and charge current from the conservation laws and chiral anomaly. We summarize our results in Section 4.

We choose the metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ and the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ with the convention $\epsilon^{0123} = 1$. For simplicity, we set the electric charge of the chiral fermion as the unit.

2. Global Equilibrium Constraints

When a fluid is in global equilibrium without external fields [32,33], the fluid four-velocity u^μ with $u^2 = 1$ should be expansion-free and shear-free, and the thermal potential $\bar{\mu} = \mu/T$, which is defined as the chemical potential μ divided by the temperature T should be constant, i.e.,

$$\Delta^{\mu\rho}\Delta^{\nu\sigma}\left(\partial_\rho u_\sigma + \partial_\sigma u_\rho\right) = 0, \quad \partial_\mu \bar{\mu} = 0 \tag{1}$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ denotes the spatial projection tensor. In conjunction with the ideal hydrodynamical equation, it is easy to verify that these above conditions are equivalent to the following equations

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad \partial_\mu \bar{\mu} = 0 \tag{2}$$

where $\beta^\mu = u^\mu / T$ can be referred to as the thermal velocity similar to the the thermal potential for chemical potential. These are just the constraint conditions which should be obeyed by the fluid in global equilibrium without external fields. When an external electromagnetic field tensor $F_{\mu\nu}$ is present, the constraint conditions are generalized to

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad \partial_\mu \bar{\mu} = -F_{\mu\nu}\beta^\nu \tag{3}$$

where the electromagnetic field should be static so as to be able to arrive at the global equilibrium. The second equation above indicates that the external electromagnetic field is balanced by the gradient of the thermal potential. In this work, we assume further that the electromagnetic field is also homogeneous, which means that $F_{\mu\nu}$ must be constant, i.e., $\partial_\lambda F_{\mu\nu} = 0$.

Now, we first review how these constraint conditions can be derived from more underlying theories. The derivation from quantum statistical theory is based on the global thermodynamic equilibrium density operator, whose details are given in [30,34]. The general covariant form of the local thermodynamic equilibrium density operator is given by

$$\hat{\rho} \;=\; \frac{1}{Z}\exp\left[-\int_\Sigma d\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \bar{\mu}\hat{j}^\mu\right)\right]. \tag{4}$$

where $\hat{T}^{\mu\nu}$ is the symmetric energy-momentum tensor operator, $\hat{j}^\mu$ the conserved current operator, Z the normalization factor such that $\mathrm{tr}\hat{\rho} = 1$, and Σ a spacelike 3-D hypersurface. In global equilibrium, the integrand should be time independent

$$\int_{\Sigma(\tau)} d\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \bar{\mu}\hat{j}^\mu\right) - \int_{\Sigma(\tau+\Delta\tau)} d\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \bar{\mu}\hat{j}^\mu\right) = 0 \tag{5}$$

and will not depend on the hypersurface Σ any more. With the assumption that the field β^μ and $\bar{\mu}$ vanish at the timelike boundary which connects two spacelike hypersurfaces, $\Sigma(\tau)$ and $\Sigma(\tau + \Delta\tau)$, and according to Gauss's theorem, the above equation implies that the integrand is divergenceless:

$$\partial_\mu\left(\hat{T}^{\mu\nu}\beta_\nu - \bar{\mu}\hat{j}^\mu\right) = \left(\partial_\mu \hat{T}^{\mu\nu}\right)\beta_\nu + \hat{T}^{\mu\nu}\partial_\mu\beta_\nu - \left(\partial_\mu\bar{\mu}\right)\hat{j}^\mu - \bar{\mu}\left(\partial_\mu \hat{j}^\mu\right) = 0 \tag{6}$$

Using the conservation equations for energy-momentum tensor $\partial_\mu \hat{T}^{\mu\nu} = F^{\nu\mu}\hat{j}_\mu$ and charge current $\partial_\mu \hat{j}^\mu = 0$ and the fact that the energy-momentum tensor is symmetric, we obtain

$$\frac{1}{2}\hat{T}^{\mu\nu}\left(\partial_\mu\beta_\nu + \partial_\nu\beta_\mu\right) - \left(\partial_\mu\bar{\mu} + F_{\mu\nu}\beta^\nu\right)\hat{j}^\mu = 0 \tag{7}$$

It is obvious that this equation always holds if the constraint condition (3) is satisfied.

The global equilibrium condition can also be derived from kinetic theory [22,35]. In equilibrium, the collision terms in the Boltzmann equation will vanish due to the detailed balancing principle and the kinetic equation will reduce to the Vlasov equation:

$$\delta(p^2 - m^2)p^\mu\left(\frac{\partial}{\partial x^\mu} - F_{\mu\nu}\frac{\partial}{\partial p_\nu}\right)f(x,p) \;=\; 0\,. \tag{8}$$

where p^μ denotes the four-momentum of the particle with mass m and we have written the Vlasov equation in Lorentz covariant form. In equilibrium, the distribution function $f(x,p)$ should depend on x, p through the argument $\beta \cdot p - \bar{\mu}$

$$f(x,p) \quad = \quad g(y), \quad y = \beta \cdot p - \bar{\mu}. \tag{9}$$

Then, the kinetic Equation (8) can be expressed as

$$\delta(p^2 - m^2)\left[\frac{1}{2}p^\mu p^\nu(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu) - p^\mu \partial_\mu \bar{\mu} - p^\mu F_{\mu\nu}\beta^\nu\right]\frac{dg}{dy} = 0\,, \tag{10}$$

Obviously, the kinetic equation always holds if the equilibrium conditions (3) are satisfied.

Now, let us consider the constraint conditions listed above in more detail. We can solve the first condition directly [36] and the general solution is given by

$$\beta_\mu \quad = \quad b_\mu - \Omega_{\mu\nu}x^\nu \tag{11}$$

where b_μ is a constant vector and $\Omega_{\mu\nu}$ is a constant antisymmetric tensor. Actually, $\Omega_{\mu\nu}$ is just the thermal vorticity tensor of the fluid (there is a minus sign difference from the usual definition)

$$\Omega_{\mu\nu} = \frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu)\,. \tag{12}$$

The second condition in (3) has a solution only if the integrability condition is fulfilled [37]. It can be obtained by differentiating both sides of the second equation in Equation (3) with ∂_ν and using the commutative property of ordinary partial derivatives

$$\partial_\nu \partial_\mu \bar{\mu} = \partial_\mu \partial_\nu \bar{\mu} = -F_{\mu\lambda}\partial_\nu \beta^\lambda = -F_{\nu\lambda}\partial_\mu \beta^\lambda, \tag{13}$$

Together with Equation (11), the above equation can be written as

$$F_\lambda{}^\mu \Omega^{\nu\lambda} - F_\lambda{}^\nu \Omega^{\mu\lambda} = 0\,, \tag{14}$$

The general solution under this integrability condition is given by

$$\bar{\mu} \quad = \quad -\frac{1}{2}F^{\mu\lambda}x_\lambda \Omega_{\mu\nu}x^\nu - F^{\mu\nu}x_\mu b_\nu + c \tag{15}$$

We can decompose the antisymmetric tensors $F_{\mu\nu}$ and $\Omega_{\mu\nu}$ with the fluid velocity u_μ as

$$F_{\mu\nu} \quad = \quad E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma}u^\rho B^\sigma\,, \tag{16}$$

$$\Omega_{\mu\nu} \quad = \quad \frac{1}{T}(\varepsilon_\mu u_\nu - \varepsilon_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma}u^\rho \omega^\sigma)\,, \tag{17}$$

where the electric field E^μ, magnetic field B^μ, acceleration vector ε^μ and vorticity vector ω^μ are given by, respectively,

$$E^\mu \quad = \quad F^{\mu\nu}u_\nu\,, \quad B^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}\,, \tag{18}$$

$$\varepsilon^\mu \quad = \quad T\Omega^{\mu\nu}u_\nu\,, \quad \omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}u_\nu \partial^x_\alpha u_\beta\,. \tag{19}$$

With this decomposition, it is easy to verify that the integrability condition (14) is equivalent to

$$E^\mu \omega^\nu - E^\nu \omega^\mu = -B^\mu \varepsilon^\nu + B^\nu \varepsilon^\mu, \quad E^\mu \varepsilon^\nu - E^\nu \varepsilon^\mu = B^\mu \omega^\nu - B^\nu \omega^\mu\,. \tag{20}$$

We show that these relations play an important role in determining the possible forms of the non-dissipative terms in the energy-momentum tensor and the charge current in global equilibrium.

3. Non-Dissipative Transport Coefficients

In this section, we apply the conservation laws and trace anomaly to constrain the possible anomalous transport coefficients in a chiral system in which only right-hand or left-hand Weyl fermions are involved. These conservation laws and the trace anomaly are given by

$$\partial_\mu T^{\mu\nu} = F^{\nu\mu} j_\mu, \quad \partial_\mu j^\mu = CE \cdot B, \quad g_{\mu\nu} T^{\mu\nu} = \tilde{C}(E^2 - B^2) \tag{21}$$

We expand the energy-momentum tensor and charge current in powers of $F^{\mu\nu}$ and $\Omega^{\mu\nu}$ or equivalently in terms of B^μ,E^μ, ω^μ and ε^μ. Because $F^{\mu\nu}$ and $\Omega^{\mu\nu}$ are both constant, it is unnecessary to consider $\partial_\mu T$, $\partial_\mu u_\nu$ and $\partial_\mu \bar{\mu}$ because all these derivatives can be expressed as the linear combination of E^μ, ω^μ and ε^μ by using the constraint condition (3), e.g.,

$$\partial_\mu T = -T\varepsilon_\mu, \quad \partial_\mu u_\nu = -u_\mu \varepsilon_\nu + \epsilon_{\mu\nu\alpha\beta} u^\alpha \omega^\beta, \quad \partial_\mu \bar{\mu} = -\frac{E_\mu}{T} \tag{22}$$

We take u^μ, T and $\bar{\mu}$ to be of the zeroth order, $F^{\mu\nu}$ and $\Omega^{\mu\nu}$ to be of the first order and so on.

Let us start with the zeroth-order $T^{\mu\nu}$ and j^μ. They are just the well-known ideal hydrodynamical results:

$$T^{(0)\mu\nu} = \rho u^\mu u^\nu - P\Delta^{\mu\nu}, \quad j^{(0)\mu} = n u^\mu \tag{23}$$

where ρ is the energy density, P the pressure and n the charge density. It is easy to verify that

$$\partial_\mu T^{(0)\mu\nu} = (\rho + P) u^\mu \partial_\mu u^\nu - \partial^\nu P = -\rho \varepsilon^\nu - T\partial \frac{P}{T} \tag{24}$$

Using the thermal identity

$$d\frac{P}{T} = n d\bar{\mu} - \rho d\frac{1}{T} \tag{25}$$

and the last equation in (22), we obtain

$$\partial_\mu T^{(0)\mu\nu} = E^\nu n = F^{\nu\mu} j^{(0)}_{s\mu} \tag{26}$$

which indicates that the energy-momentum conservation law holds automatically. It is trivial to show that at zeroth-order charge current is also conserved automatically

$$\partial_\mu j^{(0)\mu} = 0 \tag{27}$$

There is no chiral anomaly at zeroth order as there should be. For the massless fermions, the conformal symmetry holds at the zeroth order and the trace of the energy-momentum tensor must vanish, which results in the well-known relation

$$\rho = 3P. \tag{28}$$

When we go beyond the zeroth-order, we need to first pin down which frame we choose for the fluid velocity u^μ. Transport coefficients might take different forms in different frames. Some recent investigations showed that the frame choices might even influence the causality and stability of the fluid [38–43]. In our work, we use the β frame introduced in [44], In this frame, the non-dissipative coefficients in global equilibrium would take a more elegant form [45–47]. We assume the interactions which control the chiral system keep charge, parity and time reversal invariance. Then, at first order, the general expressions for the energy-momentum tensor and charge current take the following form

$$T^{(1)\mu\nu} = \lambda^{\omega}(u^{\mu}\omega^{\nu}+u^{\nu}\omega^{\mu})+\lambda^{B}(u^{\mu}B^{\nu}+u^{\nu}B^{\mu}), \quad (29)$$
$$j^{(1)\mu} = \xi\omega^{\mu}+\xi^{B}B^{\mu} \quad (30)$$

With this expression, the divergence of the current reads

$$\partial_{\mu}j^{(1)\mu} = \frac{\partial\xi}{\partial T}\partial_{\mu}T\omega^{\mu}+\frac{\partial\xi}{\partial\bar{\mu}}\partial_{\mu}\bar{\mu}\omega^{\mu}+\xi\partial_{\mu}\omega^{\mu}+\frac{\partial\xi^{B}}{\partial T}\partial_{\mu}TB^{\mu}+\frac{\partial\xi^{B}}{\partial\bar{\mu}}\partial_{\mu}\bar{\mu}B^{\mu}+\xi^{B}\partial_{\mu}B^{\mu} \quad (31)$$

Using the relations (22) and the derived relations below

$$\partial_{\mu}\omega_{\nu} = \varepsilon\cdot\omega\, g_{\mu\nu}-2\varepsilon_{\mu}\omega_{\nu}, \quad (32)$$
$$\partial_{\mu}B_{\nu} = -E_{\mu}\omega_{\nu}+\varepsilon\cdot B\, u_{\mu}u_{\nu}+\omega\cdot E\,\Delta_{\mu\nu}-(u_{\mu}\epsilon_{\nu\lambda\rho\sigma}+u_{\nu}\epsilon_{\mu\lambda\rho\sigma})u^{\lambda}\varepsilon^{\rho}E^{\sigma} \quad (33)$$

the Equation (31) can be written as

$$\partial_{\mu}j^{(1)\mu} = \left(2\xi-T\frac{\partial\xi}{\partial T}\right)\varepsilon\cdot\omega+\left(2\xi^{B}-\frac{1}{T}\frac{\partial\xi}{\partial\bar{\mu}}\right)E\cdot\omega+\left(\xi^{B}-T\frac{\partial\xi^{B}}{\partial T}\right)\varepsilon\cdot B-\frac{1}{T}\frac{\partial\xi^{B}}{\partial\bar{\mu}}E\cdot B \quad (34)$$

The fact that this result should equal the anomalous term $CE\cdot B$ from the second equation in (21) leads to the following equations:

$$2\xi-T\frac{\partial\xi}{\partial T}=0,\quad 2\xi^{B}-\frac{1}{T}\frac{\partial\xi}{\partial\bar{\mu}}=0,\quad \xi^{B}-T\frac{\partial\xi^{B}}{\partial T}=0,\quad -\frac{1}{T}\frac{\partial\xi^{B}}{\partial\bar{\mu}}=C. \quad (35)$$

The general solution for this set of equations is easy to obtain

$$\xi^{B} = -CT\bar{\mu}+bT=-C\mu+bT, \quad (36)$$
$$\xi = -CT^{2}\bar{\mu}^{2}+2bT^{2}\bar{\mu}+aT^{2}=-C\mu^{2}+2bT\mu+aT^{2} \quad (37)$$

where a and b are both integral constants. It should be noted that the temperature dependence derived from the differential equations is consistent with the direct dimension analysis. Actually, it is more convenient to determine the temperature power from dimension analysis. These results were derived from the anomalous hydrodynamics by using the principle of entropy increase [12–15]. However, it seems as though our method that we present here involves many fewer calculations. Similarly, the divergence of the energy-momentum tensor can be expressed as

$$\begin{aligned}\partial_{\mu}T^{(1)\mu\nu} &= \tfrac{\partial\lambda}{\partial T}\partial_{\mu}T(u^{\mu}\omega^{\nu}+u^{\nu}\omega^{\mu})+\tfrac{\partial\lambda}{\partial\bar{\mu}}\partial_{\mu}\bar{\mu}(u^{\mu}\omega^{\nu}+u^{\nu}\omega^{\mu})+\lambda\partial_{\mu}(u^{\mu}\omega^{\nu}+u^{\nu}\omega^{\mu})\\ &\quad+\tfrac{\partial\lambda^{B}}{\partial T}\partial_{\mu}T(u^{\mu}B^{\nu}+u^{\nu}B^{\mu})+\tfrac{\partial\lambda^{B}}{\partial\bar{\mu}}\partial_{\mu}\bar{\mu}(u^{\mu}B^{\nu}+u^{\nu}B^{\mu})+\lambda^{B}\partial_{\mu}(u^{\mu}B^{\nu}+u^{\nu}B^{\mu})\\ &= \Big[(3\lambda-T\tfrac{\partial\lambda}{\partial T})\varepsilon\cdot\omega+(2\lambda^{B}-\tfrac{1}{T}\tfrac{\partial\lambda}{\partial\bar{\mu}})E\cdot\omega+(2\lambda^{B}-T\tfrac{\partial\lambda^{B}}{\partial T})\varepsilon\cdot B-\tfrac{1}{T}\tfrac{\partial\lambda^{B}}{\partial\bar{\mu}}E\cdot B\Big]u^{\nu}\\ &\quad-2\lambda^{B}\epsilon^{\nu\alpha\beta\gamma}u_{\alpha}\omega_{\beta}B_{\gamma}\end{aligned} \quad (38)$$

where we use the second identity in Equation (20). The righthand of the energy-momentum conservation at first order in Equation (21) is given by

$$F^{\nu\mu}j^{(1)}_{s\mu} = -\xi(E\cdot\omega)u_{\nu}-\xi^{B}(E\cdot B)u^{\nu}-\xi\epsilon^{\nu\alpha\beta\gamma}u_{\alpha}\omega_{\beta}B_{\gamma} \quad (39)$$

Then, the conservation law $\partial_{\mu}T^{(1)\mu\nu}=F^{\nu\mu}j^{(1)}_{s\mu}$ requires

$$3\lambda-T\frac{\partial\lambda}{\partial T}=0,\quad 2\lambda^{B}-\frac{1}{T}\frac{\partial\lambda}{\partial\bar{\mu}}=-\xi,\quad 2\lambda^{B}-T\frac{\partial\lambda^{B}}{\partial T}=0,\quad \frac{1}{T}\frac{\partial\lambda^{B}}{\partial\bar{\mu}}=\xi^{B},\quad 2\lambda^{B}=\xi \quad (40)$$

From the last equation, we note that the coefficient λ^B has been totally determined by the coefficient ξ in the charge current. It is trivial to verify that both the second and third last equations hold automatically with the result of ξ in Equation (37). Substituting the result of λ^B into the first and second equations, we can obtain the general expression for λ. We list the solution for λ^B and λ in the following:

$$\begin{aligned}\lambda^B &= \frac{1}{2}\xi = \frac{1}{2}\left(-C\bar{\mu}^2 + 2b\bar{\mu} + a\right)T^2, &(41)\\ \lambda &= \frac{2}{3}\left(-C\bar{\mu}^3 + 3b\bar{\mu}^2 + a\bar{\mu} + c\right)T^3 &(42)\end{aligned}$$

where c is another integral constant. Similarly, the temperature dependence can also be obtained from direct dimension analysis. It is obvious that the energy-momentum tensor at first order is traceless automatically.

Now, let us move on to consider the second-order case. The charge current and energy-momentum tensor at second order take the general form

$$\begin{aligned}j^{(2)\mu} &= \left(\xi^{\varepsilon\varepsilon}\varepsilon^2 + \xi^{\omega\omega}\omega^2 + \xi^{\varepsilon E}\varepsilon\cdot E + \xi^{\omega B}\omega\cdot B + \xi^{EE}E^2 + \xi^{BB}B^2\right)u^\mu \\ &\quad + \xi^{\varepsilon\omega}\epsilon^{\mu\nu\rho\sigma}u_\nu\varepsilon_\rho\omega_\sigma + \xi^{\omega E}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho\omega_\sigma + \xi^{EB}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma, \qquad (43)\\ T_s^{(2)\mu\nu} &= \left(\lambda^{\varepsilon\varepsilon}\varepsilon^2 + \lambda^{\omega\omega}\omega^2 + \lambda^{\varepsilon E}\varepsilon\cdot E + \lambda^{\omega B}\omega\cdot B + \lambda^{EE}E^2 + \lambda^{BB}B^2\right)u^\mu u^\nu \\ &\quad + \left(\bar{\lambda}^{\varepsilon\varepsilon}\varepsilon^2 + \bar{\lambda}^{\omega\omega}\omega^2 + \bar{\lambda}^{\varepsilon E}\varepsilon\cdot E + \bar{\lambda}^{\omega B}\omega\cdot B + \bar{\lambda}^{EE}E^2 + \bar{\lambda}^{BB}B^2\right)\Delta^{\mu\nu} \\ &\quad + \tilde{\lambda}^{\varepsilon\varepsilon}\varepsilon^\mu\varepsilon^\nu + \tilde{\lambda}^{\omega\omega}\omega^\mu\omega^\nu + \tilde{\lambda}^{\varepsilon E}(\varepsilon^\mu E^\nu + \varepsilon^\nu E^\mu) \qquad (44)\\ &\quad + \tilde{\lambda}^{\omega B}(\omega^\mu B^\nu + \omega^\nu B^\mu) + \tilde{\lambda}^{EE}E^\mu E^\nu + \tilde{\lambda}^{BB}B^\mu B^\nu \\ &\quad + (u^\mu\epsilon^{\nu\alpha\beta\gamma} + u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\left(\lambda^{\varepsilon\omega}\varepsilon_\beta\omega_\gamma + \lambda^{\omega E}E_\beta\omega_\gamma + \lambda^{EB}E_\beta B_\gamma\right)\end{aligned}$$

In order to calculate the divergence of these quantities, we need other useful relations:

$$\begin{aligned}\partial_\mu\varepsilon_\nu &= \omega_\mu\omega_\nu - \varepsilon_\mu\varepsilon_\nu + \varepsilon^2 u_\mu u_\nu - \omega^2\Delta_{\mu\nu} + (u_\mu\epsilon_{\nu\lambda\rho\sigma} + u_\nu\epsilon_{\mu\lambda\rho\sigma})u^\lambda\varepsilon^\rho\omega^\sigma, &(45)\\ \partial_\mu E_\nu &= B_\mu\omega_\nu + \varepsilon\cdot E\, u_\mu u_\nu - \omega\cdot B\,\Delta_{\mu\nu} + (u_\mu\epsilon_{\nu\lambda\rho\sigma} + u_\nu\epsilon_{\mu\lambda\rho\sigma})u^\lambda E^\rho\omega^\sigma, &(46)\\ 0 &= \epsilon^{\mu\alpha\beta\gamma}\varepsilon_\alpha\omega_\beta E_\gamma = \epsilon^{\mu\alpha\beta\gamma}\omega_\alpha E_\beta B_\gamma = \epsilon^{\mu\alpha\beta\gamma}E_\alpha B_\beta\varepsilon_\gamma = \epsilon^{\mu\alpha\beta\gamma}B_\alpha\varepsilon_\beta\omega_\gamma &(47)\end{aligned}$$

All these relations can be derived from the first-order relations (22). It is easy to verify that the conservation law for the charge current $\partial_\mu j^{(2)\mu} = 0$ is satisfied automatically. Although we cannot constrain any coefficients appearing in the second-order current $j^{(2)\mu}$, we can still relate the coefficients in the second-order energy-momentum tensor $T_s^{(2)\mu\nu}$ to the ones in $j^{(2)\mu}$ through the energy-momentum conservation. Following the same step as we did at the first order, the divergence of the energy-momentum tensor reads

$$\begin{aligned}\partial_\mu T^{(2)\mu\nu} &= \mathcal{X}_1\varepsilon^2\varepsilon^\nu + \mathcal{X}_2\omega^2\varepsilon^\nu + \mathcal{X}_3\varepsilon\cdot\omega\omega^\nu \\ &\quad + \mathcal{X}_4\omega\cdot B\varepsilon^\nu + \mathcal{X}_5\varepsilon\cdot B\omega^\nu + \mathcal{X}_6\omega\cdot E\omega^\nu + \mathcal{X}_7\varepsilon^2 E^\nu + \mathcal{X}_8\omega^2 E^\nu \\ &\quad + \mathcal{X}_9 E^2\varepsilon^\nu + \mathcal{X}_{10}B^2\varepsilon^\nu + \mathcal{X}_{11}E\cdot B\omega^\nu + \mathcal{X}_{12}\varepsilon\cdot EE^\nu + \mathcal{X}_{13}\omega\cdot BE^\nu \qquad (48)\\ &\quad + \mathcal{X}_{14}E^2E^\nu + \mathcal{X}_{15}B^2E^\nu + \mathcal{X}_{16}E\cdot BB^\nu\end{aligned}$$

where the coefficients $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$, which are irrelevant to the electromagnetic field, read

$$\begin{aligned}\mathcal{X}_1 &= -\lambda^{\varepsilon\varepsilon} - \bar{\lambda}^{\varepsilon\varepsilon} - \tilde{\lambda}^{\varepsilon\varepsilon} - T\frac{\partial\bar{\lambda}^{\varepsilon\varepsilon}}{\partial T} - T\frac{\partial\tilde{\lambda}^{\varepsilon\varepsilon}}{\partial T}, \\ \mathcal{X}_2 &= -\lambda^{\omega\omega} - 2\bar{\lambda}^{\varepsilon\varepsilon} + 2\lambda^{\varepsilon\omega} - 3\bar{\lambda}^{\omega\omega} - 3\tilde{\lambda}^{\varepsilon\varepsilon} - T\frac{\partial\bar{\lambda}^{\omega\omega}}{\partial T}, \qquad (49)\\ \mathcal{X}_3 &= 2\bar{\lambda}^{\varepsilon\varepsilon} + 2\bar{\lambda}^{\omega\omega} + \tilde{\lambda}^{\varepsilon\varepsilon} + \tilde{\lambda}^{\omega\omega} - 2\lambda^{\varepsilon\omega} - T\frac{\partial\tilde{\lambda}^{\omega\omega}}{\partial T},\end{aligned}$$

the coefficients from $\mathcal{X}_4$ to $\mathcal{X}_8$ with linear dependence on the electromagnetic field are given by

$$
\begin{aligned}
\mathcal{X}_4 &= \lambda^{\varepsilon E} - 3\tilde{\lambda}^{\varepsilon E} - \lambda^{\omega B} + 3\tilde{\lambda}^{\omega B} + T\frac{\partial\bar{\lambda}^{\varepsilon E}}{\partial T} + T\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial T} - T\frac{\partial\bar{\lambda}^{\omega B}}{\partial T} - T\frac{\partial\tilde{\lambda}^{\omega B}}{\partial T} + \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon\varepsilon}}{\partial\bar{\mu}} \\
\mathcal{X}_5 &= -\lambda^{\varepsilon E} + \tilde{\lambda}^{\varepsilon E} - \tilde{\lambda}^{\omega B} - T\frac{\partial\bar{\lambda}^{\varepsilon E}}{\partial T} - T\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial T} - T\frac{\partial\tilde{\lambda}^{\omega B}}{\partial T} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon\varepsilon}}{\partial\bar{\mu}} \\
\mathcal{X}_6 &= \lambda^{\varepsilon E} + 2\bar{\lambda}^{\varepsilon E} + \tilde{\lambda}^{\varepsilon E} + 2\bar{\lambda}^{\omega B} + 5\tilde{\lambda}^{\omega B} - 2\lambda^{\omega E} \\
&\quad + T\frac{\partial\bar{\lambda}^{\varepsilon E}}{\partial T} + T\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial T} - T\frac{\partial\tilde{\lambda}^{\omega B}}{\partial T} + \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon\varepsilon}}{\partial\bar{\mu}} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\omega\omega}}{\partial\bar{\mu}} \\
\mathcal{X}_7 &= -\lambda^{\varepsilon E} - T\frac{\partial\bar{\lambda}^{\varepsilon E}}{\partial T} - 2T\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial T} - \frac{1}{T}\frac{\partial\bar{\lambda}^{\varepsilon\varepsilon}}{\partial\bar{\mu}} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon\varepsilon}}{\partial\bar{\mu}} \\
\mathcal{X}_8 &= -\lambda^{\varepsilon E} - 2\bar{\lambda}^{\varepsilon E} - 3\tilde{\lambda}^{\varepsilon E} - 2\bar{\lambda}^{\omega B} - 3\tilde{\lambda}^{\omega B} + 2\lambda^{\omega E} \\
&\quad - T\frac{\partial\bar{\lambda}^{\varepsilon E}}{\partial T} - T\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial T} + T\frac{\partial\bar{\lambda}^{\omega B}}{\partial T} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon\varepsilon}}{\partial\bar{\mu}} - \frac{1}{T}\frac{\partial\bar{\lambda}^{\omega\omega}}{\partial\bar{\mu}}
\end{aligned} \tag{50}
$$

the coefficients with double linear dependence on the electromagnetic field are

$$
\begin{aligned}
\mathcal{X}_9 &= -\bar{\lambda}^{EE} - \lambda^{EE} - 2\bar{\lambda}^{BB} - 3\tilde{\lambda}^{BB} + 2\lambda^{EB} - T\frac{\partial\bar{\lambda}^{EE}}{\partial T} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial\bar{\mu}} + \frac{1}{T}\frac{\partial\tilde{\lambda}^{\omega B}}{\partial\bar{\mu}} \\
\mathcal{X}_{10} &= \bar{\lambda}^{BB} + \tilde{\lambda}^{BB} - \lambda^{BB} - T\frac{\partial\bar{\lambda}^{BB}}{\partial T} - T\frac{\partial\tilde{\lambda}^{BB}}{\partial T} \\
\mathcal{X}_{11} &= 2\bar{\lambda}^{EE} + 2\bar{\lambda}^{BB} + \tilde{\lambda}^{EE} + 3\tilde{\lambda}^{BB} - 2\lambda^{EB} - T\frac{\partial\tilde{\lambda}^{BB}}{\partial T} - \frac{2}{T}\frac{\partial\tilde{\lambda}^{\omega B}}{\partial\bar{\mu}} \\
\mathcal{X}_{12} &= 2\bar{\lambda}^{EE} + 2\bar{\lambda}^{BB} + \tilde{\lambda}^{EE} + 3\tilde{\lambda}^{BB} - 2\lambda^{EB} - \frac{1}{T}\frac{\partial\bar{\lambda}^{\varepsilon E}}{\partial\bar{\mu}} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\varepsilon E}}{\partial\bar{\mu}} - \frac{1}{T}\frac{\partial\tilde{\lambda}^{\omega B}}{\partial\bar{\mu}} - T\frac{\partial\tilde{\lambda}^{EE}}{\partial T} \\
\mathcal{X}_{13} &= 2\lambda^{EB} - 2\bar{\lambda}^{EE} - 2\bar{\lambda}^{BB} - 3\tilde{\lambda}^{EE} - \tilde{\lambda}^{BB} + T\frac{\partial\bar{\lambda}^{BB}}{\partial T} - \frac{1}{T}\frac{\partial\bar{\lambda}^{\omega B}}{\partial\bar{\mu}}
\end{aligned} \tag{51}
$$

and the coefficients with triple linear dependence on the electromagnetic field are given by

$$
\mathcal{X}_{14} = -\frac{1}{T}\left(\frac{\partial\bar{\lambda}^{EE}}{\partial\bar{\mu}} + \frac{\partial\tilde{\lambda}^{EE}}{\partial\bar{\mu}}\right), \quad \mathcal{X}_{15} = -\frac{1}{T}\frac{\partial\bar{\lambda}^{BB}}{\partial\bar{\mu}}, \quad \mathcal{X}_{16} = -\frac{1}{T}\frac{\partial\tilde{\lambda}^{BB}}{\partial\bar{\mu}} E\cdot BB^{\nu} \tag{52}
$$

It should be noted that in order to arrive at the final result above (48), we used the following identities

$$
\begin{aligned}
\varepsilon\cdot\omega B^{\nu} &= \varepsilon\cdot B\omega^{\nu} + \varepsilon^2 E^{\nu} - \varepsilon\cdot E\varepsilon^{\nu}, \\
\omega\cdot EB^{\nu} &= E\cdot B\omega^{\nu} + \varepsilon\cdot EE^{\nu} - E^2\varepsilon^{\nu}, \\
\varepsilon\cdot BB^{\nu} &= B^2\varepsilon^{\nu} - \omega\cdot BE^{\nu} + E\cdot B\omega^{\nu}, \\
\varepsilon\cdot E\varepsilon^{\nu} &= \varepsilon\cdot B\omega^{\nu} + \varepsilon^2 E^{\nu} + \omega^2 E^{\nu} - \omega\cdot B\varepsilon^{\nu} - \omega\cdot E\omega^{\nu}
\end{aligned} \tag{53}
$$

which can be derived directly from the constraint (20). With these identities, we express the final result as the linear combination of independent terms. The source contribution from the coupling between the electromagnetic field and the charge current is given by

$$
\begin{aligned}
F^{\nu\mu} j^{(2)}_{s\mu} &= -\xi^{\varepsilon\omega}(\omega\cdot B)\varepsilon^{\nu} + \xi^{\varepsilon\omega}(\varepsilon\cdot B)\omega^{\nu} + \xi^{\varepsilon\varepsilon}\varepsilon^2 E^{\nu} + \xi^{\omega\omega}\omega^2 E^{\nu} \\
&\quad + \xi^{\omega E}(E\cdot B)\omega^{\nu} - \xi^{\varepsilon E}(\varepsilon\cdot E)E^{\nu} + (\xi^{\omega B} - \xi^{\omega E})(\omega\cdot B)E^{\nu} \\
&\quad + \xi^{EE}E^2 E^{\nu} + (\xi^{BB} - \xi^{EB})B^2 E^{\nu} + \xi^{EB}(E\cdot B)B^{\nu}
\end{aligned} \tag{54}
$$

Then, from the conservation law $\partial_{\mu} T^{(2)\mu\nu} = F^{\nu\mu} j^{(2)}_{s\mu}$, we obtain the equations that could determine or constrain these coefficients. It is convenient to decompose these equations into three groups: Group I includes the coefficients for the pure ε^μ and ω^μ term in the energy-momentum tensor,

$$\mathcal{X}_1 = 0, \quad \mathcal{X}_2 = 0, \quad \mathcal{X}_3 = 0 \tag{55}$$

Group II contains the mixed terms between the electromagnetic field and vorticity field in the energy-momentum tensor

$$\mathcal{X}_4 = -\zeta^{\varepsilon\omega}, \quad \mathcal{X}_5 = \zeta^{\varepsilon\omega}, \quad \mathcal{X}_6 = 0,, \quad \mathcal{X}_7 = \zeta^{\varepsilon\varepsilon}, \quad \mathcal{X}_8 = \zeta^{\omega\omega} \tag{56}$$

and Group III involves the pure electromagnetic terms in the energy-momentum tensor,

$$\begin{aligned} \mathcal{X}_9 &= 0, \ \ \mathcal{X}_{10} = 0, \ \ \mathcal{X}_{11} = \zeta^{\omega E}, \ \ \mathcal{X}_{12} = -\zeta^{\varepsilon E}, \ \ \mathcal{X}_{13} = \zeta^{\omega B} - \zeta^{\omega E}, \\ \mathcal{X}_{14} &= \zeta^{EE}, \ \ \mathcal{X}_{15} = \zeta^{BB} - \zeta^{EB}, \ \ \mathcal{X}_{16} = \zeta^{EB} \end{aligned} \tag{57}$$

We note that if we know the coefficients in the energy-momentum tensor, we can directly obtain the coefficients in the charge current from Group II or Group III. At second order for the chiral fermions, the energy-momentum tensor would include trace anomaly, which can lead to extra constraint identities, referred to as Group IV

$$0 = \lambda^{\varepsilon\varepsilon} + 3\bar{\lambda}^{\varepsilon\varepsilon} + \tilde{\lambda}^{\varepsilon\varepsilon}, \tag{58}$$
$$0 = \lambda^{\omega\omega} + 3\bar{\lambda}^{\omega\omega} + \tilde{\lambda}^{\omega\omega}, \tag{59}$$
$$0 = \lambda^{\varepsilon E} + 3\bar{\lambda}^{\varepsilon E} + 2\tilde{\lambda}^{\varepsilon E}, \tag{60}$$
$$0 = \lambda^{\omega B} + 3\bar{\lambda}^{\omega B} + 2\tilde{\lambda}^{\omega B}, \tag{61}$$
$$\tilde{C} = \lambda^{EE} + 3\bar{\lambda}^{EE} + \tilde{\lambda}^{EE}, \tag{62}$$
$$-\tilde{C} = \lambda^{BB} + 3\bar{\lambda}^{BB} + \tilde{\lambda}^{BB} \tag{63}$$

From Group I together with the first two equations in Group IV, we note that only three coefficients are independent. From the naive dimension analysis, we know that these coefficients in Group I must take the form of T^2. Choosing $\lambda^{\varepsilon\varepsilon}$, $\lambda^{\omega\omega}$ and $\bar{\lambda}^{\omega\omega}$ as independent variables, we obtain

$$\tilde{\lambda}_s^{\varepsilon\varepsilon} = 0, \tag{64}$$
$$\bar{\lambda}_s^{\varepsilon\varepsilon} = -\frac{1}{3}\lambda_s^{\varepsilon\varepsilon}, \tag{65}$$
$$\tilde{\lambda}_s^{\omega\omega} = -\lambda_s^{\omega\omega} - 3\bar{\lambda}_s^{\omega\omega}, \tag{66}$$
$$\lambda_s^{\varepsilon\omega} = -\frac{1}{3}\lambda_s^{\varepsilon\varepsilon} + \frac{1}{2}\lambda_s^{\omega\omega} + \frac{5}{2}\bar{\lambda}_s^{\omega\omega} \tag{67}$$

Once these coefficients are known, from Group II and Group IV, together with the naive dimension analysis $\lambda^{\varepsilon E}, \bar{\lambda}^{\varepsilon E}, \tilde{\lambda}^{\varepsilon E}, \bar{\lambda}^{\omega B}, \tilde{\lambda}^{\omega B}, \lambda^{\omega E} \propto T$, we find that $\zeta^{\varepsilon\varepsilon}$,$\zeta^{\omega\omega}$ and $\zeta^{\varepsilon\omega}$ in $j^{(2)\mu}$ satisfy the following constraint

$$\zeta_s^{\varepsilon\varepsilon} - \zeta_s^{\varepsilon\omega} - \zeta_s^{\omega\omega} = -\frac{1}{T}\frac{\partial\bar{\lambda}_s^{\varepsilon\varepsilon}}{\partial\bar{\mu}_s} + \frac{1}{T}\frac{\partial\bar{\lambda}_s^{\omega\omega}}{\partial\bar{\mu}_s} + \frac{1}{T}\frac{\partial\tilde{\lambda}_s^{\omega\omega}}{\partial\bar{\mu}_s}, \tag{68}$$

which indicates that only two of $\zeta^{\varepsilon\varepsilon}$,$\zeta^{\omega\omega}$ and $\zeta^{\varepsilon\omega}$ are independent. Still, from Group II with known $\zeta^{\varepsilon\varepsilon}$, we have

$$\bar{\lambda}_s^{\varepsilon E} = \frac{1}{2}\left(\zeta_s^{\varepsilon\varepsilon} + \frac{1}{T}\frac{\partial\bar{\lambda}_s^{\varepsilon\varepsilon}}{\partial\bar{\mu}_s}\right), \tag{69}$$

which further leads to

$$\bar{\lambda}_s^{\omega B} = \frac{1}{2}\left(\zeta_s^{\varepsilon\omega} - 2\bar{\lambda}_s^{\varepsilon E}\right), \tag{70}$$

Among the other transport coefficients for the mixed terms in the energy-momentum tensor, we find that only one transport coefficient is independent. We choose $\lambda_s^{\omega B}$ as the independent one and from Group II and the middle two equations in the trace constraint equations, we express other coefficients as the following

$$\tilde{\lambda}_s^{\omega B} = -\frac{1}{2}\left(\lambda_s^{\omega B} + 3\bar{\lambda}_s^{\omega B}\right), \tag{71}$$

$$\bar{\lambda}_s^{\varepsilon E} = -\frac{1}{2}\left(\lambda_s^{\omega B} + \bar{\lambda}_s^{\omega B}\right), \tag{72}$$

$$\lambda_s^{\varepsilon E} = -\left(3\bar{\lambda}_s^{\varepsilon E} + 2\tilde{\lambda}_s^{\varepsilon E}\right), \tag{73}$$

$$\lambda_s^{\omega E} = \frac{1}{2}\left(2\bar{\lambda}_s^{\omega B} + 4\tilde{\lambda}_s^{\omega B} - \frac{1}{T}\frac{\partial\tilde{\lambda}_s^{\omega\omega}}{\partial\bar{\mu}_s}\right) \tag{74}$$

From the last equations in Group III, it is straightforward to obtain

$$\tilde{\lambda}_s^{BB} = -\int T\xi_s^{EB} d\bar{\mu}_s, \tag{75}$$

$$\bar{\lambda}_s^{BB} = -\int T\left(\xi_s^{BB} - \xi_s^{EB}\right) d\bar{\mu}_s, \tag{76}$$

where $\int T\xi^{XX} d\bar{\mu}$ denotes the undetermined integral and possibly includes arbitrary functions with temperature dependence. Then, from Group III together with the trace anomaly in Group IV, the other coefficients can be totally determined by

$$\lambda_s^{BB} = -\tilde{C} - 3\bar{\lambda}_s^{BB} - \tilde{\lambda}_s^{BB}, \tag{77}$$

$$\tilde{\lambda}_s^{EE} = -\frac{1}{2}\left(\xi_s^{\omega B} - 2\tilde{\lambda}_s^{BB} + \frac{1}{T}\frac{\partial\bar{\lambda}_s^{\omega B}}{\partial\bar{\mu}_s} + \frac{2}{T}\frac{\partial\tilde{\lambda}_s^{\omega B}}{\partial\bar{\mu}_s}\right), \tag{78}$$

$$\bar{\lambda}_s^{EE} = -\tilde{\lambda}_s^{EE} - \int T\xi_s^{EE} d\bar{\mu}_s, \tag{79}$$

$$\lambda_s^{EE} = \tilde{C} - 3\bar{\lambda}_s^{EE} - \tilde{\lambda}_s^{EE}, \tag{80}$$

$$\lambda_s^{EB} = \tfrac{1}{4}\left(\xi_s^{\omega B} - 2\xi_s^{\omega E} + 4\bar{\lambda}^{EE} + 4\tilde{\lambda}^{EE} + 4\bar{\lambda}_s^{BB} + 4\tilde{\lambda}_s^{BB} - 2T\tfrac{\partial\bar{\lambda}_s^{BB}}{\partial T} + \tfrac{1}{T}\tfrac{\partial\bar{\lambda}_s^{\omega B}}{\partial\bar{\mu}_s} - \tfrac{2}{T}\tfrac{\partial\tilde{\lambda}_s^{\omega B}}{\partial\bar{\mu}_s}\right) \tag{81}$$

Three independent equations are not used and remain as the constraint conditions:

$$0 = \bar{\lambda}_s^{EE} + \lambda_s^{EE} + 2\tilde{\lambda}_s^{BB} + 3\bar{\lambda}_s^{BB} - 2\lambda_s^{EB} + T\frac{\partial\bar{\lambda}_s^{EE}}{\partial T} + \frac{1}{T}\frac{\partial\tilde{\lambda}_s^{\varepsilon E}}{\partial\bar{\mu}_s} - \frac{1}{T}\frac{\partial\bar{\lambda}_s^{\omega B}}{\partial\bar{\mu}_s}, \tag{82}$$

$$0 = \bar{\lambda}_s^{BB} + \tilde{\lambda}_s^{BB} - \lambda_s^{BB} - T\frac{\partial\bar{\lambda}_s^{BB}}{\partial T} - T\frac{\partial\tilde{\lambda}_s^{BB}}{\partial T}, \tag{83}$$

$$\xi_s^{\varepsilon E} = -2\bar{\lambda}_s^{EE} - 2\bar{\lambda}_s^{BB} - \tilde{\lambda}_s^{EE} - 3\tilde{\lambda}_s^{BB} + 2\lambda_s^{EB} + T\tfrac{\partial\bar{\lambda}_s^{EE}}{\partial T} + \tfrac{1}{T}\tfrac{\partial\bar{\lambda}_s^{\varepsilon E}}{\partial\bar{\mu}_s} + \tfrac{1}{T}\tfrac{\partial\tilde{\lambda}_s^{\varepsilon E}}{\partial\bar{\mu}_s} + \tfrac{1}{T}\tfrac{\partial\bar{\lambda}_s^{\omega B}}{\partial\bar{\mu}_s}. \tag{84}$$

It should be noted that we have eliminated the partial derivative on temperature from the naive dimension analysis for the pure ε, ω terms and mixed terms between ε, ω and E, B, whereas we kept the partial derivative for the pure E, B terms in Equations (80)–(83). This is because the pure E, B terms in the energy-momentum tensor may include another regularization scale due to ultraviolet divergence and the naive dimension analysis would be broken, whereas there is no such complexity for the pure ε, ω terms and mixed terms. This point was demonstrated by the direct calculation given in [37]. We checked that all these second-order results were totally consistent with the results obtained from other approaches [37,48,49].

4. Summary

When a system is in global equilibrium under an electromagnetic field, only constant vorticity tensor is allowed when there is no gravity field involved. The electromagnetic and vorticity field must fulfill some constraint conditions. It turns out that these constraint conditions can be applied to determine the non-dissipative anomalous coefficients together with the energy-momentum conservation, chiral anomaly and trace anomaly.

At zeroth order, we find that the energy-momentum conservation and charge conservation hold automatically and trace vanishing leads to the well-known relation between the energy density and pressure. At first order, from the chiral anomaly and energy-momentum conservation, all the coefficients can be totally determined up to some integral constants, which is what the hydrodynamic method achieved from the second law of thermodynamics. The trace of the energy-momentum tensor always vanishes at first order. At second order, we find that the charge conservation holds automatically and we cannot say anything about the transport coefficients relevant to the charge current. However, we can relate these transport coefficients in the charge current to the ones in the energy-momentum tensor by using the energy-momentum conservation law and find that once we obtain the coefficients in the energy-momentum tensor, the coefficients in the charge current can be derived directly. We find that among the coefficients relevant to the pure vorticity tensor, in the energy-momentum tensor there are only three coefficients which are independent and the other four coefficients can be expressed as the linear combination of these three coefficients. We present the formulas which express the coefficients in the mixed terms from the electromagnetic and vorticity field as well as the ones associated with the pure vorticity terms in the energy-momentum tensor and charge current. Furthermore, we determine the coefficients relevant to the pure electromagnetic field in the energy-momentum tensor from the charge current associated with the electromagnetic field and the energy-momentum tensor associated with the vorticity field. All these results do not depend on any specific interactions and are very general. They are intended to be very helpful in determining the second-order anomalous transport coefficients in various chiral systems.

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